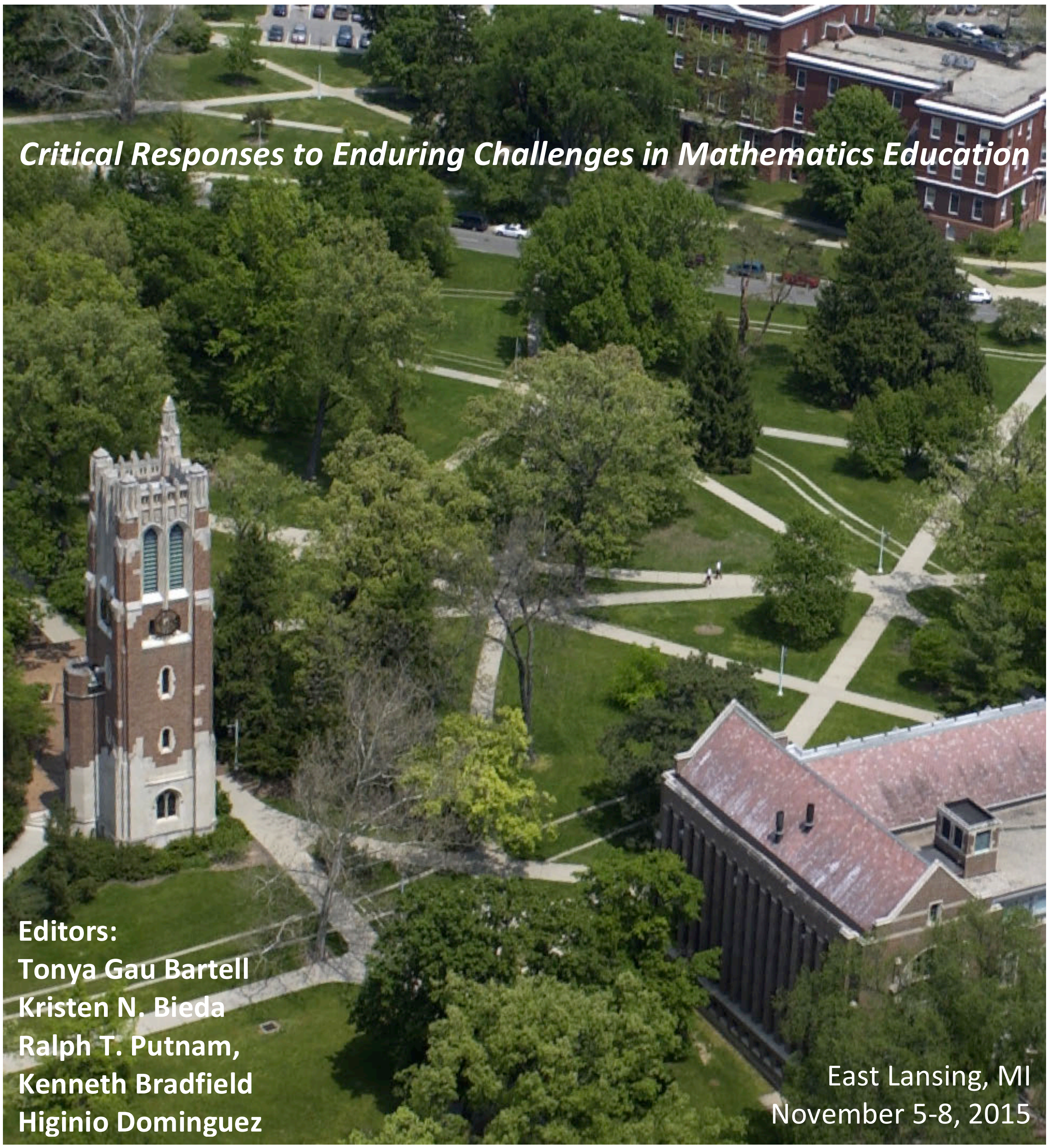
PMENA-37

Proceedings
Thirty-Seventh Annual Meeting

of the North American Chapter of the International Group for the Psychology of Mathematics Education



Critical Responses to Enduring Challenges in Mathematics Education

Editors:
Tonya Gau Bartell
Kristen N. Bieda
Ralph T. Putnam,
Kenneth Bradfield
Higinio Dominguez

East Lansing, MI
November 5-8, 2015

MICHIGAN STATE UNIVERSITY

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PME-NA History and Goals

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of the International Group of Psychology of Mathematics Education. The first PME-NA conference was held in Evanston, Illinois in 1979.

The major goals of the International Group and the North American Chapter are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Membership is open to people involved in active research consistent with PME-NA's aims or professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and click on "Membership" at the left of the screen.

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We also acknowledge and are grateful for the volunteer support provided by many others during the conference including faculty and graduate students from Michigan State University, CREATE for STEM and the Connected Mathematics Project

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Preface

On behalf of the 2015 PME-NA Steering Committee, the 2015 PME-NA Local Organizing Committee, and Michigan State University, we welcome you to the 37th Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter held on the campus of Michigan State University in East Lansing, Michigan. The year 2016 marks 30 years since the conference was hosted by Michigan State University, and we hope it is the best PME-NA yet!

The theme of this year's conference is Critical Responses to Enduring Challenges in Mathematics Education. The theme of this year's conference invites mathematics education scholars to reflect upon and critically respond to enduring challenges in teaching and learning mathematics for all students. To ignite discussion within the field, we organized the conference around four focal enduring challenges, each of which is featured in one of four plenary talks: teaching as responsive to various conceptions of mathematics (Plenary #1, Dr. Jerry Lipka, Dr. Melfried Olson, Dora Andrew-Ihrke, Evelyn Yanez); addressing the needs of marginalized populations in school mathematics (Plenary #2, Dr. Robert Q. Berry III); (3) the impact of teacher evaluation and high-stakes assessment in teaching (Plenary Panel: Craig Huhn, Lisa Jilk, Marty Schnepp, Marie Smerigan); the role of assessment in teaching and learning (Plenary #4: Dr. Malcolm Swan). Our hope is that the conference will catalyze collective reflection, collaborative inquiry, and discussion about various means for responding to and addressing these, and other, challenges.

For this year's conference, we received 528 submissions. The overall acceptance rate was 64%. We are pleased to offer 92 Research Report sessions, 96 sessions, 136 Posters, and 12 Working Groups in our conference program.

We wish to acknowledge and thank the many people who generously volunteered their time in preparation for this conference to ensure a high-quality program, including proposal authors, reviewers, Strand Leaders, the PME-NA Steering Committee, and the 2015 PME-NA Local Organizing Committee. In particular, we want to highlight the extremely generous financial and personnel support provided by the MSU CREATE for STEM Institute. We are deeply grateful for the hard work and support of both Dr. Robert Geier and Sue Carpenter at the Institute. We also wish to thank the generous financial support of the MSU Program in Mathematics Education (PrIME), the MSU College of Education, and the MSU Department of Teacher Education.

Finally, we wish to thank this year's attendees for their hard work, passion and enthusiasm for addressing enduring challenges in mathematics education. You have inspired us to do our best to provide a venue for exchanging ideas and promoting ongoing reform in mathematics education. We hope you will enjoy the conference! Go Green!

Tonya Bartell & Kristen Bieda
Conference Co-Chairs

Contents

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Chapter 1

Plenary Papers

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

INDIGENOUS KNOWLEDGE PROVIDES AN ELEGANT WAY TO TEACH THE FOUNDATIONS OF MATHEMATICS

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This unlikely cast of characters, by working collaboratively in a trusting learning community, was able to identify an approach to teaching rational numbers through measuring from the everyday practices of Yup'ik Eskimo and other elders. "The beginning of everything," as named by a Yup'ik elder, provided deep insights into how practical activities were conceptualized and accomplished by means of body proportional measuring and nonnumeric comparisons. These concepts and practices shed light on the importance of measuring as comparing and the importance of relative units of measure, and helped us imagine a way to establish an alternative learning trajectory and school-based curriculum that begins with the insights gained from Yup'ik and other elders. This approach may well provide teachers a way to teach aspects of elementary school mathematics in an integrative and elegant way.

Keywords: Measurement, Number Concepts and Operations, Curriculum

Introduction

More than 30 years ago, Evelyn Yanez, Dora Andrew-Ihrke, and Jerry Lipka almost blindly began a journey to map the landscape of Yup'ik elders' knowledge, practices, and conceptions of mathematics. Evelyn and Dora were students in the Cross-Cultural Education Development Program (X-CED), University of Alaska Fairbanks, in Dillingham, Alaska, and Jerry was a field faculty member. Little did we know that insights gained from our long-term collaboration with Yup'ik elders and our more recent work with Indigenous Knowledge (IK) holders from the Yap Outer Islands, Greenland, Kamchatka, and Norway would result in a way to teach aspects of the foundation of elementary school mathematics. Nor did we realize that this would become our life's work.

From 1995 to approximately 2010, we worked with a dedicated group of Yup'ik elders and teachers. The most significant outcome of this phase of our collaborative work was the publication of the *Math in a Cultural Context* series—ten elementary school culturally based mathematics modules and ten accompanying storybooks. During this time, we also developed a culturally based and reform-oriented pedagogical approach (Lipka, with Mohatt & the Ciulistet, 1998), which incorporated aspects of Sternberg's theory of intelligence (Sternberg, Lipka, Newman, Wildfeuer, & Grigorenko, 2006). We shared our curricular and pedagogical approach in many teacher training institutes and workshops. This was our response to identifying conceptions and contexts of mathematical behavior and thinking based on Indigenous Knowledge and applying them to a classroom context. Quantitative and qualitative journal articles documented both teachers' pedagogical practices while implementing MCC curriculum and outcome studies on the efficacy of the curriculum (Kisker, Lipka, Adams, Andrew-Ihrke, Yanez, & Millard, 2012; Lipka, Hogan, Webster, Yanez, Adams, Clark, & Lacy, 2005).

Yet, we were not satisfied, as we were beginning to perceive a more cohesive approach to teaching elementary school mathematics based on Indigenous Knowledge (Lipka, Andrew-Ihrke, & Yanez 2011; Lipka, Wong, & Andrew-Ihrke, 2013). Could we identify and understand the embedded and encoded mathematics in everyday Yup'ik activities in a systematic way? We wondered what it is that enables Yup'ik and other elders to accomplish an array of everyday tasks without overburdening memory. Yup'ik people are generalists; individuals have needed to make clothing, build houses, fish racks, smokehouses, and kayaks, and orient themselves in various weather conditions when traveling on the tundra, in forests, and on rivers and open seas. Were there concepts that cohesively supported the performance of these tasks?

Our latest grants have enabled us to continue this exploration into mathematical concepts embedded in the everyday activities of Indigenous Knowledge holders among the Yup'ik and other cultural groups across the Arctic as well as an "outlier" group from Yap State in the Federated States of Micronesia. This work has allowed us to imagine what an elementary school mathematics curriculum would look like if developed from key aspects of Indigenous Knowledge. We began this phase of our research aware of the importance of symmetry and measuring in the everyday activities of Yup'ik elders (Lipka, Andrew-Ihrke, & Yanez, 2011). Measuring is the major conception of "mathematics" that elders have identified and use in their everyday activities (Lipka, with Mohatt & the Ciulistet, 1998; Lipka, Wong, & Andrew-Ihrke, 2013).This paper reports on what we have learned to date.

From the outset of our work, because numbers and operations represent such an overwhelming part of the curriculum, we were initially unable to see how a cultural group such as Yup'ik, which does not prize numbers in its daily activities, could provide us with key insights on how to teach the foundations of school mathematics. Yet, the information that follows is what we found, and it is what was right in front of us for years. Even though there are fundamental epistemological differences between Yup'ik practical activity and school mathematics, the embedded mathematical principles in everyday activity can generalize to the teaching of school mathematics. Because their measuring approach occurs in a nonnumeric environment, the embedded mathematical principles provide a generalized model for teaching aspects of rational number reasoning and other aspects of mathematics. In fact, rational number learning (fractions, ratios, and scaling) has been identified as a difficult topic for most U.S. students (Confrey, Maloney, Wilson, & Nguyen, 2010; Wu, 2011; Lamon, 1999), and rational number reasoning is considered a key concept in students' mathematical education (National Mathematics Advisory Panel, 2008).

This article describes how measuring can be perceived as a central and integrative concept across a wide range of everyday activities conducted by Yup'ik and other Indigenous Peoples. We will identify and describe a few cultural activities that highlight underlying generative cultural and mathematical principles. We briefly describe these principles and will argue that the principles embedded in activity provide an alternative pathway to teaching the foundations of mathematics. The key curricular and teaching examples demonstrated in this paper from Indigenous Knowledge connect measuring with the elders' halving algorithm and demonstrate how this can be an exemplar to teach place value in base 2.

Brief Methodological Considerations

Math in a Cultural Context (MCC) is a long-term project at the University of Alaska Fairbanks; federally funded grants have supported this work. A cohort of Yup'ik students enrolled in the X-CED Program and coauthor Jerry Lipka began working with elders in the late 1980s as a first step in understanding and connecting their everyday knowledge to elementary school teaching. This unlikely long-term collaboration (well documented in *Transforming the Culture of Schools: Yup'ik Eskimo Examples)* occurred among Yup'ik elders, Yup'ik teachers and now-retired Yup'ik teachers, and academics (mathematicians, mathematics educators, linguists, cultural anthropologists, and educators) to make teaching elementary school math more cohesive, accessible, and relevant. Critical to our long-term work has been the establishment of trust, respect, and continuity, as we have been working with some elders for over twenty years and, in the process, have become "elders" ourselves. Evelyn Yanez, a coauthor and retired Yup'ik teacher and current MCC faculty, stated that "elders trusted us enough to give us their stories and knowledge so that it may go into the future like an arrow. They knew we would prepare books for future generations; they were excited about sharing their work" (personal communication, June, 2015). The meetings were important to the elders as the following anecdote describes. Lily Gamechuk, an elder from Manokotak, Alaska, and now deceased, came to a meeting in Fairbanks some years ago. She needed to go to the hospital but refused to go until she told her story to the gathered group. Elders would often state to us how MCC was one of the few programs in which their knowledge counted.

Over these many years, insiders and outsiders have met three or four times a year for three to five days per meeting, when possible through grant funding. These meetings included storytelling, describing and performing everyday tasks such as making clothing and patterns, and discussing and simulating star navigation. Less often, we would practice these skills and knowledge *in situ*. However, more remarkable, our elder meetings at times have become Socratic in character as the discursive activities of the entire group demonstrate and refine the growing understanding of connections between Yup'ik cosmology, epistemology, and practice as a system, and as we relate this Indigenous Knowledge System to school mathematics. Koester (2014) describes the process as "getting to 'mathematical foundations' from oral accounts of activities, a discursive process that invokes metaphor, symbols, and various forms of discursive displacement."

Jerry Lipka, Principal Investigator of MCC and first author of this paper, cannot stress enough the absolute importance of the long-term collaborative inquiry, which has allowed exploration of topics in depth and in new ways and has allowed opportunity for serendipitous events to occur, ones from which completely new lines of inquiry developed. For example, a single geometrical construction performed by Dora Andrew-Ihrke, which she was not going to share because she "thought everyone knew this," led to a generalized way to construct many different planar shapes as well as three-dimensional shapes. (It is beyond the scope of this paper to explore in depth the methodological considerations or the geometrical constructions.)

The Beginning of Everything: Connecting Everyday Activity and Mathematical Reasoning

In one of our first meetings with Yup'ik elders many years ago, we asked Lily Gamechuk, a respected elder from Manokotak, Alaska, to share with the group how she made clothing. Lily asked one of the Yup'ik teachers-in-training in the room to stand. Without any instruments, and without touching her, Lily "measured" her. The measuring took place in her mind's eye. Minutes later, Lily had made a complete outfit out of butcher paper including a dress, belt, hat, and boots. The only instrument she used was a pair of scissors. How did she do this? How did she measure? She never told us directly, as it was more important from her point of view that we learn this skill in our way. What mental operations did she employ to transfer her visual perception to the practice of cutting and sewing proportionally? Little did we know that understanding how Yup'ik elders performed such everyday math would transform our own thinking.

In a recent elders' meeting with the Yup'ik cohort, Raphael Jimmy, an elder of approximately 90 years from Mountain Village, Alaska, slowly raised his hands above the table lifting them at eye level so that we could all see that he had crossed his left and right index fingers, forming right-angled axes referred to here as a plus sign: "+" (Figure 1). Simultaneously, he stated, "What was once hidden is now revealed" (personal communication, November, 2013).

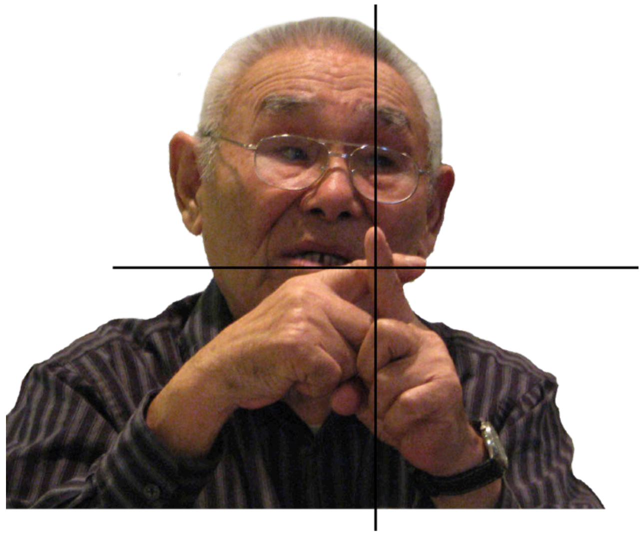


Figure 1: The beginning of everything

Mr. Jimmy explained that his crossed fingers represent "the beginning of everything." In a practical sense, this meant that the embodied abstraction was the starting point for many, if not most, practical activities. Even a length or a line segment has an implied center; folding material in half establishes a line of symmetry and two equal parts. Once Mr. Jimmy shared this concept, other members in our group realized that they too use this concept, heretofore unnamed, in their everyday constructions. We slowly realized that the concept reflected a culturally preferred way of perceiving, thinking, and performing across a wide variety of activities. Other members in our research group had also described this process, but until that moment, the process had gone unnamed. We have begun to observe the importance of "the beginning of everything" in other cultural groups with whom we are working, most notably the Yap Outer Islanders from the Federated States of Micronesia.

The cultural and mathematical activity-generating concepts signaled by Mr. Jimmy's action of crossing his fingers include the Yup'ik concepts of *qukaq* [center] and *ayagneq* [a place to begin], *avek* [halving], *tapluku* [to fold it, partitioning material], and *ayuquk* [testing and verifying congruence and symmetry or identifying equality]. This cluster of concepts and actions relate space, locating, and measuring space by invoking a line of symmetry that emerges through the action of folding material in half, or as a mental image in which the whole contains its parts, and a process of verifying equalities (is this side equal to the other side?). These words and actions establish a center and a place to begin many different projects performed by Yup'ik elders. The following few examples from Yup'ik cultural activity will illustrate how ubiquitous this concept is.

Although Dora Andrew-Ihrke, a long-term Yup'ik colleague and coauthor of this paper, did not have a name for "+," she uses the concept in her geometric constructions, such as for making a square out of irregular uneven material and in numerous other projects. Through body proportional measuring and folding, she establishes the center of the material. The "+" becomes the inner structure for a square, as shown in Figure 2. (The drawn lines are for demonstration purposes.)

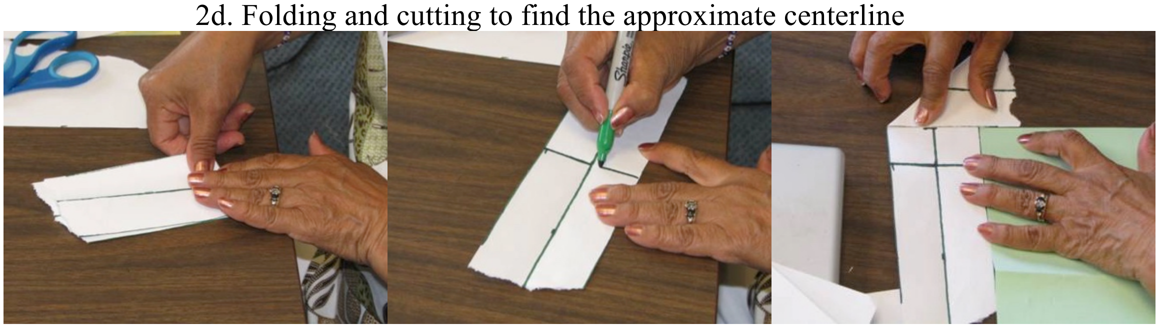
The square and its center are co-constructed, as Dora refers to the initial center point as "fake," meaning approximate. The square and its center become "real" when she verifies the congruence of each fold (vertical, horizontal, and diagonals).

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Dora uses these same processes and the concept of the orthogonal center to create patterns. She folds a square in half from top to bottom and side to side twice. This produces smaller squares, all the while verifying that she has maintained the orthogonal center. Figure 3 shows an example of this

2a. Irregular material 2b. One fold creates an edge; 2c. Measuring from the edge

2d. Folding and cutting to find the approximate centerline



2e. Perpendicular fold 2f. The approximate center of the square is established and Dora continues to construct the square

Figure 2: Geometric constructions using the concept of "+"

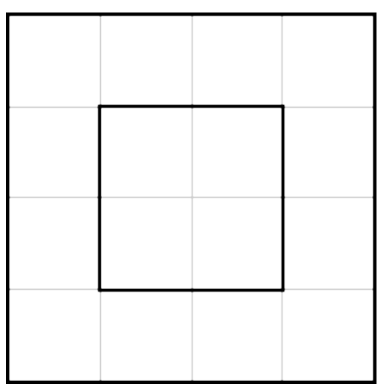


Figure 3: An example of geometrical similarity and scaling

similarity and scaling. Scaling is invoked through many everyday activities within the Yup'ik culture and is part of our school-based program.

The square is Dora's central geometrical shape, which she transforms through symmetrical folding into a circle or other planar shapes (Lipka, Wong, & Andrew-Ihrke, 2013). Aspects of the Yup'ik orientation system guide the transformation of a square into a circle by folding along lines of symmetry oriented by the winds, then the in-between winds, and once more between-those-winds until there are 16 points on the square (Figure 4).

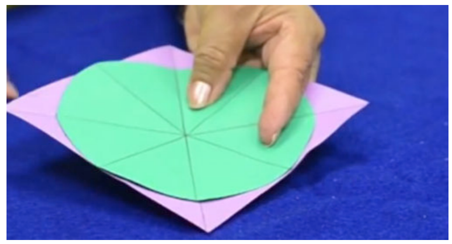


Figure 4:

When the square has been transformed into a many-sided polygon, Dora cuts a circle out of the square; the circle is inscribed in the square as shown in Figure 4 (Lipka, Andrew-Ihrke, & Yanez, 2011). All of these actions occur through the axial center.

Mrs. Nanalook, a respected elder from Manokotak, Alaska, has been a working member of the Yup'ik elders' cohort for a long time. Recently, she showed us step-by-step how she begins to weave a grass basket (Figure 5). A detailed ethnographic description is beyond the scope of this paper, but by referring to Mrs. Nanalook's key movements around the center point (see photographs in Figure 6), you can imagine how she begins weaving a grass basket.

Mrs. Nanalook takes a blade of grass, and if it is wide, she cuts it in half along its line of symmetry. As shown in Figure 6, she takes the two parts of the blade of grass and forms a cross, positioned in reference to the basket maker. Her actions of going up [*quletmun*] and going down [*acitmun*] along the vertical axes are coordinated by her actions of going left-to-right or right-to-left on the horizontal axes [*canitmun*]. She further orients the grass's motion in reference to her body as the grass is going inside toward her body [*ilutmu*n] and is going away from her body [*elatmun*]. The upward and downward and sideway folds form the loop at the center of the basket [*qukaq*]. The process is repeated along the horizontal plane, as these motions going inside/inward, and, going outward form another loop. The process is repeated six times until the beginning of the grass basket has been constructed.

Motions around the center, coupled with specialized demonstratives that aid orientation and location in reference to the speaker/weaver, reveal aspects of this generalized system. The Yup'ik language is exceedingly rich in demonstrative words that describe locating and orienting differentiating space (see Jacobson, 1984, pp. 653–662). Underlying this system is the central, bodily-situated orthogonal axis that functions as a cultural code of practical action and supports a wide range of activities from locating one's own self on the tundra or in the bays and surrounding waters, to orienting patterns and weaving the structure of a grass basket, to assisting a seamstress in measuring, cutting, and sewing clothing. In fact, even the Yup'ik counting system—base 20 and sub-base 5—uses movement across the spatially oriented four sectors of the body in relation to the axial center.

Figure 5: Yup'ik coiled grass basket

6a. Establishing the center 6b. Going downward 6c. Going horizontally

6d. Establishing the center of the basket

Figure 6: Mrs. Nanalook demonstrates basket weaving

The following examples are from both the Yapese and the Yup'ik context. These examples were chosen because they reveal how actions around the center (line of symmetry) contain both practical knowledge, used in making everyday products, and mathematical knowledge, applicable to teaching the foundation of mathematics. The illustration in Figure 7 shows Larry Raigetal, a Lamotrek (Yap Outer Islands) knowledge holder who lives on the main island of Yap, the state capital of fourteen widely dispersed atolls. Larry demonstrates to members of our research/study group how he transforms a coconut leaf into a "ruler." (Typically, a master canoe builder would use a pandanus leaf [personal communication, Cal Hachibmai, August 20, 2015].)

Larry first measures the desired length of the leaf by holding it between his thumb and index finger, stretching it over the top of the rest of his fingers and down the side of his hand until he reaches his wrist (proportional to his body) (Figure 8). He then constructs the tool by folding the measured leaf in half, and then he folds from one edge to the center and folds in half again. He repeats this process on the other side of the leaf and then folds each of the four segments in half again until there are eight segments in total. Master carvers use this measuring tool to build boats, and such tools are used to build traditional houses. We have observed others in the Yap Outer Islands as well as in Chuuk (also part of the Federated States of Micronesia, 1500 kilometers away from Yap, with a

Figure 7:

distinct although related language and culture) use repeated halving to create tools and build canoes and houses, and as a way to fashion loom-woven cloth. For the Yap Outer Islands, this process is well documented (Alkire, 1970).

Figure 8: Measured proportional to the hand

Dora Andrew-Ihrke similarly learned how to fold material as a young girl from her mother, Annie Andrew, a well-known seamstress. Mrs. Andrew demonstrated and explained to Dora that there are two kinds of folds: easy and difficult. Easy folds are the same as those demonstrated by Larry, using either serial or recursive half-folds, producing 2, 4, 8, and so on, number of parts. Difficult folds are odd folds that include 6 equal parts, as 6 equal parts are constructed by multiplying an odd and even number.

When Dora folds a strip of paper into three equal parts, she follows her mother's folding algorithm and principle—always use the simplest fold, the half-fold. The main difficulty we had in understanding this halving algorithm was in moving away from our habituated way of seeing three parts of the whole as thirds, when in fact Dora was consistently expressing the relationship between two parts: "Is this half equal to that half?" (See Figure 9).

Stating Mrs. Andrew's folding algorithm more formally, it is n-1, when n is an odd number. For elementary school classroom purposes, we demonstrate this process using small numbers as a model for the algorithm and managing the number of folds. Thus, with the single-digit prime numbers 3, 5, and 7 using the n-1 algorithm, the number of folds reverts to the simple half-fold. Subtracting 1 from those primes results in 2, 4, and 6 parts each, and these specific examples are achievable through the half-fold. The process is illustrated and described in Figure 9.

The halving algorithm represents movements around the center, as shown in previous examples. This algorithm provides a crucial part of Indigenous Knowledge that transfers to the teaching of

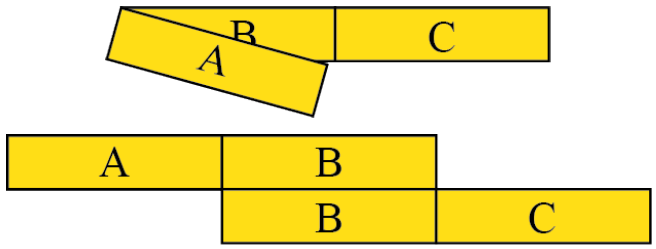


Figure 9:

school mathematics. Examples described below further develop the connection between Indigenous Knowledge and its potential in teaching rational numbers in a school context. Applying the n-1 folding algorithm to create three equal parts, we can either estimate Length A or Length C as the estimated third part. In the diagram shown in Figure 9, we estimated Length C as one-third of the whole. This leaves segment n-1, which is folded in half to create Length A and Length B. As illustrated, the halving algorithm is used twice.

When Lengths A and B were folded on top of each other, Dora treated them as a single entity and again referred to the notion: Are Lengths A and B with A folded on top of B equal to C? When we understood these actions and descriptions from her Yup'ik perspective, the centrality and the power of this halving algorithm (binary operation) became clear from within a Yup'ik cultural context. This realization opened a new and deeper understanding of how Yup'ik practical knowledge relates to fundamental aspects of numbers and relations.

The last cultural example is a variation of the halving algorithm, which both Mrs. Andrew and Dora use in their constructions. We observed other Yup'ik elders using the following algorithm, but clearly, Mrs. Andrew and Dora have refined this process at an expert level. In fact, other Yup'ik people learn from them. To find the difference between two quantities, for example Length A and Length B, Dora aligns them and then folds back the difference, Length C, so that Length A minus Length C equals Length B. (We have demonstrated this process in multiple workshops and institutes by comparing the length of Jerry's foot with that of Dora's. By finding the difference using the method shown in Figure 10, Dora eventually establishes a common unit that can measure both sets of feet.)

The difference between Length A and Length B is C. Dora uses C as the divisor. She now divides (or measures) Length A and Length B by the divisor, C. She folds until there is no remainder. In this simplified example, there is no remainder after two folds. C is now established as the common unit that can be used to measure Lengths A and B. Algebraically, B = 2C and A = 3C. This algorithm was developed in an Indigenous culture, but it is essentially Euclid's famous algorithm (https://en.wikipedia.org/wiki/Euclidean_algorithm), used to find the greatest common divisor of two numbers as well as the greatest common factor.

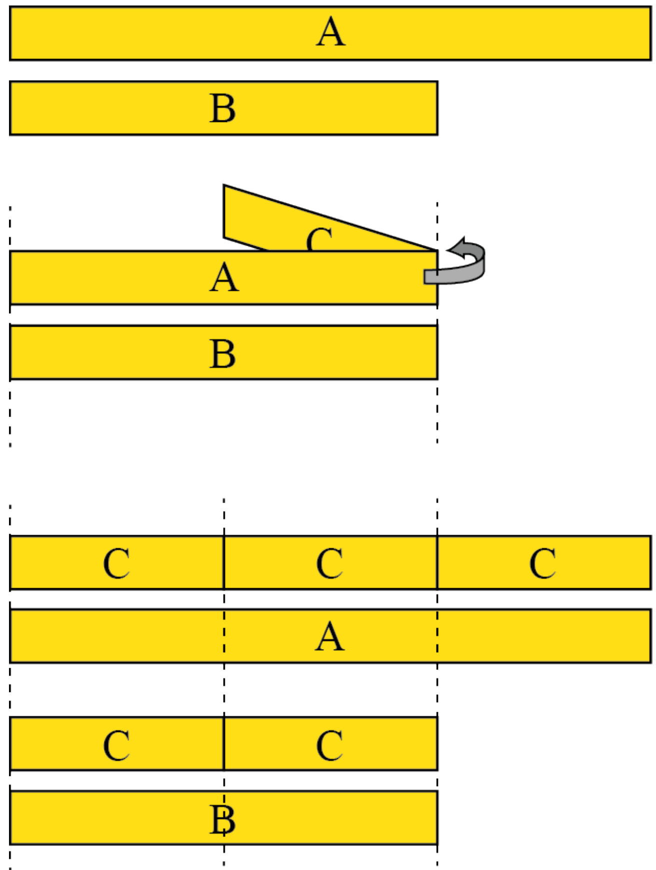


Figure 10:

Discussion: Indigenous Knowledge as a Basis for Teaching the Foundations of Mathematical Thinking

The above-described examples represent only a fraction of our data and provide evidence that in an Indigenous cultural context, measuring—as a means of comparing objects in practical activity—models the concept of ratios, expressed as rational numbers. When Dora creates a square from irregular material, or when Larry creates a measuring tool, both the square and the tool are constructed based on body measures and bodily techniques that establish ratios and proportions. The resulting products are made in proportion to the body of the person creating them. Everything is balanced between the user and the crafted product or clothing. The products and processes reflect relational thinking and relative units.

Mr. Jimmy identified the beginning of everything as a cultural code, which Mrs. Nanalook enacted while weaving her grass basket. This culturally grounded code is transferrable to the teaching of rational numbers in the elementary school. Dora's mantras are part of that cultural code: "What you do to one side you must do to the other side," and "Is this side equal to the other side?" Measuring as comparing (including body proportional measuring) reflects Lockhart's (2012) concept of measuring:

> How are we going to measure the length of two sticks? Let's suppose (for the sake of argument) that the first stick is exactly twice as long as the second stick. Does it matter how many inches or centimeters they come out to be? I certainly don't want to subject my beautiful mathematical universe to something mundane and arbitrary like that. For me, it is the proportion (that 2:1 ratio)

> that's the important thing. In other words, I'm going to measure these sticks relative to each other. (Lockhart, 2012, p.32)

We believe that it is through *measuring as comparing* that we can establish an integrative approach to teaching aspects of rational numbers. Measuring proportionally is about quantitative comparisons, comparing two or more lengths (or measurable attributes of quantities such as length, area, mass, and volume). A set of seemingly simple principles reflects the generative cultural practices and ways of thinking, which we then use to establish a mathematical starting point for developing school-based mathematics. The generative principles and processes can be expressed as:

- measuring as comparing
- relative units, ratios and rational numbers
- symmetry, halving, and verifying
- scaling

Applying Indigenous Knowledge to the Development of Rational Number Reasoning

These generative principles are the active ingredients that we have identified, and based on the knowledge of these principles and processes, we have begun the development of supplemental curriculum materials and a learning trajectory. Our work is informed by the *Measure Up* program, University of Hawaiʻi at Mānoa's Curriculum Research Development Group. The *Measure Up* program follows the experimental curriculum developed in Russia by Elkonin-Davydov (Dougherty & Simon, 2014), in which first grade students explore quantitative relationships through nonnumeric comparisons as a way to establish early algebraic reasoning including generalizing. Elkonin-Davydov's curriculum was influenced by Vygotsky's cultural-historical-activity theory (Engeström, Miettinen, & Punamäki, 1999). According to Davydov (2008), "the basic task of the school mathematics curriculum is to bring the students to the closest possible understanding of the conception of real number." Davydov goes onto to state, "the properties of quantities are discovered when a person works with real lengths, volumes, weights, time intervals, and so on (even before these are expressed in numbers)" (Davydov, 2008, p. 148). Moxhay (2008), who adapted the Elkonin-Davydov curriculum, and implemented and assessed it in the Portland School District in Maine, noted:

> All the children are exposed to the same, very high level of mathematical content, which at first view looks like high-school mathematics (use of algebraic notation, counting in bases other than 10). In particular, Davydov's curriculum has the goal of developing, in all students, a scientific, or theoretical, *concept of number,* from the very beginning of Grade 1. (Moxhay, 2008, p.2)

The developmental psychologist Sophian, at the University of Hawaiʻi at Mānoa, who collaborated with the *Measure Up* program, makes a persuasive argument for why the comparison-of-quantities approach should be considered a legitimate alternative to the "counting first approach." Sophian contrasts the concept of number and quantity as follows:

> In order to clarify the contrast between these two perspectives, the concept of *number* needs to be differentiated from that of *quantity.* In the senses most pertinent to the present discussion, *Webster's New World Dictionary* (Neufeldt & Guralnik, 1994) defines number as "a symbol or word, or a group of either of these, showing how many or which one in a series"; and quantity as … "that property of anything which can be determined by measurement." (Sophian, 2008, p. 3)

She notes an ontological difference between number and quantity by explaining that quantity is associated with physical things which can be compared in a variety of ways, while numbers are not physical things but are a mental operation (Sophian, 2008). She cites the work of Gal'perin and

Georgiev (1969), that numbers arise from the comparison of a quantity and a unit, and that different numbers occur if the unit size changes (Sophian, 2008). This approach leads to the recognition that "numerical values are essentially representations of the relation between the quantity they represent and a chosen unit" (Sophian, 2008, p. 7). The Davydov approach in some ways mirrors the cultural practices of some Indigenous People. Davydov takes measuring as comparing, as an alternative approach to establishing the foundations of mathematical thinking in a school context.

Measuring as Comparing Connects to Properties of Equality

Introduction to early algebraic thinking via the comparison of nonnumeric quantities. The measuring approach has allowed us to reverse and refine the sequence of introducing early algebraic thinking prior to the concept of number. In our approach, students compare and explore with quantities, understand, and slowly formalize algebraic operations (addition, subtraction, division, and multiplication) through the comparison of nonnumeric quantities, experimenting with length segments represented by strips. Elders' comparison of quantities lends itself directly to classroom application. Such experimentation organically leads to generating, representing, and verifying the basic algebraic properties of inequality and equality such as those noted below:

Properties of Inequality

- $quantity\ a < quantity\ b$
- $quantity\ b > quantity\ a$

Properties of Equality

- **Reflexive**: $a = a$
- **Symmetric**: $if\ a = b, then\ b = a$
- **Transitive**: $if\ a = b\ and\ b = c, then\ a = c$
- **Substitution**: $if\ a = b, then\ b\ can\ be\ substituted\ for\ a\ in\ any\ expression$
- **Commutative Property of Addition**: $a + b = b + a$
- **Associative Property of Addition and Multiplication**: $(a + b) + c = a + (b + c)$
- **Division Property**: $if\ a = b\ then\ \frac{a}{c} = \frac{b}{c}$

A Curricular Example: Halving, Comparing, and Place Value

The *Measuring Proportionally* curriculum construction process follows the lessons that we learned from working with Indigenous cultures. Measuring as comparing, the binary nature of the halving algorithm, and the concept and process of measuring by *dividing a quantity by a unit of measure* is ideally suited for modeling place value understanding. We believe that Indigenous practices that highlight halving and comparing lend themselves to modeling place value through base 2. The process of "halving," which is intuitively accessible to young students, becomes a powerful algorithm for furthering students' understanding of numbers. Research has shown that many students do not have a generalized understanding of place value systems (Venenciano & Dougherty, 2014). Recent work suggests that for students to understand a positional place value system, they need to compare base 10 to other systems (Schmittau & Morris, 2004; Slovin & Dougherty, 2004). We use measuring, halving, and properties of equality, particularly equivalence substitutions, as a way to conceptually develop this understanding. We describe this process below.

Place value example. Before engaging in these activities, students would have had ample opportunity to measure lengths by units of measures; to generate numbers from comparing a quantity and a unit of measure; and to explore through project materials many of the properties of inequality and equality enumerated above. Students are provided with an unnamed length, but for purposes of this discussion, we skip the processes involved in naming and representing quantities. We will name this example Length D. Students will be given a few different strips equal in length to Length D. They will be asked to keep Length D whole. With the other strips, through a series of recursive folds (1, 2, and 3 folds), they will create strips with 2, 4, and 8 parts. The strips will be labeled as shown in Figure 11.

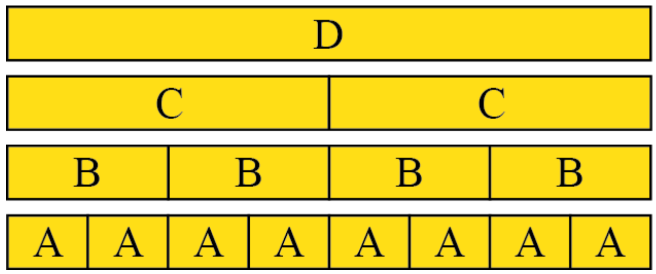
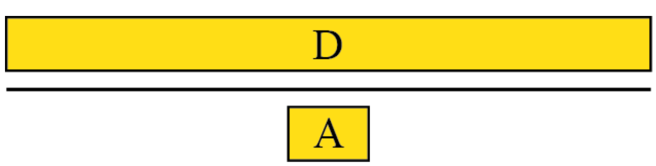


Figure 11:

Students will generate numeric values as they measure a quantity such as Length D by a unit of measure such as Length A. After they have had a chance to explore and communicate mathematically, students will be told that they are developing a Measuring Kit (Figure 12) based, in part, on the ways that Larry and Dora create their own measuring tools. Each Measuring Kit will contain only one of each length. Students will be challenged to measure various objects simply by using these lengths.

Figure 12:

Students will create a table of values by using Length A to measure the other quantities. This method results in $\frac{D}{A} = 8$, $\frac{C}{A} = 4$, $\frac{B}{A} = 2$, and $\frac{A}{A} = 1$.



$= \frac{D}{A} = 8$ and so on

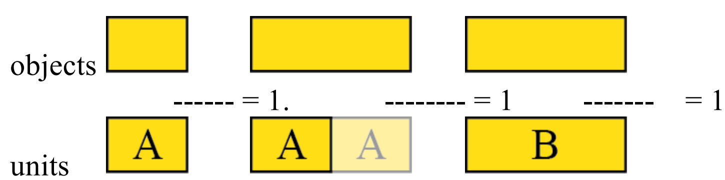
Students can either create or account for the units using a recording table as they measure the specific objects. There are only two numbers in this system: a unit of measure is used or a unit of measure is not used. Classes that have piloted these lessons have used 0 to indicate that they did not use a particular length, and they have used 1 to indicate that they did use a particular length. Some classes used “Y” for *yes* (they used a particular length) and “N” for *no* (they did not use a particular length). Each of these aspects of the activities on place value makes for interesting classroom discussions.

We challenge students to measure a set of objects. These objects are equivalent in base 10 lengths ranging from 1A to 16A. As students use their measuring kits to measure objects, they are faced with a few challenges, such as how to measure when your measuring kit has only one of each unit, and how to create a new unit of measure for your measuring kit when all of the units in the kit have been used. To facilitate solving these challenges, students receive a recording table (Table 1) and objects to be measured. For purposes of this presentation, we limited the number of objects to be measured.

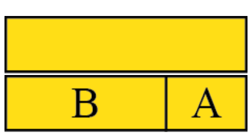
Table 1. Recording Table

Object to be measured	D	C	B	A
				1
			1	0
			1	1
		1	0	0

Length A (the length of A equals the length of the first object) is used to measure the first object one time. Students record 1 in the A column to show that one Length A was used. However, Length A is too small to measure the second object. Two Length A's would be needed to measure the second object. When students try to place 2 in the A column, they realize that there is space for one Length A and there is only one Length A in the kit. They can immediately observe that two Length A's = the second object, and that two Length A's = Length B. They can substitute two A's for Length B. Students record that they used 1 Length B and 0 Length A's. The specific measuring code for the second object is 10 (read as "one zero").



The next object is measured by (B + A) units, and thus it is recorded in the table as 11 (read as "one one").



The process continues as described above. Students measure multiple objects designed so that they continually experience the process of adding a unit, substituting the next unit from the tool kit, and moving from one place value column in a right-to-left direction.

Students continue this process until they reach a quantity that is exactly one unit longer than their entire set of measuring tools concatenated. They are then challenged to create a new unit of measure, Length E, that follows the system's exponential relationships: Length B is two times longer than

Length A; Length C is two times longer than Length B; Length D is two times longer than Length C. Now the students need to realize that Length E is twice as long as Length D. With these concrete objects and the structure of the recording table, the students are learning through measuring, building on prior knowledge related to properties of equality and equivalence substitutions. Only two numbers are allowed in each column (0 and 1), and each column to the left increases by a power of 2. In fact, this example models any positional number place value system, and may enhance understanding of base 10.

Discussion and Conclusion

This paper provides a glimpse into a long-term and continuing collaboration between Indigenous Knowledge holders—retired Yup'ik teachers who bridge Indigenous Knowledge and Western schooling—and academicians. We believe that the insights gained from everyday Indigenous activity and the ongoing intensive discussions among the Yup'ik cohort have enabled us to see how Indigenous Knowledge could have relevance for modern Western schooling. Measuring as comparing, and Dora's and Mrs. Andrew's folding algorithm, have implications for teaching aspects of the foundation of mathematics that go well beyond the scope of this paper. Our school-based curriculum development approach is attempting something not often done in academia or in the teaching of school mathematics: to use Indigenous Knowledge as a starting point for developing an approach to learning a core academic subject. Here we are building an elementary mathematics supplemental curriculum and learning trajectory from the perspective of Indigenous Knowledge. *Measuring Proportionally* is aided by insights gained from allied programs such as *Measure Up* and the work of Davydov and his colleagues. The halving algorithm and Euclidean-like way of generating the greatest common divisor are key aspects of our approach learned from Indigenous People, which distinguishes this approach from the work spearheaded by Davydov and his followers.

Yup'ik and Micronesian elders, in particular, use a process of *measuring as comparing and halving* to create tools and subunits across a wide spectrum of everyday activity. We are applying this process to our mathematical and pedagogical approach to teaching elementary school mathematics. Because the same processes are used in making the simplest of comparisons, such as direct comparisons of two quantities, to modeling place value, to comparing fractions and creating common units, our approach supports a cohesive way to teach aspects of elementary school mathematics. Similarly, our approach provides teachers with an integrative way of teaching foundations of mathematics. Measuring as comparing nonnumeric quantities establishes early notions of algebraic reasoning, and establishes relational units (as an early form of ratios). Proportional measuring and symmetrical folding are used in constructing geometrical shapes. The approach is both horizontally and vertically integrative. The same processes and halving algorithm used to compare two length quantities are also used to model and teach division of fractions, proportions, and scaling. Similarly, these cultural/mathematical principles are applied to Dora's and other elders' everyday construction of geometrical shapes and designs. Constructing a square and transforming a square into other planar shapes through symmetrical folding links the elders' knowledge to aspects of teaching geometry in school, including geometrical similarity (Lipka, Andrew-Ihrke, & Yanez, 2011). Scaling is invoked through many everyday activities within the Yup'ik culture and is taught in our program in the same fundamental ways as has been described for the comparison of quantities.

We believe that the mathematical pedagogical approach being developed by this project has potential to provide students and teachers with an elegant way to teach the foundations of mathematical thinking. This remains an empirical question.

The distance that this program has traveled—from its exploratory beginnings with elders to a more systematic collaborative study of everyday activity—continues to amaze and inspire those of us who are working in it.

Acknowledgments

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ADDRESSING THE NEEDS OF THE MARGINALIZED STUDENTS IN SCHOOL MATHEMATICS: A REVIEW OF POLICIES AND REFORMS

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Introduction

An examination of past research, policies, and reforms in mathematics education suggests that there have always been, and remain, tensions in conceptualizing the aims and goals of mathematics teaching and learning. These tensions have focused on addressing three concerns: (1) what mathematics students should learn; (2) how students should be taught mathematics; (3) who is qualified to teach mathematics? When examining how responses to these questions have addressed the needs of learners who are identified as marginalized (defined here as Black[1], Latin@[2], Indigenous, and poor) there is a constant pattern in which they are routinely given the least access to advanced mathematics content, the fewest opportunities to learn through methods other than memorizing facts and mimicking teacher-modeled procedures, and the least access to well-prepared mathematics teachers (Berry, Ellis, & Hughes, 2014). As a result, these learners experience the following conditions: a) reduced access to advanced mathematics courses that prepare them for higher education and improved career options; b) routine exposure to activities that focus primarily on rote, decontextualized learning through drill and practice with little to no engagement that promote reasoning and use mathematics as a tool to analyze social and economic issues, critique power dynamics, and build advocacy; and c) less access to qualified teachers of mathematics who both understand mathematics deeply and understand their students' cultural and community context deeply in order to give learners access to mathematical knowledge (Ellis, 2008; Flores, 2007; Gutiérrez, 2008; Martin, 2007). The effect of these conditions on marginalized learners' attainment in mathematics demonstrates well that such an approach constrains outcomes to a narrow range of proficiencies focused on basic skills.

While the disproportionality and conditions of marginalized learners is a cause for concern, it is important to understand that addressing the needs of these learners may not have been the primary goal of prior policies and reforms in mathematics education. Berry, Ellis, and Hughes (2014) argued that prior policies and reforms in mathematics education have failed due to having been developed to address the needs and interests of the larger dominant culture[3], not those of marginalized learners. In fact, many past policies and reforms in mathematics teaching and learning have come at the expense of the needs and interests of marginalized learners by framing policies and reforms based on economic, technological, and security interests of the dominant culture. There are statements such as "Mathematics has become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate: Equity has become an economic necessity" (NCTM, 1989, p. 4). This situates equity in mathematics education as serving economic interests of the dominant culture by situating participation as supporting the drivers of an economy. A consequence of this framing is that participation in mathematics education is based on ensuring that the dominant culture's economic, technological, and security interests are met rather than addressing the needs of learners. Examining the convergence of interests allows us to understand the motivating factors for policies and reforms that might lead to fortuitous benefits for marginalized students.

Derrick Bell, a former attorney with the National Association for the Advancement of Colored People (NAACP) during the Civil Rights Era, employed his interest-convergence principle to explain how the United States Supreme Court issued the landmark ruling in *Brown v Board of Education of*

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Topeka, Kansas (*Brown I*) in 1954. The Supreme Court's ruling in the *Brown* case revoked the "separate but equal" doctrine, which legally sanctioned segregation in public education and all aspects of daily life. Bell (2004) argued that the *Brown* decision was not the result of America coming to terms with its democratic ideals or moral sensibilities. Rather, the Supreme Court was more interested in providing "immediate credibility to America's struggle with communist countries to win the hearts and minds of emerging third world people" than in doing what was morally right (p. 233). Thus, under the interest-convergence principle, the *Brown* decision is best understood as progress requiring the coincidence of a pressing issue, more than a commitment to justice (Donnor, 2005).

Brown provided the impetus for legislation, such as the Elementary and Secondary Education Act (ESEA) of 1965 and it reauthorizations Improving America's School Act of 1994 and No Child Left Behind Act (NCLB) of 2001 (Zion & Blanchett, 2011). These along with other legislations and mandates established requirements that address the need to ensure that all students in the United States are provided equal educational opportunities. Although not fully realized, the requirements of these legislative acts and mandates created pressure to address the historical inequity in educational opportunity, achievement, and outcomes. Zion and Blanchett (2011) argued that the reason why large scale improvement in outcomes for all students have yet to be realized is that the problem has not yet been framed appropriately. The problem must be framed as part of the history and legacy of racism, and as an issue of civil rights and social justice, viewed through a critical lens. This article use a critical lens to apply the interest-convergence principle informed largely by the work of legal scholar, Derrick Bell (1980 & 2004), to examine motivating factors of policies and reform efforts in mathematics education. Specifically, this article makes the case that policies and reforms in mathematics education were not designed to address the needs of marginalized learners; rather these policies and reforms are often designed and enacted to protect the economic, technological, and social interests of the dominant culture.

Theoretical Framework: Interest-Convergence Framework

Social institutions are set up by those in power and are organized to support and value the types of cultural and social capital held by those in power (Bourdieu & Passeron, 1990). Schools are institutions where power is controlled by the dominant culture's interests. In many schools we find values of individualism and independence, self-direction, competitiveness, decontextualized teaching, and passive methods of communication and learning (Stein, 2004). For many marginalized students, they must choose to engage using the values of the dominant culture or choose to resist becoming a part of the value set (Zion & Blanchett, 2011). Policies and reforms in education, and those particularly geared to marginalized learners, often portray these learners as deficient or in need of "fixing" to be more align with the values of the dominant culture (Stein 2004). Stein (2004) described the language of education policies as positioning marginalized people as being culturally deprived and deficiencies for marginalized learners are within their cultures, families, and communities. Consequently, policies and reforms frame marginalized students as problems to fixed through labels (i.e. Title I students; culturally deprived) then propose policies and reforms that are in the interests of those in power. That is, if marginalized students adopt the values of the dominant culture, then the economic, technological and security interest of those in power are maintained. Policies and reforms are more about the dominant culture's interest and less about needs and interests of marginalized students.

Interest-convergence is an analytical viewpoint for examining how policies and reforms are dictated by those in power to advance their political, social, and economic interests (Donnor, 2005). Bell's (1980; 2004) interest-convergence principle theorizes that any empowered group will not help any disempowered group unless it is in their best interest to do so. For Bell, the historical advancement of Black people's needs and interests is a result of being fortuitous beneficiaries of measures directed at furthering aims other than racial equity and social justice (Bell 2004). Bell states, "Even when interest-convergence results in an effective racial remedy, that remedy will be abrogated at the point that policymakers fear the remedial policy is threatening the superior societal status of Whites, particularly those in the middle and upper classes" (Bell, 2004, p. 69). Interest-convergence has its theoretical grounding in critical race theory (CRT) which draws from a broad literature in law, sociology, history, education and women's studies (DeCuir & Dixson, 2004; Ladson-Billings & Tate, 1995; Matsuda, Lawrence, Delgado, & Crenshaw, 1993; Solorzano & Yosso, 2001). Historically, its roots can be traced to legal studies. With respect to CRT's use in education, as Solorzano and Yosso (2002) explained, "critical race theory in education is a framework or set of basic insights, perspectives, methods, and pedagogy that seeks to identify, analyze, and transform those structural and cultural aspects of education that maintain subordinate and dominant racial positions in and out of the classroom" (p. 25). In education, interest convergence provides framework to discuss power dynamics as framed by systemic interests and a loss–gain binary (Milner, 2008). Interest-convergence has been used to examine policies and practices related to teacher education programs (Milner, 2008); practices for STEM education serving marginalized learners at universities (Barber, 2015); intercultural movements in multicultural education (Caraballo, 2009); inclusion and equity in special education (Zion & Blanchett, 2011); intercollegiate athletics (Donner, 2005); the development of historically Black colleges and universities (Gasman & Hilton, 2012); and postsecondary access for Latino immigrant populations (Alemán & Alemán, 2010). This body of work provides the lens for using the interest-convergence principle to examine the motivating factors for policies and reforms in mathematics education to understand whose interests are served and the resulting fortuitous beneficiaries.

Historical Perspectives in School Mathematics

In their review of the history of school mathematics, Ellis and Berry (2005) noted a tension that reforms in mathematics education focused on efficiencies with an emphasis on procedural learning coupled with a belief that mathematics beyond arithmetic should be reserved for those deemed capable of advancing to such heights. Efforts to improve mathematics education,

> …situated many learners in an a priori deficit position relative to disembodied mathematical knowledge—meaning learning mathematics was taken to be harder for certain groups of students due to their backgrounds and/or innate abilities—and failed to acknowledge the importance of mathematics for all students. (Ellis & Berry, pp. 10-11)

Throughout this history, systems of standardized assessment were developed as a means to justify the separation of students within and between schools by race, class, and ethnicity. The use of assessments to stratify was built on the assumption that a distribution of mathematical ability exists that can be fairly measured and meaningfully interpreted as the basis for separating students and providing unequal access to opportunities to learn mathematics. The conflation of this with societal beliefs about race and intelligence cannot be overlooked; the interest of those with power was preserved. This article documents some policies and reforms along the historical trajectory in mathematics education using the lens of interest-convergence to examine whose interests are served and whether there were any fortuitous benefits for marginalized students.

New Math, Sputnik, & Brown

The launch of the first artificial satellite, *Sputnik*, on October 4, 1957, by the Russians gave impetus to the drive to improve mathematics education in America. The launching of *Sputnik* brought heightened concern about America's national security as well as concern that America was lagging behind the Russians in mathematics and science. Influenced by the launch of *Sputnik*, federal funds for mathematics education became available through the National Defense Education Act (NDEA) of 1958 (Walmsley, 2003). Title V of the NDEA Act laid the groundwork for gifted programs and began the trend of using standardized testing in schools to measure competency (Walmsley, 2003). NDEA provided funds to identify "best and brightest" young scientific minds and was designed to fulfill defense interests in mathematics, science, engineering and foreign languages. The appeal to identify "best and brightest" was built on protecting national security and defense interests (Tate, 1997).

Approximately three years prior to the launching of *Sputnik*, the United States Supreme Court issued the landmark ruling in *Brown v Board of Education of Topeka, Kansas* which revoked the "separate but equal" doctrine. Black parents and community leaders sought desegregation based on the assumption that better school resources accompanied schools were White children were taught and that better resources provided greater opportunities. The *Brown* decisions occurred in the midst efforts to reform what mathematics should be taught and how it should be taught. This reform, the "new math" reform, offered teaching new mathematics content as well as new pedagogical approaches (Walmsley, 2003). One main idea of "new math" was to reduce focus on the drill and practice approach with approaches where students could develop conceptual understanding of mathematics. These pedagogical approaches included the use of manipulatives, guided-discovery learning teaching practices, and the spiral curriculum (Walmsley, 2003; Willoughby, 2000).

When we consider that many schools were still segregated and the process of desegregation was slow, and that schools serving Black children often received used textbooks handed down from schools serving white students (Snipes & Waters 2005), the reforms of "new math" did very little to address the needs of marginalized children, specifically Black children (Tate, 2000). That is, Black children did not have access to new content nor experienced the pedagogies to teaching associated with the "new math" reform. Within the interest-convergence framework, this era was characterized "benign neglect" (Tate, 2000; p. 201) for marginalized students because the needs and interest of marginalized students were largely ignored. This does not imply that these learners did not have access to quality teaching in segregated schools, in fact, there is a body of research that suggest that many teachers in segregated schools "made do" with substandard materials and provide high quality teaching (Foster, 1997; Siddle-Walker, 2000; Snipes & Waters, 2005; Standish, 2006). Rather, the "new math" reforms focused on identifying the best and brightest while ignoring the needs of marginalized students.

Great Society & Segregation

During the mid-1960s, President Lyndon Johnson had a vision for a "Great Society," which was an effort to address issues of civil rights, poverty, economic inequities, health care, housing, jobs, and education (Levitan & Taggart, 1976). Title I was enacted through the Elementary and Secondary Education Act of 1965 which allocated federal funds to schools with high concentrations of poverty in order to improve the educational opportunities of poor students (Wong & Nicotera, 2004). The Civil Rights Act of 1964 forbade job discrimination and the segregation of public accommodations; the Voting Rights Act of 1965 suspended use of literacy tests, other voter qualification tests, and stopped poll taxes; and the Civil Rights Act of 1968 banned housing discrimination and extended constitutional protections to Native Americans on reservations. These legislative acts provided greater opportunities for marginalized people but the activities of the Civil Rights movement facilitated these acts. In response to the radical protests of this period, the interest of those motivated by America's image to the world converged with the interests of the Civil Rights movement (Bell, 1980).

In 1966, the *Equality of Educational Opportunity,* commonly referred to as the Coleman Report, argued that school resources had little effect on student achievement and that student background and socioeconomic status are much more important in determining educational outcomes (Coleman, Campbell, Hobson, McPartland, Mood, & Weinfeld, 1965). The Coleman report was a challenge to President Johnson's policies on education that increased spending. One finding that received significant attention from policymakers was that peer effects had a significant impact on student achievement, meaning the background characteristics of other students influenced student achievement. This finding was interpreted to mean that marginalized children, specifically Black children, would have higher test scores if a majority of their classmates were white (Wong & Nicotera, 2004). This finding coupled with the tensions of desegregation was a catalyst for busing that occurred in many places in the United States. Busing was an effort to desegregate schools and marginalized children were more likely to be bused than white children. Thus, many marginalized children were displaced from their neighborhoods (Doughty, 1978). Busing was a policy sought to "fix" marginalized students because it existed primarily to assist these students by allowing them entrance to perceived superior schools that served white students; thus allowing marginalized students to receive the benefits of peer effect as described in the Coleman report. By displacing marginalized students from their communities, it positioned these students' communities as problem centers rather than as resources. In an effort to desegregate schools busing and peer effect policies served the interests of those who had power to make decisions about which children will be displaced from their home communities. Further, these policies assumed that immersion into schools serving white students will help marginalized students with better achievement and reap values that served the interests of those in power.

While busing sought to desegregate at the school-level, we must consider what happened at the classroom-level. In schools where significant numbers of marginalized children were bused, these children experienced resegregation for their mathematics instruction. In fact 70% of school districts had racially identifiable classrooms as a result of resegregation (Doughty, 1978). That is, because of the development of and placement in low-level mathematics courses, marginalized children were placed in mathematics classrooms that denied them access to high-level mathematics content and these students were in segregated classroom within integrated schools. Not only were students resegregated for mathematics instruction but a disproportionate number of marginalized students were placed in special education programs. Doughty (1978) estimated that 91% of Black children in special education programs during this period were incorrectly assigned on the basis of low expectations and inaccuracies in IQ scores. The misuse of a standardize test had negative consequences for many marginalized students. Michelson (2001) argued that resegregation in classrooms through tracking undermined any potential benefits of school-level desegregation. Given the consequences of resegregation at the classroom-level, it is plausible to consider that desegregation as policy for reform was a facade to hide the interests of those who wanted to maintain segregation but appease the interests who fought for desegregation.

Schools are as segregated today as they were in the 1960s and 1970s, and many schools are rapidly resegregating (Garda, 2011). In 2014, the percentage of public school students who were considered to be part of a racial or ethnic minority group was greater that the percentage of students considered being white (Hussar & Bailey, 2013). Yet, white children are the most racially isolated group of students in the United States; they have little contact with students from other ethnic groups. Nearly half of white students attend schools that are more than 90% white and approximately one-third of white students attend schools that are more than 95% white (Garda, 2011). These statistics suggest that *Brown* did not permanently integrate schools in the long-run; in fact the intended goal of racial balance and desegregation of *Brown* has not been realized.

“Back to Basics”

In the late 1960s and early 1970s, the “back to basics” reform movement in mathematics emerged in response to the perceived shortcomings of “new math” (Burrill, 2001). During this period, the National Science Foundation discontinued funding programs focused on “new math,” and there was a call to go back to the “core curriculum” which was understood to be basic skills in mathematics. The “back to basics” movement called for teaching basic mathematics procedures and skills and was closely connected to the minimum competency testing movement used extensively by states in the 1970s and 1980s (Resnick, 1980; Tate, 2000). Testing had a significant impact on the mathematics content that was taught and the methods used to teach mathematics. Typically, students were taught mathematics content deemed important for passing tests. Although the emphasis on skills did result in slightly improved standardized test scores for marginalized children, it did not adequately prepare these students for mathematics coursework requiring higher levels of cognition and understanding (Tate, 2000). Thus, marginalized students were underrepresented in the upper achievement distribution and in upper-level mathematics courses (Tate, 2000).

Considering the impact of desegregation and resegregation at the classroom-level coupled with an emphasis on testing, it plausible that the pedagogies and the curriculum offerings during the “back to basics” reform were similar for marginalized students during the “new math” reform. The pedagogies of “back to basic” were already apart of marginalized students’ mathematical experiences. The growing emphasis on testing during this period was used to legitimize the perception that many marginalized students are not capable of rigorous studies in mathematics (Perry 2003). The “back to basics” movement provided more focus of using achievement tests to pathologize marginalized students as being inferior, deficit and deviant. We find the first of many research studies focusing on the achievement gap in mathematics during this period describing marginalized students as deficient and in need of fixing (Perry, 2003). The analysis of achievement gap language advantaged the values of the dominant culture and ignored the ongoing experiences resegregation of marginalized students. If one considers the context of the late 1960s and 1970s and the persistent limited educational opportunities available to marginalized children, discussion of an achievement gap serves to reinforce an ideology about marginalized children’s intellectual inferiority. The focus on testing served the interest of those who focused on efficiency and stratifying learners to identify the “cognitive elite” (Hernstein & Murray, 1994) to protect the interest of those with power.

Increased Enrollments in Upper-Level Mathematics Courses

In 1983, the National Commission on Excellence in Education issued a report titled, *A Nation at Risk: The Imperative for Educational Reform*. The report suggested that education reform is necessary because competitors throughout the world are overtaking America’s preeminence in commerce, industry, science, and technology. Furthermore, the report stated, “If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war” (p. 1). The inflammatory rhetoric of *A Nation at Risk* heightened concerns about national security and America was lagging in mathematics and science when compared internationally. *A Nation at Risk* stated that through educational reform, American children’s promise of economic, social, and political security in the future would be earned by meritocratic ideals of effort, competence, and informed judgment.

As a reaction to *A Nation at Risk* Many states placed Algebra I as a high school graduation requirement. Between 1982 and 1992, students enrolled in Algebra I increased from 65 to 89 percent, in Algebra II from 35 to 62 percent, and in calculus from 5 to 11 percent (Raizen, McLead, & Rowe, 1997). Planty, Provasnik, and Daniel (2007) reported that the percentage of graduates who completed a semester or more of Algebra II rose from 40 percent in 1982 to 67 percent in 2004. This evidence suggests that the average number of mathematics courses at or above Algebra I taken by high school students has increased. The increased focus on maximizing students' performance on standardized tests has led schools to rethink course-taking patterns (Kelly, 2009). While there were increases in the enrollments rates of all students, the enrollment rates for marginalized students in higher-level mathematics courses was still relatively low (Jetter, 1993). Black students to be more likely to be enrolled in Algebra I and geometry but less likely enrolled in higher-level courses (Jetter, 1993). In the early 1990s, Bob Moses, a leader in the civil rights movement, argued that access to algebra is the "new" civil rights (Jetter, 1993). Moses contended that algebra served as a curricular gatekeeper tracking numerous students out of many in-school and out-of-school opportunities. Within an interest-convergence framework, Algebra I as a graduation requirement was to improve America's standing on internationally comparisons which lead to increase enrollments in for some marginalized students in high-level mathematics courses. These increased enrollments supported the economic, social, and political interests of those in power.

The increased enrollment in the upper-level mathematics courses did not influence instructional methodologies to meet the increase in the diverse learning needs of children (Porter, Kirst, Osthoff, Smithson, & Schneider, 1993). That is, for marginalized learners, instruction focused primarily on the acquisition of skills. Additionally, much of the increase in mathematics course enrollment occurred by simply placing students in Algebra I tracks. When the increased enrollment in mathematics courses seemed an insufficient means for increasing student achievement, policymakers and reformers began to investigate notions of systemic reform within the larger education system (Raizen, McLead, & Rowe, 1997). In the fall of 1989, President George H. W. Bush, the nation's governors, and other leaders held an educational summit in Charlottesville, Virginia. One result of this meeting was a call for national standards. Participants at the 1989 National Education Summit made a commitment to make U.S. students first in the world in mathematics and science by the year 2000.

Standards Movement

In 1980, NCTM put forth its *Agenda for Action*, which diverged from "back to basics." The *Agenda for Action* put forward recommendations that broadens the notion of basic skills as the acquisition of skills toward focusing on problem solving, use of technology, called for measures other than conventional testing, and an effort to meet students' diverse ne advocated for pedagogy and curriculum to accommodate the diverse needs of the student population. While the *Agenda for Action* was not a standards document, it was the foundation for the first standards document, *Curriculum and Evaluation Standards for School Mathematics* (*CSSM*), developed by NCTM (1989).

CSSM provided broad frameworks for mathematics content and processes across grade bands. Emphasis was placed on an inquiry-based approach to mathematics teaching and learning. The inquiry-based approach supported conceptual understanding as a primary goal and algorithmic fluency would follow once conceptual understanding was developed. Critics of *CSSM* argued that the primary goal of conceptual understanding through an inquiry-based approach did not help children acquire basic skills efficiently nor learn standard algorithms and formulas (Klein, 2003). *CSSM's* release came at a time when there were calls for national mathematics standards and it received support from the U.S. Department of Education and the National Science Foundation (NSF). Through the 1990s, NSF supported the creation and development of commercial mathematics curricula aligned to the standards in *CSSM.* Critics of the curricula objected to the inquiry-based approaches, claiming that not enough emphasis was placed on acquisition of basic skills and general mathematics principles (Klein, 2003). Tension between proponents and opponents of *CSSM* resulted in the "Math Wars." There were proponents for improving mathematics instruction for marginalized children on both sides of the "Math Wars." The primary tensions focus on mathematics content, pedagogical approaches, and student achievement, with both sides agreeing that reform is necessary for America's economic, technological, and security interests.

There is a long history indicating that during times of reform, the interests and needs of marginalized children are in many ways dismissed. The tensions of the "Math Wars" appear to have an underlying narrative focusing on the nation's technological interests, social efficiency, and perpetuation of privilege. There are intense debates focusing on curriculum, teaching, and assessment but little debate focusing on understanding the realities of children's lives. For marginalized learners, issues of race, racism, identity, and conditions were not under consideration in the "Math Wars." There is evidence from the *CSSM* to suggest that the standards were moving towards a democratic vision by including "for all" language. However, critics of the "for all" language argue that the use does not delve into serious considerations of the social and structural realities faced by Black children; rather the language suggests a myopic focus on modifying curricula, classrooms, and school cultures (Martin, 2003). Consequently, the underpinning of the "for all" message has done little to understand the variables that impact mathematics teaching and learning for marginalized learners.

The *CSSM* outlined four social goals for schools: (a) mathematically literate workers, (b) lifelong learning, (c) opportunity for all, and (d) informed electorate. These four goals derived from the fact that at the time society was moving towards an increase in technologies. These goals situated social justice issues in school mathematics within the framework of economic competition and national technological interests. Positioning social justice in mathematics education within the framework of economic competition and national technological interests situates mathematics education as being primarily utilitarian. Using the utilitarian perspective situates, increasing marginalized learner's participation in mathematics education is based on ensuring the economic and social interests of those with power. Consequently, the interests of marginal learners' needs are not given careful consideration; it was the interests of the broader American contexts that drove the implementation of standards. Within this context, mathematics is always situated as a utilitarian area of study, and the focus of mathematics education is on the needs of national interests rather than the needs of a democratic society.

NCTM revised it standards document in 2000 through the release of the *Principles and Standards for School Mathematics* (*PSSM*). The *PSSM* revision provided slightly greater emphasis on the importance of algorithms and computational fluency. *PSSM* was received as more balanced than *CSSM*, which led to some calming of but not an ending to the "Math Wars." *PSSM* highlighted equity as one of its six principles for school mathematics by stating that equity requires: (a) high expectations and worthwhile opportunities, (b) accommodating differences to help everyone learn mathematics, and (c) resources and support for all classrooms and all students. These points situate equity in a broad context but fail to recognize issues of social justice or understanding social, economic, and political context in which mathematics is learned. Martin (2003) is critical of *PSSM's* Equity Principle for not providing a sense of equity that considers the contexts of students' lives, identities of students, and conditions under which mathematics is taught and learned. He states:

> …the Equity Principle of the *Standards* contains no explicit or particular references to African American, Latino, Native American, and poor students or the conditions they face in their lives outside of school, including the inequitable arrangements of mathematical opportunities in these out of school contexts. I would argue that blanket statements about *all* students signals an uneasiness or unwillingness to grapple with the complexities and particularities of race, minority/marginalized status, differential treatment, underachievement in deference to the assumption that teaching, curriculum, learning, and assessment are all that matter. (p. 10)

Too often, race, racism, social justice, contexts, identities, conditions, and others are relegated as issues not appropriate for mathematics education when in fact these issues are central to the learning and teaching of mathematics for marginalized students.

No Child Left Behind & Common Core

In 2002, President George W. Bush signed the Elementary and Secondary Education Act better known as the No Child Left Behind Act (NCLB) into law with the declared intention of helping "all students meet high academic standards" (NCLB, 2002). NCLB required states to implement student testing, collect and disseminate subgroup results, ensure teachers are highly qualified, and guarantee that all students achieve academic proficiency by 2014. States were required to use sanctions to hold schools and districts accountable for their success in meeting adequate yearly progress (AYP) goals, set by the states, for both overall performance and performance in each subgroup. Similar to previous reforms, NCLB motives are cast in the interests of improving America's standing international measures and the interests of the future of the dominant culture maintaining power. NCLB narrows the definition of achievement, thus focusing on measureable outcomes and applying technical solutions, such as setting Standards and using tests to measure attainment of the standards. NCLB stated a desire to close the "achievement gap between high- and low-performing children, especially the achievement gap between minority and non-minority students and between disadvantaged children and their more advantaged peers" (NCLB, 2002; Sec 1001). To achieve these goals NCLB takes a position in instruction and measurement by insisting that instructional approaches be deemed based on "scientifically based research" (Stein, 2004). However, it is not clear in NCLB what constituted "scientifically based research." Given the language of measurement focused on closing the achievement gap and the language of standards, it is plausible that "scientifically based research" instructional approaches are approaches that can be measured and quantified. Through standards and assessment there are restrictions in the autonomy of teachers. Consequently, teachers are restricted in the ways to meet the needs and interest of students. NCLB assumed that members within a subgroup have static identities that are quantifiable in term of race, class, gender, language, etc. (Gutierrez, 2008).

NCLB unintentionally create incentives that encourages states to lower their academic standards, promote school segregation and the push out of poor and minority students, and reinforces the unequal distribution of good teachers (Ryan, 2004). The category of "failing school" was based on whether or not schools met a set of performance standards drafted by the states which were grounded in test-based measures of academic proficiency, states had incentives to lower their standards so that fewer schools are identified as failing (Ryan, 2004). Stein (2004) found that many states lowered performance standards rather than raised standards. To improve the chances that a particular school or schools within a district make AYP, administrators have an incentive to minimize the numbers of marginalized students. Since marginalized students traditionally do not perform as well as white and more affluent peers on standardized tests, the incentives to exclude these students are grounded on improving status (Ryan, 2004). Attaching consequences to test results creates incentives for teachers to avoid schools that are likely to not meet AYP. Thus, teaching will be less attractive in those schools where teachers must spend a great deal of time preparing for the tests.

In mathematics education, it is likely that marginalized students' needs and interests were served minimally by NCLB. As with previous reforms, these students most likely experienced mathematics as procedural and rote. This time of instruction appears to be consistent across all reforms. Similar to other reforms, the emphasis on testing was used to legitimize negative perception about marginalized students' capabilities in mathematics. As noted earlier, under NCLB schools serving predominately marginalized students were likely to be deemed as failing, have the least access to good teachers, and were stigmatized. The emphasis on testing does not recognize the external factors that contribute to testing results which could lead to a labeling that impacts the quality of mathematics that students experience. For students, in schools or subgroups labeled as failing the experience is focused on increasing test scores and for students in schools or subgroups that consistently meet AYP the experience may be more enriching.

When considering an interest-convergence perspective, we must consider whose interests are served under NCLB. If we consider one context in which NCLB was developed, there was public pressure on legislatures for competition in public schools through school choice and vouchers. These pressures lead to increase in public options for schooling, including the increase in charter schools. The interests of proponents of school choice were served. A consequence of NCLB has been the narrowing of the conception of what constitutes a ―good school. Judging schools simply based on standardized test scores had implications beyond the schools themselves. In a study examining property values and schools designated as failing because of NCLB, Bogin and Nguyen-Hoang (2014) found that the "failing" designations significantly decrease home prices and that low and moderate income neighborhoods were negatively impacted. Public schools are primarily funded through property taxes in most areas. This means that in neighborhoods with higher property values, the schools in that area get more resources. Similar to the peer finding of the Coleman report, proponents of school choice supported giving students options to be in "good" schools with the idea that placement in "good" schools would have positive effects. This conclusion ignores the external factors that contribute to failing schools.

A factor that contributed to the creation of the Common Core State Standards (CCSS) was the "known-yet-unacknowledged failure of No Child Left Behind" (Schneider, 2015; p. 20). One complication of NCLB was the lack of consistency among states and what constituted AYP. As a result there was agreement among the Council of Chief State School Officers (CCSSO) that consistent standards were necessary. In 2010 the *Common Core State Standards for Mathematics* (CCSSM) were released by the National Governors' Association and CCSSO. The mission statement for the Common Core makes clear these reforms are emerging from the same interests of college and career readiness by positioning American students to be able to compete in a global economy (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010)

Nowhere to be found is mention of the gross inequities within society that continue to be reflected in students' educational outcomes. Framing the reform from the position of economic interest, diminishes the needs of learners to focus primarily on the acquisition of mathematics content and practices.

Conclusion

The interest convergence principle provides an intellectual and political frame within which to question the motivating factors of policies and reforms in mathematics education to understand how they are designed to address the needs of marginalized learners. Employing the interest-convergence principles, I raise the following questions: (1) Whose interests are served by policies and reforms in mathematics education? (2) Where are opportunities for convergence?

Whose interests are benefited by policies and reforms in mathematics education? The review of policies and reforms suggest that economic, technological and security interests were the drivers of many policies and reforms. It is difficult to argue against ensuring students' competitive place in the global marketplace. However, a careful look at policies and reforms focused on labeling and identifying the "best and brightest," identifying high achievers, stratifying students based on characteristics, or identifying "failing" schools. Such labeling and identifying determines groups or populations as having merit and others as being deficit. Policies and reforms have typically not attended to or appreciated the social realities and needs of marginalized students in ways that lead to improvements in their life circumstances. This critique suggests that marginalized students voices and experiences within a broader context are either missing or are situated within deficit perspectives in mathematics education research, policy, and reform. In fact, patterns over time suggest that marginalized students have experienced mathematics instruction as a focusing on the acquisition of facts over the entire history discussed in this article. Thus, policies and reform have had little to no impact on the type of instruction these children receive.

Where are opportunities for convergence? The stage is set for convergence to occur, given the growing body of research focused on context, identities and conditions and the interests of practices and process in mathematics teaching and learning. There is a growing body of research that positions marginalized students (Berry, 2008; Boaler, 2014; Jett, 2011; Gutiérrez, 2010; Martin, 2000; Noble 2011; Stinson, 2010; Thompson & Lewis, 2005, Walker, 2006). This body of research considers issues of race, racism, contexts, identities, and conditions as variables that impact the mathematical experiences of marginalized. This body of research challenges the dominant discourse and pushes the field of mathematics education to consider sociological, anthropological, and critical theories. It encourages researchers to consider outcomes other than achievement as the primary measure of success. One finding that we find from this research is that educators must create opportunities for students to experience mathematics learning using the resources they bring to classrooms; teachers must know and understand learners' identities, histories, experiences, and cultural contexts and consider how to use these to connect students meaningfully with mathematics. There is a need of policies and reforms that focus on leveraging communities and community-members. Mathematics teaching and learning not only occurs in classrooms but also occurs in other spaces. By leveraging these resources, we situate mathematics teaching and learning as a way to structure experiences that are contextual and provide opportunities for exchange in mathematical ideas. The use of context in mathematics education can help learners to recognize and build upon the cultural and social resources they bring to the mathematics classroom.

Endnotes

[1]I use the term Black to acknowledge the Black Diaspora and to highlight that Black people living in North America have ancestry dispersed around the world. Black learners who attend schools and live in North America are racialized in similar ways regardless of country of origin.

[2]I borrow Latin@ from Rochelle Gutiérrez (2013) who stated that the use of the "@ sign to indicate both an "a" and "o" ending (Latina and Latino). The presence of both an "a" and "o" ending decenters the patriarchal nature of the Spanish language where is it customary for groups of males (Latinos) and females (Latinas) to be written in the form that denotes only males (Latinos). The term is written Latin@ with the "a" and "o" intertwined, as opposed to Latina/Latino, to show a sign of solidarity with individuals who identify as lesbian, gay, bisexual, transgender, questioning, and queer (LGBTQ)" (p. 7).

[3]A dominant culture is one that is able, through economic or political power, to impose its values, language, and ways of behaving. This is often achieved through oppression and political suppression of other sets of values and patterns of behavior.

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DESIGNING FORMATIVE ASSESSMENT LESSONS FOR CONCEPT DEVELOPMENT AND PROBLEM SOLVING

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Formative assessment is the process by which teachers and students gather evidence of learning and then use this to adapt the way they teach and learn. I describe a design research project in which we integrated formative assessment strategies into lesson materials that focus on developing students' conceptual understanding and their capacity to tackle non-routine problems. A theoretical framework for assessment task design is presented, together with an analysis of research-based principles for formative assessment lesson design. Particular aspects are highlighted: the roles of pre-assessment, formative feedback questions and sample work for students to critique. While there are some early signs that these lessons provide an effective model for teachers to introduce formative assessment into everyday classroom practice, the materials require a radical shift in the predominant culture within most classrooms.

Keywords: Assessment and Evaluation; Design Experiments

Introduction

There is little doubt that assessment has a profound impact on the nature of student learning, and that this is often detrimental in nature. Our assessment practices have the potential to convey our valued learning goals to students, but this is often unrealized because the tasks and methods we use do not reflect these values. It has been found, for example, that even when teachers clearly acknowledge the importance of eliciting students' understanding and of giving useful, qualitative feedback, the tests they use encourage 'rote and superficial learning' and appear more concerned with grading and record keeping than with developing learning (Black & Wiliam, 1998). The poor design of summative, high-stakes tests must take some of the blame for this. These are designed to be cheap, predictable and simple to grade and, in consequence, focus on fragments of mathematical performance. Policy makers tend to ignore their powerful backwash effect and continue to claim that tests are merely measuring instruments (ISDDE, 2012).

Assessment needn't be this way. High quality assessment, focused on important mathematics, can be a powerful lever for positive change. This requires a radical shift away from multiple choice, computer-based assessments of procedural knowledge toward assessments that focus on the mathematics we care about - understanding, reasoning and problem solving. More substantial assessment tasks are required and scoring must begin to assess the quality of students' extended reasoning. (This is possible even in high stakes assessment when human judgment, rather than machine scoring, is allowed to have a role. Point scoring rubrics of chains of reasoning, long established in other subjects, can give reliable scores on mathematics tests. Reliable qualitative methods, such as adaptive comparative judgment, are also now recognized as a possible way forward (Jones, Pollitt, & Swan, 2015). Further, when teachers are involved in scoring, suitably organized, it can have considerable value for professional development.)

In this paper, however, I have insufficient space for a thorough discussion of high stakes assessment. Instead I wish to focus on the potential of classroom assessment to produce significant and substantial student learning gains. This potential was brought to our attention by the research reviews of Black, Wiliam and others (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Black & Wiliam, 1998; Black & Wiliam, 1999). In their original definition, the term 'formative assessment' is taken to include:

> … all those activities undertaken by teachers, and by their students in assessing themselves, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged. Such assessment becomes 'formative assessment' when the evidence is actually used to adapt the teaching work to meet the needs. (Black & Wiliam, 1998, p. 140)

This definition is wide-ranging, and includes both pre-planned and incidental assessment activities, such as diagnostic tests, oral questioning, collaborative tasks and observation of students. Improving the nature and focus of teacher-student and student-student communication is central. Most importantly, however, it must lead to adaptive action, not just the reteaching of the material concerned.

Since their work was published, this definition has often been mutated to mean more frequent testing, scoring and record keeping. In the UK, for example, one government initiative, "Assessing Pupil Progress" (APP) degenerated into the atomized profiling of pupils. This involved teachers in monitoring work, keeping files on pupils and regularly assessing progress against detailed criteria. Teacher workload was significantly increased and many teachers did not use the feedback to improve instruction. Recognizing such mutations, Black and Wiliam refined their definition a little differently in a later paper, laying more emphasis on the *agents* in the process: teachers, learners and peers, and the requirement for each of these agents to make *effective use* of the evidence obtained:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited. (Black & Wiliam, 2009, p. 9)

The interaction between these agents and the three main aspects of formative assessment: identifying where learners are in their learning, where they are going, and how to bridge the gap have been clearly articulated by Wiliam, and Thompson (2007), see Table 1. Within the matrix formed, are their five "key strategies" of formative assessment.

Table 1: Key Strategies of Formative Assessment

	Where the learner is going	**Where the learner is right now**	**How to get there**
Teacher	1. Clarifying learning intentions and criteria for success	2. Engineering effective class-room discussions and other learning tasks that elicit evidence of student understanding	3. Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	4. Activating students as instructional resources for one another	
Learner	Understanding and sharing learning intentions and criteria for success	5. Activating students as the owners of their own learning	

Black and Wiliam launched programs of work that aimed at engaging teachers in these key strategies, but found that regular meetings over a period of years were needed to enable a substantial proportion of teachers to acquire the "adaptive expertise" (Hatano & Inagaki, 1986) needed for self-directed formative assessment. This is clearly an approach that is challenging to implement on a large scale.

The Mathematics Assessment Project

In 2009, the Bill & Melinda Gates Foundation approached us at Nottingham to develop a suite of "formative assessment lessons" to form a key element in the Foundation's program for "College and Career Ready Mathematics" based on the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). In response, the Mathematics Assessment Project (MAP) was designed to explore how far well-designed teaching materials can enable teachers to make high-quality formative assessment an integral part of the implemented curriculum in their classrooms, even where linked professional development support is limited or non-existent. The lessons are thus designed, not only to provide teachers with diagnostic information, but to enable them use it to move each student's reasoning forward.

To date, we have designed and developed about a hundred formative assessment lessons to support US Middle and High Schools in implementing the new Common Core State Standards for Mathematics (CCSSM). Each lesson consists of student resources and an extensive teacher guide. The data we have does appear to support the assertion that these lessons have enabled teachers to integrate the key strategies for formative assessment, as identified in Table 1, into their normal teaching. The research-based design of these lessons, now called Classroom Challenges, forms the focus of this paper.

A Design-Based Methodology

Our methodology for lesson design was based on design research principles, involving theory-driven iterative cycles of design, enactment, analysis and redesign (Barab & Squire, 2004; Bereiter, 2002; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; DBRC, 2003; Kelly, 2003; van den Akker, Graveemeijer, McKenney, & Nieveen, 2006). In contrast with much design research, we worked to ensure that the products were robust in large-scale use by fairly typical end-users. This is, in fact why some prefer the term "engineering research" to design research (Burkhardt, 2006). Each lesson was developed, through two iterative design cycles, with each lesson trialed in three or four US classrooms between each revision. This sample size enabled us to obtain rich, detailed feedback, while also allowing us to distinguish general implementation issues from more idiosyncratic variations by individual teachers. As we were designing at a distance, revisions had to be based on structured, detailed feedback from experienced local observers in California, Rhode Island and the Midwest. We obtained approximately 700 observer reports of lessons, from over 100 teachers (over 50 schools) using these materials. We also observed many of the lessons first-hand, in UK schools.

In order for feedback to be useful in the revision process it had to be specific and reliable, based on a *detailed description* of what happed in each lesson. To meet this challenge, a protocol was developed. Two design questions permeated the protocol: How well did the materials communicate the formative assessment strategies to the teacher? How far was the learning experience profitable for students? The protocol was in three parts. The first part was descriptive, asking for the context, the nature of the students, the environment, the support given to the teacher, followed by a vivid description of the course of the lesson, illustrated by a sample of student work of varied quality. Significant events that might inform the designer were noted. The second part was analytical. Observers were asked for: their overall impressions; deviations from the lesson plan; quality of teacher questioning; quality of student reasoning, explanations, discussion and written work. They were also asked to provide evidence of learning. They were specifically asked about the relevance of the formative assessment opportunities. The third part sought the teacher's views, through an interview after the lesson. Teachers were asked about their lesson preparation, their views on the lesson plan, the lesson and the response of students, and implications for professional development. In developing 100 Classroom Challenges over the course of the project, about 700 such reports were obtained and discussed by the design team. This process enabled us to obtain rich, detailed feedback, while also allowing us to distinguish general implementation issues from idiosyncratic variations by individual teachers. On this basis the lessons themselves were revised, and ultimately published on the web: http://map.mathshell.org.uk/materials/index.php.

Theoretical Framework for Assessment Task Design

Our first priority was to clarify the learning intentions for *Classroom Challenges*. The CCSSM make it clear that the goals of the new curriculum are to foster a deeper, connected conceptual understanding of mathematics, along with the strategic skills necessary to tackle non-routine problems. A particular emphasis is the development of mathematical *practices* that should permeate all mathematical activity. We rapidly found it necessary to distinguish between tasks that are designed to foster conceptual development from those that are designed to develop problem-solving strategies. In the former, the focus of student activity is on the analysis and discussion of different *interpretations* of mathematical ideas, while in the latter the focus is on discussing and comparing alternative *approaches* to problems.

The intention was that concept lessons might be used partway through the teaching of a particular topic, providing the teacher with opportunities to assess students' understanding and time to respond adaptively. Problem solving lessons were designed to be used more flexibly, for example between topics, to assess how well students could select already familiar mathematical techniques to tackle unfamiliar, non-routine problems and thus provide a means for improving their strategic thinking.

The validity of any assessment scheme lies in the design of the tasks, which should reflect the intentions of the curriculum in a balanced way. We therefore begin by describing our task design framework. This is followed by a review of the research we used to design the formative assessment lesson structures within which the tasks are embedded.

(i) Assessment Task Genres for Concept Development

The tasks we selected for concept *Classroom Challenges* were designed to foster collaborative sense-making. Sierpinska (1994) suggests that people feel they have understood something when they have achieved a sense of order and harmony, where there is a sense of a 'unifying thought', of simplification, of seeing an underlying structure and that in some sense, feeling that the essence of an idea has been captured. She lists four mental operations involved in understanding: "identification: we can bring the concept to the foreground of attention, name and describe it; discrimination: we can see similarities and differences between this concept and others; generalisation: we can see general properties of the concept in particular cases of it; synthesis: we can perceive a unifying principle." To this, we would add the notions of representation. When we understand something, we are able to represent it in a variety of ways: verbally, visually, and/or symbolically. In the light of this, we developed four 'genres' of tasks for our concept development lessons (Table 2).

Space dictates that we only provide a few examples. For *Classify and define,* students were typically invited to sort a collection of cards showing mathematical objects using their own, or given criteria. The results of their sorting were then offered to other students, who would reconstruct the criteria that had been used. The objects ranged from geometric shapes to algebraic functions. As Zaslavsky (2008) has shown, this is a powerful way of enumerating properties of mathematical objects. Occasionally, students were presented with a mathematical object and were invited to list as many of its properties as possible. The task then became: "do any of these properties, taken individually, *define* the object?" or "do any *pairs* of these properties define the object?" (Figure 1). This resulted in a search for justifications and counterexamples. (This could be very demanding. For example, consider the pair of statements: "When $x = 0$, $y = 0$"; "When x doubles in value, y doubles in value". Do these statements *define* proportion? If not, then find a function that satisfies these statements but is not a proportion). Seeking definitions in this way lies at the very heart of mathematical activity (Lakatos, 1976).

Table 2: Assessment Task Genres for Concept Development

Assessment task genres	**Sample classroom activities.**
Classify and define mathematical objects and structures.	Identifying and describing attributes and sorting objects accordingly. Creating and identifying examples and non-examples. Creating and testing definitions.
Represent and translate between mathematical concepts and their representations.	Interpreting a range of representations including diagrams, graphs, and formulae. Translating between representations and studying the co-variation between representations.
Justify and/or prove mathematical conjectures, procedures and connections.	Making and testing mathematical conjectures and procedures. Identifying examples that support or refute a conjecture. Creating arguments that explain why conjectures and procedures may or may not be valid.
Identify and analyze structure within situations	Studying and modifying mathematical situations. Exploring relationships between variables. Comparing and relating mathematical structures.

Mathematical object	*A square*	*A proportional relationship exists between two continuous variables x and y.*
Properties	Four equal sides Two equal diagonals Four right angles Two pairs of parallel sides Four lines of symmetry ….	The graph of y against x is linear. $y \div x$ always gives the same result. When $x = 0$, $y = 0$ When x doubles in value, y doubles in value When x increases by equal steps then so does y ….

Figure 1: Observe, Classify and define: Listing properties and building definitions

For *represent and translate,* we developed activities that require students to translate between numerical, verbal, graphical, algebraic and other representations. Typically, groups of students were given collections of cards that they were asked to sort according to whether or not the cards convey equivalent representations. Common misinterpretations were foregrounded by including translations that are commonly confused. For example, students were given a collection of four money cards ($100; $150; $160; $200) and a collection of ten 'arrow' cards showing percentage increase and decrease (e.g. "up by 25%"; "down by 25%). They were asked to place the money cards in a square formation and place the percentage cards between them in appropriate places (Figure 2 shows just one side of the 'square'). Typically, students considered "up by 25%" and "down by 25%" to be inverse statements and placed them together between the money cards $160 and $200. Subsequently, the teacher introduced further arrow cards showing "decimal multipliers" (e.g. x 1.25; x 0.8). As students place these, they checked both with a calculator and by relating them to the percentage cards

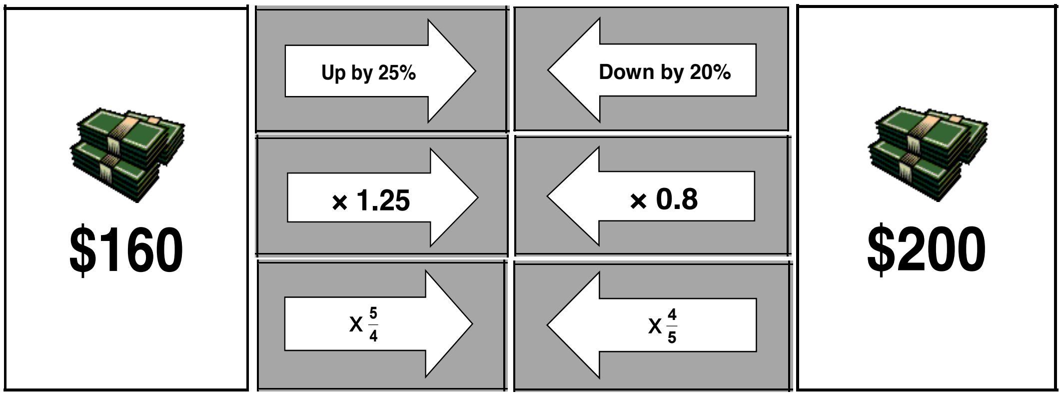


Figure 2: Represent and translate: Percentage increase and decrease.

already in position. This created "cognitive conflict" and discussion as inconsistencies were found. Later, further cards were added, as shown. Connections were drawn between all these representations and generalizations were made.

For *justify or prove* category, we designed collections of conjectures, and it was the students' task to determine their domains of validity. Figure 3 illustrates a typical selection of such assertions.

Pay rise Max gets a pay rise of 30%. Jim gets a pay rise of 25%. So Max gets the bigger pay rise.	**Fractions** If you add the same number to the numerator and denominator of a fraction, the fraction will increase in value.
Area and perimeter When you cut a piece off a shape you reduce its area and perimeter.	**Right angles** A pentagon has fewer right angles than a rectangle.
Diagonals The diagonals of a quadrilateral divide the quadrilateral into 4 equal areas.	**Right triangle** If a right-angled triangle has integer sides, the incircle has integer radius.

Figure 3: Justify or prove: A selection of conjectures to test.

Normally, a set of cards was related to a single mathematical topic, and contained some commonly held beliefs. Students were instructed: *"If you consider a statement to be always true or never true, then try to explain clearly how you be sure. If you think a statement is sometimes true, then try to describe all the cases when it is true and all the cases when it is false."* Thus students had first to identify the variables involved and then test the assertion by constructing examples and counterexamples. In some cases a formal proof could be sought. When students became stuck, the teacher pointed them toward particular cases to test. For example, in *Diagonals*, students often claimed that the statement is true for squares, but not for rectangles. The teacher needed to prompt them to re-consider and then go on to study a wider range of quadrilaterals to try to find *all* cases where the statement was valid.

Finally, we turn to *identify and analyze structure.* When students had tackled a conventional word problem, for example, they were invited to analyze its structure and in so doing construct further problems. The problem was rewritten as a list of variables together with their original values, including the solution to the original problem (see Figure 4). The task was to first describe how each variable might be obtained from the others, then to explore the effect of changing variables systematically. So the teacher erased the profit and asked: "How may this be constructed from the other variables?" (60x4-50 or $p=ns-k$). Then the profit was reinstated and the selling price was erased. How might this be found? ($s=(p+k)/n$). After working through each variable separately, the teacher considered variables in pairs. Suppose *both n* and *p* are erased? How will the profit depend on the number of cards made? Students could then generate a table and/or graph. Finally students might be asked to erase all values and describe the general structure algebraically ($p=ns-k$). This strategy could easily be used whenever students tackle word problems in order to focus more explicitly on structural relationships.

(ii) Assessment Task Genres for Problem Solving

These lessons were designed to assess and improve the capability of students to solve multi-step, non-routine problems and to extend this to the formulation and tackling of problems from the real world. We define a problem as a task that the individual wants to tackle, but for which he or she "does not have access to a straightforward means of solution" (Schoenfeld, 1985). One consequence of this definition is that it is pedagogically inconsistent to design problem-solving tasks for the

Making and Selling Candles

Original word problem

A student wants to earn some money by making and selling candles.
Suppose that he can make 60 candles from a $50 kit and that these will each be sold for $4.
How much profit will be made?

Rewritten problem

		k
The cost of buying one kit	**$**	**50**
		n
The number of candles that can be made with the kit		**60**
		s
The price at which each candle is sold	**$**	**4**
		p
Total profit made if all candles are sold.	**$**	**190**

Figure 4: Identify and analyze structure: Working with word problems

purpose of practicing a procedure or to develop understanding of a particular concept. In order to develop strategic competence, students must be free to experiment with a range of approaches. They may or may not decide to use any particular procedure or concept; these cannot be pre-determined. Problem solving is contained within the broader processes of mathematical modelling. Modelling additionally requires the formulation of problems by, for example, restricting the number of variables and making simplifying assumptions. Later in the process, solutions must be interpreted and validated in terms of the original context. Some task genres and sample classroom activities for strategic competence are shown in Table 3.

Table 3: Task Genres for Problem Solving Lessons

Assessment task genres	Sample classroom activities.
Solve a non-routine problem by creating an extended chain of reasoning.	Selecting appropriate mathematical concepts and procedures. Planning an approach. Carrying out the plan, monitoring progress and changing direction, where necessary. Reflecting on solutions; examining for reasonableness within the context. Reflecting on strategy; where might it have been improved?
Formulate and interpret a mathematical model of a situation that may be adapted and used in a range of situations.	Making suitable assumptions to simplify a situation. Representing a situation mathematically. Identifying significant variables in situations. Generating relationships between variables. Identifying accessible questions that may be tackled within a situation. Interpreting and validating a model in terms of the context.

The essence of a task in this category is that it should be amenable to a variety of alternative approaches, so that students may learn from comparing these approaches. An example of each type is given in Figure 5. The first is a pure mathematics ‘puzzle’ type problem set in an artificial context, that of a playground game. The second, a modelling task, is taken from a real-life context and involves the student in making simplifications and assumptions. Both however may be tackled in a variety of ways. *The playground game* may be tackled by practical drawing and measuring; by repeated use of Pythagoras’ theorem; and also by ‘pure, non-quantitative, geometric reasoning’. *Having Kittens* may be modelled with a wide variety of representations, and therein is its educational value.

The Playground Game

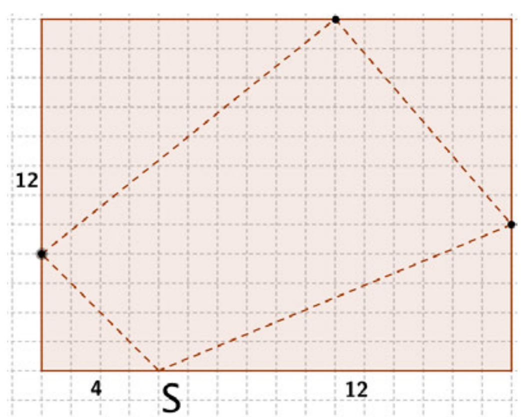
This is a plan view of a 12 meter by 16 meter playground.

The children start at point S, which is 4 meters along the 16-metre wall.

They have to run and touch each of the other three walls and then get back to S.

The first person to return to S is the winner.

What is the shortest route to take?



Having Kittens

Here is a poster published by an organization that looks after stray cats.

Cats can't add but they do multiply!
In just 18 months, this female cat can have 2000 descendants.

Figure out whether this number of descendants is realistic.
Here are some facts that you will need:

Length of pregnancy
About
2 months

Age at which a female cat can first get pregnant
About
4 months

Number of kittens in a litter
Usually
4 to 6

Average number of litters a female cat can have in one year
3

Age at which a female cat no longer has kittens
About
10 years

Figure 5: Tasks for assessing and improving problem solving processes.

Research-based Principles for Formative Assessment Lesson Design

Having discussed the mathematical focus of the tasks we used, we now turn our attention to how these tasks were incorporated into formative assessment lessons.

The principles that underpinned the design of our lessons were rooted in our "Diagnostic Teaching" program of design research in the 1980s. This was essentially formative assessment under another name (See e.g. Bell, 1993; Swan, 2006a). In a series of studies, on many different topics, we began to define an approach to teaching that we showed were more effective, over the longer term, than either expository or guided discovery approaches (Bassford, 1988; Birks, 1987; Brekke, 1987; Onslow, 1986; Swan, 1983). This approach consisted of four phases. The first involved offering a task designed that would expose students' existing conceptual understanding and make students aware of their own intuitive interpretations. The second involved the provocation of cognitive conflict by asking students to compare their responses with those of their peers or by asking them to repeat the task using alternative representations and methods. This feedback generated 'cognitive conflict' as students began to realize and confront the inconsistencies in their own and each others' interpretations and methods. Considerable time was then spent reflecting on and discussing the nature of this conflict and students were encouraged to write down the inconsistencies and possible causes of error. The third phase was whole class discussion aimed at resolving conflict. During this phase the teacher would introduce the mathematician's interpretation. Finally, new learning was 'consolidated' by using the newly acquired concepts and methods on further problems. Students were also invited to create and solve their own problems within given constraints, analyze completed work and diagnose causes of error for themselves.

From these studies it was deduced that the value of diagnostic teaching appeared to lie in the extent to which it assessed, identified and focused on the intuitive methods and ideas that students brought to each lesson, and created the opportunity for discussions between students; the greater the intensity of the discussion, the greater was the impact on learning. This is a clear endorsement of the formative assessment practices described in Table 1.

More recently, these results have been replicated on a wider scale. UK government funded the design and development of a multimedia professional development resource to support diagnostic teaching of algebra (Swan & Green, 2002). This was distributed to all Further Education colleges, leading to research on the effects of implementing collaborative approaches to learning in 40 classes of low attaining post 16 students. This again showed the greater effectiveness of approaches that assess and address conceptual difficulties through student-student and whole class discussion (Swan, 2006a, 2006b; Swan, 2006c). A particular design feature of these lessons was the use of a pre- and post-lesson assessment task that would allow both the teacher and the student to assess growth in understanding. The government, recognizing the potential of such resources, commissioned the design of a more substantial multimedia professional development resource, ‘Improving Learning in Mathematics’ (DfES, 2005). This material was trialed in 90 colleges, before being distributed to all English FE colleges and secondary schools. This material provided many of the resources that where subsequently redeveloped for the Mathematics Assessment Project.

In addition to our own research, we drew inspiration from the ways in which other researchers have structured the design of lessons. These include the Lesson Study research in Japan and the US (Fernandez & Yoshida, 2004; Shimizu, 1999). In Japanese classrooms, lessons are often structured into four phases: *hatsumon* (the teacher gives the class a problem to discuss); *kikan-shido* (the students tackle the problem in groups or individually); *neriage* (a whole class discussion in which alternative strategies are compared and contrasted and in which consensus is sought) and finally the *matome*, or summary, where teachers comment on the qualities of the approaches used. Formative assessment is clearly evident in the way in which the teacher carefully observes students working during the *hatsumon* and *kikan-shido* phases, and selects the ideas to be discussed in the *neriage* stage. The *neriage* phase is considered the most crucial. This term also refers to kneading or polishing in pottery, where different colours are blended together. This serves as a metaphor for the selection and blending of students’ ideas. It involves great skill on the part of the teacher, as she must assess student work carefully then select and sequence examples in a way that will elicit fruitful discussions.

Other researchers have adopted similar models for structuring classroom activity. They too emphasize the importance of: anticipating student responses to demanding tasks; carefully monitoring student work; discerning the value of alternative approaches; purposefully selecting ideas for whole class discussion; orchestrating this discussion to build on the collective sense-making of students by careful sequencing of the work to be shared; helping students make connections between and among different approaches and looking for generalizations, and recognizing and valuing and students’ constructed solutions by comparing this with existing valued knowledge (Brousseau, 1997; Chazan & Ball, 1999; Lampert, 2001; Stein, Eagle, Smith, & Hughes, 2008).

In order to illustrate how these principles, together with the key strategies in Table 1, have influenced the design of our lessons, we now illustrate the design of complete lessons.

Examples of Formative Assessment Lessons

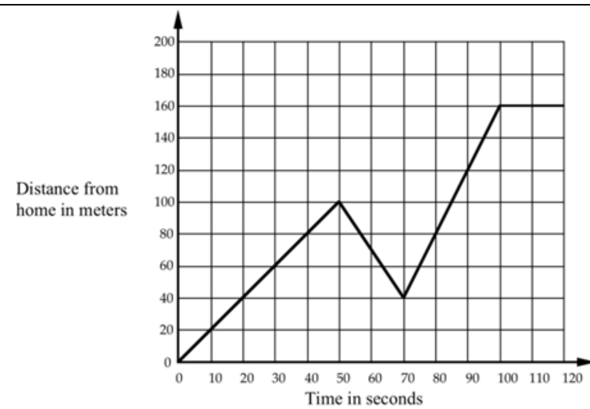
We now illustrate how this research has informed the lesson structure of the *Classroom Challenges*, integrating the formative assessment strategies of Table 1. A complete lesson guide for this and the other lessons may be downloaded from http://map.mathshell.org.

A Concept Development Lesson

The objective of this lesson is to provide a means for a teacher to formatively assess students’ capacity to interpret distance-time graphs. The lesson is preceded by a short diagnostic assessment, designed to expose students’ prior understandings and interpretations (Figure 6). We encourage teachers to prepare for the lesson by reading through students’ responses and by preparing probing questions that will advance student thinking. They are advised not to score or grade the work. Through our trials of the task, we have developed a "common issues table" that forewarns teachers of some common interpretations students may have, and suggests questions that the teacher might pose to advance a student's thinking. This form of feedback has been shown to more powerful than grades or scores, which detract from the mathematics and encourage competition rather than collaboration (Black et al., 2003; Black & Wiliam, 1998). Some teachers like to write their questions on the student work while others prepare short lists of questions for the whole class to consider.

Journey to the bus stop

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day.



1. Describe what may have happened. Include details like how fast he walked.
2. Are all sections of the graph realistic? Fully explain your answer.

Issue	*Suggested questions and prompts*
Student interprets the graph as a picture For example: The student assumes that as the graph goes up and down, Tom's path is going up and down *or* assumes that a straight line on a graph means that the motion is along a straight path.	If a person walked in a circle around their home, what would the graph look like? If a person walked at a steady speed up and down a hill, directly away from home, what would the graph look like? In each section of his journey, is Tom's speed steady or is it changing? How do you know? How can you figure out Tom's speed in each section of the journey?
Student interprets graph as speed–time The student has interpreted a positive slope as 'speeding up' and a negative slope as 'slowing down'.	If a person walked for a mile at a steady speed, away from home, then turned round and walked back home at the same steady speed, what would the graph look like? How does the distance change during the second section of Tom's journey? What does this mean? How can you tell if Tom is traveling away from or towards home?

Figure 6: Initial assessment task: *Journey to school,* and an extract from the 'Common issues table'

The lesson itself is structured in five parts:

1. **Make existing concepts and methods explicit**. An initial task is offered with the purpose of clarifying the learning intentions, making students aware of their own intuitive interpretations, creating curiosity and modeling the level of reasoning to be expected during the main activity (Table 1, strategy 1). The teacher displays the task shown in Figure 7 and

asks students to select the story that best fits the graph. This usually results in a spread of student opinions, with many choosing option B. The teacher invites and probes explanations, and labels the diagram with these explanations, but does not correct students, nor attempt to reach resolution at this point.

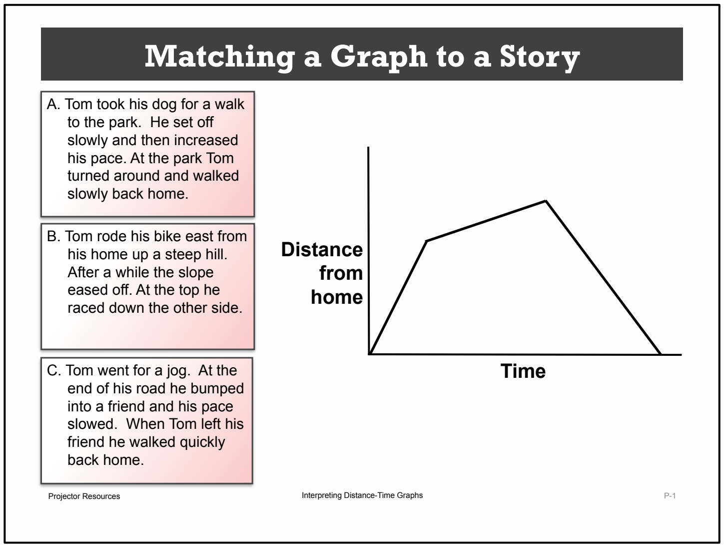


Figure 7: Introductory activity: Interpreting distance-time graphs

2. **Collaborative activity: Matching graphs, stories and tables**.
This phase is designed to create student-student discussions in which they share and challenge each others' interpretations (Table 1, strategy 2). Each group of students is given a set of the cards shown in Figure 8. Ten distance/time graphs are to be matched with nine 'stories' (the tenth to be constructed by the student). Subsequently, when the cards have been discussed and matched, the teacher distributes a further set of cards that contain distance/time tables of numerical data. These provide feedback by enabling students to check their own responses (by plotting if necessary), and reconsider the decisions that have been made. Students collaborate to construct posters displaying their reasoning. While students work, the teacher is encouraged to ask the pre-prepared questions from the initial diagnostic assessment (Table 1, strategy 3).
3. **Inter-group discussion: Comparing interpretations.** Students' posters are displayed, and students visit each other's posters and check them, demanding explanations for matches that do not appear to be correct (Table 1, strategy 4).
4. **Plenary discussion.** Students revisit the task that was introduced at the beginning of the lesson and resolution is now sought. Drawing on examples of student work produced during the lesson, the teacher draws attention to the significant concepts that have arisen (e.g. the connection between speed, slopes on graphs, and differences in tables). Further questions are posed to check learning, using mini-whiteboards. "Show me a distance time graph to show this story"; "Show me a story for this graph"; "Show me a table that would fit this graph". (Table 1, strategy 2)
5. **Individual work: Improving solutions to the pre-assessment task.** Students now revisit the work they did on the pre-assessment task. They describe how they would now answer the task differently and write about what they have learned. They are also asked to solve a fresh, similar task (Table 1, strategy 5).

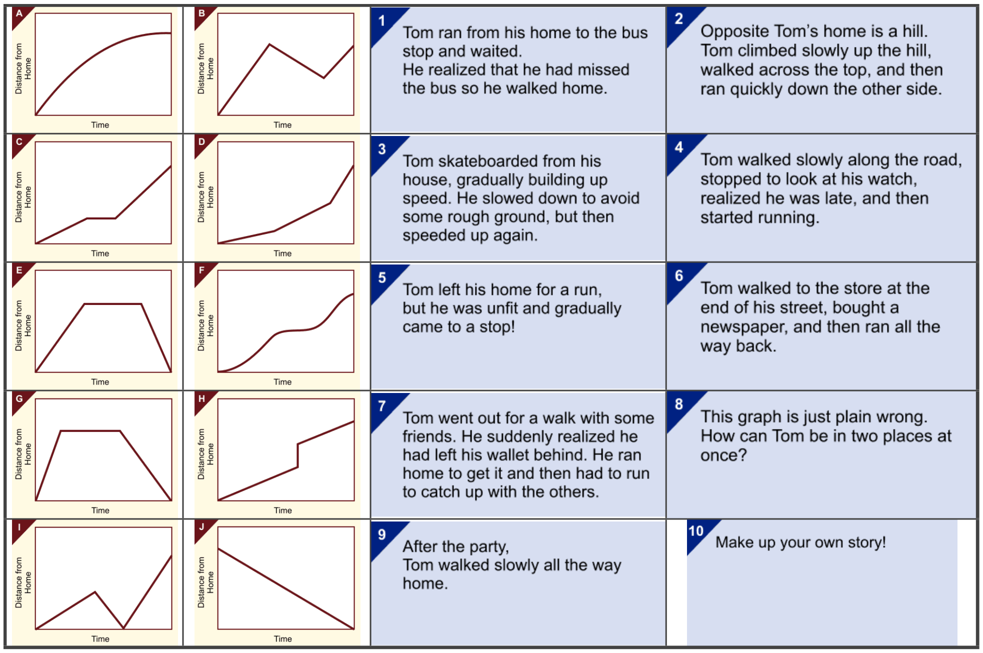


Figure 8: Matching cards: Graphs and stories.

A Problem Solving Lesson

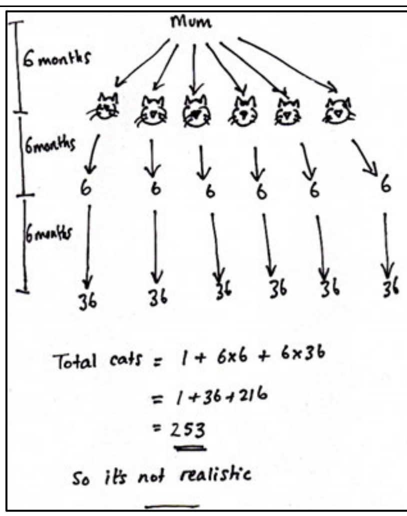
The problem solving lessons were constructed in a similar way, but with a different emphasis. Teachers found it very difficult to interpret, monitor and select students' extended reasoning during a problem-solving lesson. We therefore decided again to precede each lesson with a preliminary assessment in which students tackle the problem individually. The teacher reviews a sample of the students' initial attempts and identifies the main issues that need addressing. This time the issues focus on *approaches* to the problem. If time permits, teachers write feedback questions on each student's work, or alternatively prepare questions for the whole class to consider. Figure 9 illustrates some of the common issues and suggested questions for the task "*Having Kittens*" (Figure 5).

Issue	*Suggested questions and prompts*
Has difficulty starting	Can you describe what happens during first five months?
Does not develop a suitable representation	Can you make a diagram or table to show what is happening?
Work is unsystematic	Could you start by just looking at the litters from the first cat? What would you do after that?
Develops a partial model	Do you think the first litter of kittens will have time to grow and have litters of their own? What about their kittens?
Does not make clear or reasonable assumptions	What assumptions have you made? Are all your kittens are born at the beginning of the year? Are all your kittens females?
Makes a successful attempt	How could you check this answer using a different method?

Figure 9: An extract from the 'Common issues table' for *Having Kittens*

Now we come to the lesson itself. While the precise structure is problem-specific, these lessons are generally structured as follows:

1. **Introduction**. The teacher re-introduces the main task for the lesson and returns students' work along with the formative questions. Students are given a few minutes to read these questions and respond to them, individually (Table 1, strategy 3)
2. **Group work: comparing strategic approaches**. The students are asked to work in small groups to discuss the work of each individual, then to produce a poster showing a joint solution that is better than the individual attempts. Groups are organised so that students with contrasting ideas are paired. This activity promotes peer assessment and collaboration. The teacher's role is to observe groups and challenge students using the prepared questions and thus refine and improve their strategies (Table 1, strategy 2).
3. **Inter-group discussion: comparing strategic approaches**. Depending on the range of approaches in evidence, the teacher may at this point ask students to review the strategic approaches produced by other groups in the class, and justify their own. (Most will not have arrived at a solution by this stage). If there is not a sufficient divergence of methods, or more sophisticated representations are not becoming apparent, then the teacher may move directly to the next stage. (Table 1, strategy 4)
4. **Group work: critiquing pre-designed 'sample student work'**. The teacher introduces up to four pieces of "sample student work", provided in the materials (Figure 10). This work has been chosen to highlight significant, alternative approaches. For example, it may show different representations of the situation. Each piece of work is annotated with questions that focus students' attention. (E.g. "What has each student done correctly? What assumptions have they made? How can their work be improved?") This intervention is discussed further in the following section.
5. **Group work: refining solutions**. Students are given an opportunity to respond to the review of approaches. They revisit the task and try to use insights to further refine their solution (Table 1, strategy 4).
6. **Whole class discussion: a review of learning**. The teacher holds a plenary discussion to focus on the processes involved in the problem, such as the implications of making different assumptions, the power of alternative representations and the general mathematical structure of the problem. This may also involve further references to the approaches in the sample student work.



Questions for students

- What has Wayne done correctly?
- What assumptions has he made?
- How can Wayne's work be improved?

Notes from the teacher guide

Wayne has assumed that the mother has six kittens after 6 months, and has considered succeeding generations. He has, however, forgotten that each cat may have more than one litter. He has shown the timeline clearly. Wayne doesn't explain where the 6-month gaps have come from.

Figure 10: Sample work for discussion, with commentary from the teacher guide.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

The above lesson description contains many features that are not common in mathematics teaching, at least in the US and UK. There is a strong emphasis on the use of preliminary formative assessment, which enables the teacher to prepare for and adapt interventions to the student reasoning that will be encountered. Students spend much of the lesson in dialogic talk, focused on comparing mathematical processes. The successive opportunities for refining the solution enable students to pursue multiple methods, and to compare and evaluate them. Finally, designed ‘sample student work’ is used to foster the development of *critical competence*. This aspect has become the focus of our recent research, and we now draw out some of the issues this raises.

Students Assessing Student Work

In Cobb’s terms, the products of design research are ‘humble’ theories that guide future designs (Cobb et al., 2003). As we have worked through successive refinements, many of the findings from the data have been incorporated into the designs themselves. Below we just one of the features of these lessons that we are continuing to study further (Evans & Swan, 2014); that of students critiquing pre-designed ‘sample student work’.

Researchers (e.g. Stein et al., 2008) have emphasised the importance of students assessing approaches to cognitively demanding tasks, but this has proved difficult for teachers to put into practice, particularly for problem solving, where student reasoning is extended, complex and often poorly articulated. In a busy classroom, teachers find it difficult to observe, interpret and select suitable work for sharing. In whole class discussions we frequently observe students presenting posters of their reasoning, to a sea of incomprehension. Teachers also find it difficult to quickly recognize and make connections between students’ ideas and draw out significant learning points. It is therefore understandable that, in practice, the sharing of ideas often degenerates into mere ‘show and tell’, with participation prioritized over learning (Stein et al., 2008).

In response to this challenge we are researching the potential uses of pre-designed ‘sample student work’ to focus classroom discussion on key concepts and processes, while at the same time developing critical competence. We construct this work by analyzing a sample of genuine student responses to a problem, then identifying conceptual difficulties or problem solving strategies that will provide significant learning opportunities for students. When problem solving, for example, very few students autonomously decide to employ an algebraic method (Treilibs, 1979). Given choice they tend to resort to more secure numerical or graphical methods. For this reason we may include an algebraic method among the sample work so that students will be confronted with methods they may not yet have considered. We present this work in clear, legible, handwritten form, to suggest that the work is tentative, open for criticism and improvement. We have found that students feel more able to criticize such work than the work of peers, where social pressures often come into play.

We have found that pre-designed sample student work has many potential uses. In problem solving, for example, it can be used to encourage a student that is stuck in one line of thinking to consider others, to enable comparison of alternative representations and to focus on the identification of modeling assumptions. In concept learning it may be used to draw attention to common mathematical misconceptions and alternative interpretations. Perhaps most importantly, the sample work may provide an opportunity for ‘clarifying our learning intentions and criteria for success’ (Table 1, strategy 1). By assessing the work of others, students become more aware of the criteria by which their own work is judged. Thus, for example, by asking students to compare four methods and judge which is most ‘powerful’, ‘clear’, or ‘elegant’, then they may come to understand what such terms may mean.

In our classroom observations (in the UK and the US), however, we found that there were frequent problems with implementation (Evans & Swan, 2014). These included: students commenting superficially, focusing merely on presentation and clarity; students being given insufficient time to engage with the reasoning presented in the work; students spending time correcting errors rather than focusing on strategy; students not using the work to improve their own solutions; students failing to make comparisons between approaches. In response, we established the following guidelines for the design of sample work:

- Discourage superficial analysis by students, by stating explicitly the purpose of the sample work, and by asking specific questions that relate to this purpose;
- Encourage holistic comparisons by making the sample work short, accessible and clear, and by excluding procedural and other errors that distract attention away from the identified purpose;
- Leave the work unfinished, so that students have to engage with the reasoning in order to complete it;
- Sequence the distribution of the sample student work so that successive pairwise comparisons of approaches may be made;
- Offer students sufficient time and opportunity to incorporate what they have learned from the sample work into their own solutions;
- Offer the teachers support for the whole class discussion so that they can identify and draw out criteria for the comparison of alternative approaches.

When these guidelines were followed, however, we found that critiquing work provides the potential to refocus students' attention away from 'getting answers' towards 'thinking about reasoning' and a deeper awareness of the learning intentions of the teacher and the criteria for success.

Concluding Remarks

In this brief paper, I have attempted to describe how systematic design research has enabled us to tackle a significant pedagogical problem: how might we enable teachers to embed formative assessment practices into their normal classroom practice? I have discussed the five strategies described by Black and Wiliam and shown how these have been integrated into the structure of the *Classroom Challenges*. In particular, I have attempted to show how:

- Learning intentions and criteria for success may be clarified by making use of task genres that *require* the mathematical practices that we seek to foster; by sharing these intentions and modeling the reasoning required at the beginning of lessons; and by encouraging students to focus on criteria for success as they critique and evaluate the work of others.
- Evidence of student understanding may be elicited through: pre-assessment tasks that offer students opportunity to engage with a problem individually, before group discussion takes place; and through group activities that require shared resources and dialogic talk in which students share interpretations and strategies. These give the teacher opportunities to reflect on student reasoning and to plan and make appropriate interventions.
- Common issues tables may be used to help teachers plan appropriate feedback that will prompt students to reconsider their thinking and move them forward.
- Students may become instructional resources for one another as they work collaboratively and review and comment on the work of their peers.
- Students may take a greater responsibility for their own learning as they become more aware of what they have learned and what they still need to learn through reflection at the end of lessons and through the matching of their own responses to the designed sample student work.

Of course, we realize that however carefully we design lesson structures, each classroom is unique and teachers will modify what we offer in their own way. Early evidence of their impact is, however, encouraging. Drawing on a national survey of 1239 mathematics teachers from 21 states, and interview data from four sites, Research for Action (RFA, 2015), found that a large majority of teachers reported that the use of the *Classroom Challenges* had helped them to implement the Common Core State Standards, raise their expectations for students, learn new strategies for teaching subject matter, use formative assessment; and differentiate instruction.

The National Center for Research on Evaluation, Standards and Student Testing (CRESST) examined the implementation and impact of *Classroom Challenges* in 9th grade Algebra 1 classes (Herman et al., 2015). This study used a quasi-experimental design to compare student performance with *Classroom Challenges* to a matched sample of students from across Kentucky comparable in prior achievement and demographic characteristics. On average, study teachers implemented only four to six *Challenges* during the study year (or 8-12 days), yet, relative to typical growth in math from eighth to ninth grade, the effect size for the *Classroom Challenges* represented an additional 4.6 months of schooling. Although teachers felt that that the *Challenges* benefited students' conceptual understanding and mathematical thinking, they reported that sizeable proportions of their students struggled, and it appeared that lower achieving students benefitted less than higher achievers. This they suggested, may be due to the great difference in challenge and learning style required by these lessons, compared with their previous diet of procedural learning.

Finally, Inverness Research (IR, 2015) in 2014 surveyed 636 students from 31 trial classes (6th grade to High School) across five states in the US. They found that the majority of students enjoyed learning math through these lessons, reported that they understood it better, had increased in their participation, listening to others, and in explaining their mathematical thinking. About 20%, however, remained unaffected by or disaffected with these lessons. This was because they didn't enjoy working in groups, they objected to the investigative approach, and/or they felt that these lessons were too long, or too difficult

In conclusion, it does appear that the *Classroom Challenges* provide a model for teachers as they attempt to introduce formative assessment into their everyday classroom practice, but they do require a radical shift in the predominant culture within many classrooms. The potential for improving learning through the integration of these formative assessment practices into everyday teaching is, however, clear. This project has shown that classroom materials with this focus can help teachers make it a reality in their classrooms. How far teachers transfer this approach into the rest of their teaching is the focus of ongoing research.

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Chapter 2

Special Session

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

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This session will provide information about current funding opportunities in mathematics education research and development from the National Science Foundation. The following programs of particular interest to PME-NA participants will be included in the discussion: EHR Core Research, Discovery Research K-12, and CAREER. Program officers will also provide guidance and suggestions for preparing proposals for a variety of programs.

Currently, the National Science Foundation has a number of programs in the Education and Human Resources Directorate that provide opportunities for funding for the mathematics education research and development across K-12, undergraduate, and graduate education. In this session, program officers will provide information about current program announcements as well as suggestions for preparing proposals. The following programs of particular interest to PME-NA participants will be included in the discussion: EHR Core Research, Discovery Research K-12, and CAREER. Investigators should also refer to other opportunities in the EHR directorate for funding. Investigators should refer to the NSF website (www.nsf.gov) for current solicitation information when preparing proposals for submission. Investigators should also refer to the *Common Guidelines for Education Research and Development* when preparing materials and designing their projects (2013). In this session, we will present an overview of current programs (20 minutes) followed by time for questions from participants (20 minutes). New faculty and graduate students are particularly encouraged to attend.

Chapter 3

Curriculum and Related Factors

Research Reports

Brief Research Reports

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Poster Presentations

COMMUNICATIONS IN CURRICULUM MATERIALS AND TYPES OF STUDENT AUTONOMY PROMOTED

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The interaction between types of communication and socio-mathematical norms on students' learning autonomy in five curriculum programs was investigated. Uni-directional, contributive, reflective, and instructive types of communications were present in curriculum programs studied. Investigations in Number Data and Space and Math Trailblazers provided opportunities for students to explain, justify, and compare solution strategies. Math in Focus and Scott Foresman Addison Wesley-Mathematics required explanations, justifications, and comparisons of solution strategies, but those were mainly provided by the teacher. The former and latter programs potentially foster intellectual autonomous and intellectual heteronomous learning in students, respectively, while Everyday Mathematics almost equally supports both.

Key words: Curriculum Analysis; Classroom Discourse; Curriculum

Classroom discourse is critical for sharing, reflecting upon, refining, supporting, and extending students' mathematical ideas (NCTM, 2014), and the ways in which teachers organize classroom discussion influence student thinking. Curriculum materials (CMs) communicate to teachers in ways that can position students as largely independent learners, as learners depending heavily on an authority, or as both types of learners during enactment. Yackel and Cobb (1996) described students who "make sense of explanations, compare solutions, and make judgments about similarities and differences" (p. 466) as intellectually autonomous and otherwise as intellectually heteronomous(i.e., relying heavily on an authority to lead and determine mathematically appropriate ways to act). These descriptions by Yackel and Cobb imply that kinds of engagement opportunities offered to students in the classroom determine the degree of independence provided to students in the learning process. Yackel and Cobb also recommended teachers establish what they call socio-mathematical norms—"normative aspects of mathematical discussions that are specific to students' mathematical activity" (p. 458) in their classroom so that students can decide for themselves what is mathematically appropriate and acceptable with minimal mathematical input from the teacher. Little is known about the interaction between communication types and socio-mathematical norms embedded in CMs and their impacts on students' autonomy. This study examines CMs for types of communication they promote and the kind of learning autonomy fostered. To investigate this, we asked a research question: *What are potential impacts of communication types and socio-mathematical norms embedded in written lessons?* Three reasons suggest that investigating the potential impacts of the interaction between communication types and socio-mathematical norms might benefit the mathematics education community.

First, students need to develop authority in the mathematics they learn, and teachers are in the position to foster desired "students' mathematical authority" (Stein, Engle, Smith, & Hughes, 2008, p. 332). Teachers might be able to do this when curriculum designers embed such practices in CMs. Examining the impact of communication types embedded in written lessons might explain the kind of mathematical authority teachers are expected to promote in students.

Second, Hiebert, Morris, Berk, and Jansen (2007) argued that teachers ought to learn from teaching in order to improve their practice. A possible way of doing this is by communicating to teachers through CMs what counts as a mathematical explanation and justification, which has possibility to develop "teachers' mathematical knowledge for teaching" (Ball, Thames, & Phelps, 2008, p. 394). As teachers understand what counts as mathematical explanation and justification, they might be able to orchestrate classroom interaction to develop mathematical authorities students need to be successful.

Third, Shulman (1986) recommended that teachers should understand both substantive and syntactic structures of mathematical ideas students are to learn. An understanding of ways CMs communicate, what is communicated, and the potential impact on students' mathematical authority may provide insights into whether teachers' learning of substantive and syntactic structures of mathematics is promoted.

Theoretical Perspectives

Researchers have investigated classroom discourse to identify ways in which teachers communicate and how these might affect students' mathematical thinking. Brendefur and Frykholm (2000) identified four hierarchical types of communication between two preservice teachers and their students: *uni-directional*, *contributive*, *reflective*, and *instructive*. They defined uni-directional communication as dominated by teachers; contributive as limited to sharing of ideas with minimal attention given to discussing them; reflective communication makes shared ideas objects of discussion; and instructive communication focuses on students' thought processes that reveal their strengths and limitations. Brendefur and Frykholm argued that classifying these four types is important because they affect classroom norms.

Cobb, Yackel and Wood (1989) and Yackel, Cobb, and Wood (1991) identified examples of socio-mathematical norms in classrooms as explanations, justifications, and argumentation. Yackel and Cobb (1996) argued that socio-mathematical norms focusing on "what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant … acceptable mathematical explanation and justification" (p. 461) create additional learning opportunities for students and can develop their mathematical thinking.

CMs can contribute to teachers' classroom practice by promoting productive communications among teachers and students (Ball & Cohen, 1996; Davis & Krajcik, 2005) and supporting them to establish appropriate socio-mathematical norms in the classroom. As such, it is natural to ask whether CMs explicitly support teachers for effective classroom discourse, in terms of socio-mathematical norms and different types of communication. Because of the critical role CMs can play in influencing classroom practice, we drew on the works of Brendefur and Frykholm (2000) and Yackel and Cobb (1996) to identify types of communications embedded in CMs and potential impacts of communications types and socio-mathematical norms.

Methods

Data for this study were drawn from five curriculum programs used in a larger project, *Improving Curriculum Use for Better Teaching (ICUBiT)*. We selected written lessons from each of the five curriculum programs: (a) *Investigations in Number, Data, and Space*(*Investigations*);(b)*Everyday Mathematics* (*EM)*; (c) *Math Trailblazers* (*MTB*); (d) *Scott Foresman Addison Wesley-Mathematics* (*SFAW-Mathematics*), and (e)*Math in Focus*(*MiF*). The first three programs are reform-oriented and their designs were funded by the National Science Foundation; the fourth is commercially developed and widely used in U.S. classrooms; and the fifth, originally from Singapore, is gradually and steadily gaining prominence in U.S. classrooms. Teacher's guides of these five curriculum programs were analyzed, focusing on the guidance provided in the main portion of lessons—devoted to the main content, excluding routine practice. Fifteen written lessons in grades 3-5 (five per grade) randomly selected from each program were analyzed.

The main parts of each lesson were coded sentence by sentence and the analysis went through five stages. First, each sentence was associated with one of the following codes: (1) Uni-directional – directly speaking to teachers or students (through teacher); (2) contributing –explaining and demonstrating possible ideas; (3) reflecting – engaging students in making sense and generating meaning through use of representations, strategies, and discussions; and (4) instructing –posing situations that encourage students to compare and make judgments. Second, each sentence was again coded for socio-mathematical norms (i.e., forms of students' engagements): (1) making sense of explanations; (2) comparing solutions and strategies; (3) making judgments about similarities and differences; and (4) providing explanations, demonstrations, and justifications. Third, we made a summary for the number and percentage of sentences for each type of communication. Fourth, within each type of communication, we used socio-mathematical norms promoted to determine the degree of possible students' autonomy fostered in each program. When CMs communicate to teachers to engage students in making comparisons of solutions and strategies, providing explanations, demonstrations, or justifications, students appear to be supported to be more intellectually autonomous. Otherwise, they may be moving toward the intellectually heteronomous end, depending on the teacher or an authority to understand and articulate mathematical concepts or ideas. Fifth, for each program, we identified patterns in types of communication promoted and described the kinds of autonomy fostered, as described above, and compared the five programs in terms of kinds of communications and the type of student learning autonomy the programs tend to support.

Results

Table 1 shows that all four types of communications are present in all five curriculum programs, but with varying emphasis. For each program, the most dominant type of communication is *uni-directional*. *Contributive* communication is the least emphasized

Table 1: Percentages of Communication Promoted

Types of Communication	*INVESTIGATIONS*	*SFAW-MATHEMATICS*	*EM*	*MTB*	*MiF*
Uni-directional	60.0	80.5	76.9	84.1	70.2
Contributive	9.0	3.5	4.7	4.6	10.1
Reflective	16.1	5.6	10.7	6.2	6.4
Instructive	14.9	10.4	7.7	5.1	13.3
Grand Total	**100**	**100**	**100**	**100**	**100**

type in all of these programs, although *MiF* allocated a greater portion of its guidance to this type of communication, followed by *Investigations*. However, most of the contributions in *MiF* are from the teacher, whereas in *Investigations* they are from students. The proportion of sentences allocated for *reflective* communication is greatest for *Investigations* and least for *SFAW-Mathematics*. *Investigations* provide students opportunities to provide explanations, justifications, comparison of solution methods, and strategies presented, whereas in *SFAW-Mathematics* these are often provided by teachers. The proportion of sentences allocated for *instructive* communications is greatest for *Investigations* and least for *MTB*. In spite of this proportional difference, *Investigations* and *MTB* lessons guide teachers to use students' thinking to shape subsequent lessons. In other words, what students struggle with is used to design subsequent lessons.

The emphasis on types of communication propagated by each curriculum program indicates the kind of autonomy nurtured in student learning. Figure 1 shows a continuum from intellectually heteronomous to intellectually autonomous and the location of each curriculum program used for this study along this continuum.

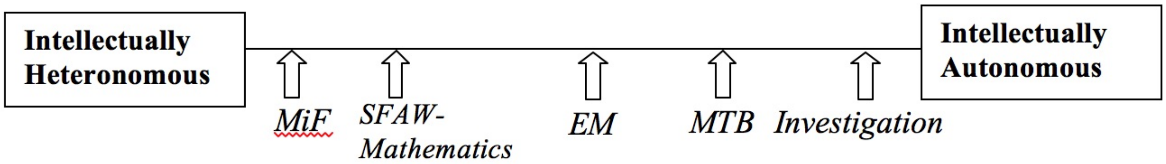

Figure 1: Intellectually Heteronomous-Intellectually Autonomous Continuum.

Intellectually Autonomous

All types of communications in *Investigations* provide students with opportunities to *justify*, *demonstrate*, and *make sense*. Contributive communications in *Investigations* position students to provide their solution strategies and methods, *justifying* and *demonstrating* to others. For example, *Investigations* provides an anticipated student's explanation of how a 120-degree angle is formed as "I put four 30-degree angles (on shape O) together to make 120, because 4 times 30 is 120," which also includes a *justification* of why the strategy and reasoning is mathematically appropriate. Students are also expected to demonstrate their solutions in class with others, as *Investigations* directs teachers to "ask students to demonstrate each of their strategies with the Power Polygon pieces they used." As students provide explanations, justifications, and demonstrations, other students are provided with opportunities to make sense, compare, and identify similarities and differences with theirs.

Instructive communication in *Investigations* provides questions to teachers to determine whether students are *making sense* and *justifying* their solutions. For example, "Do students correctly identify each angle? Do they use what they know about the measure of other angles, such as right angles, to help them find the measure of new angles?" These questions when posed to students provide opportunities to compare and make judgments. Therefore, *Investigations* seems to position students to rely more on their capability, thereby making them intellectually autonomous.

In *MiF*, instructive communication emphasizes *comparing solution methods* to determine which is simpler. During Guided Practice, *MiF* directs teachers to "Have students work in pairs to solve the problem. Have each pair choose a method to solve it. Then have students explain why their method is simpler." In this task, students are required to compare solution methods presented to them by the teacher and determine which is more efficient.

In *EM*, reflective communication suggests that students compare their work, such as "before turning in their work, have students compare their answers with a partner." This has the potential of moving students beyond just comparison of answers to reflecting on their solution strategies in case their answers are different or they have different solution methods to understand mathematical ideas in them. In *MTB*, reflective communication demands that students support their thinking. Questions are provided that asks for mathematical justifications from students for what decision they make. For example, questions such as, "Which fraction is larger: $\frac{1}{6}$ *or* $\frac{1}{4}$? How do you know? Show me with circles," are suggested to teachers to pose to students. *MTB* further asks teachers to "tell students that they are going to defend their choice to the class." These questions could potentially modify students' mathematical understanding by causing them to reflect on a representation and provide substantive arguments to support their reasoning. *MTB* directs teachers to position students to be reflective by letting them know that whatever choice they make needs to be defended. This, in a way, increases the "cognitive demand" (Stein, Grover, & Henningsen, 1996) of the tasks students are engaged with.

MTB supports the use of multiple strategies in classrooms but these come from students, in contrast to *MiF*. For example, *MTB* provides sentences such as, "Did anyone think of it in another way? Can you use another tool?" These different strategies contributed by students could enable them to build their confidence in doing mathematics. It also provides multiple entry points by which students come to understand a mathematical idea rather than depending on the unique approach usually provided by teachers. It also communicates to students that in the absence of the teacher's direct guidance, they could devise valuable and appropriate solution strategies and move their own learning forward.

In some cases, the authors of *SFAW-Mathematics* recommend that teachers provide reflective opportunities to students by asking questions such as, "Suppose the survey question in Example A had been asked to 50 people at a dog show. Would the sample represent the entire population well? Explain." Such questions position students to think about what a sample of a population means and whether the selected sample can adequately represent a population. It also positions students with opportunities to reflect on how a sample can be selected so that it adequately represents that population from which it is drawn. An important idea students must understand before making such a reflection is, what is a population? The authors of *SFAW-Mathematics* direct teachers to explain what an entire population is, in case students do not understand it (example provided under intellectual heteronomous). Although reflective opportunities are provided to students, the authors of *SFAW-Mathematics* often take away such moments and immediately hand them over to the teacher.

IntellectuallyHeteronomous

In *MiF*, contributive communication dominates, but solution strategies and methods presented are mainly from the teacher. For example, sentences such as "Show students how to rename a mixed number as an improper fraction using multiplication and addition with $3\frac{1}{2}$as an example" are used often, and this is followed by the methods which must be demonstrated by the teacher. In *MiF*, reflective communication emphasizes *comparison* of methods used, but the comparison is done mainly by teachers. For example, teachers are asked to "compare this method to the method in the previous *Learn*. Lead students to see that both methods involve multiplication followed by addition." The teacher is expected to *make judgments about similarities and differences*, and learning opportunities for students to develop these understandings themselves are lost. This implies that in *MiF*, students are not provided with opportunities to make sense of solution strategies presented and provide justifications why the methods work. Therefore, students seem to be frequently positioned in a way that makes them rely heavily on an authority, mainly the teacher.

In some situations, *EM* solely positions teachers to be the main authority in class, providing vital mathematical definitions and explanations, and demonstrating solution methods to students. For example, *EM* directs teachers to "explain that a map scale is a tool that helps to estimate real distances between two places shown on a map by relating the distances on the map to distances in the real world." Also, *EM* asks teachers to "model the following solution methods in your discussion." At other times, *EM* provides a step-by-step approach for teachers to lead students through to their solution. The methods teachers are asked to model above for students illustrate this approach. Although these are instructive, most of it comes from the teacher, communicating to teachers that students must depend on them.

In *SFAW-Mathematics*, instructive communication is mainly for teachers to direct students on what to do or provide explanations of mathematical concepts for student learning. The authors of *SFAW-Mathematics* speak to teachers about possible modifications that should be made to foster students' learning of mathematics. For example, they communicate to teachers that "if students do not understand what is meant by the entire population, explain that this is the whole group of people being considered by those who are conducting the survey. For example, it might be everyone living in the United States over the age of 18." Also, the authors of *SFAW-Mathematics* identify possible errors students might make and suggest ways teachers might engage students in fixing them, instead of recommending an instructive approach that positions students to depend on the teacher's authority. For example, *SFAW-Mathematics* authors suggests that "if students cannot decide whether statements are fact or opinion, then point out that words such as best, good, and favorite are clues that a statement is probably an opinion." In this way, the teacher is positioned as the main classroom authority students must look up to in times of difficulties or challenges on certain mathematical ideas. As such, these students might be given the signal that the teacher must evaluate every response as correct or incorrect before they can proceed. Students exposed to this kind of approach might not develop the ability to determine for themselves whether an approach or argument is mathematically sound and accurate.

Discussion/Significance

Many researchers have described curriculum materials as carriers of educational reform because whatever ideas for improvement are conceptualized, they must pass through these materials to get into classrooms. One of the goals of reform efforts has been to make students independent learners. In other words, reform efforts seek to develop students' understanding of mathematical concepts as they reason and provide justification for why an explanation is accurate, decide for themselves what is mathematically correct and acceptable, and critique the reasoning of others students (NCTM, 2000) to learn independently and be certain progress is made. In order for this to happen, teachers need to create such a learning environment. NCTM (2000) recommended that teachers should create environments "in which intellectual risks and sense making are expected" (p. 197).

In this study, some curriculum materials have been found to communicate to teachers in ways that make intellectual risk taking a substantial part of classroom discourse, while others have minimally done so. For example, the authors of *Investigations* and *MTB* have provided teachers with ways students might be thinking about particular mathematical concepts and the depth of the ideas embedded in students' thoughts. The authors of these two curriculum programs have also provided questions teachers might ask to help engage or motivate students to take intellectual risks. In other words, these two programs have communicated the risk-taking environments by providing what students may say and how ideas presented can be deliberated upon. These programs have opened up opportunities for students to justify whatever they say and embrace critique from their peers. These two programs have the potential to empower teachers to support students in figuring out their difficulties and making judgments while depending less on the teacher. Hence, when teachers implement suggestions from curriculum programs similar to these two, focusing on learning goals, students in such classrooms are likely to develop intellectually autonomous habits of mind, monitoring their own progress and making substantial claims of what they understand and what challenges them.

Other curriculum programs such as *MiF* and *SFAW-Mathematics* have attempted to provide opportunities for explanations and justification, but mainly by the teacher. As mentioned above, these programs often require teachers to explain mathematical ideas and strategies to students as well as provide justifications for why approaches they demonstrate are mathematically accurate. These programs implicitly communicate to teachers to take absolute control of the mathematics taking place in their classes. As such, students are to follow, depending heavily on the teacher to "show" them what is correct to copy. Such students are likely to depend greatly on the teacher or any other authority, possibly making them intellectually heteronomous. This does not mean students taught this way might never become intellectually autonomous, but that it might take them a much longer time, thereby achieving this retrospectively and delaying learning of other mathematical concepts. Cumulatively, delays of this sort might have a long-term negative impact on students' learning, causing students to be discouraged about mathematics.

Although many studies have emphasized the kinds of resources CMs make available to teachers (e.g., Ball & Cohen, 1996; Davis & Krajcik, 2005), the study reported here extends our understanding of the kinds of resources that CMs offer to teachers. In addition to overall organization, development of content, and how this content should be taught (e.g., Davis & Krajcik, 2005),mathematics CMs also offer ways teachers might communicate to students to support their learning of mathematical content. Brendefur and Frykholm (2000) identified four types of communication teachers use in their classroom to interact with students. The study reported in this paper found that these four types of communications are provided in all CMs mentioned above. Although Brendefur and Frykholm did not indicate the kind of curriculum materials used by teachers in their study, it is possible the teachers may have been drawing on resources/guidance from their CMs to communicate with their students. Teachers are therefore encouraged to pay close attention to ways CMs communicate with them and how CMs expect them to communicate with students in order to promote student learning, as the communication types identified by Brendefur and Frykholm have been found to create learning opportunities for students if judiciously used.

This study found that CMs can foster both kinds of learning autonomy, although some may strongly propagate one type or the other or even both in some significant proportion (like *EM*). Reform efforts (e.g., NCTM, 2014) support intellectual autonomy for students by recommending that effective teaching provide opportunities for students to share ideas, clarify mathematical understanding, justify their approaches, and also critique ideas of others. This recommendation underscores the importance of autonomous learning that students might achieve and teachers need to support to effectively make student learning of mathematics autonomous. One way to support teachers to gain such knowledge to promote students' autonomous learning of mathematics is by embedding such moves in the teachers' guide. Although some curriculum programs have begun doing this, as the results of this study indicate, it is important to note that curriculum designers may make use of this finding to develop better ways of promoting students' learning autonomy and communicate these to teachers effectively. Hence, these findings are potentially useful for curriculum designers to deliberately position students on the path of intellectual autonomy.

These findings are also potentially useful for teacher educators. Although during teacher training preservice teachers often take a look at curriculum programs, they have done so mainly in the light of preparing lessons to teach during field work. Rarely have preservice teachers been focused on examining the curriculum in the direction of promoting students' learning autonomy. Teacher educators might use these results to engage their preservice teachers in examining curriculum programs for how teachers are positioned to promote learning autonomy in students beyond simply asking students to engage with tasks. This can potentially support teachers to gain skills in assessing curriculum materials for use in schools based on which kind of learning autonomy is fostered.

Providing students opportunities to share their solutions, explain their work, make comparisons, and justify their thinking might not automatically develop the needed intellectual autonomy for students. Although this study has revealed the relationship between types of communications and desired intellectual autonomy and identified the pathway to sustained intellectual autonomy, the following questions still need to be answered: How can teachers effectively foster students' intellectual autonomy in the classroom environment of sharing, comparing, explaining, and justifying reasoning? This question is important because teachers must be able to assess and conclude that their students can work independently and still make sense. Further research involving many classroom observations over an extended period of time might provide results that can fully describe and characterize intellectual autonomy.

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CURRICULUM METAPHORS IN U.S. MIDDLE SCHOOL MATHEMATICS

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We describe two metaphors that we hope can be used to better understand the contemporary mathematics curriculum context in U.S. middle schools, to see how this new context is both similar to and different from prior curriculum contexts. We explain the role and positioning of middle school mathematics curriculum materials over the last century or more and build from learning theory to develop the metaphors. The first metaphor, curriculum as delivery mechanism, builds from technical rational or scientific discourses and encompasses perspectives that are so pervasive they are often unstated and unquestioned. The second metaphor, curriculum as epistemic device, posits that role of curriculum is to provoke interactions that generate understanding. In this metaphor, the role of tasks in curriculum materials is to provoke and progressively refine student thinking, individually and collectively.

Keywords: Curriculum Design; Middle School Mathematics; Learning Theory

Purpose

The U.S. curriculum context is rapidly changing with the increased use of digital and open source materials, the introduction of the Common Core State Standards, and the expansion in the numbers of individuals and organizations involved in curriculum development. In an effort to better understand how this new context is both similar to and different from prior curriculum contexts, we explore two curriculum metaphors that encompass broad trends in the role and positioning of middle school mathematics curriculum materials over the last century or more. The first metaphor, *curriculum as delivery mechanism*, encompasses perspectives that are so pervasive they are often unstated and unquestioned. This dominant metaphor appeals to modernist discourses of science and technical rationalism that permeate educational and other policy contexts (Datnow & Park, 2009) in spite of decades of work to deconstruct such discourses (cf. de Alba et al., 2000). Language referencing the dominant modernist or scientific perspective is often used to market products ostensibly developed using rigorous scientific methods. We deconstruct this metaphor and its manifestations in curriculum materials to highlight its influence on learning opportunities in middle school classrooms. Our goal is to highlight the nature of teachers' work and decisions when working with these materials and the impact of those decisions on the educational experiences of middle school mathematics students. We then describe an alternative metaphor, *curriculum as epistemic device*, which is implicitly if not explicitly evident in some curriculum programs, including those developed as a result of National Science Foundation (NSF) funding in the 1990s and 2000s.

Perspective: Curriculum as Tool

In order to explore how curriculum design gets taken up by various stakeholders, we employ a perspective that considers human cognition in terms of action that is mediated by resources and tools situated in particular contexts (Engestrom, 1999; Pea, 1993; Wertsch, 1998). That is, human knowledge and understanding are manifest in action, which is mediated by the available tools and resources in ways that align with a person's goals and purposes. This contrasts with the idea of cognition as something that strictly happens inside the head of individuals and which has universal attributes. The context in which human action / cognition takes place influences the nature and goals of activity, division of labor, mediating artifacts, and discourse channels (Engestrom; Gee, 1999). With respect to curriculum, resources are deployed or mobilized in relation to the characteristics of the teacher and the curriculum program, prior teacher-curriculum interactions, and the curriculum goals of the teacher (Brown, 2009). Given the situated view of cognition from which we operate, we see teachers as curriculum designers, in that they exercise agency as they draw from resources in curriculum programs to design lessons, inevitably altering the resources in ways big and small (Ben-Peretz, 1990; Remillard, 2005).

The nature of curriculum materials influences the actions of the teacher and transforms both the goals and the activities of the teacher, while simultaneously the teacher employs the resources within curriculum programs according to a given history (personal, organizational, and political) and context that mediate how the resources are taken up (Remillard, 2005). Curriculum resources include representations of mathematics, representations of mathematical tasks and instructional activities, articulation of instructional goals, recommendations for lesson structure and requisite materials, and so forth. Because the enacted curriculum is a dynamic and interactive process of co-constructed activity between teachers and students (Ball & Cohen, 1996), curriculum resources are transformed as they are enacted in classrooms, with the enacted curriculum varying in ways both anticipated and unanticipated by the designers or teacher. The intended written curriculum is thus an inert form of curriculum that becomes 'lived' when enacted in classrooms (Guedet, Pepin, & Trouche, 2012).

Methods/Modes of Inquiry

This paper is primary conceptual and theoretical in nature; nevertheless, we describe the process by which we came to consider these metaphors. First, we turned to prior research which suggests that, in most U.S. middle school classrooms, the typical lesson consists of teachers explaining a topic, modeling a particular procedure or skill, and then having the students independently work on sets of problems around that problem or skill, with minimal solving of complex or novel problems (Jacobs, et al., 2006; Stigler & Hiebert, 1999). This pattern reflects the presentation of mathematics in most middle school curriculum materials, which follow a similar explain-model-independent practice-problem solving (predictably applying the skill just practiced) sequence. This pattern also reflects the delivery metaphor.

The curriculum as epistemic device metaphor emerged from a five-year study of teachers using the Connected Mathematics Program (CMP) curriculum program (Lappan et al., 1998, 2006), conducted by the lead author. In this study, two of the teachers used the materials in ways quite distinct from their counterparts (Author, 2009, 2011a, 2011b, 2011c). These two teachers attended closely to student thinking, typically using the initial tasks in an instructional sequence to elicit students' informal reasoning. Subsequent tasks were used to refine and develop students' reasoning and the language used to describe mathematical concepts and relationships. In short, these teachers used the tasks in the curriculum materials as a means of eliciting and refining students' reasoning, language, and strategies.

A current project involving all four authors and a national sample of middle school teachers using materials from six different curriculum programs has provided corroborating evidence that these metaphors can be used to describe the practices of many of these teachers. However, the metaphors needed a stronger conceptual treatment in order to be useful for developing analytic categories to describe teachers' understanding and use of curriculum materials. In order to develop the metaphors, we have been reviewing literature on the history of mathematics curriculum trends in the U.S., on teachers' understanding and use of curriculum materials, and on theories of learning. Our ongoing synthesis of these literatures is presented below.

Dialogic and Monologic Functions of Text

The two curriculum metaphors are distinguished largely by the extent to which their primary goal is to transmit information or to promote dialogue. Wertsch and Toma (1995), citing Lotman's work, discuss the dual functions of text as monologic and dialogic. The monologic function follows the delivery or conduit metaphor (Reddy, 1979, as cited in Wertsch and Toma), in which the main functions of text are encoding, transmission, and decoding. The monologic function is the primary function of curriculum as delivery mechanism, with an implication that messages as encoded in text or curriculum can be delivered with fidelity. Lotman describes the dialogic function of text in terms of text as *thinking device* in which the "main structural attribute of a [dialogic] text is its internal heterogeneity" (Lotman, p. 37, as cited in Wertsch and Toma) and second is that of a *generator of meaning*. Internal heterogeneity refers to the extent to which different approaches or interpretations are afforded. These different approaches or interpretations are generators of meaning when their differences and underlying similarities are made explicit, in the process generating new understanding or meaning in that community. We thus consider that the dialogic function of curriculum materials is to promote *interactions that generate understanding* and consequently emphasize the term interactions rather than dialogue in our discussion below. Although all texts simultaneously have monologic and dialogic functions, Wertsch and Toma (1995) state that "communication models based on the unidirectional transmission of messages cannot be amended in any simple way to deal with the issue of texts as thinking devices" (p. 166), suggesting that there need to be a priori decisions made about which function to emphasize. Thus, curriculum can be thought of as text that primarily serves as a generator of interactions or as a conduit to transmit information or knowledge, with the design situated near one pole or the other.

Two Contrasting Curriculum Metaphors

Below, we unpack each metaphor in terms of its broad historical and epistemological foundations. We then describe how each metaphor is manifest in existing U.S. middle school curriculum programs. We then describe each metaphor in terms of task design, how teachers are positioned, and the underlying principles of curriculum development. First, we discuss the nature of tasks and how they provide opportunities to initiate dialogue and then we describe how teachers are positioned in this metaphor. We conclude with implications regarding learning experiences engendered by materials from each metaphor, the curriculum features essential to each metaphor, and the kinds of teacher understanding entailed by each metaphor.

Curriculum as Delivery Mechanism Metaphor

The first metaphor, *curriculum as delivery mechanism*, speaks to the perspective that the primary goal of curriculum is to transmit information and knowledge. In the U.S., this perspective has a rich and long tradition. Curriculum as delivery mechanism stems from the technical rational approach to curriculum that has been the predominant curriculum perspective over the last century in the U.S. and was first espoused in the early 1900s (Kliebard, 1975). Early adherents to the technical rational (also termed scientific) approach, such as Bobbitt (1918, 1924) and Charters (1923) describe curriculum development as entailing a highly detailed analysis of disciplinary experts' knowledge and performance, rather than activity from the perspective of the child. Rigorous analysis of expert knowledge and task analysis of expert performance ostensibly are performed to identify the discrete bits of knowledge and skills that, when mastered, constitute competence in a discipline (e.g., the mastery perspective). Of this approach Gravemeijer (2004) states:

> Older design principles take as their point of departure the sophisticated knowledge and strategies of experts to construe learning hierarchies… The result is a series of learning objectives that can make sense from the perspective of the expert, but not necessarily from the perspective of the learner. (p. 106)

The discrete bits of knowledge have the added property of being more easily measured than broader, more complex, and consequently ill-defined knowledge (Eisner, 1967). This property extends the appeal of the technical rational approach to those interested in developing psychometric methods to assess student learning and thus to policy makers who wish to use assessments of student achievement data to gauge teacher and school effectiveness (Datnow & Park, 2009).

This approach appeals to modernist notions of scientific advancement by ostensibly employing disciplinary rigor. Although actual attempts to develop curriculum using this approach have been critiqued as impractical and ultimately subjective (Eisner, 1967; Kliebard, 1975), claims of scientifically-based approaches persisted through the 1900s and are currently evident in the marketing of publisher-developed curriculum programs (defined as those programs developed by large publishers, according to perceived market demand) and neo-liberal educational policies related to assessment and accountability. For example, see the debates around the use of terms such as 'scientifically-based research,' 'evidence-based,' 'high-quality,' or 'rigorous' (Darling-Hammond & Youngs, 2002; No Child Left Behind, 2002; Schoenfeld, 2006; U.S. Department of Education, 2002, 2003). Mainly, the modernist or scientific discourse is used to market an approach or product, while the scientific or rigorous nature of the process is rarely explained or examined in detail.

Curriculum development in the technical rational approach embodies the delivery metaphor, in which knowledge can be detached from an authority or expert (i.e., textbook, teacher) and transmitted to the novice learner (student), what Jackson (1986) calls the mimetic tradition. In the technical rational approach, expertise flows directly from the expert or authority to the learner, allowing those far from classrooms to exert control over content (Datnow & Park, 2009), thus minimizing the role of the teacher. Schoenfeld (2006) describes what he terms traditional U.S. curriculum materials in the following way:

> For most of the 20th century, the dominant perspective on learning in most fields, and specifically in mathematics, was that learning is the accumulation of knowledge; that practice solidifies mastery; and that knowledge is demonstrated by the ability to solve particular (well-studied) classes of problems. (p. 15)

The mastery perspective in this metaphor focuses at the scale of lesson or topic, with mastery expected on one topic before proceeding to the next. Furthermore, the transmission approach inherently entails a deficit view of the learner. The curriculum design is based on explaining and modeling concepts and procedures, which presumes that the learner has minimal understanding of the subject matter or intuitive understandings on which to base instruction. The treatment of language in curriculum materials from the delivery metaphor mirrors the treatment of mathematics content. There is typically an emphasis on early formalization and precision, with little validation of less formal or everyday terminology. In general, terms are defined and explained before students have had opportunities to explore the content.

The delivery approach is so pervasive that there is typically a minimal effort to explain the learning model beyond appeal to a mastery perspective. That is, there is little overt description of an instructional philosophy or theory of learning in the curriculum materials, especially publisher-developed programs. Furthermore, authorship of the materials is often anonymous, with only a listing of the experts consulted during the development process. Distinctions between curriculum programs developed through this approach usually entail the scope and sequence of content, the aesthetics of the materials, and the ancillary materials that are emphasized by publishers to market the materials.

Dissatisfaction with materials developed from this perspective has been longstanding, widespread, and multifaceted, with critiques focusing on the passive nature of student activity (National Council of Teachers of Mathematics, 1989), the coherence and rigor of the materials (Schmidt, McKnight, & Raizen, 1997), unexamined issues of power and identity (c.f. Gutierrez, 2002), and the limited role of the teachers as presenters of content (Confrey, et al., 2008). This dissatisfaction has led to numerous attempts to transform mathematics curriculum, including systematic and well-funded efforts, such as those associated with the New Math era and the programs funded by the National Science Foundation in the 1990s.

Curriculum as Epistemic Device

We conceptualize an alternative metaphor as *curriculum as epistemic device*, in which the primary goal of curriculum is to provoke interactions that generate understanding. In this metaphor, the role of tasks in curriculum materials is to provoke and progressively refine student thinking, individually *and* collectively, as opposed to serving as a delivery mechanism for content. This conceptualization of curriculum design builds from a notion of text as *thinking device* that promotes dialogic interaction (Wertsch & Toma, 1995).

A primary characteristic that shapes task affordances in this metaphor is the potential for heterogeneous approaches that vary in terms of their entry points and sophistication, or what has been called *low-threshold, high ceiling* tasks (Myers, Hudson, & Pausch, 2000). Myers and colleagues described software interfaces in terms of being *low-threshold and high ceiling*, meaning that the software was relatively easy to learn at a basic level but could be used to accomplish complex and difficult problems. This idea can be applied to tasks that are accessible to intuitive approaches while also allowing for the possibility of more abstract or symbolic approaches. Comparing intuitive approaches to more abstract or symbolic approaches creates opportunities for making connections that promote conceptual understanding.

Kapur and Bielaczyc provide insights into the dialogic potential or affordances that result from low-threshold, high ceiling tasks (Kapur, 2008; Kapur & Bielaczyc, 2012). Such tasks afford opportunities for students to initially attempt a problem before encountering challenges that require additional personal and collective resources. Kapur and Bielaczyc refer to the phenomenon of allowing students to reach the limit of their current resources and understanding before seeking assistance as *productive failure*. Productive failure typically entails the use of informal approaches and invented representations or over-generalized application of previous skills that are eventually contrasted with properties of more productive representations or efficient approaches, which helps students to better understand the conceptual properties of a given representation or approach. Students who were allowed to reach productive failure on complex tasks called on their individual and collective epistemic resources in ways that helped them connect their evolving mathematical understandings to more conventional and efficient representations and approaches. Students' *epistemic resources* include their intuitive forms of reasoning, their invented representations, and informal language (Hammer, Elby, Scherr, & Redish, 2005; Kapur & Bielaczyc, 2012).

The discussion above emphasizes the emergent and localized construction of knowledge that is associated with dialogic curriculum and instruction. When tasks promote interactions that generate sense-making and afford opportunities for students to draw on and coordinate their epistemic resources, local and idiosyncratic forms of knowledge are more likely to be emphasized as sense-making resources. This process has been conceptualized as *knowledge building* by Scardamalia and Bereiter (2006). They state that schools should focus on the emergent and collective development of understanding in a knowledge-building community so that student thinking is viewed in terms of its epistemic value – its ability to advance the knowledge of the community. This contrasts with the process of evaluating student thinking with respect to conventional knowledge, as is typically done in classrooms. The emergent and collective development of understanding in a knowledge-building community is facilitated when participants see how their ideas build from one another and how they are positioned with respect to more conventional or expert knowledge. Students' solutions serve as e*pistemic artifacts* (Sterelny, 2005, as cited in Scardamalia and Bereiter) that serve to advance the understanding of mathematics in the classroom community.

In summary, the metaphor of curriculum as epistemic device focuses on the dialogic affordances of tasks that offer the potential for interactions that promote understanding through their accessibility, ambiguity, and connections to big mathematical ideas. These low-threshold high-ceiling tasks allow for students to draw on their epistemic resources in ways that contribute to the collective and emergent development of mathematical understanding. This stands in contrast with the more remote authority manifest in the transmission or delivery metaphor.

In this metaphor, teachers are positioned as orchestrators of mathematical discussions (O'Connor & Michaels, 1996). In order for the dialogic affordances of tasks to be mobilized, teachers need to recognize the heterogeneous approaches and the relations between those approaches in order to support the development of dialogue around those approaches. Furthermore, teachers need to understand how student reasoning develops across instructional sequences, which stands in contrast to the much more local conception of mastery in the delivery metaphor. Similarly, students are positioned as active intellectual contributors with challenging *epistemic roles* (O'Connor & Michaels, 1993, 1996).

The development of curriculum in this metaphor departs from the technical-rational or scientific approach in the delivery metaphor. Gravemeijer (1994) describes curriculum development as integrating elements of research and design, in part by conducting design experiments that inform the development of curriculum materials. These design experiments (Cobb et al., 2003) focus on developing *local instruction theories* (Gravemeijer, 2004) that are situated within particular instructional sequences. The instructional sequences are enacted in classrooms, generating data and insights that are used to revise the sequence. This process involves intensive observations of how student thinking is elicited and refined over the sequence, positioning students as key resources not only in classroom enactments but also as dynamic agents in the design experiment. Another feature of curriculum development is the notion of *progressive formalization* (Bransford, et al., 2000) in which instruction elicits and builds from students' informal or pre-formal thinking, which is then progressively refined toward more formal mathematical representations and terminology. Gravemeijer argues that teachers' ability to recognize and build from student thinking is related to their understanding of how that thinking is situated within a broader instructional sequence.

The curriculum as epistemic device approach is based on interactions that promote sense-making or understanding, which inherently involves heterogeneous voices (Wertsch & Toma, 1995), including those based in everyday or informal language. The instructional sequences provide opportunities for this language to be revisited and revised, and a primary role of the teacher is to facilitate the process of language development. Thus, the view on language development in this metaphor contrasts sharply with the delivery metaphor.

Conclusion

We elaborate two metaphors, in part to draw distinctions between two approaches historically evident in U.S. middle school mathematics curriculum materials but also to develop an analytic lens for looking at new materials and technologies related to the rapid and comprehensive move to digital forms of curriculum resources. In many cases, new digital materials have intensified features that follow the delivery metaphor and accompanying technical rational basis. A few prominent programs, such as Khan Academy, deliver mathematical explanations in new platforms, while others situate traditional content in learning management systems (Author, 2014). Few programs as yet provide the potential to elicit and make public student thinking in ways that utilize those approaches as epistemic devices in classrooms.

Moving forward, it will be important to analyze curriculum programs and enactments of those programs with respect to which metaphor prevails. Given the emphasis on knowledge creation in civic and economic life, it is imperative that curriculum resources and associated instructional systems help teachers recognize and build from student thinking in mathematically productive ways.

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GENERALIZING AVERAGE RATE OF CHANGE FROM SINGLE- TO MULTIVARIABLE FUNCTIONS

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This paper explores students' ways of thinking about the average rate of change of a multivariable function and how they generalize those ways of thinking from rate of change of single-variable functions. I found that while students thought about the average rate of change of a multivariable function as the change in the independent quantity with respect to the changes in the dependent quantities, they had difficulty determining a process to assign a value to that rate of change. Most tried to represent the average rate of change as a singular expression, generalizing the Δy/Δx expression to create expressions of the form Δz/[Δx and Δy], yet did not appear to have a sense of what they believed they were measuring. This suggests that quantitative reasoning, or lack thereof, was at the heart of the students' generalizations .A pedagogical implication of this research is that students' natural tendency to try to determine a singular expression for the average rate of change of a multivariable function could serve as useful as motivating the need to hold a variable fixed.

Keywords: Advanced Mathematical Thinking; Cognition; Post-Secondary Education

Introduction

While it is clear to experts how multivariable calculus topics are mostly natural extensions of single-variable calculus topics, how students come to see the relationship between ideas like function and rate of change in single- and multivariable calculus is not well understood. Though some recent advances have been made with regard to student thinking about these ideas, these studies are only preliminary (Kabael, 2011; Martinez-Planell & Trigueros, 2013; Trigueros & Martinez-Planell, 2010; Yerushalmy, 1997). Multivariable functions are used extensively in the sciences, engineering, statistics, and higher mathematics. It is imperative that we learn more about student understanding of multivariable functions, as they serve as the foundational tools by which students make sense of and represent relationships between quantities in complicated systems in these fields. I focus in particular on average rate of change in three dimensions for two reasons. First, a coherent image of it is necessary to understand the mathematical construction of instantaneous rate of change. While average rate of change (e.g. average speed) exists in the real world, instantaneous rate of change is a mathematical construction that relies on anticipating the result of taking averages rates of change over infinitesimally small intervals. Second, since multivariable calculus topics like rate of change build on single-variable calculus topics, I think it is important to study students' understanding of the multivariable topic with respect to their understanding of its single-variable counterpart. That is, I focus on how students generalize ideas, which yields insight into both what students understand about a new idea and how they use prior knowledge in the process of making sense of new content. To address these two aims, I sought to answer the following research question: *How do students think about the average rate of change of a multivariable function and how is this generalized from their understanding of the average rate of change of a single-variable function?*

Background Literature

Findings from literature indicate many difficulties that students have understanding rates of change. Thinking about rate of change as a measurement of how fast quantities are changing is foundational to calculus, yet many students have difficulty reasoning about rate in this way (Rasmussen, 2001; Thompson & Silverman, 2008). Thompson (1994) proposed that understanding constant rate of change depends on coordinated understandings of respective accumulations of accruals in relation to total accumulations. The understanding of constant rate of change he described entails quantities having covaried (Saldanha & Thompson, 1998; Thompson, 2011). This is not at all obvious to students. Rather, some students interpret the average rate of change of a single-variable function as the arithmetic mean of some number of instantaneous rates of change (Bezuidenhout, 1998; Dorko & Weber, 2013). Students may also struggle with computing $\Delta y/\Delta x$ to estimate the rate of change of linear and non-linear functions (Orton, 1983). Thompson (1994) and Zandieh (2000) have suggested that these difficulties may be attributed to not conceiving of rate of change as a ratio, but instead thinking of it as the steepness of a function.

To the author's knowledge, there are two studies about students' thinking about slopes and rate of change in three dimensions. McGee and Moore-Russo (2014) found that at the beginning of instruction about slope in three dimensions, students asked to determine the slope between the points (1,2,1) and (3,2,5) computed $m = \Delta y/\Delta x$. Some of these students found it difficult to understand that this formula did not work in three dimensions. Once students accepted that $m = \Delta y/\Delta x$ did not work in three dimensions, they agreed that the "rise" of the slope should be Δz, but were not sure what the "run" was. Weber (2015) also investigated how students conceived of rate of change in three dimensions, and found that students often sought a way to combine the rates of change of two independent quantities into a single rate of change. My work extends these studies' findings by focusing on students' generalizations of rate of change in addition to their conceptions of it.

Theoretical Perspective

My theoretical perspective drew from two sources, both with constructivist underpinnings. First, I used Ellis' (2007) framework for studying generalization. She defines generalization as the influence of prior activity on novel activity, even if the student's action in the new activity is not mathematically correct. It is important to note that by framing my work in terms of Ellis' framework, I implicitly adopt an actor-oriented perspective (Lobato, 2003) for characterizing students' ways of thinking, which sets aside normative notions of correctness and allows us to focus on *how* students make sense of a situation rather than the *outcome* of that sense-making. Ellis' framework characterizes students' generalizing activity in terms of *generalizing actions* and *reflection generalizations*, which we explain later in the methods. Second, I drew from Thompson's (1990) work on quantitative reasoning, which characterizes a specific way of conceiving of situations. Two key ideas from his work that I draw on here are quantity (a measurable attribute of an object), and quantification (the process of assigning a value to that attribute), which help to explain the different meanings for average rate of change that students generalized, and the different ways in which they attempted to calculate average rate of change.

Data Collection and Analysis

I conducted hour-long semi-structured interviews with eleven students currently enrolled in integral calculus at a large university in the Pacific Northwest. While eleven students is not a particularly large sample size, I believe that these results have what Maxwell (1996) calls *face generalizability*, or there is "no obvious reason *not* to believe that the results apply more generally" (p. 97, emphasis original). That is, even a case study of a few students is likely to generate results that apply more generally. Interviews were recorded with LiveScribe technology, which gives a synched record of students' written work and talk. Interviews were subsequently transcribed for use in coding. In this paper, I focus on students' responses to the following two tasks:

[Q1] Let $V(\mathrm{s}) = s^3$ represent the volume of a cube. What is the average rate of change of the volume of the cube if the length of its sides increases by one?

[Q2] Let $A(\mathrm{L,W}) = LW$ represent the area of a rectangle. What is the average rate of change of the area of the rectangle if the length increases by one and the width increases by two?

I chose area and volume as contexts for talking about average rate of change because I hypothesized that they would be novel contexts and hence allow me to observe students in the process of generalizing. If students became stuck, I asked them one or both of the following problems, which we thought would be more familiar:

[QA] What is the average rate of change of $f(x) = x^2$ over the interval [0,5]?

[QB] Suppose you are in a car and you travel from mile 324 to mile 360 in one hour. What is your average speed?

The V(s) and f(x) function tasks (and other similar area and volume tasks not reported on here) allowed me to observe how students thought about the average rate of change of a single-variable function, while the $A(L,W) = LW$ task (and other similar area and volume tasks not reported on here) allowed me to observe how students thought about the average rate of change of a multivariable function. Comparing students' responses from the single-variable to the multivariable tasks allowed me to analyze how students generalized their thinking about average rate of change from one setting to the other.

I used Ellis' (2007) generalization taxonomy as an analytic framework. This taxonomy distinguishes between *generalizing actions*, or students' mental activity as they generalize as inferred through their activity and talk, and *reflection generalizations,* or students' final statements of generalization. In this paper, I focus only on students' generalizing actions (Table 1) because these reveal student thought during the process of understanding a situation, while reflection generalizations are often summary statements of a generalization. Analysis consisted first of reading the transcripts and coding instances of generalization. I then re-read those instances, and coded them based on the categories shown in Table 1. Due to space limitation, I give examples for only the categories that appeared in this study. The examples in Table 1 are from Ellis' (2007b) paper.

Table 1. Ellis' Generalization Taxonomy (adapted from Ellis, 2007a; 2007b) (Ellis, 2007)

Generalizing Actions		
Type I: Relating	1. Relating situations: The formation of an association between two or more problems or situations.	Connecting back: The formation of a connection between a current situation and a previously-encountered situation. (Example: Realizing that "this gear problem is just like the swimming laps problem we did in class!")
		Creating new: The invention of a new situation viewed as similar to an existing situation.
	2. Relating objects: The formation of an association between two or more present objects.	Property: The association of objects by focusing on a property similar to both. (Example: Noticing that two equations in different forms both show a multiplicative relationship between x and y).
		Form: The association of objects by focusing on their similar form. (Example: Noticing that "those two equations both have one thing divided by another")
Type II: Searching	1. Searching for the same relationship: The performance of a repeated action in order to detect a stable relationship between two or more objects.	
	2. Searching for the same procedure: The repeated performance of a procedure in order to test whether it remains valid for all cases.	
	3. Searching for the same pattern: The repeated action to check whether a detected pattern remains stable across all cases.	
	4. Searching for the same solution or result: The performance of a repeated action in order to determine if the outcome of the action is identical every time.	

Type III: Extending	1. Expanding the range of applicability: The application of a phenomenon to a larger range of cases than that from which it originated. (Example: Having found that the difference between successive y-values is constant for $y = \mathrm{m}x$ equations, applying the same rule to $y = \mathrm{m}x + \mathrm{b}$ equations)
	2. Removing particulars: The removal of some contextual details in order to develop a global case.
	3. Operating: The act of operating upon an object in order to generate new cases.
	4. Continuing: The act of repeating an existing pattern in order to generate new cases.

Results

I found that students primarily thought about the average rate of change as a ratio of changes (the measurement process or quantification), which measured some aspect of an object (either the graph or the growing rectangle). The students made sense of the average rate of a multivariable function by connecting back to prior situations; expanding the range of applicability of average rate of change as the ratio of the change in the independent quantity with respect to the change in the dependent quantity; and relating objects based on their form and property. For example, V6 tried to create an expression to represent her understanding of the average rate of change as meaning "how much is the area growing in regards to the changes in the length and the width". V6 struggled with what to put in the denominator of the expression, and concluded that 3 (the sum of ΔL and ΔW) made sense (Figure 1). The crossed-out part in V6's work is $[(L+1) - L] + [(W+2) - W]$, which indicates that she sought to make an expression of the form ΔA/[an expression she believed combined ΔL and ΔW]. After telling the interviewer that having two different variables was confusing her, V6 crossed out her first expression and wrote 3 because "the total change you're adding one to the length and two to the width and so the total change is three". It was unclear whether she recognized that her original denominator simplified to 3, or if she were not paying attention to that expression and instead thinking about a way to combine ΔL and ΔW.

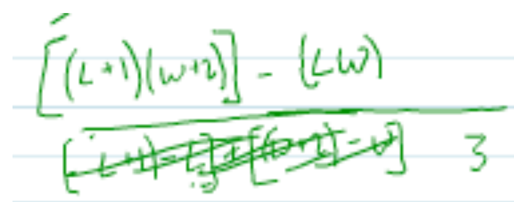

Figure 1: V6's Average Rate of Change of A(L,W)

V10 also constructed an expression of the form ΔA/[an expression she believed combined ΔL and ΔW], choosing to put coordinate pairs in the denominator (Figure 2). She began by writing what she believed to be the expression for the average rate of change of $f(\mathrm{x})$ [see top left of Figure 3] and then tried to construct a version for the average rate of change of $f(\mathrm{L,W})$ [top right of Figure 3], saying "I guess it would just translate over into two variables like that." This statement indicates that she related average rate of change in 2D and average rate of change in 3D as similar situations, and the arrow between the expressions indicates generalization based on the expressions' form (relating objects: form; see Table 1). The lower half of Figure 3 shows V10's attempt to determine an average rate of change for some actual length and width measures (it was unclear why she switched from multiplying the length and width in the crossed-out [(9)(5)] – [(8)(4)] to adding the dimensions in [(9 + 5)] – [(8 + 4)]). It is also unclear why V10 wrote $f'(\mathrm{x}_1)$, $f'(\mathrm{x}_0)$, $f'(L_1,W_1)$, and $f'(L_0,W_0)$, but used $f(L_1,W_1)$ and $f(L_0,W_0)$ in her computation. Other students also thought that average rate of change involved derivative values rather than function values. One student (V11) explained, "I envision derivatives when anything involving change occurs."

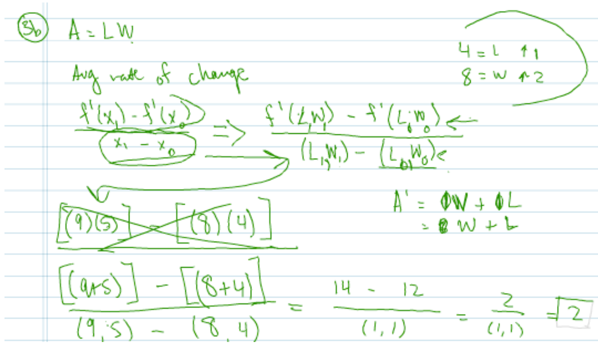

Figure 2: V10's Average Rate of Change for the Area of the Rectangle

While most students identified the average rate of change as meaning the change in area with respect to the change in the length and the width, V10 was the only student who came close to explaining that average rate of change is a *constant* rate. She said

> *V10*: A rate of change would be like at, like at any point it could be a different rate of change, I guess, and then the average rate of change would be more just like the mean rate of change. So like even if they, there might be some different, like some difference, some small difference between every point, you could take the average to try to predict a certain amount to add every time.

I interpret V10's comment about finding 'a certain amount to add every time' as evidence of thinking of average rate of change as a constant rate.

Discussion

Recall that I sought to answer the following question: How do students think about the average rate of change of a multivariable function and how is this generalized from their understanding of the average rate of change of a single-variable function? I found that students think of the average rate of change of a multivariable function as involving changes in three different quantities, and that they attempt to find a single expression to represent it. They generalize the form Δy/Δx to Δz/[a combination of Δx and Δy]. Weber (2015) also found that students' first inclination for the rate of change of a multivariable function is to combine the rates of change of the independent variables.

There is not sufficient evidence to claim that the students think about the average rate of change of a function (multivariable or not) or as a quantification of how variables vary. Indeed, they primarily focused on the structural aspects of calculating the rate of change. Very few students hinted at any notion of many variables varying simultaneously, or that the average rate of change is a constant rate of change. This finding further suggests that rate of change, average or instantaneous, is not about the variation in quantities for students, which led to some of the surprising approaches they used to "find" the average rate of change. In many cases, the students did not have a robust enough image average rate of change as a measure of covarying quantities to support the important conceptual issues one encounters in measuring rate of change in three dimensions.

My finding that students attempted to find the average rate of change of A(L,W) by creating a three-space version of $\Delta y/\Delta x$ is similar to McGee and Russo's (2014) finding that students initially try to find the slope between (x_1, y_1, z_1) and (x_2, y_2, z_2) by computing $m = \Delta y/\Delta x$. A difference is that McGee and Russo's students ignored the z coordinate entirely, while our students created expressions of the form Δz/[some combination of Δx and Δy]. In both cases, however, students tried to leverage their understanding of the 3D scenario by connecting back to their knowledge of slope in 2D. They paid particular attention to the structure of slope as a change in one variable divided by the change in a second variable, and tried to create an expression analogous to $\Delta y/\Delta x$. While this study only reports on the behavior of eleven students, that my finding is similar to what McGee and Moore-Russo (2014) observe supports that the way the students in this study thought and acted with respect to multivariable functions may be representative of students in general.

Students' search for a single expression was largely generalizing by *relating objects* (see Table 1). Generalization by focusing on objects seems to play an important role in students' thinking not only in this setting, but in other cases in which students try to generalize an idea from a familiar $f(x)$ context to the unfamiliar $f(x,y)$ context (e.g., Dorko & Weber, 2014). I hypothesize two reasons for students' attempts to create a single expression. One is that before multivariable calculus most of the concepts students deal with can be represented by a single expression or formula (piecewise defined functions being one exception). I hypothesize that students thus come to expect that there exists one expression for everything. A second reason is that students may not have considered the implications of being in three-space, namely that having two independent variables often necessitates holding one variable constant so that one can talk about the change in the other variable. Supporting this hypothesis, Kabael (2011) found that students' schema for R^3 is critical for their construction of multivariable functions. I think that students' experience in trying to determine what went in the denominator of their expressions could be pedagogically useful as motivating the need to hold a variable fixed. That is, instructors could begin instruction about rates of change of multivariable functions by giving students tasks such as the ones used in this study, letting students discover that it is difficult to talk about two changes at once, and then introduce the idea of talking about a rate of change in a direction. An additional pedagogical implication for precalculus courses and lower-division mathematics is to find other situations in which a single expression is inadequate. This might prevent students from coming to believe that there exists a single equation that describes any given situation.

Regardless of students' difficulty determining what to put in the denominator, it is notable that students tried to compute such an expression in the first place. That is, thinking of the average rate of change as the change in the function values over the changes in the independent variables indicates that multivariable calculus students conceive of average rate of change as a ratio, an understanding that is not always present in single-variable calculus students (Orton, 1983; Thompson, 1994; Zandieh, 2000). In particular, students in this study did not talk about rates of change as "steepness". That students in this study largely generalized their understanding of average rate of change based on the notion of rate-as-ratio reinforces the importance of students developing a robust understanding of slope as a ratio in algebra.

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CATEGORIZING STATEMENTS OF THE MULTIPLICATION PRINCIPLE

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The multiplication principle is a fundamental principle in enumerative combinatorics. It underpins many of the counting formulas students learn, and it provides much-needed justification for why counting works as it does. However, given its importance, the way in which it is presented in textbooks is surprisingly varied. In this paper, we document this variation by presenting a categorization of statement types we found in a textbook analysis. We also highlight mathematical and pedagogical implications of the categorization.

Keywords: Advanced Mathematical Thinking; Curriculum Analysis; Post-Secondary Education

Introduction and Motivation

Consider the following three statements of the multiplication principle[1](MP), seen in Figures 1, 2, and 3. Given that these statements are all meant to describe the same fundamental issue in counting, a number of questions naturally arise. Does Mazur's statement include anything that Roberts and Tesman's does not? Is Bona's set-theoretic statement equivalent to the others? If so, what are pedagogical consequences of such variation? These questions serve as motivation for better understanding how the MP is presented in the current generation of textbooks and what implications such varying formulations might respectively entail. In this paper, we report on a textbook analysis in which we examined statements of the MP, providing a categorization of statement types intended to illuminate mathematical and pedagogical issues related to the MP.

> ***Product Rule:*** *If something can happen in n_1 ways,* ***and*** *no matter how the first thing happens, a second thing can happen in n_2 ways,* ***and*** *no matter how the first two things happen, a third thing can happen in n_1 ways,* ***and*** *..., then all the things together can happen in $n_1 \times n_2 \times n_3 \times \ldots$ ways.*

Figure 1: Roberts & Tesman's (2003) statement of the MP

> ***The Product Principle:*** *In counting k-lists of the form (l_1, l_2, ...,l_k), if*
> 1. *there are c_1 ways to specify element l_1 of the list, and each such specification ultimately leads to a different k-list; and*
> 2. *for every other list element l_i, there are c_i ways to specify that element no matter the specification of the previous elements l_1,...,l_{i-1}, and that each such specification of li ultimately leads to a different k-list,*
>
> *then there are $c_1c_2...c_k$ such lists.*

Figure 2: Mazur's (2009) statement of the MP

> ***Generalized Product Principle:*** *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then the number of k-tuples (x_1, x_2,...,x_k) satisfying $x_i \in X_i$ is $|X_1| \times |X_2| \times \ldots \times |X_k|$.*

Figure 3:Bona's (2007) statement of the MP

TheMP is a fundamental aspect of combinatorial enumeration. It is generally considered to be foundational to many of the major counting formulas students learn and is called by some "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009). Mazur (2009) notes that the MP is "quite flexible and perhaps the most widely used basic rule in combinatorics" (p. 5). Even more, the MP can provide a much-needed source of justification for why many common counting formulas work as they do. For key concepts in other domains (such as limit, derivative, the fundamental theorem of arithmetic, etc.), there tend to be clear, agreed upon, consistent definitions provided in textbooks. However, we have found in our experiencethat textbooks vary widely in how they present the MP. Given the importance and the prevalence of the principle, and given the apparent lack of consistency with which it is presented, we decided to study how the MP is treated in a sample of postsecondary Combinatorics, Discrete Mathematics, and Finite Mathematics textbooks. We answer the following research questions:

1. How is thestatement of the multiplication principle presented in postsecondary Combinatorics, Discrete Mathematics, and Finite Mathematics textbooks?
2. What mathematical issues arise in comparing and contrasting different statements of the multiplication principle?

Literature Review

Counting Problems are Important but are Difficult to Solve

Counting problems foster rich mathematical thinking, and they have a number of important applications. However, correctly solving counting problems is challenging, and there are many studies that report on students' difficulties with counting (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981). Brualdi (2004) says, "The solutions of combinatorial problems often require *ad hoc* arguments sometimes coupled with use of general theory. One cannot always fall back onto application of formulas or known results" (p. 3). Within the last couple of decades, a number of researchers have investigated reasons for students' difficulties and have made progress toward better understanding students' combinatorial reasoning and activity (e.g., Eizenberg & Zaslavsky, 2004; English, 1991; 1993; Maher, Powell, & Uptegrove, 2011; Tillema, 2013). In spite of such work, however, student difficulties with counting persist.

There is a growing body of research suggesting that students may benefit from explicitly thinking about the outcomes they are trying to count. Lockwood (2014) has proposed a set-oriented perspective toward counting, which entails viewing the activity of solving counting problems as inherently involving structuring and enumerating a set of outcomes. The work herein contributes to current literature that frames sets of outcomes as an indispensable aspect of students' counting. In addition, previous work (Lockwood, Swinyard, & Caughman, 2015) has demonstrated the importance of the MP in counting, and the lack of a well-developed understanding of the MP appeared to be a significant problem and hurdle for the students. It is important to note that the MP as a principle of counting is different than the operation of multiplication. We have found in our experience that students can easily assume that they completely understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of multiplication are problematic. We are concerned by the lack of attention students give to the MP, and we argue that the MP is worthy of further investigation. While some researchers have discussed multiplication within combinatorial contexts (Tillema, 2013), there have not yet been studies that explicitly target the MP, and more attention must be paid to the role of multiplication *in counting*.

Textbook Analyses as Insight into How Concepts are Presented

According to Thompson, Senk, and Johnson (2012), "Begle (1973) found that the textbook is 'the only variable that on the one hand we can manipulate and on the other hand does affect student learning' (p. 209)" (p. 254). Thompson et al., go on to point out that textbooks "help teachers identify content to be taught, instructional strategies appropriate for a particular age level, and possible assignments to be made for reinforcing classroom activities" (p. 254). In light of this, a number of researchers have examined textbooks in order to get a better sense of how ideas are presented to students (e.g., Mesa, 2004). At the post-secondary level, this has been seen in the domain of linear algebra (Cook & Stewart, 2014; Harel, 1987), trigonometry (Mesa & Goldstein, 2014), and abstract algebra (Capaldi, 2013). We follow such researchers in using textbooks to gain insight into how mathematical ideas are presented. A potential limitation of this study is that we are simply looking at textbooks, and we cannot make claims about how ideas in textbooks are actually taught to students by an instructor or are understood by students. Nonetheless, a textbook analysis provides an efficient snapshot of how experts in the field of combinatorics define and frame a foundational concept like the MP.

Theoretical Perspective

Structural vs. Operational Conceptions

In Sfard's (1991) presentation of the dual nature of mathematical conceptions, she highlights a relationship between structural and operational conceptions. This dual nature is reflected in the idea that mathematical conceptions can, on the one hand, be considered as *objects* (a structural conception), but that those same conceptions might also be able to be thought of as *processes* (an operational conception). It is interesting that, in her original descriptions of these ideas, Sfard mentions an analysis of textbook definitions:

> The careful analysis of textbook definitions will show that treating mathematical notions as if they referred to some abstract *objects* is often not the only possibility. Although this kind of conception, which from now on will be called *structural*, seems to prevail in the modern mathematics, there are accepted mathematical definitions which reveal quite a different approach. (p. 4, emphasis in original)

Sfard goes on to say that "The latter type of description speaks about *processes, algorithms, and actions* rather than being about objects. We shall say therefore, that it reflects an *operational conception* of a notion" (p. 4, emphasis in original). Sfard (1991) also emphasizes the complementary relationship between the structural and operational conceptions, noting that, "the ability of seeing a function or a number both as a process and as an object is indispensable for a deep understanding of mathematics, whatever the definition of 'understanding' is" (p. 5). This suggests that there could be benefits to having both structural and operational notions of a concept like the MP, something we address in our results and discussion.

Methods

In order to create a broad list of textbooks that were used in postsecondary Finite Mathematics, Discrete Mathematics, and Combinatorics courses, we compiled a list of universities in the union the top 25 ranked universities, the top 25 ranked graduate mathematics programs, the top 10 ranked liberal arts colleges, and the universities with the 10 largest undergraduate populations (National Universities Rankings, n.d., Math, n.d., and National Liberal Arts Colleges Rankings, n.d., respectively). This represented 52 schools in 26 states. We then identified and added to the list the largest university in the remaining 24 states. In total, we surveyed 76 universities representing all 50 states and including some of the top universities in the country.[2]

For each of these 76 universities, we identified courses from the university catalogs and found titles of the required texts from the department's website, the university bookstore, or online course pages. We found textbooks from 70 of these universities. In total, we found three textbooks from one university, two textbooks from 22 universities, and one textbook from 47 universities. We thus identified textbooks for a total of 94 courses at these 70 universities. Multiple universities used many of the textbooks, and so this search yielded a total of 32 textbooks. We also examined relevant textbooks within our own personal and university libraries, and this added 32 textbooks not yet on the list. Therefore, in total we had a set of 64 textbooks, which both provides a sense of how students are being exposed to the MP and also gives a relatively comprehensive picture of ways in which the MP is presented in textbooks. Our analysis and results are based on all 64 of these textbooks. Six textbooks did not include a statement of the MP, and some textbooks included multiple statements, and thus we analyzed a total of 73 statements of the MP in these 64 texts.

Analyzing the Textbooks. Once the list of textbooks was compiled, we digitally scanned the sections of each text that introduced the multiplication principle, including any worked examples and *narratives* (the text surrounding the principle, see Thompson, et al., 2012) that accompanied the statement itself. The authors each independently examined the narrative portion of the texts, including the statements of the MP and any worked examples, recording phenomena and developing categories for what we observed. This is in line with Strauss and Corbin's (1998) constant comparative method of qualitative analysis, where our data consist of the textbook sections. Following the creation of codes, for the sake of reliability, each author analyzed the entire set of texts separately, and we then met and discussed all of the codes until consensus was reached. We also addressed our second research question by more deeply examining mathematical properties of the statements via carefully reviewing and discussing the statements.

Results

Due to space, we share only two aspects of our findings. In this section, we first provide a categorization of statements of the MP (which resulted from investigating the extent to which statements themselves reflect structural versus operational conceptions) and report on the frequencies of statement types. Then, we demonstrate the value of this categorization by highlighting a mathematical implication that emerged from an articulation of the statement types.

Structural versus Operational Conceptions Reflected in the Overall Statements of the MP

Drawing heavily on Sfard (1991), we found that the statements of the MP could be categorizedinto three types: *structural statements*, *operational statements*, and *bridge statements*. Broadly, these three statement types differ in terms of what they state the MP is counting. Structural statements characterize the MP as counting objects (without specifying a process to construct those objects), while operational statements characterize the MP as counting ways to complete a process (without specifying the outcomes of that process). Bridge statements provide a link between the two – they frame the MP as counting objects, but they also specify the counting process that would generate the objects. Thus, in order to code the statements, we looked to see how the statement frames what the MP is counting. Table 2 provides the codes, what we took to be criteria for a statement to receive that code, and an example of a textbook whose statement reflects that code.

Table 2: Criteria for Statement Types

Code	Criteria
Structural	The statement characterizes the MP as involving counting objects (such as lists or *k*-tuples)
Operational	The statement characterized the MP as determining the number of ways of completing a counting process
Bridge	The statement simultaneously characterizes the MP as counting objects and specifies a process by which those objects are counted

Structural Statements. To be coded as a structural statement, a statement had to describe counting a set of objects, without any mention of a process that would generate that set. For example, notice that Bona's structural statement (Figure 3) has characterized the MP as a statement about *k*-

tuples (ordered sequences of length k), which are objects with an inherent structure. The MP describes the total number of k-tuples from k sets, and it is simply expressed by product of cardinalities of k sets. There is no connection made between those k-tuples and a process that would generate them; the statement is simply presented set-theoretically.

Operational Statements. In contrast to Bona's structural statements, the operational statements frame the MP not as counting structural outcomes, but rather as counting ways of completing a process (and a process is clearly articulated in the statement). Roberts and Tesman (2003) provide a statement (Figure 1) that we coded as *operational*, describing the MP in terms performing a task with t successive operations, and the MP provides the number of ways of completing a task. Notice that the nature of what is being counted – the result of the MP is not the number of objects, but rather it is the number of ways of completing a process. Note these two types of statements naturally reflect the duality between structural and operational conceptions that Sfard (1991) proposes. We also note that some textbooks provided both structural and operational in their narratives.

Bridge Statements. A statement like Mazur (Figure 2) on the one hand reflects a structural framing of the MP (the objects being counted are k-lists), but the statement also explicitly describes an operation for how to construct those objects. Such statements, which we call bridge statements, simultaneously both count objects and describe a process by which to count or construct those objects. In Mazur's case, the k-listshe describes are the same object as Bona's (2007) k-tuples. However, unlike Bona, notice that Mazur (Figure 2) describes an operational process that explains how to generate the objects (k-lists) that are being counted – specifically, he describes, "there are c_1 ways to specify element l_1 of the list." The presence of this explicit connection between the structural and operational framings of the statement led us to code this statement by Mazur as a bridge statement.

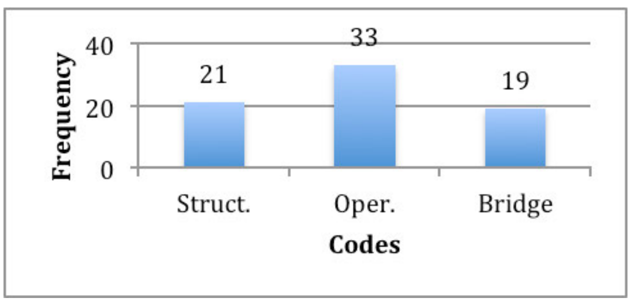

Table 3: Frequencies of structural, operational, and bridge statements (n = 73)

Frequencies. For coding statements at this level, the unit of analysis was a statement of the MP. For any given formulation of a statement, the codes of *structural, operational*, and *bridge* are mutually exclusive, so a statement was coded with exactly one of these codes. Because some textbooks had multiple statements (while some did not include statements), we coded a total of 73 statements across the 64 textbooks. Table 3 shows the respective frequencies of structural, operational, and bridge statements, using the total number of statements as the total frequency.

From Table 3, we observe that operational statements were the most frequent, comprising 45% of the total statements, but each type of statement was represented. These findings convey the variation among statements, supporting the notion that this fundamental counting idea is not presented consistently across textbooks. Through our analysis we also found wide variation in the language used among statements and the representations that accompanied statements, although we do not share those findings here due to space.

Mathematical Implications of Different Statement Types

In this section we address one mathematical implication of different statement types, making a case for what we might gain from a categorization of statement types. As we have noted, the majority of the statements in textbooks are operational, framing the MP in terms of counting the number of ways of completing counting processes that have some number of successive stages. A significant issue with these statements is that they make no claim about whether that total number of ways to complete the process are in a one-to-one correspondence with the desirable set of outcomes, and in fact they make no explicit connection to the overall outcomes of the procedure at all. Given our prior focus on sets of outcomes and their importance (e.g., Lockwood, 2013; 2014), the lack of explicit attention to outcomes is concerning.

For example, Roberts and Tesman's (2003) statement (Figure 1) is strictly operational, and we see that the MP yields the number of ways for "all the things together" to happen, but the statement says nothing about the total number of outcomes. We contrast this with Tucker's bridge statement (Figure 4), which describes a process by which to generate outcomes, not the number of ways to complete the process. In fact, Tucker goes so far as to state that, as a condition of implementing the MP, the "distinct composite outcomes must all be distinct."

> ***The Multiplication Principle:*** *Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has* $r_1 \times r_2 \times \ldots \times r_m$ *different composite outcomes.*

Figure 4 – Tucker's (2002) statement of the MP

In many simple problems, using a strictly operational statement type is not problematic, and any differences between the operational and bridge statements may seem immaterial. For instance, consider the question "*Suppose we flip a coin 10 times in a row. How many possible ways are there to do this?*" Here, we can solve the problem by thinking of ten successive, ordered stages, and each stage has two different possibilities (heads or tails). For both statements, the product that yields the total number of ways for "all the things to happen together" (Roberts and Tesman) is the same as the number of the total "different composite outcomes of the procedure" (Tucker). This "Coin Flips" problem is one in which both types of statements can be applied and yield the correct answer to the counting problem. The number of ways to complete the procedure is in a one-to-one correspondence with the number of desirable outcomes.

However, not all counting problems can be solved in such a straightforward manner. To detail our discussion of this issue, we turn to a "Words" problem presented by Tucker (2002, p. 172): *How many ways are there to form a 3-letter word using the letters a, b, c, d, e, and f, if the word must contain e and repetition of letters is allowed?* In applying an operational statement, we note that the first "thing" that will happen is to decide where to put the *e* that must be in the password, and there are 3 choices for this (the first, second, or third position of our word). The second "thing" is to choose a letter for the leftmost available position, and there are 6 choices, because repetition of letters is allowed. Then, the third thing to do is to choose a letter for the last remaining position, and again there are 6 choices. By an operational statement of the MP, then there are 3*6*6 = 108 ways of completing the process. This is, in fact, true, and there is no claim being made about what this means in terms of distinguishable desirable outcomes.

However, a key aspect of counting is that there is a relationship between a counting process and outcomes associated with that process (Lockwood, 2013). In this "Words" problem, it is true that there are 108 possible ways to complete the three-stage counting process. However, the outcomes of that process are not all distinct: notice that many of the outcomes – those involving 2 or 3 *e*s – would appear more than once in the list of 108 ways to complete the process. Our counting process generated some of the same outcomes more than once. If we simply wanted to count possible ways to complete a procedure, this would not be an issue. However, counting involves specifying the cardinality of the set of outcomes – determining exactly how many of something satisfies certain constraints. Therefore, the fact that in using an operational statement, we are counting ways to complete the procedure, and not actually determining the number of distinct outcomes, is problematic. By only counting ways of completing a counting process, without tying that to outcomes, there is a danger of overcounting when the ways of completing that process are not in one-to-one correspondence with the desirable set of outcomes.

Discussion and Conclusion

By drawing on Sfard's (1991) work in identifying and describing structural, operational, and bridge statements of the MP, we have demonstrated the different ways that the MP may be presented. Operational statements frame the MP as counting the *number of ways* to complete some process, procedure, or sequence of tasks. To us, this reflects viewing the act of counting as involving completing counting processes, but not necessarily about determining the total size of a set of outcomes. Lockwood (2014) has previously demonstrated the value of what is called a *set-oriented perspective*, which frames counting as being about determining the cardinality of a set of outcomes. This stands in contrast to how many of the operational statements of the MP situate the activity of counting. We feel that our findings suggest that, in fact, counting is not always framed as inherently involving counting sets of things, and structural and bridge statements might more naturally align with a set-oriented perspective.

Finally, a major pedagogical implication of our study is that the MP is much more nuanced than instructors and students give it credit for. Given its foundational place in counting, we need to help students focus more on understanding the details of the MP. Because there are clearly a variety of ways to present and talk about the MP, we feel that teachers of counting need to be very explicit with students about what exactly the MP is saying. Instructors could offer multiple statements of the principle and have a clear discussion of what a given statement in terms of ways of completing a counting process versus determining number of distinct outcomes of that process, as we discussed. In addition, instructors should very clearly explain how overcounting can occur in counting situations that involve multiplication. Regardless of which type of statement a student (or an instructor) prefers or which statement their particular book uses, students must be faced with the potential to overcount, and it may be up to the instructor to share this, especially if the book does not address it explicitly.

Endnotes

[1]We follow a number of authors by referring to the principle as the "multiplication principle" throughout the paper, even though the textbooks we surveyed had many different names for it.

[2]There are two ways in which we limited our search. We did not include universities outside of the United States to limit the scope and because we did not feel equipped to linguistically analyze textbooks in other languages. We also did not examine probability textbooks, again to limit the scope of the study, primarily because we suspect that reasoning about multiplication in probability contexts may fundamentally differ from strictly combinatorial contexts.

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EXAMINING K-12 PROSPECTIVE TEACHERS' CURRICULAR NOTICING

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This paper explores the construct of curricular noticing, defined as the act of teachers making sense of the complexity of content and pedagogical opportunities in written or digital curricular materials (Dietiker, Amador, Earnest, Males, & Stohlmann, 2014), and reports the results of four exploratory studies aimed to examine the Curricular Noticing framework. Taken together, these studies capture work done with 62 PSTs in elementary and secondary mathematics methods courses at four universities. Findings illuminate what PSTs attend to in curriculum materials and how they interpret and respond to these materials. Irrespective of level (i.e., elementary, secondary) and materials, PSTs can learn to notice aspects of curriculum materials in order to make decisions about what to do and how to do it, and activities within methods courses can facilitate this development.

Keywords: Curriculum; Teacher Education-Preservice; Teacher Knowledge; Instructional Activities and Practices

Curriculum materials are integral to mathematics instruction. In fact, more than 80% of K- 12 teachers use a textbook or curricular program for mathematics instruction (Banilower, Smith, Weiss, Malzahn, Campbell, & Weis, 2013), though such materials greatly vary in design and philosophy. According to Brown and Edelson (2003), curriculum materials have the most direct influence on what teachers actually plan for and enact in their classrooms and, although research does describe what teachers do with materials, we do not necessarily know the process of how teachers make decisions about what to do and how to do it (Stein, Remillard, & Smith, 2007), and we know even less about how prospective teachers make sense of curriculum materials or use them when enacting instruction.

In this paper, we consider how to make such work explicit through curricular noticing. We define curricular noticing as the process through which teachers make sense of the complexity of content and pedagogical opportunities in written or digital curricular materials. In the following sections we briefly present what researchers have learned thus far about teachers' interactions with curriculum materials, describe our framework and how this contributes to this literature, and present a snapshot of this framework in use by describing four individual studies. We conclude with implications.

Teachers' Use of Curriculum Materials

Research on teachers' use of curriculum materials has presented us with a foundation for describing what teachers do with materials. In the midst of planning and enacting instruction, teachers engage in a variety of activities with curriculum. Remillard (2005) describes the teacher-curriculum relationship as a dynamic transaction in which teachers "participate with" the materials. The socio-cultural conception of this relationship emphasizes the fact that both the teacher and the curriculum influences what and how curriculum materials are used. Using this conception, researchers have outlined ways in which teachers participate with curriculum. This includes the activities teachers engage in such as "reading, evaluating, and adapting" (Drake & Sherin, 2009) and what Brown (2009) describes as "offloading, adapting, and improvising." This research has provided us with a sense of what teachers do with curriculum materials, but we still know little about the process of how teachers make decisions about what to do. To understand how teachers make these decisions we turn to describe the Curricular Noticing Framework.

Theoretical Framing of Curricular Noticing for Mathematics Teaching

Curricular noticing (CN) draws upon the extensive work in professional noticing of children's mathematical thinking (PNCMT), a core instructional activity that is integral to ambitious teaching (Philipp, 2014). PNCMT describes a three-part process of making decisions based on student thinking: *attending to*, *interpreting*, and *responding* to children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010). This process illuminates the phases of work involved in how teachers may leverage children's mathematical thinking. Unless teachers can recognize the complexity of students' mathematical thinking (which includes the diverse strategies and rationales of student ideas), they cannot use this information to inform their decisions.

We argue that this noticing framework may be productively applied to yet another dimension of classroom instruction, the use of curriculum materials. Unlike PNCMT, which focuses on student thinking, CN focuses on curriculum materials. Like PNCMT, we draw upon constructs that illuminate aspects of the work of teaching with curriculum materials: attending, interpreting, and responding. We define each of these aspects in the context of curriculum below.

Attending. Looking at, reading, and recognizing aspects of curricular materials

Interpreting. Making sense of that to which the teacher attended

Responding. Making curricular decisions based on the interpretation (e.g., generating a lesson plan, a visualization, or enactment)

Figure 1 further depicts some of the activities in which teachers might engage in each of the phases of CN.

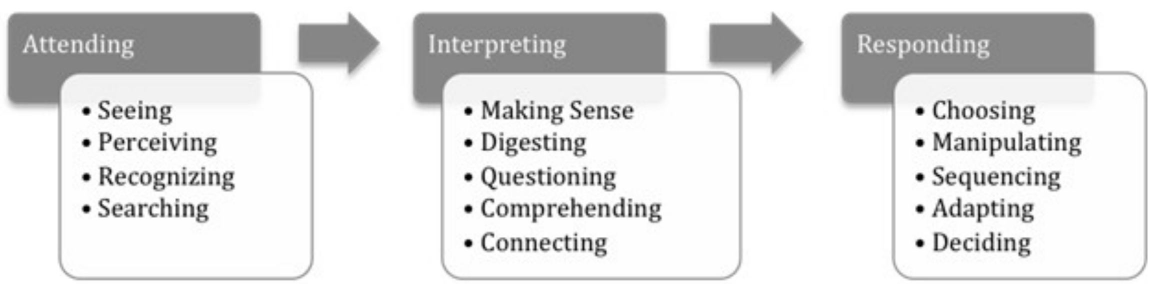

Figure 1. Activities embedded within each of the dimensions of Curricular Noticing.

CN and PNCMT have important commonalities, two of which we highlight here related to (1) the role of tasks and (2) supporting PSTs. First, both CN and PNCMT treat task selection as a necessary and critical component of ambitious teaching. While there have been varied empirical techniques in research on PNCMT, much of this underscores the role of teachers' attending to the mathematics of the present task and interpreting how students interact with the mathematics of that task, and in some cases how to then strategically respond with a new problem, task, or lesson. We see tasks as a critical component of CN as well. Second, both constructs allow the field to consider methods to support PSTs. Cultivating PNCMT practices has been identified as a mechanism to provide PSTs with opportunities to understand student-centered teaching and develop the pedagogical content knowledge necessary for effective and high-leverage instruction (Hill, Ball, & Schilling, 2008; Jacobs et al., 2010). Similarly, we see CN as inextricably linked to these efforts to support PSTs. As teachers make decisions in order to support children's mathematical thinking, curricular materials – specifically teachers' interaction and understanding of the complexity and opportunities reflected in such materials – influence their decisions. In practice, teachers participate or collaborate with curricular materials (Remillard, 2005). Noticing, therefore, is related to both the teacher-student dimension and teacher-curriculum dimension of instructional practices.

We see the CN framework providing a lens for examining not only what teachers do with curriculum materials, but how they do it. Specifically, how may we describe the mechanisms that determine how teachers make particular decisions in their practice? For example, we know some teachers adapt curriculum materials (Brown, 2009; Drake & Sherin, 2011); at the same time, we do not yet know how teachers come to such decisions to adapt. The CN framework allows us describe how teachers make the decision to adapt by considering how teachers' attention and interpretation may lead to such adaptations. We see these actions instead highlighting, more specifically, how teachers are interacting with the text when engaging in the reading and evaluating process, which in turn impacts the responses they make. As described below in our work with PSTs, we argue the phases of CN provide a useful framework for empowering teachers' decision-making, as each phase can be an explicit object of inquiry and development and productive engagement in the phases can help teachers make more informed decisions about how to use their curriculum materials.

Four Studies Aimed at Examining PSTs' Curricular Noticing

Here we present the methods and findings from four independent and exploratory studies, each of which examined CN. The first two studies focused on mathematical tasks and what PSTs can *attend* to in the tasks in order to grapple with identifying affordances and constraints based on the characteristics of these tasks. The second two studies focus on PSTs' *attention to* and *interpretation of* multiple sets of curriculum materials in order to make decisions (or, *respond*) about adoption and lesson planning. Each methods course supported PSTs in inquiry-oriented and student centered mathematics instruction. Studies explored the character of the three phases of noticing, and how these manifest in the context of curricular materials.

Study 1: Noticing Curricular Task Design Features

In Study 1 the task design features to which secondary mathematics PSTs attend to was explored. PSTs (n = 8) in two groups were given one of two versions of the same challenging optimization task, with each version reflecting common presentations in textbooks. One version was open-strategy, prompting students to make a prediction, work together as a group to find a solution, and justify that the result was indeed the optimal solution. The other version was closed-strategy, prompting students to test two possible solutions and use the results of those tests to choose new possible values to test. After each group worked together for 10 minutes on their version of the task, the whole class discussed what mathematical challenges they encountered and what strategies they had used. When the different versions were revealed, they had another five minutes with their group to read through the other version and consider the differences it would have had on their experience solving the tasks.

The comparison of tasks afforded reflections on the interpretation of task design. The whole class held a discussion about what differences they noticed between the two versions of the task. While the mathematical goal of the task (solving for an optimal value) and the context (locating a stereo on a cabinet) were the same, the way the task prompted students to engage resulted in different experiences. Overall, five themes of task design were noticed and mentioned: (1) students pointed out the way in which the design enables or prevents students from following "gut reactions," affecting how the students may engage with the mathematical content of the task. The open-strategy prompt enabled these gut reactions to be followed-up while the given-strategy version encouraged abandonment of potentially fruitful reactions. A teacher with the closed-strategy task commented: "Something interesting was when we were first starting the task, I felt like my gut reaction was to write an equation and graph it to find a minimum, but we were like, 'that's not what we were supposed to do.'" (2) The PSTs pointed to what they called the "heart of the problem," which was taken as the core mathematical point of the task. Fundamentally, the two versions provided different glimpses of what mathematical ideas were in play. Several prospective teachers were disturbed how the design of the task could "obscure" important mathematical ideas. (3) The PSTs explained how the purpose of their work depended on the version they used. The groups that started with the open-strategy task reported testing a point to see what it would tell them, while the closed-strategy group limited to a purpose of following directions. (4) The PSTs noticed the degree to which the task held students accountable for the mathematics. For example, one task explicitly asked “how can you be sure you found the best answer?” while the other just asked for the answer. (5) Several PSTs noted the design constrains what mathematical ideas there is to talk about. They noted that when a strategy is given, the group works in parallel and limited discussion to the verification of answers or how to perform a procedure. They also noted that the opportunity for discussion as a whole class was greatly enhanced when multiple strategies were supported. Findings indicate that such a comparison of tasks afforded interpretations of task design. Next steps will explore how to leverage such interpretations to support teacher decision-making.

Study 2: Noticing Mathematical and Pedagogical Opportunities in Curricular Tasks

The focus of Study 2 was on elementary PSTs’ noticing in the context of fractions, an area of mathematics that is notoriously hard-to-learn and hard-to-teach (Lamon, 1996; Saxe, et al., 2005). In order to empower a teacher to make productive choices in implementing a fractions task, that teacher needs to know something about mathematical properties embedded in—and often hidden in—traditional task design. For example, consider an area model for ¼. The canonical representation features a rectangle or circle divided into four equal sections with one of the four equal parts shaded. Such routine design may obscure two important properties in determining fractional quantities: the role of equal parts and the role of defining the unit (or whole). Study 2 explored how to support PSTs’ *interpretations* of such mathematical properties in routine tasks through the use of nonroutine tasks in the methods course. A premise was that sustained discussion involving a nonroutine task may thereby support teachers’ noticing of—in particular, interpreting—a routine task in terms of core mathematical properties that typically remain hidden.

PSTs (n = 18) were administered a pretest one month prior to and a posttest one month after intervention, each featuring routine and nonroutine fractions representations. In the intervention, all PSTs were asked to identify mathematical properties of two tasks with area models. Task A featured a routine, equally partitioned model with 1/6 shaded. Task B featured a nonroutine, unequally partitioned model with ⅛ shaded. In class, PSTs were asked to analyze each task and anticipate student responses. Results of activities using both routine and nonroutine tasks indicated the vast majority of PSTs did not originally identify *equal parts* or *defining the unit* as mathematical properties of the routine task, yet the majority did so with the nonroutine task.

While a pre-test showed PSTs did not identify equal parts or defining the whole as important mathematical components, a pre-post comparison confirmed PSTs interpreted both routine and nonroutine routine tasks according to these mathematical ideas after intervention. Results of this exploratory study suggest that nonroutine tasks may support PSTs’ interpretations of important underlying mathematical properties of tasks they are likely to encounter in curricular materials.

Study 3: Using a Tool to Examine PSTs Attention to and Interpretation of Curriculum Materials

Within the context of the second of two secondary mathematics methods course, Study 3 examined how PSTs (n=17) evaluated content related to quadratics in three different textbooks. In the first few weeks of a 15-week semester, PSTs were asked to examine the teachers’ guides from Algebra I textbooks in three curricular series: Prentice Hall (PH), The CME Project (CME), and The College Preparatory Mathematics Program (CPM). PSTs were asked first to determine what was similar and different between the three sets of curriculum materials and then to determine which text, if given the option, they would choose to use in their classroom and why. Each PST turned in a written response to these questions. For the next eight weeks PSTs used the CCSSM Curriculum

Analysis Tool (CCCAT, Bush, 2011) to analyze the materials with respect to 1) content, 2) practices, and 3) equity, special needs, and technology. The tool required PSTs to use a rubric to rate each text and to provide qualitative descriptions. Following this analysis, PSTs responded to the same questions from the beginning of the semester. Each pre- and post-tool response was read multiple times to generate initial codes. Each response was then read again and codes were assigned to these responses.

The post-tool responses indicated that, if given the chance, 76% would choose to adopt CPM (compared to 72% before using the CCCAT), 6% would choose CME (no change), and 18% would choose PH (compared to 22%). One important note is that students engaged in this assignment in the second methods course and in the first methods course had engaged in cursory examinations of curriculum materials (without attention to particular content) and also taught from reform-oriented materials in a micro-teaching setting. It is likely that these previous experiences impacted students' choices of materials.

Although there was not much of a difference regarding which text PSTs chose to adopt after engaging with the CCCAT, there was a shift in the reasoning used by PSTs when discussing their choice. In their pre-tool responses, PSTs' responses were quite general in nature and included the general approach of the materials, whether the materials had good or bad teacher resources, the tools included in the materials such as calculator and manipulatives, and the clarity of layout for students. After using the CCCAT, their evaluations were more detailed and they described different aspects of the materials. On average, PSTs wrote 32% more (as measured by number of sentences) in their post-tool response and included more examples from the materials (mostly to illustrate features that they liked). Six out of the 10 most frequent reasons were explicitly aligned to aspects that PSTs were asked to use when evaluating texts using the CCCAT. PSTs made reference to the CCSS Mathematical Practices and the balance between procedural and conceptual opportunities and when referring to the teacher resources described in detail the supports for assessment, differentiated instruction and working with ELL students.

They also commented more on the ways in which technology was integrated, meaning whether it seemed to be an integral part of the text rather than just naming what tools were used in the text. All of these aspects were explicitly addressed by the CCCAT. In addition, however, PSTs also discussed aspects that were not an explicit object of analysis in the CCCAT. PSTs discussed the types of participation structures that were emphasized in the materials, whether detailed lesson plans or suggestions were provided to teachers, the cognitive demand or richness of the tasks, the flexibility (or often lack of flexibility) of the text and the level of planning needed in order to be successful in using the textbook. Although potentially helpful in being able to apply the CCCAT, these aspects were not explicitly addressed, meaning that PSTs were not asked to attend to these aspects in the same ways they were the others. Results indicate that the CCCAT may have aided in shifting what curricular features PSTs attended to and how they then interpreted those features.

Study 4: Responding with Curricular Materials

This study focused on understanding how elementary PSTs made decisions about intended lesson plans as they interacted with multiple curricular resources to further understand the reasons behind their instructional decisions. In this process, close attention was given to what (i.e., the content) the PSTs *attended* to, how they attended to this content (e.g., the degree to which they selected to include in the content in their lesson design), and how they *interpreted* this content with respect to teaching the intended learning outcomes. Finally, there was a focus on how the PSTs *responded* to the selected curricular components to create a plan for teaching with an emphasis on PSTs' decision-making process.

PSTs (n=19) were provided with Grade 6 teacher materials for a lesson on the division of fractions from four curricular programs. PSTs were tasked with using components of any of the resources to write out a detailed lesson plan that would address the following standard: “Apply and extend previous understanding of multiplication and division to divide fractions by fractions” (CCSSM, 2010, p. 42). PSTs were required to provide rationale for their decisions regarding including or excluding particular curricular resources. Following the design of their lesson plan, PSTs were prompted to respond to questions about their use of the resources, motivation for using particular materials, reasons for not including particular materials, and an overall rationale for their decision-making with respect to curricular materials. Often, the PSTs cited their own experiences, belief, and knowledge for their rationale.

Findings indicated that the PSTs noticed curricular components that aligned with their personal conceptualizations about effective mathematics teaching. PSTs considered problems with authentic contexts and problems that involved students through some tangible manipulative to be exemplary components that aligned with effective pedagogical practices. In addition, PSTs based their selections on preconceived notions of what a lesson on the division of fractions should include. For many, they considered how they personally learned the division of fractions and then searched through the materials until they found something closely aligned with their preconceived method of how a lesson on this topic would be taught. In this case, the Curricular Noticing Framework afforded opportunities for understanding how PSTs conceptualized division of fractions after reading multiple curricular resources. Following this, the decisions the PSTs made about how they would respond, or teach the lesson, became transparent. Consequently, past prior experience with the specific content influenced their decision-making process. In contrast, many PSTs documented that the mathematics content was advanced and they had to grapple with the concept of division of fractions before they were able to consider how they would plan a lesson on the topic for sixth grade students.

Implications

Use of the CN Framework identifies the conceptual work involved in translating curricular materials to classroom practice. Our studies offered glimpses of the character of the phases of curricular noticing. We reflect here on both the individual studies described above and what looking across the set of studies as a whole helps us understand about CN.

First, in Study 1, the analysis of different versions of the same mathematical task enabled PSTs to recognize design features of mathematical tasks and connect them to the potential affordances and constraints of teaching with the tasks. PSTs developed a potential lens to critique the tasks of their mathematical curriculum materials. That is, by attending to one aspect of a mathematical task (e.g., how students are held accountable for reasoning), teachers can scrutinize that dimension of the task design (i.e., define how and to what degree students are accountable for mathematical reasoning) and can decide to adapt their task to enhance this quality (i.e., add “Explain how you know” to a task statement).

Second, involving the use of routine and nonroutine problems to support noticing of mathematical and pedagogical opportunities, Study 2 indicated that PSTs may benefit from task exploration that problematizes the big mathematical ideas (in this case involving fractions) embedded in the routine tasks they are likely to encounter in curricular materials. Furthermore, exploration of such tasks may highlight (Goodwin, 1994) the mathematical aspects that are indeed critical to notice in order to choose tasks that anticipate and respond to student thinking.

Third, in Study 3, there was a shift in the reasoning used by PSTs when discussing their choice. This shift suggests that the tool supported PSTs in being able to attend to and interpret curricular features in order to articulate reasons for curriculum evaluations.

Finally, Study 4 indicated that PSTs noticed opportunities that aligned with their concepts about effective mathematics teaching. Many PSTs had some idea of what the target lesson should include and then wrote a lesson plan irrespective of the relation to materials. Findings suggest that PSTs may benefit from further instruction on how to use curricular materials.

These studies are not without limitations. First, the studies were humble in scope, both in terms of working with a single mathematics methods course and without consideration the multiple dimensions involved with CN. As a result, we do not yet fully understand how to support each element of noticing - attending to, interpreting, responding- for PSTs. Second, the contexts for each of the methods courses is varied in terms of level (i.e., elementary, secondary), number of PSTs, and grain-size of materials (i.e., tasks, lessons, units), thereby limiting our capacity to compare across studies.

Despite these limitations, however, the four studies offer a glimpse into the work involved in CN and identify an exciting and important arena in the work of teaching. As mentioned, we do not and cannot know the materials to which PSTs will have access once they have jobs. Yet, we can be confident that most will have some form of curricular materials. Our goal is to understand how to enable PSTs to become strategic and productive users of curricular materials regardless of what those materials are, thereby supporting them to make informed curricular decisions as they teach their students. The first two studies show that by focusing on aspects of tasks that PSTs learned to attend to the mathematics and pedagogical opportunities afforded by or constrained by the tasks. The second two studies indicated that providing PSTs with different curriculum materials and focusing either on lessons or units not only provided PSTs with the opportunity to attend to various mathematical and pedagogical opportunities within the materials, but also required them to interpret the materials in order to respond in some way to a particular question (i.e., What materials would you adopt?) or take a particular action (i.e., plan a lesson).

Taken together these studies indicate that, irrespective of level (i.e., elementary, secondary) and materials, that PSTs can learn to notice aspects of curriculum materials in order to make decisions about what to do and how to do it and that activities within methods courses can facilitate this development. In future research, we hope to further reveal how our framework may be strategically implemented in methods coursework or professional development to support teacher decision-making.

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CURRICULAR TREATMENT OF FRACTIONS IN JAPAN, KOREA, TAIWAN, AND THE UNITED STATES

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This paper investigates how the selected three East Asian countries—Japan, Korea, and Taiwan—introduce and develop ideas related to fractions and fraction addition and subtraction compared to the Common Core State Standards of Mathematics and EngageNY. Looking at curricular approaches used across countries can provide a better picture of what is of importance in instruction aimed at developing students' mathematical proficiency. Understanding how the aforementioned three Eastern Asian materials treat fractions will offer both mathematics teachers and teacher educators some concrete images of the visions of the Common Core State Standards of Mathematics and specific ideas on teaching and learning of fractions.

Keywords**:** Curriculum Analysis; Elementary School Education; Number Concepts and Operations

Introduction

Teaching and learning of fractions in middle grades remain a major challenge for many teachers and students. Developing deep understanding of fractions, which has been identified as a foundation for algebra (Math commission), is a major focus of the Common Core State Standards: Mathematics (CCSSM, Common Core State Standards Initiatives, 2010). The authors of the CCSSM examined the mathematics standards from the high-achieving countries including Asian countries because one finding from the previous cross-national studies is that, in general, United States students do not perform as well as the Asian students in mathematics especially Hong Kong, Japan, Korea, Singapore, and Chinese Taipei (Taiwan) (Mullis, et al., 2008).

The observed performance differences among students in different countries might be attributed to variations in mathematical curricula (Reys, Reys, & Chavez, 2004). A growing body of research has begun to investigate the content of mathematics textbooks as a possible factor for the achievement gaps as reflected in the large international assessments such as Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA). As Kilpatrick, Swafford and Findell (2001) pointed out, "what is actually taught in classrooms is strongly influenced by the available textbooks" (p. 36).

The purpose of this study is to examine how curriculum materials from the selected three East Asian countries—Japan, Korea, and Taiwan--introduce and develop ideas related to fractions and fraction addition and subtraction compared to the recommendations from the CCSSM. In this study, we seeks to understand intended students' learning opportunities in three Asian countries by analyzing how textbooks from these three countries develop a mathematics topic known to be challenging to school children. In particular, we compare and contrast the treatment of fraction concepts and fraction addition and subtractions in the three Asian textbooks compared to that in *EngageNY*. *EngageNY* is curriculum modules and resources in preK-12 developed by New York State Education Department to support teachers implement key aspects of the CCSSM (https://www.engageny.org/). The research questions that guide this study are: 1) What are the similarities and differences of the intended learning progressions of fraction concepts development among the three Asian curricula and those recommended by CCSSM? and 2) What are the similarities and differences in the development of fraction addition and subtraction fluency among the three Asian curricula and those presented in *EngageNY*?

Cross-national comparative studies in the teaching and learning of mathematics provide unique opportunities to understand the current state of students' learning and to explore how students' learning can be improved (Stigler & Hiebert, 1999; Son & Senk, 2010).

Theoretical Background

Teaching and learning fractions has traditionally been problematic. Prior research identified one of the predominant factors contributing to the complexities of teaching and learning fractions lies in the fact that fractions comprise a multifaceted construct (Lamon, 2007). Kieran(1976)articulated that fractions consist of five subconstructs — part-whole, measure, quotient, operator, and ratio. Behr, Lesh, Post and Silver (1983) further developed Kieren's ideas and proposed a theoretical model linking the different interpretations of fractions to the basic operations of fractions as shown in Figure 1.

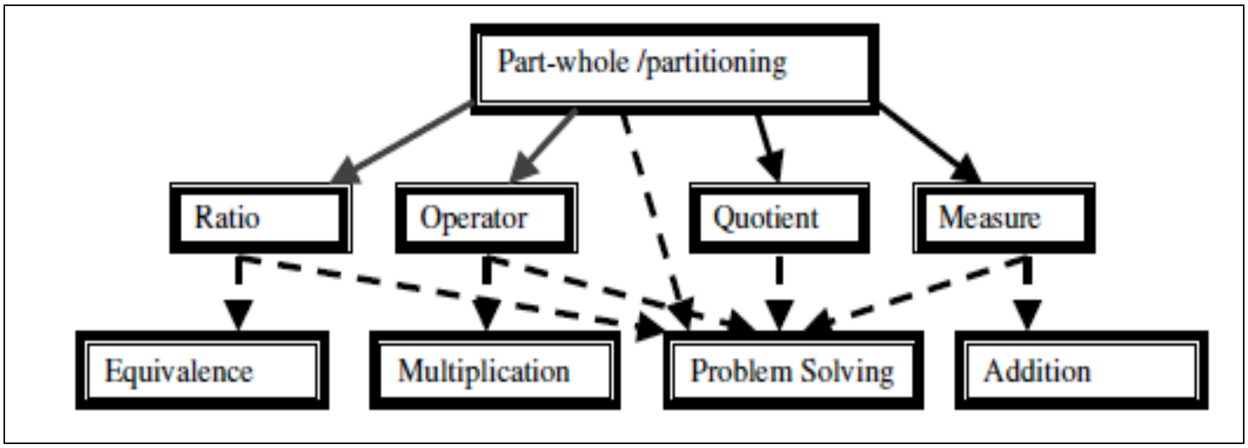


Figure 1. Five subconstructs of fractions and their relationships (Behr, et al., 1983)

According to Behr, et al., the part-whole subconstruct of rational numbers is fundamental for developing understanding of the four subordinate constructs of fractions. Moreover, the operator and measure subconstructs are helpful for developing understanding of the multiplication and addition of fractions, respectively. Although there is a consensus that fraction instruction that focuses solely on the part-whole subconstruct is limiting, there are many unanswered questions about how to incorporate these subconstructs in a mathematics curriculum (Lamon, 2007).

Research on Fractions and Fraction Addition and Subtraction

As mentioned before, over the last three decades researchers and scholars have identified several factors contributing to students' difficulties in learning fractions. The NCTM (2000) provides the following instructional guidelines in developing deep understanding of fractions: (1) begin with a simple contextual task and (2) have students explore each of the operations using a variety of representations and models. We took account of them in our analytical framework. In particular, as word problems serve as a way to contextualize mathematical operations (Carpenter et al., 1999), this study investigated how the meanings of fraction addition and subtraction are addressed and developed through the word problems in each curriculum. The Cognitively Guided Instruction framework (CGI) and CCSSM's problem types, which categorizes addition and subtraction word problems based on the semantic structure of problems shown in Table 1, was utilized for the analysis.

Table 1: Four problem types for addition and subtraction (from Carpenter, et al., 1992)

Problem Type	Unknown Factors		
Join (Add to)	**(Result Unknown)** Connie had 5 marbles. Juan gave her 8 more marbles. How many marbles does Connie have altogether?	**(Change Unknown)** Connie had 5 marbles. How many marbles does she need to have 13 marbles altogether?	**(Start Unknown)** Connie had some marbles. Juan gave her 5 more. Now she has 13 marbles. How many marbles did Connie have to start with?
Separate (Take from)	**(Result Unknown)** Connie had 13 marbles. She gave 5 to Juan. How many marbles does Connie have left?	**(Change Unknown)** Connie had 13 marbles. She gave some to Juan. Now she has 5 marbles left. How many marbles did Connie give to Juan?	**(Start Unknown)** Connie had some marbles. She gave 5 to Juan. Now she has 8 marbles left. How many marbles did Connie have to start with?
Part-Part-Whole (puttogether/take apart)	**(Whole Unknown)** Connie has 5 red marbles and 8 blue marbles. How many marbles does she have altogether?		**(Part Unknown)** Connie has 13 marbles: 5 are red, and the rest are blue. How many blue marbles does Connie have?
Compare	**(Difference Unknown)** Connie has 13 marbles. Juan has 5 marbles. How many more marbles does Connie have than Juan?	**(Bigger Unknown)** Juan has 5 marbles. Connie has 8 more than Juan. How many marbles does Connie have?	**(Small Unknown)** Connie has 13 marbles. She has 5 more marbles than Juan. How many marbles does Juan have?

Method

The primary data source of this study includes the national curriculum guidelines and selected textbook from the three Asian countries. CCSSM and fraction modules from *EngageNY* were analyzed. This study applied the content analysis method to analyze the problems presented in the mathematics textbooks (Confrey & Stohl, 2004). The data analysis of this study went through several iterations with respect to the following aspects:

- Examine the overall curricular flow on fractions – what topics are introduced in which grade
- Analyze in details how textbooks develop the concept of fractions, paying specific attention on fraction subconstructs
- Analyze in details how textbooks develop addition/subtraction of fractions:
- Word problem types, e.g., add to, take from, put together/take apart, compare.
- Types of representations

Note that the textbook series from each country were analyzed in their respective languages by the three authors who are native speakers of the respective languages. Because problem contexts and fraction subconstructs are not always visually verifiable, we needed to calibrate our coding of these two factors. We used the English translation of the Japanese series (Fujii & Iitaka, 2012) and analyzed them independently. We then compared our analysis, and whenever there was a discrepancy in our analyses, we discussed the particular instance until a consensus was reached.

Summary of Findings

Overall Curricular Flow on Fractions

Table 2 shows the overall curricular flow on fractions across the three Asian countries as well as the CCSSM. All Asian curricula shared similar overall flows with some embedded variations. Japanese curriculum introduces the initial concept of fractions in grade two with a brief introduction of the concepts of 1/2 and 1/4. In third grade, the focus is on the continued development of fraction concepts along with the introduction of the addition and subtraction of fractions with the same denominator. Fraction concepts are further extended in grade four to the improper fractions and mixed fractions. The quotient meaning of fractions is introduced in Grade 4 in the Korean and the Taiwanese curriculum, while the idea is introduced in Grade 5 in the Japanese curriculum.

In the Japanese and the Korean curricula, simple cases of equivalent fractions are discussed in Grade 4 even though the formula for creating equivalent fractions is not discussed until Grade 5, while the Taiwanese curriculum addresses this topic in Grade 4. All three curricula took time develop fraction addition and subtraction over three grades: first with proper fractions with the same denominators, then with improper fractions or mixed numbers with the same denominators. Fraction addition and subtraction with unlike denominators follows the discussion of equivalent fractions in the fifth grade curricula. In all three curricula, multiplication of fractions are also discussed in multiple grades. Both the Japanese and the Korean curricula formally discuss multiplication of fractions by whole numbers in Grade 5 and multiplication by fractions in Grade 6. The Taiwanese curriculum follows the same sequence but one grade level earlier. The Japanese and the Korean curricula follow the similar sequence with division while the Taiwanese curriculum discuss division of fractions only in Grade 6.

Table 2:Curricula Flow on Fractions in curriculum in three countries

	Japan	Korea	Taiwan	CCSSM
Fractions as equal shares	2	3	3	1/2/3
Fraction as number	3/4	¾	3/4	3/4
Comparison	3/4/5	3/4/5	3/4	3/4
Equivalent fractions	4/5	4/5	4	3/4
Fractions as quotients	5	4	4	5
Addition/subtraction	3/4/5	3/4/5	3/4/5	4/5
Multiplication	5/6	3/5	4/5	4/5
Division	5/6	5/6	6	5/6

The overall flow in the three Asian curricula is fairly similar to the overall flow in the CCSSM. In the CCSSM, simple fraction ideas are introduced in Grades 1 and 2 through equal partitioning of geometric shape like circles and rectangles. This approach is similar to the Japanese curriculum in Grade 2. The formal introduction of fraction as numbers occurs at Grade 3 for all curricula and CCSSM. However, while the three Asian curricula discuss addition and subtraction soon after they start the formal instruction of fractions in Grade 3, the CCSSM delays the discussion of addition and subtraction until Grade 4. Although the timing of addition/subtraction instruction is different, the three Asian curricula and the CCSSM all rely heavily on the measure subconstruct of fractions, that is, fractions are collections of unit fractions. While multiplication of fractions in the CCSSM follows the same sequence as the three Asian curricula, that is, multiplication of fractions by whole numbers first, then multiplication by fractions, the way division of fractions is developed in the CCSSM is different. Unlike the Asian curricula which first discusses the division of fractions by whole numbers then division by fractions, the CCSSM discusses division of unit fractions by whole numbers and whole numbers by unit fractions in Grade 5 before discussing division of fractions in general in Grade 6.

The Development of Fraction Subconstructs and Contexts/Models Used

Table 3 summarizes which of the five subconstructs offractions are present in different grade levels of themathematics curricula from the three Asian countries and from *EngageNY*.It is clear from this table that there are variations in the ways different fraction subconstructs are used in the three Asian curricula. However, one commonality is that the foundational role the part-whole subconstruct appears to play in the three curricula. Moreover, the part-whole and the measure subconstructs are the two primary subconstructs undergirding the initial instruction of fractions, including addition and subtraction of fractions, in the three curricula. This approach is similar to the way the *EngageNY* (and the CCSSM) introduces fractions, starting with the partitioning of wholes in Grades 1 and 2, and then in Grade 3, developing the understanding of non-unit fractions as collections of unit fractions. Students in *EngageNY* are then expected to use that knowledge to think about fraction equivalence, ordering, and addition and subtraction of fractions in Grades 4 and 5.

Table 3: Fraction subconstructs that appeared in the Asian curricula and in EngageNY

Grades	Japanese	Korean	Taiwanese	EngageNY (US)
2	**Part-whole**			**Part-whole**
3	Part-whole **Measure**	**Part-whole** **Measure** **Operator**	**Part-whole** **Measure**	Part-whole **Measure**
4	Part-Whole Measure	Part-whole Measure **Quotient**	Part-whole Measure **Quotient**	Part-Whole Measure **Operator**
5	Part-Whole **Quotient** Measure	Part-whole Measure Quotient Operator	Part-whole Measure Quotient **Operator**	Part-Whole Measure **Quotient** Operator
6	Part-Whole **Operator** **Ratio**	Part-whole Measure Operator **Ratio**	Part-Whole Measure Quotient **Ratio**	Part-Whole Measure Operator **Ratio**

Note: A sub-construct that is newly addressed in each grade is bolded in Table 3.

While all five subconstructs of fractions are present in each curriculum, there exist different emphases on the operator and ratiosubconstructsamong the four curricula. The Japanese curriculum emphasizes a unitary view of fractions thus putting more emphasis on the measure subcontstruct than the othersubconstructs. The primary representation is linear (either tape or number line) (see example below).

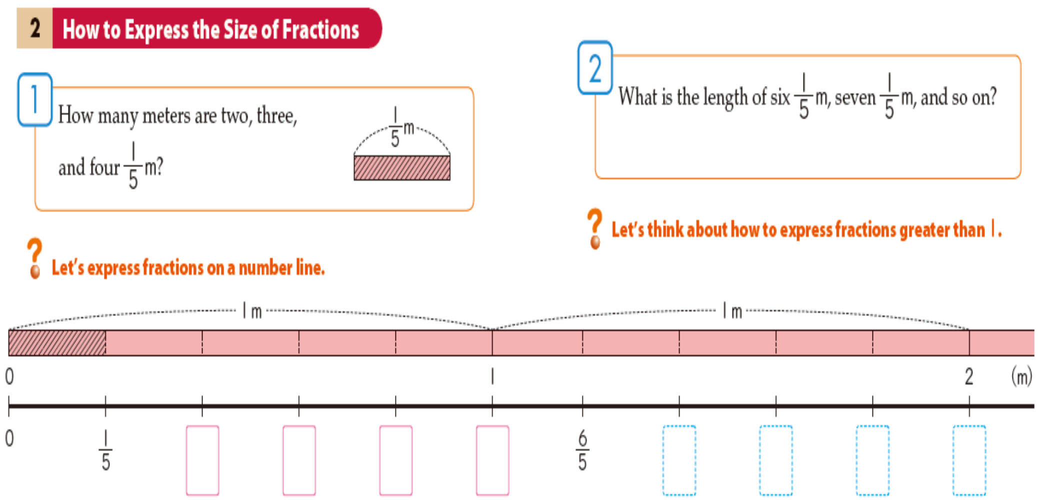


Figure 2. Typical representations used in the Japanese textbooks

One unique feature of Korean curriculum is to introduce operator construct much earlier than Japanese and Korean curricula. Below is an example from the 3rd Grade Korean textbook asking students to find what is ¾ of 8. A fraction as operator typically implies partitioning followed by iterating: for example, 3/4 as operator (3/4 of 8) implies first partitioning the object into four equal parts and then making three copies of (iterating) one of those parts as shown below. In addition, the Korean curriculum addresses the meaning of fractions as operator first in grade 3 and then in grade 5 and 6 as multiplication of fractions is introduced and developed. Furthermore, the ratio subconstruct is used to promote the concept of equivalence and, subsequently, the process of finding equivalent fractions. Thus the Korean curriculum appears to intentionally introduce students to the variety of subconstructs sooner than the other two curricula.

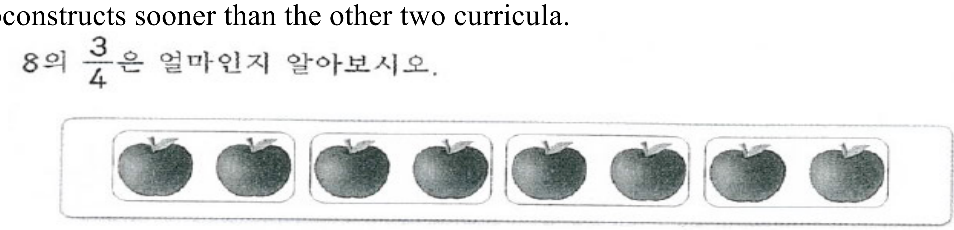


Figure 3. An example problem introducing operator construct in the Korean curriculum

How are addition and subtraction of fractions introduced and developed?

Table 4 describes the types of word problems presented in the mathematics curricula from the four countries. We found that all three Asian curricula included relatively small number of word problems as they discussed addition and subtraction of fractions. Moreover, the problem types found are generally simpler types such as Join/Separate-Result-Unknown and Part-Part-Whole-Whole-Unknown. However, *EngageNY* and Taiwanese textbooks include other types of word problems such as Separate-Initial\Change-Unknown and Part-Part-Whole Part Unknown, which emphasize the relationship between fraction addition and subtraction. Note that the Compare-Smaller-Unknown type was only found in the *EngageNY*

In addition, we found that a significant percentage of pure computation exercises of fraction additions and subtractions with different denominators are included in each curriculum with the following number and percentage: Korean 76 (76%), Japan 29 (74%), Taiwan 24 (75%), and EngageNY (66%).

Table 4:Types of addition and subtraction word problems in four curricula

Join Result Unknown *JKTUS*	Join Change Unknown *TUS*	Join Initial Unknown
Separate Result Unknown *JKTUS*	Separate Change Unknown *TUS*	Separate Initial Unknown *TUS*
Part-Part-Whole Whole Unknown *JKTUS*	Part-Part-Whole Part Unknown *TUS*	
Compare Difference Unknown *JKT US*	Compare Smaller Unknown US	Compare Larger Unknown *T*

Note: J stands for Japanese textbooks with K for the Korean texts, T for the Taiwanese texts, and US for *EngageNY*.

While discussing addition and subtraction, all three curricula incorporated various models including linear, area, and discrete. Below is an example from the 3rd Grade Taiwanese textbook using linear model. The question asked how much would left if Wei-Ting had a 7/10 meter rope and used 3/10 meters for her art project. The students are encouraged to think "7/10 meter is 7 units of 1/10 meter, used 3 units of 1/10 meter. So left will 4 units of 1/10 meter, which is…."

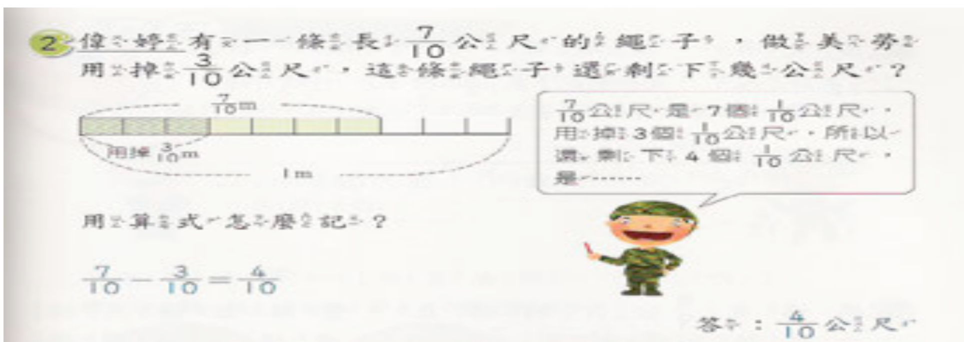
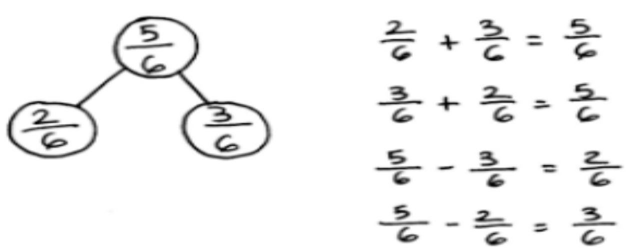


Figure 4. Typical representations used in the Taiwanese textbooks

Similar to the three Asian textbooks, modules from EngageNY incorporated various models. The most frequently model is a tape diagram, followed by a number line and area model. One unique model (or an approach) used in EngageNY is using a number bond to find the sum or the difference as shown below.



Discussion

The findings of this study showed more similarities than differences among the three Eastern Asian curricula in terms of overall flow, approach, and grade level expectations. Moreover, as far as the early discussion of basic fraction concepts, there is a significant alignment between the three Asian curricula and the CCSSM. However, some of the differences may have significant implications. For example, we noted that how the Korean curriculum incorporates all five fraction subconstructs from early grades while the Japanese curriculum takes a much more deliberate pace to introduce the subconstructs beyond part-whole and measure. Further research in the way students in these countries understand fractions may tell us how their learning may be impacted by the curricular decision. Research has long reported that many students and even teachers have difficulty understanding fractions (Ball, 1990; Behr et al. 1992; Ma, 1999).An increased understanding how fractions and fraction addition and subtraction are introduced and developed in other countries provides us with a tool to critically reflect on our current practices and that can help us to improve the quality of both curriculum materials and fraction instruction.

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STUDYING THE ARC OF LEARNING IN MIDDLE SCHOOL MATHEMATICS CURRICULUM MATERIALS

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Researchers and policy writers have advocated for the importance of curricular coherence. The purpose of this study is to move beyond surface-level features of coherence by attending to the mathematics embedded within problems and investigating how key mathematical concepts are developed across sequences of problems into mathematical storylines.

Keywords: Curriculum; Curriculum Analysis; Learning Trajectories; Middle School Education

Introduction

The importance of coherence in mathematics curriculum policy has been well documented in the literature. For example, in 2000, the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) emphasized that curriculum is "more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades" (p. 14). In the development of the Common Core State Standards (The National Governors Association Center for Best Practices and The Council of Chief State School Officers, 2010), attending to coherence was key in the development of standards through "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (p. 4). According to Trafton, Reys, and Wasman (2001), "If students are to think mathematically and use mathematics as a tool for solving problems, coherence [in curriculum materials] is crucial, and establishing connections among the big ideas of mathematics fosters coherence" (p. 260).

The purpose of this study is to understand how mathematical understanding embedded in problems is connected and how each problem contributes to the coherence of a mathematical storyline. This is important to mathematics educators because it can inform (a) how teachers understand the development of long-term mathematical goals as supported by the daily lessons, (b) how student thinking and learning is targeted and how that thinking and learning might unfold within and across mathematics units, and (c) the mathematical, pedagogical, and assessment decisions teachers make when planning or enacting lessons that respond to students' mathematical conceptions. This study addresses the major PME-NA 2015 conference theme.

Theoretical Framework and Related Research

To make explicit the mathematical storylines embedded within curriculum materials, we use the *Arc of Learning* framework to describe the implicit learning progressions in the sequence of mathematics problems (Phillips, Gilbertson, Grant, & Stewart, 2014). According to the National Research Council (2007), learning progressions are defined as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (p. 214). Encompassing learning progressions, the Arc of Learning framework and its phases embody Freudenthal's view that "students should be given the opportunity to reinvent mathematics by organizing or mathematizing either real world situations or mathematical relationships and processes that have substance for them" (Cobb, 2008, p. 105). This is a central tenet in the Realistic Mathematics Education (RME) instructional theory, where mathematics is interpreted as a human activity. From this perspective, realistic "refers more to the intention that students should be offered problem situations which they can imagine" (Van den Heuvel-Panhuizen, 2003, p. 10). From this perspective, contextual problems are defined as situations that are experientially real to the student, which can include pure mathematical problems (Gravemeijer & Doorman, 1999). In RME, "context problems are intended for supporting a reinvention process that enables students to come to grips with formal mathematics" (Gravemeijer & Doorman, 1999).

In our work, the Arc of Learning framework expands on the work in RME and consists of five phases or stages, moving from informal to formal mathematics. These phases describe a process for students to develop their mathematical understandings as they explore problems over time. In the *Introduction (Setting the Scene)* phase, students explore problems that reveal mathematical theme and where the problems informally highlight the key mathematical concepts. The problems also provide an opportunity to assess what students bring to the lesson in terms of the goals of the unit. In the *Exploration(Mucking About)* phase, students explore problems that establish a platform for developing key aspects of concepts and strategies. Students consider and explore a context that students can use to build, connect, and retrieve mathematical understandings. In the *Analysis(Going Deeper)* phase, students explore problems often with a variety of contextual situations and examine nuances in key aspects of the core mathematical ideas. Students make connections between concepts and representations. In the *Synthesis* (*Looking Across*) phase, students explore problems and consolidate and refine their emerging mathematical understanding(s) into a coherent structure. They recognize core ideas across multiple contextual or problem situations. Students begin to generalize their mathematical ideas and strategies. In the *Abstraction (Going Beyond)* phase, students explore problems in which they make judgments about which representations, operations, rules, or relationships are useful across various contexts. Students look back on prior learning to generalize, extend, and abstract the underlying mathematical structure. The tasks provide opportunities for assessing student understandings at a more general level.

Methodology

Since little is known regarding how to deeply characterize coherence for particular topics in curriculum materials, the purpose of this study is to examine the mathematics embedded within sequences of problems and to make explicit the mathematical storyline of curriculum materials. The research question that guides our study investigates how the key mathematical concepts and methods (i.e., unit/chapter objectives)in middle school curriculum materials were developed within sequences of problems. For each curriculum, Grade 7 units or chapters that shared similar big mathematical ideas in probability, proportionality, and linearity are selected. The unit of observation is the mathematical unit/chapter of the curriculum materials. Homework tasks are not examined in this study. Four middle school CCSS-oriented curriculum programs chosen for the study are *Connected Mathematics*, *Mathematics in Context*, *College Preparatory Mathematics*, and *Big Ideas MATH*. This represents a range of materials of middle school programs including commercially developed and NSF-funded materials.

In our analysis, we use the Arc of Learning framework to code each mathematics problem or lesson activity located within the selected units/chapters. The researchers, using the phase description of the Arc of Learning described earlier, code each mathematics problem or lesson activity located in the student text. Codes include the Arc of Learning phases of Introduction, Exploration, Analysis, Synthesis, or Abstraction. Coding is based on information from the student text and accompanying teacher's guide notes that pertain only to the unit of analysis. Independent codes conducted by the researchers are aggregated together to form final code(s) decisions for each mathematics problem. If agreement in codes is not immediate, the research group discusses the accompanying evidence for the code and determine final consensus.

Findings

The analysis of the research study is currently ongoing. Findings of the study will report on the overall development of the mathematical ideas embedded across a sequence of problems for each curriculum unit. For example, Table 1 presents preliminary findings that describe the development of understanding similarity from the *Stretching and Shrinking* unit of *Connected Mathematics 3*. Figure 1 shows how each problem contributes to the development of similarity.

Table 1: The development of understanding similarity for *Stretching and Shrinking*

Introduction (Setting the Scene)	Emergence of initial ideas for the meaning of similarity, including informal ideas from everyday life Emergence of initial ideas of mathematical similarity and tools for creating similar shapes. Some beginning ideas of similarity such as angle, lengths, and area relationships may emerge Assessment of students' understanding of ratio and rates as in "for every" and corresponding angles, and sides
Exploration (Mucking About)	Continue building a working definition for similarity using contexts Emergence of more details of similar and non-similar shapes (e.g., what stays the same, what changes) with attention to measurements using tools such as angle rules, rules, and lengths on a grid Begin a formal description of similar and non-similar shapes using features of side lengths, angle measures, perimeters, or areas
Analysis (Going Deeper)	Refinement of similarity definition regarding lengths and angle relations between similar and non-similar shapes Scale factor and its relationship to side lengths are refined Scale factor and its relationship to areas is strengthened The ratio of two lengths within a shape is equivalent to the ratio of the corresponding lengths in a similar shape
Synthesis (Looking Across)	Build on prior knowledge, experiences, and dispositions of learning from earlier problems to determine scale factors and ratios of adjacent side lengths. Find missing measurements of different situations Discuss similarity in which direct measurement is difficult and collect data. Discuss accuracy of that approximate data
Abstraction (Going Beyond)	Generalize understandings of similar shapes and strategies beyond contextual situations

Implications and Future Work

Unpacking the mathematical storylines over sequence of problems as arcs of learning provide teachers, teacher educators, mathematics education researchers,curriculum developers and administrators effective tools to characterize deeply grounded and connected learning. Attention to mathematical storylines is important because the development of ideas over time istypically not transparent (Davis and Krajcik, 2005; Remillard & Bryans, 2004). This work suggests that the Arc of Learning can be a potentially useful tool as it provides (a) an overall and detailed view of development of mathematical ideas will become more transparent and (b) support in describing student learning goals both at a lesson level at a unit level. Such a curriculumawareness will provide a knowledge space and support mechanism to make curricular decisions thatare more deliberate with their actions (Remillard, 2000). Future research is needed that explores how the Arc of Learning can be utilized in teacher planning and enactment, as a context for developing teacher knowledge in professional learning situations, and to impact student learning.

Stretching and Shrinking ***Connected Mathematics 3***	**Introduction (Setting the Scene)**	**Exploration (Mucking About)**	**Analysis (Going Deeper)**	**Synthesis (Looking Across)**	**Abstraction (Going Beyond)**
1. Enlarging and Reducing Shapes					
1.1 Solving a Mystery	1.1				
1.2 Scaling Up and Down	1.2	1.2			
Mathematical Reflection		MR			
2. Similar Figures					
2.1 Drawing Wumps		2.1	2.1		
2.2 Hats off to the Wumps			2.2		
2.3 Mouthing Off and Nosing Around			2.3		
Mathematical Reflection			MR		
3. Scaling Perimeter and Area					
3.1 Rep-Tile Quadrilaterals			3.1		
3.2 Rep-Tile Triangles			3.2	3.2	
3.3 Designing Under Constraints				3.3	
3.4 Out of Reach				3.4	
Mathematical Reflection				MR	
4. Similarity and Ratios					
4.1 Ratios Within Similar Parallelograms			4.1		
4.2 Ratios Within Similar Triangles			4.2		
4.3 Finding Missing Parts				4.3	
4.4 Using Shadows to Find Heights				4.4	
Mathematical Reflection					MR
Looking Back				LB	LB

Figure 1: The role of each problem in the mathematical storyline of similarity

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TEACHING STRATEGIES AND THEIR ASSOCIATED LEARNING OPPORTUNITIES

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This study compared the effects of two teaching strategies on learning opportunities and learning outcomes for students enrolled in different sections of a community college precalculus course taught by the same instructor. Features from Chval, Chavez, Reys, and Tarr (2009) were used to guide the instructional strategies used in one section while lectures were used in the other section. Results show that reform-based teaching methods can provide richer learning opportunities if exploration is supplemented with high-level tasks (Stein, Grover, & Henningsen, 1996). We determined that all students benefit from experience with all types of tasks, but focus should be given to high-level tasks.

Keywords: Curriculum; Post-Secondary Education

There exists a plethora of research centered on curriculum evaluation at the K-12 level (National Research Council [NRC], 2004; Tarr, et al., 2008). However, curriculum research at the postsecondary level is few and far between. This paper is a comparative study that analyzes two sections of precalculus at a Southeastern community college being taught by the same instructor. One section was taught using lecture-based methods and the other section using reform-based methods. The purpose of this study is to delve into how learning opportunities are impacted by inherent differences between teaching methods at the postsecondary level, and to provide a foundation for curriculum enactment research in that setting.

Framework

The premise of our study hinges upon the fact that teaching methods used in each group are in fact different. Accordingly, a crucial component for this study is *presentation fidelity* (Holstein & Keene, 2013). Presentation fidelity describes the level of faithfulness that a teacher maintains to the pedagogical approach suggested by developers of curricular materials. For the present report, we modify this definition and use presentation fidelity as a means to measure how strict the teacher follows the teaching methods prescribed for each group. Features of standards-based instruction (Chval, Chavez, Reys, & Tarr, 2009) guided the reform-based methods. The lecture-based methods are based on an unpublished set of expectations of a college lecture created by the community college where the study took place.

The purpose of this study is to examine how learning opportunities are impacted by inherent differences between teaching methods at the postsecondary level. Therefore, once we laid the foundation for the two different teaching methods we needed to choose a framework to analyze learning opportunities. We chose the mathematical tasks framework (Stein, Grover, & Henningsen, 1996) to code learning opportunities. The authors identify four types of mathematical tasks within two levels of cognitive demand, high and low. Tasks involving *memorization* (MEM) of mathematical facts and completing *procedures without connections* (PNC) to deeper concepts are considered to be low-level. On the other hand, tasks that include completing *procedures with connections* (PWC) to deeper concepts or *doing mathematics* (DM) are high-level. This framework has been used by its developers to assess the learning opportunities afforded to students based on the level of mathematical tasks in which they engage.

In the present study we aim to address two research questions: 1) what effects do reform- and lecture-based teaching have on students' performance?, and 2) how do learning opportunities differ based on teaching method when the teacher and course are the same?

Methods

This quasi-experimental comparative study examines two sections of precalculus at a large community college in Southeastern United States. This study took place at the beginning of the fourth unit of instruction, approximately 75 percent of the way through the semester. Students were studying rational functions at the time. For students within this course, emphasis is placed on intercepts, asymptotes, and points of discontinuity. All observations were video recorded.

Participants

Each section was taught using a different teaching method, which we used to make distinctions between the two groups. One section was taught using lecture-based teaching methods (LG, *N=24*), primarily consisting of lecture and whole class discussion. The second section was taught using reform-based teaching methods (RG, *N=29*), where students worked in small groups using discovery and inquiry to investigate topics. The same instructor, who is one of the authors, taught each section.

To compare the two groups we considered several factors. First, we considered the average test scores on three unit assessments that took place prior to the study. These three scores were consistent between the two groups (*p-values=0.15; 0.50; 0.93*). Second, we found the proportion of students in each group who needed to take remedial mathematics courses prior to enrolling in the course (RG: 13 of 29; LG: 14 of 24; *p-value = 0.33*). Thus, we concluded the groups were academically comparable.

Materials

Students in RG were given a handout that included exploration tasks, and used pre-constructed GeoGebra files during each class. These files utilized sliders that were dynamically linked to functions, tables, and graphs being explored. In RG, students were encouraged to take their own notes based on conjectures they made as they explored. On the other hand, students of LG were not given the handout but instead the teacher wrote many notes on the board that students could transcribe while the whole class discussed mathematical concepts involved.

Data Collection and Analysis

Lesson plans for each group were developed to comply with the pre-selected teaching methods. Following each class, we met to discuss how faithful the teacher was to the method designated for that group. To assess the level of presentation fidelity achieved during each class we created a qualitative scale consisting of four levels: almost always, frequently, rarely, and never.

To respond to our first research focus, we evaluated student scores on a post-assessment. To do so, a two-sample *t*-test was used to determine if there was a significant difference of average score between the two groups. To address our second research question, we analyzed video recordings of instructional time and considered students' engagement with the types of tasks.Further, on the post-assessment we categorized each question as, MEM, PNC, PWC, or DM so that an item-by-item analysis could be connected to learning opportunities. Additionally, some questions on the post-assessment required that students provide explanations, coded as either full (F), partial (P), or no explanation (N), which were used to enhance our discussion on learning opportunities.

Results

When determining presentation fidelity, we coded the teacher at 'almost always' for all criteria for each class in both groups, thus the two sections were in fact unique.

Post-assessment

The post-assessment was given in the class following the three classes that were videotaped. Items 2, 4, 6, 7, 8c, and 8d were items that contained an explanation component. We independently coded post-assessments for correctness and explanation, and then discussed. There was a maximum score of 14; results between the two groups were not significantly different (RG: 7.59 [3.08]; LG:

8.21 [3.02]; *p-value = 0.28*). Therefore based on overall scores of the post-assessment we conclude that teaching method did not have an effect on student performance. However, an item-by-item analysis follows.

In table 1 we report on the mean (and standard deviation) disaggregated by item within each group (only items that were significantly different are reported). Items reported in table 1 were scored with a 0 (incorrect) or 1 (correct). Next to each item number is the type of task (Stein et al., 1996) coded for that specific item. Table 2 describes the proportion of questions answered with full (F), partial (P), or no explanation (N) on the post-assessment.

Table 1: Post-assessment scores disaggregated by item

	Question – Type			
Group	2 – MEM*	6 – MEM***	7 – DM*	8c – PWC***
RG	0.45 (0.51)	0.31(0.47)	0.28 (0.45)	0.69 (0.47)
LG	0.71 (0.45)	0.67 (0.48)	0.08 (0.28)	0.33 (0.48)

Table 2: Proportions of F, P and N for each item by group

	Question – Type											
	2 – MEM			6 – MEM			7 – DM			8c – PWC		
Group	F	P	N	F***	P***	N	F***	P***	N**	F***	P	N***
RG	0.38	0.34	0.28	0.17	0.59	0.24	0.31	0.00	0.69	0.66	0.07	0.27
LG	0.58	0.17	0.25	0.58	0.04	0.38	0.04	0.54	0.42	0.21	0.08	0.71

*$p < 0.10$; **$p < 0.05$; ***$p < 0.01$

Learning Opportunities

Items 2 and 6 were coded as MEM, and in both cases LG did significantly better than RG. Items 7 and 8c were coded as DM and PWC items, respectively. In both cases RG did significantly better. It is clear from these results that because LG was instructed on “rules” of rational functions and engaged primarily in low-level tasks (PNC) throughout the classes they did stronger on items of comparable caliber. Likewise, RG engaged primarily in high-level tasks (DM) throughout the classes and therefore, did stronger on DM or PWC items.

Reform-based. Students were doing mathematics almost exclusively because all classes were student-lead and exploration-based. Small groups explored with the software and modified their conjectures about when holes occur versus vertical asymptotes. Similar activity transpired in groups during each class throughout the study; hence, students of RG were *doing mathematics.*

Students periodically engaged in PWC, the secondary type of task, because of their use of multiple representations to engage with rational functions (Stein et al., 1996). It is worth noting that there was no occasion where we coded tasks as PNC or MEM in RG. We therefore were able to conclude the most common types of tasks for the unit in RG were high-level, primarily DM tasks.

Lecture-based. Alternatively, all lecture-based classes were unique in their structure and the types of tasks in which students actually engaged. The teacher began class one by defining rational functions and outlining procedures to follow to gather information about characteristics of rational functions. These procedures were algorithmic and were focused on the correct answer instead of deepening mathematical understanding; hence deserving a code of PNC (Stein et al., 1996). This style was prominent in class one. However, since students engaged with technology to view graphical and algebraic representations, the secondary level of tasks was coded as PWC. Class three had a comparable lesson to class one so it was coded similarly. It should be noted that some tasks were coded as MEMin class three because students were asked to provide information based on “rules” learned in one of the previous classes (e.g., students’ hearing and subsequently memorizing phrases such as “vertical asymptotes trump holes”).

Class two was slightly different. The primary task was still PNC yet the secondary task was coded as MEM and the tertiary task was coded as DM. Students were instructed on rules to determine if a discontinuity was a hole or a vertical asymptote (MEM); however, students were also given an introductory task where they were given certain criteria and asked to create a rational function to meet those criteria (DM). Consequently, even though all four types of tasks appeared in LG every class, we concluded low-level tasks, primarily PNC were the most common types of tasks.

Discussion

In this section we provide an elaboration on the results presented in the previous section. To begin, there was no significant difference in mean scores on the post-assessment between the two groups. This is consistent with the students' prior unit assessments, so it is not very surprising, and indicates that a reform-based teaching method did not have an effect (at least in this case) on community college students' performance. However, when an item-by-item analysis of the post-assessment was conducted we see that there were many significant differences between the two groups. We will focus this section on those differences and then suggest areas for future research.

Recall that the most common level of tasks in which students engaged in the two groups was low-level in LG and high-level in RG. Students in RG did not experience low-level tasks during any of their classes. On the other hand, students in LG were able to experience all four types of tasks during every class. Students in LG outperformed RG students on memorization tasks, such as item 6. This implies that all students benefited from both low- and high-level tasks, but low-level tasks should not be prevalent during instructional time. Students in RG did significantly better than students in LG on item 8c in terms of both performance and explanation. In RG the teacher and the group never specifically went through finding y-intercepts of rational functions, indicating that students in RG were able to realize that y-intercepts cannot exist when there is a vertical asymptote of $x=0$. On the other hand, LG students were not able to explain that there was no y-intercept for the given function. These results further justify that all students should be exposed to all types of tasks, but high-level tasks should prevail during instruction. This answers our second research question.

In conclusion, curriculum research at the postsecondary level is lacking in the literature. Our findings can act as a basis for future research endeavors. However, some limitations of our study are we only studied one teacher working with 53 students and our assessment may not have been reliable. Nevertheless, during our investigation of the effects on learning opportunities, we see that reform-based teaching methods can provide students with richer learning opportunities if student exploration is supplemented with high-level tasks. However, all students benefit from experience with all types of tasks, but focus should be given to tasks that require doing mathematics and procedures with connections.

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USE OF WRITTEN CURRICULUM IN APPLIED CALCULUS

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In order to evaluate a curriculum, it must be known how students are using it. However, the opportunities for use afforded to students by curriculum are changing with technology. The purpose of this study was to discover how undergraduate students in applied calculus use the written curriculum in their course which included an online homework system. The study drew on three data sources: (a) online survey, (b) observations, and (c) interviews. Students' survey responses indicated that, while not many students referenced the textbook, those that did preferred to use textbooks for finding worked problems, finding formula or definitions, and doing homework problems. The observations and interviews revealed that students responded to negative feedback by checking computations and for formatting errors.

Keywords: Curriculum; Post-Secondary Education; Technology

Use of Curriculum

Researchers across disciplines have recognized the importance of curricular use. Lee, McNeill, Douglas, Koro-Ljungberg, and Therriault (2013) studied undergraduate engineering students' use of textbooks during problem solving, and Peng (2009) studied accounting students required to use an online homework system. In mathematics education research, Williams and Clark (2012) looked at student actions broadly, including "what resources they used such as tutors, textbooks, online notes, and study groups" (p. 184), also considering the use of an online homework system and textbooks. The only resource which received significant use was the online homework, and that only for the completion of homework; students did not read the online textbook.

Over the last two decades, many mathematics departments implemented online homework systems as part of their curricula (Hirsch & Weibel, 2003). Beginning in the 1990s, colleges and universities posited that online homework system in mathematics classes enhanced students' learning in their first-year courses (Kehoe, 2010; Hauk & Segalla, 2005; Zerr, 2007). In these studies learning was measured by students' self-reporting on surveys and performance on standardized assessments. Studies on the effectiveness of online homework determined either that there is no significant difference in standardized-assessment performance between online homework system and traditional paper homework (e.g., Hauk & Segalla, 2005), or that the use of online homework system has a small positive effect on performance compared to traditional paper homework (e.g., Burch & Kuo, 2010). Until we know how students are using the online homework systems, we cannot be certain how the features support student performance or learning.

Methods

In order to build on this research, we investigated the question: How do undergraduate students in an applied calculus course at a large Midwestern university use the written curriculum in their course?We define the *written curriculum* as the collection of instructional materials endorsed by the course coordinator to be used by students. It may include textbooks, documents, and online homework systems. By *use of curriculum* we mean any action involving the written curriculum with the intention of accomplishing a goal (e.g., using the textbook to find a similar problem on which to model a solution or using the online homework system to satisfy a course requirement).

Setting

This study took place in an applied calculus course at a large Midwestern university. The syllabus listed the textbook and described the required online homework system (WebAssign). This particular instance of WebAssign allowed students three tries before deducting points for each question, required answers to be in one of a few acceptable formats, and included resources called Read It and Watch It for some problems in which students could reference a relevant section of the textbook or watch a short instructional video, respectively.

Procedure

When recruiting participants, we randomly selected instructors and then recruited in each of the two sections they taught. In this way all the students taking the applied calculus course were potential participants. Students were invited to volunteer to complete the online survey and participate in the observation and interview portion of the project. Thirty-five students completed the online survey and eight students were additionally observed and interviewed. Participants will be referred to by gender-preserving pseudonyms.

Survey

The survey was designed and distributed using Qualtrics software. Responses served as a baseline for descriptions of students' access to and use of curriculum. Questions were designed to explore three different curricular materials: online homework systems, textbook (electronic or printed), and exams from a previous semester. For each of these materials, questions focused on exploring students' use and/or goals for use.

Observation/Interview

Observations of the students using the written curriculum and an interview of those students provided the main sources of data. We observed eight participants working on their homework assignment or studying for an exam for 30 minutes. Immediately after finishing the observation, each student was interviewed for up to 30 minutes. We implemented a guided interview (Patton, 2002). There were some questions which were created before the interview and others were informed by the observation.

Findings

Use of Textbooks

Of the 35 participants who responded to our survey, 34 (97%) had access to the textbook with either a physical copy, digital copy, or both. Of these 34, only 15 (44%) participants reported referencing the textbook at least once per week (see Table 1). No participants reported referencing the textbook more than 5-6 times per week.

Of the eight participants we observed and interviewed, three had only a physical copy of the textbook, two had only the e-book, two had access to both, and one had no access to the textbook at all. Two of the participants who did have access never used it (from Brendan: "It's been sitting in my room collecting dust"), and the remaining participants reported rarely using it. Brendan initially tried to use the textbook to find similar problems to those on the homework, but found, "It never corresponded with the problems I was having." Another participant reported that, "When we were told we had to get it, I thought we would be assigned problems out of it, but we're not."

Table 1: Students' Frequency of Use of Textbooks

Frequency	Responses	Percentage (out of 34 students)
0 times	19	56%
1-2 times	10	29%
3-4 times	4	12%
5-6 times	1	3%
7 or more times	0	0%

Use of Online Homework System

Each of the 35 participants reported having access to WebAssign. Most students reported using WebAssign for working on current homework (97%). A majority of students use WebAssign for looking ahead to future homework assignments (68%). Students also reported consulting WebAssign when studying for exams (65%), learning new material (94%), and studying for quizzes (88%). Kristen reported that when using WebAssign to study for exams, she would "try to think of problems I had difficulty with and maybe go back through them." Cathy reported: "Just before the exam I will do the WebAssign again, go through all of them."

We also asked students to rank the strategies they were likely to employ when WebAssign marked an answer incorrect. Students reported being most likely to look for mistakes in their work, closely followed by trying the same answer in a different format. Students were much less likely to reference their textbooks or give up.Of the eight participants we interviewed, seven reported checking their own computations, four reported seeking the advice of peers (be it through personal communication or WebAssign's online forum) or seeking the help of their instructor. Two reported consulting extracurricular online resources (e.g., Khan Academy) before resorting to the textbook.

Use of Previous Exams

Another curricular resource available to students was the exams used during the previous semester the course was offered. Perhaps not surprisingly, 82% of the participants in our study reported using these previous exams primarily to study for exams, with no other reason being reported by more than 18% of participants. Each of the eight participants we interviewed reported using the previous exams only to study for midterms and the final exam. Manny reported using the previous exams to gauge how many of each type there would be. Ben reported that he wanted to "see what kind of questions would come up on the exam." Brendan wanted to use the previous exams to gauge the difficulty of upcoming exam problems and how many of each type would be used, but felt that they did not do this sufficiently well.

We observed Brendan when he was using the previous exams to study for the final exam. He would highlight questions he wanted to review again later. Manny reported that he would take the practice exam under exam-like settings: "I keep a clock with me, I remove everything out, I just keep calculator, I do have a pencil with eraser." After working through the entire exam, he would check the answers. Esther also reported taking the practice exam "like I was taking the exam." Ben and Cathy would treat the previous exam as a set of sample problems and checked the answers after each question.

Discussion and Conclusion

Once a major pillar of written curriculum, our study shows that the textbook no longer holds a place of privilege (cf. Lee et al., 2013). Less than half of the participants surveyed reported accessing the textbook weekly. With few exceptions, the class met three times a week and there were six homework assignments due per week. Thus, for most students, neither lectures nor homework consistently prompted use of the textbook, a finding consistent with Williams and Clark (2012).

Nevertheless, it may be that the textbook was used irregularly by the majority of students; the lone participant interviewed without it revealed that there were a few times he wished he had purchased it.

The lack of textbook use may be partially attributable to its integration into this particular course. The interviews uncovered a student who tried to use the textbook, but found it unhelpful. In another interview, a student attributed the lack of use to the fact that problems were not assigned directly from the book. In fact, none of the assessments for the course were directly related to the textbook. Those students who were using the textbook were using it merely as a reference for worked examples, formulas, and definitions, rather than as a way to learn material, a finding consistent with those by Lee et al. (2010), Williams and Clark (2012), and Lithner (2003).

We conclude that course coordinators should look for more connections between parts of curriculum, including the features of the online homework system and use of the textbook. In this study we found that students did not use the textbook often, and, if they did use it, it was to look for worked examples, formulas, or definitions. The most prominent motivating factor seemed to be the bonus points they get for completing assignments early. Also troubling, students' expressed a persistent concern about being able to correctly format of the online homework answers. If students do not trust the feedback from the online homework systems, its value as a learning tool, that is, its status as curriculum must be critically examined.

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EXAMINING TEACHERS' CONCEPTUALIZATIONS OF CURRICULAR MATERIALS IN THE PLANNING PROCESS WITH DISCOURSE ANALYSIS

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Keywords: Curriculum; Middle School Education

It has long been established that there is a difference between written curriculum and enacted curriculum. Teachers play an intermediary role between written curriculum and enacted. This suggests curriculum developers' intentions do not directly determining what occurs in the classroom and therefore what students learn (Tarr, et al., 2008). This poster is focused on one aspect of how written curriculum is turned into enacted curriculum, teachers' planning process.

This study is a part of a larger NSF funded study (grants DRL-746573 and DRL-1222359) that utilized discourse analysis in the tradition of Gee (2014) by analyzing contextual/situated meaning to examine the relationship between teachers' conceptualization of curriculum during the planning process and features of the curriculum resources used. The primary data source included interview data from 14 middle school teachers from the same state. Teachers participated in a staged lesson plan (SLP), a semi-structured interview where teachers were provided with materials from a curriculum that was philosophically different from the curriculum they currently enact and asked to plan a hypothetical lesson. For example teachers who used a traditional curriculum were given reform-based materials to plan from and vice-versa.

Analysis of data included open coding of interview transcripts and provided resource materials. Remillard's (2005) work was used as a framework to categorize how teachers conceptualized curricula. The framework was based on ways researchers have conceptualized mathematics curricula, namely as a resource to be: followed with fidelity/subverted, drawn from as one of multiple resources, treated as a text subject to interpretation, or utilized as a tool whose use is determined dialogically between the materials, teacher, and students.

Findings included that when deciding how to use "new" curricular materials three features served as catalyst points for teachers' decision making. The curricular features included teachers' interpretation of how activities in the lesson related back to the stated goal/standard, physical layout of the materials including sequencing, and teachers' anticipated student reactions to the materials/instructional activities. Analysis also indicated that the most difficult decisions for teachers focused on relating the standard/objective of the lesson to the lesson activities. The confusion teachers evidenced was attributed to differences in curricular materials. Most notably, reform-based materials provided discussions in teacher resources that tied objectives and guiding questions to the problem contexts students engaged with, whereas traditional resources did not. The lack of connections between objectives and activities made envisioning how materials could be used in the classroom more difficult and confusing for teachers.

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CONTRASTING MATHEMATICAL PLOTS: A STUDY OF "IDENTICAL" MATHEMATICS LESSONS

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Efforts to analyze the enactment of curricular materials have largely used tasks as their unit of curricular analysis (e.g., Henningsen & Stein, 1997). While the quality of tasks is clearly central to student learning, this level of analysis ignores the role that the sequencing of tasks plays in the way that students experience the lesson. Examining the sequence of tasks enables the examiner to see how the posing and resolution of questions over time shapes the experience of the learner. Interpreting a math lesson as a story (Dietiker, 2013) allows for this type of analysis. This poster uses Author's mathematical story framework to address the research question: When enacted lessons based on the same written materials are interpreted as mathematical stories, what are the variations in how the content unfolds and how do these variations potentially impact students' mathematical experiences? For this study, student mathematical experiences can include emotional responses to the story, opportunities for continued investigations, the closure of explorations, engagement with new content, etc.

As part of a larger, ongoing study, this poster presents contrasting case studies of two different enactments of content that are based on the same textbook lesson. Two teachers were recorded teaching a lesson on using substitution to solve systems of linear equations from the same curriculum materials (blinded here, as an author of the materials is also an author on this proposal). These were veteran teachers with three or more years of experience teaching with the selected textbook and were chosen to represent a range of geographic and demographic sites. The recorded videos and interviews with the teachers were analyzed by the research team for the mathematical plot of the lesson (Dietiker, 2013). The questions found in these enactments, though often similar in form, emerged in different orders and are addressed in different ways. This poster presents an analysis that helps us understand and describe how the unfolding of content may result in different mathematical experiences for the students.

Note: An author of this poster receives research funding from College Preparatory Mathematics (CPM), which is developing products related to research described in this poster. This author also receives book royalties from CPM for the textbooks used in the study. The terms of this arrangement have been reviewed and approved by Boston University in accordance with its conflict of interest policies.

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ANALYSIS OF QUADRATIC EQUATIONS IN THE ALGEBRA I AND ALGEBRA II MODULES FROM ENGAGENY (EUREKA MATH)

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The Common Core State Standards of Mathematics (CCSS) highlight the importance of gaining a conceptual understanding of key ideas by students (CCSS0, 2010, Key Shifts in Mathematics). One key idea in the CCSS for secondary mathematics is the knowledge of solving quadratic equations. The New York State Education Department (NYSED) developed EngageNY, which is a collection of curriculum modules and resources in PreK-12 aligned with the CCSS, to support teachers implementing key aspects of the CCSS. What is not known, however, is how well the EngageNY modules address the key shifts in Mathematics stated by the Common Core Initiative. In this study, we seek to answer the following research question: How many and what types of problems in solving quadratic equations are presented in the Algebra 1 and Algebra II modules on EngageNY.
The analysis in this paper compiled information on: the number of lessons (horizontal analysis) and location of lessons (vertical analysis) in the Algebra I and Algebra II modules on EngageNY that involved the knowledge of solving quadratic equations. Individual lesson content, cognitive demand and response type of each task relating to the solving of quadratic equations were also compiled. Individual lesson tasks were coded using the framework drawn from Stein, Grover and Henningsen (1996) and Son and Senk (2010). Of the five Algebra I modules on EngageNY, three modules contained lessons involving the solving of quadratic equations for a total of 14 lessons and 141 tasks. Of the four Algebra II modules on EngageNY, only 1 module contained lessons involving the knowledge of solving quadratic equations for a total of eight lessons and 70 tasks. With regard to lesson context, 110 of the lessons were purely mathematical (78%) while only 31 were coded as real-world examples (22%) for Algebra I. In Algebra II, 67 of the tasks were purely mathematical (96%). In terms of the cognitive demand of mathematical tasks, we found that 26 of the tasks were coded as memorization (18%) and 79 were coded as procedure without connections (56%) for Algebra I. Thirty-three of the tasks were coded as procedure with connections (23%), while only 3 were coded as doing mathematics (2%). For Algebra II, none of the tasks were coded as memorization and 47 were coded as procedure without connections (67%). Response type was another area of weakness for the tasks in the EngageNY Algebra I modules. One hundred twenty one tasks required students to give an answer only (86%) while only 20 tasks required an explanation (14%) for Algebra I. In Algebra II, 59 of the tasks required an answer only (84%) and 11 tasks required an explanation (16%). Although New York State has touted the EngageNY modules as having a level of difficulty equal to the Common Core expectations, our study shows that the intended curriculum of the CCSS is not being fully met by the module lessons provided on EngageNY for solving quadratic equations.

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RIGOR, RELEVANCE, AND RELATIONSHIPS: PRESERVICE TEACHERS' PREPARATION ON PROJECT BASED LEARNING

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Keywords: Curriculum; Instructional Activities and Practices; Teacher Education-Preservice

Several national initiatives have sought to reform mathematics teaching to prioritize critical thinking and reasoning in order to address concerns about the mathematical underachievement of US students (NMAP, 2008). One inquiry-based instructional approach that has become increasingly popular is PBL. Unlike units in which a project is used as a culminating experience, PBL poses a realistic situation at the beginning of a unit and uses the need to create a deliverable product to drive the course content through an extended inquiry process. Students who learn in PBL environments learn rigorous mathematics, address relevant problems or topics, and build relationships with one another. The question that motivates this study is: *What are the successes and challenges PSTs encounter as they design and implement math PBL units?*

Theoretical Framework

The framework used to evaluate the rigor and relevance of PBL units is the Six A's: Authenticity, Academic Rigor, Applied Learning, Active Exploration, Adult Connections, and Assessment Practices (Markham, Larmer, & Ravitz, 2003). This framework guided the analysis of the units and how they were implemented.

Methods & Data Sources

This exploratory study involved ten math PSTs in an accelerated, teacher residency program, earning a Masters of Arts in Teaching and a teaching licensure in 12 months. This study employed axial coding techniques (Strauss & Corbin, 1990). Data sources included before- and after-implementation reflections, student-generated artifacts, and original and revised unit plans.

Results

PBL is an innovative instructional method designed to provide an authentic purpose for students to learn and engage with mathematics. However, implementing PBL instruction is no easy feat. The implementation of project based instruction in mathematics classrooms requires a shift from more traditional teaching practices to create classroom cultures that focus on flexible and robust understandings of math. This departure from conventional modes of teaching involves more than just a change in teachers' knowledge; it requires teachers (and students) to reconceptualize what it means to teach and learn, and that they create new and different opportunities for learning in and out of classrooms.

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INVESTIGATING ALGEBRA PROGRAMS: INDIANA AS A CASE STUDY

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The *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) and separate individual state standards position algebra as a critical component of the secondary curriculum. The issue of access to algebra has also garnered much attention, as it is a gatekeeper for future success in mathematics and post-secondary schooling (Adelman, 1999), as well as access to it having been identified as a civil right (Moses, Kamii, Swap, & Howard, 1989).

The current accountability culture (e.g., No Child Left Behind, 2001) and attempted national coherence of Algebra I standards (e.g., NGA & CCSSO, 2010) present persistent challenges in mathematics education. It is important to investigate policy changes made in response to the high-stakes accountability measures within this culture as well as the state of curricular flux in this course that is required by numerous states for graduation.

Our driving question is: "What policies shape Algebra I programs at the state level?" To answer this, we consider Indiana as a case study, investigating: How do schools determine when students take and who is teaching Algebra I? Who has influence in making these decisions? To answer our research questions, we gained a broad picture of programs across the state through surveying the 292 public non-charter school districts in Indiana, asking quantitative and qualitative questions. The survey was emailed to the superintendent of each district, who then answered or distributed the survey to someone with a stronger knowledge base of the Algebra I program structures.

With a response rate over 40%, our survey suggests school personnel are ready and willing to participate in a discussion on algebra policy. On average, 57% of respondents' Algebra I students were 9th graders, with 8th graders making up 22% of respondents' Algebra I students, as expected. The most common criteria that influence when students take Algebra I were found to be age/grade level (79% of respondents), grades in previous math classes (75%), current math teacher recommendations (68%), and test scores (60%). Our survey revealed that roughly half of high school mathematics teachers and roughly one third of middle school teachers in Indiana teach Algebra I, with respondents indicating the most powerful voices in deciding who teaches Algebra I as being administrators (95% of respondents), department head (64%), and individual teachers (59%). These survey results are being used to construct interview protocols aimed at discovering the individual features of representative Algebra I programs.

This study seeks to add to the current conversation about Algebra I across the state and nation by reporting trends and patterns found throughout an entire state. We posit that being familiar with the policies that are in place will aid in understanding Algebra I programs across different schools, districts, and states.

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REFRAMING PARENTS' CONCERNS OF MATH CURRICULUM CHANGE

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My son in Grade 5 still doesn't know the answer to 6 x 8. Why aren't kids today memorizing their times tables?

As mathematics educators, we've been asked this question dozens of times. Put on the defensive, we try to justify current strategy-based approaches and explain the importance of sense-making over memorizing supported by research (e.g., Boaler, 2002; Thompson, et al., 2013). Unfortunately, our efforts often inflame parents further. Despite 80 years of research that consistently demonstrates the detrimental effects of memorization and rule-based approaches to mathematics learning, the tug-of-war between "new math" and "back to the basics" continues to be waged. In Canada, efforts to communicate advantages of current curriculum reforms to parents have largely failed. This failure leads to conflicting pedagogical agendas at school and at home or simply the disengagement of parents in supporting their children's learning.

The reported research was framed by the questions: What are parents' experiences with and perceptions of curriculum change? What is the nature of their concerns? With our long term goal to identify fruitful ways to communicate reform to parents and help them re-engage in their children's learning, the aim of the study was to establish theoretically and methodologically parents' experiences in their own histories of learning mathematics, their perceptions of current approaches, and concerns or conflicts between their past experiences and current perceptions.

Phenomenography (Marton & Booth, 1997) provided a theoretical grounding and methodological approach, where researchers develop a framework of related categories empirically to illustrate the range of participants' perceptions of their experiences. Located in a large Canadian city and in rural communities, 96 parents across socioeconomic levels volunteered to participate. Qualitative data included 12 focus group sessions and follow up interviews with 36 of the parents, selected for the diversity and intensity of their perspectives. Categories of collective experience were generated and further substantiated through rich description of data exceprts.

The results reveal two categories of collective concerns held by parents and educators: (1) the opportunity for students to reach expected learning goals, such as mastering computational skills, developing problem solving ability, and becoming functionally numerate citizens; and (2) ensuring adequate supports are in place so that learning goals are accessible and attainable, such as teaching expertise matched to learning expectations and clear, sequential resources accessible to teachers and parents.

Our findings allow us to lay aside the rhetoric of oppositional stances in approaches to learning mathematics to identify and describe commonalities. Doing so helps us establish effective avenues of communication with educational stakeholders and empower parents to re-engage in supporting their children's learning of mathematics.

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STUDENT WORK AS A CONTEXT FOR STUDENT LEARNING

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Extensive research (e.g., Silver & Su, 2010) has attended to the affordances of professional learning experiences in which teachers collaboratively examine artifacts of students' mathematical work to develop mathematical and pedagogical knowledge and to reflect on classroom practice. Less attention has been placed on the student practice of examining student work and how this impacts their mathematical understanding and reasoning (Rittle-Johnson & Star, 2011). The purpose of this study is to focus on the use of student work as a context for student learning. This is relevant to the PME-NA conference audience as student work can provide students with potentially different opportunities to examine varied mathematical conceptions by focusing on the reasoning, strategies, methods, and understandings of others.

Our research study is framed by *student*, *teacher*, and *curriculum* generated student work. First, many teachers collect and use work generated by students to develop mathematical ideas. Second, the teacher anticipates a strategy that highlights an important insight or misconception. If students do not produce this strategy themselves, the teacher implants the strategy in the class discussion. Third, student work may be embedded in mathematics curriculum materials. Our research is guided by the question: What are the intended purposes of the student work embedded within the written curriculum materials for learning mathematics?

To address this goal, we analyzed three middle school mathematics curricula: *Connected Mathematics*, *College Preparatory Mathematics*, and *Big Ideas*. The unit of analysis is each mathematics problem containing embedded student work. Curriculum-embedded student work includes references within the student text to how a person thought about a mathematical context or problem and requires that students analyze, evaluate, generalize, critique, and/or reflect on another's mathematical thinking. Analysis of the student work focuses on the existence, nature (e.g., location, frequency, problem type such as error analysis), and intended purpose.

Results of this study explicate existing curriculum-embedded opportunities for students to engage in the analysis, evaluation, generalization, critique, and/or reflection on another's mathematical thinking. This work has implications for classroom practice because it serves as a mechanism to ensure students have access to opportunities for learning key concepts and strategies. Future work includes examining how teachers use curriculum-embedded student work and its impact on student learning outcomes. In addition, future research includes examining the impact of curriculum-embedded student work on teacher pedagogical content knowledge.

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A FRAMEWORK FOR FORMATIVE ASSESSMENT AS AN ONGOING PROCESS OF DAILY CLASSROOM PRACTICES

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Assessment and evaluation remain an enduring challenge in mathematics education. Paralleling the advent of standards-based curricula that advocates for reform-based mathematics instruction, formative assessment challenges conventional notions of assessments as paper-and-pencil tests used to measure student achievement at the end of instruction. Black and Wiliam (1998) used the term, assessment, to refer to "all those activities undertaken by teachers – and by their students in assessing themselves – that provide information to be used as feedback to modify teaching and learning activities. Such assessments becomes formative when the evidence is actually used to adapt the teaching to meet students' needs" (p. 140). In this study, formative assessment is characterized as an ongoing process that occurs before, during, and after instruction by which teachers anticipate student strategies and solutions, gather and analyze evidence of student learning, and adapt their teaching to meet students' needs and to help students develop their own reflective habits of mind.

To provide attention to the complexities of daily formative assessment that occurs in mathematics classrooms, a group of curriculum developers, teachers, and mathematics education researchers have created a framework for formative assessment. The following research question guided our study: What is a framework that describes formative assessment opportunities and related teacher practices during planning, teaching, and reflecting?

Based on the research literature and best practices, a framework was developed that includes suggested questions, examples, and strategies for formative assessment practices. The resulting framework is organized around two dimensions. One dimension captures key practices of anticipating student strategies and solutions, gathering and analyzing evidence, and adapting teaching. The other dimension is the Launch–Explore–Summarize instructional model. For example, the teacher's role in the Launch is to assess students' prior knowledge and its relationship to the challenge of the problem. This contrasts with the teacher's role in the Explore where she attends to the needs of individual or groups of students as they work on the problem. Building on the Launch and Explore, the teacher orchestrates discussions during the Summarize phase to develop students' understandings as they relate to the mathematical goals of the lesson.

The framework suggests that formative assessment is an integral part of daily teaching practice and can be used as a lens to broaden teachers' interpretations of formative assessment. Future research studies may use this framework to investigate, characterize, and report on teachers' formative assessment practices. In addition, professional learning experiences can be designed using the framework to make the ongoing daily practice of formative assessment more accessible to mathematics teachers so that they can improve their classroom instruction.

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TURKISH 8TH GRADERS' MATHEMATICS SUCCESS ON TIMSS IN RELATION TO NATIONAL HIGH SCHOOL PLACEMENT TESTS

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Keywords: Assessment and Evaluation

This study examined the mathematics items in two national assessments conducted in Turkey, namely the OKS 2007 (Assessment of Secondary Educational Institutions) and the SBS 2011 (Secondary Education Placement Test), in terms of content and cognitive domains. The main focus was to interpret the gains in Turkish 8th graders' mathematics scores on TIMSS between 2007 and 2011 in terms of the OKS 2007 and SBS 2011 assessments.

The data for this study was derived from the Turkish 8th graders' mathematics scores in TIMSS 2007 and 2011 (Martin, Fullis, & Foy, 2008; Mullis, Martin, Foy, & Arora, 2012) and the mathematics questions in OKS 2007 and SBS 2011 assessments. For this study, the questions in OKS 2007 and SBS 2011 assessments were categorized into one of four content domains (numbers, algebra, data & chance, and geometry) and one of three cognitive domains (knowing, applying, and reasoning) according to the TIMSS 2011 framework.

Although Turkish 8th graders' overall average mathematics score increased from 432 to 452 from 2007 to 2011 in TIMSS assessment, there was no significant difference between given years in numbers content (Mullis et al., 2012). On the other hand, performance improved on the other content domains, especially geometry (Mullis et al., 2012) where the score increased 43 points (Mullis et al., 2012).

One of the likely reasons for the lack of improvement on the numbers domain of TIMSS is that the percentage of content items in number decreased from 39% to 20% between OKS 2007 and SBS 2011. Along with this decrease, there is evidence that teachers decreased emphasis in the area. In contrast, the percentage of geometry items increased from 35% in 2007 to 50% in 2011. Thus the change in focus of the Turkish assessments in relation to TIMSS, where the emphasis on each of the content domains was constant, is likely a major reason for the increase in geometry performance relative to number.

With respect to the cognitive domains in the assessments, it was found that there was an increase in the percentage of items in the knowing domain (28% to 55%) and a decrease in the percentage of items in the other two domains between the OKS 2007 and SBS 2011. Relative to PISA, TIMSS is more of a factual and knowing-based assessment (Kloosterman, Roach, & Pérez, in press) and thus emphasis on the knowing domain in the Turkish assessments could also have impacted the strong gains of Turkish students on TIMSS.

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RE-EXAMINING THE VALIDITY OF WORD PROBLEM TAXONOMIES IN THE COMMON CORE ERA

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Background and Research Questions

Through studies conducted in the latter half of the twentieth century across many different languages and cultural norms, the semantic structure of word problems has been found to influence problem difficulty for young children, and word problems have been organized into taxonomies based on semantic structure and difficulty (Fuson, 1992; Verschaffel, Greer, & DeCorte, 2007). In the current era, the Common Core State Standards for Mathematics (CCSSI, 2010) explicitly incorporate the full range of word problems found in the established taxonomies.

This change in the standards presents an interesting opportunity to explore questions of whether there is a fundamental way that children think about word problem types that makes some problem types inherently more difficult than others, or whether the differences in difficulty are manufactured by relative amounts of exposure and opportunities to learn.

Using data generated from interviews and written assessments involving more than 2,000 students during the 2013–2014 school year, our investigation is driven by the following two research questions:

- With respect to the semantic structure of word problems, do the textbooks used in classrooms of students in our sample afford different opportunities to learn than the textbooks used in classrooms of students in the samples of students in the 1970s and 1980s?
- Do the previously identified patterns in relative problem difficulty based upon the semantic structure of word problems continue to be valid for first grade students?

We consider this study to be an important preliminary investigation into a larger question concerning the influence of semantic structure and opportunities to learn on the relative difficulty of word problems for young learners of mathematics.

Acknowledgements

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TEACHERS' PERCEPTIONS OF MATHEMATICS STANDARDS: A COMPARISON OF PSSM AND CCSSM

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Keywords: Standards; Teacher Knowledge; Teacher Beliefs; Middle School Education

Introduction and Method

In this study, we engaged middle grades mathematics teachers in a comparative study of national and quasi-national standards documents, the CCSSM and the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics* (PSSM). Findings indicate tensions exist between what teachers' desire in standards documents and what standards authors provide. Nine middle grades teachers who were part of an existing professional development community that focused on increasing mathematical knowledge for teaching, participated in a focus group that reviewed and compared the PSSM and CCSSM. The focus group was conducted near the end of the academic year in which CCSSM was adopted by the state. The teachers were randomly divided into two focus groups, and were presented with three blinded sets of parallel excerpts from each standards document that represented similar content and foci. The excerpts included a topic from the algebra and geometry content standards, and the Problem Solving Process Standard from PSSM and the Standard for Mathematical Practice (SMP)Make Sense of Problems and Persevere in Solving Them from CCSSM. During the focus groups, the teachers were asked to compare the relative clarity and usefulness of the blinded excerpts. The discussions were video recorded and transcribed for inductive analysis. Open coding (Ryan & Bernard, 2000) was used to identify themes. All discrepancies in coding were discussed until consensus was reached. The data were then analyzed for similarities and differences (Charmaz, 2006) using a constant comparative method.

Findings and Implications

Based on teacher responses, the following findings were organized by three broad themes: 1) *formatting*, 2) *implementation of standards*, and 3) *characteristics of ideal standards.* The documents examined by the teachers in this study represent the intended curriculum; however, teachers' indicated expectations of support that are traditionally found in a written curriculum (Reys & Reys, 2010). The ideas expressed by teachers in these groups suggest that practitioner consensus regarding features of standards has yet to be conceived which manifests as tension between desired characteristics that, at times, appear in conflict with one another.

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THE EFFECTS OF VISUAL REPRESENTATIONS AND INTEREST-BASED PERSONALIZATION ON SOLVING MATHEMATICS STORY PROBLEMS

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This study examined how (1) visual representations including illustrations and diagrams and (2) personalization to students' out-of-school interests, affects 7th grade students' performance on and perceptions of math story problems. Considerable research shows that learners benefit from visual representations, and effectively using visual representations is one way in which to improve problem solving. Research also suggests that even shallow attempts to personalize problem contexts to student interests in areas like sports or shopping can promote learning and interest. In prior research we found no effect for personalization or decorative illustrations for 7th graders solving percent problems, but diagrams with mathematical information were associated with higher performance. Here, we seek to replicate the results for a new content area and examine student perceptions of easiness and interestingness for the problems they solve.

Participants were 179 7th grade students attending a suburban middle school; they solved a worksheet containing 8 story problems on which diagrams, illustrations, or personalization were manipulated. Students were significantly more likely to get a problem correct if it had a diagram alone than if it had no visuals (*Odds* = 1.66, $p = 0.014$). Students were not more likely to get a problem correct if it contained an illustration ($p = 0.225$) or a diagram with an illustration ($p = 0.378$) compared to no visuals, and students were not more likely to correctly solve personalized problems than non-personalized problems ($p = 0.526$). In the models predicting students' ratings of easiness (1-5 scale), problems with a diagram alone were rated as significantly easier compared to: no visuals ($B = 0.17$, $SE = 0.07$, $p = 0.013$), illustration alone ($B = 0.19$, $SE = 0.07$, $p = 0.006$), and a diagram with an illustration ($B = 0.20$, $SE = 0.07$, $p = 0.006$). Problems with an illustration were rated as significantly harder than problems without an illustration ($B = 0.11$, $SE = 0.05$, $p = 0.029$). Personalization was not related to easiness ($p = 0.53$). In the models predicting students' ratings of interestingness (1-5 scale), problems with a diagram alone were rated significantly less interesting compared to: no visuals ($B = -0.16$, $SE = 0.07$, $p = 0.017$), illustration alone ($B = -0.20$, $SE = 0.07$, $p = 0.002$), and a diagram with an illustration ($B = -0.27$, $SE = 0.07$, p < .001). Personalized problems were significantly more interesting than non-personalized problems ($B = 0.46$, $SE = 0.05$, $p < .001$).

In both this study and our prior study, performance was highest on problems that had personalization, a diagram, and no illustration. The combination of personalization and diagrams may be ideal to promote engagement while scaffolding performance for struggling learners. We found that diagrams improved performance and made problems seem easier but students found them boring. Personalization did not affect performance or easiness, but made problems more interesting. Illustrations did not impact performance, but students felt they made problems harder. This study provides guidance to inform curriculum design in math education.

DEVELOPING EFFECTIVE CURRICULUM MATERIALS FOR PROFESSIONAL DEVELOPMENT

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Keywords: Teacher Education-Inservice; Curriculum

This study examined the effectiveness of the curriculum materials for a series of grade 8 mathematics teacher workshops. In recent decades, educational researchers have tried to find effective ways to improve teachers' professional knowledge (Desimone, 2009). Educational "field has acknowledged a need for more empirically valid methods of studying professional development" (p. 181). The development of computer information communication technology (ICT) provides an informal and integral tool for teaching and learning.

The objectives of the materials were to integrate technology, mathematics content, pedagogical content knowledge and teaching resources in a collaborative environment of learning. The core features in this grade 8 mathematics program concentrate on Ten Dimensions (McDougall, 2004) for teachers' professional development. The Ten Dimensions are: (1) program scope and planning, (2) meeting individual needs, (3) learning environment, (4) student tasks, (5) constructing knowledge, (6) communicating with parents, (7) manipulatives and technology, (8) students' mathematical communication, (9) assessment, and (10) teacher's attitude towards and comfort with mathematics. The research questions are: (1) how do curriculum materials meet the needs of participants? And (2) how do workshops in mathematics content, use of technology and pedagogical strategies affect teacher's attitudes and beliefs about mathematics teaching?

The participants were 29 middle school teachers and 8 principals from an urban school district. The teachers took part in four full day workshops and were provided teaching materials through a two-way website. The participants selected five dimensions as core objectives for the workshops. The curricula materials and delivery for this teacher program identified a few key factors: teachers' needs based on former research, their feedback after each workshop, curriculum content selection, style of content delivery, and when and how to distribute workshops. The participants were interviewed individually and completed two surveys at the beginning and end of project in the academic year. This survey is a Likert-like six scale with 20 questions (McDougall, 2004). A paired T-test was employed to test significance for each of 20 questions to see the effects. Finally, the quantitative results were confirmed by the qualitative study.

The results showed that the tailored curriculum material designed by focusing on the Ten Dimensions framework is an effective approach for in-service teacher programs. The participants increased their knowledge in their targeted objectives of this program. The participants changed their beliefs about their scope and planning skills, their ability to design interesting students' tasks, their ability to communicate with parents and peer teachers, and the effective use of manipulatives and technology in classes. The participants were satisfied with the materials of mathematics content, use of technology for teaching and communication, and assessment activities conducted in the program. The findings of this program for design of curriculum content can be used in the similar educational contexts.

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Chapter 4

Early Algebra, Algebra, and Number Concepts

Research Reports

Brief Research Reports

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Poster Presentations

NUMBER LINE ESTIMATION WITH NEGATIVES

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When to introduce negative integers to children is an important issue in school mathematics; delaying their introduction can lead to lasting misconceptions such as one cannot subtract a larger whole number from a smaller. Yet understanding negatives involves a complex extension of whole-number knowledge. It is not known whether this extension is only possible after whole-number concepts are learned or whether simultaneous acquisition of positive and negative integer concepts is possible. This study used an established whole-number intervention (playing linear board games), extended to include negatives, with kindergartners and first graders. Performance placing integers on empty number lines provided evidence of students' understanding of integer concepts.

Keywords: Number Concepts and Operations; Cognition; Elementary School Education

Purpose of Study

One of the enduring challenges students face when learning number concepts is determining how to revise and build on their whole number understanding to include new numbers. In particular, incorporating negative integers into their number system is challenging, and many students will continue to assert that you cannot subtract a larger number from a smaller one, even if they can solve other problems with negative integers (Murray, 1985). At a basic level, when learning about negative integers, students must extend their backward counting sequence to below zero, using the positive number names with the word "negative" before them. Likewise, they must reinterpret the meaning of the minus sign to mean "negative" when attached to a numeral and referring to the numbers less than zero (Vlassis, 2004).

When given the opportunity to explore negative numbers, even first graders were able to talk about their values (Bofferding, 2014) and use them in arithmetic problems (Behrend & Mohs, 2005/2006). Other researchers have identified kindergarteners who were able reason about negative numbers (e.g., Bishop et al., 2010). However, questions still remain about the extent of knowledge possible for young students, whether typical kindergarteners can learn about negative numbers, and what types of activities might support their understanding. We explore these issues in this paper.

Theoretical Framework

According to Case's (1996) theory of Central Conceptual Structure for Number, before the age of four, children have two cognitive structures for number. The first allows them to count a set of objects, and the second allows them to make visual comparisons of sets of objects. However, they cannot use counting to help determine which set has more or less; these cognitive structures remain separate. Around the beginning of kindergarten, children begin to coordinate the two structures and can reason that adding one object to a set corresponds to moving up one number in the counting sequence. They also learn to map the numerals to quantities and number words. By first grade, these structures are often fully integrated if students have had supportive numerical experiences (Griffin, Case, & Capodilupo, 1995). This integration is referred to as a mental number line. As mentioned previously, to extend their mental number line to include negative integers, students must accept that there are numbers less than zero and learn the new notation (i.e., the importance of the negative sign), number names, how they are ordered, and their values.

One experience that helps students develop their whole-number mental number line is playing linear board games (Ramani & Siegler, 2008). On a simple board game labeled with squares from 1

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

to 10, preschoolers counted on as they moved toward the finish. The experience of seeing and saying the number sequence helped the children progress in their ability to identify the numerals and determine which number is bigger. Further, they outperformed a control group on a series of number line estimation tasks, which were the main measures of interest. When given a number line marked with 0 and 10 and asked to mark where numbers 1-9 go, the students who played the game were more likely to space the numbers evenly. Therefore, a linear model more completely explained their plots (based on R^2 values) and slope values of their lines were nearer to one. The results indicated that playing the board game helped the children develop a mental number line for whole numbers (Ramani & Siegler, 2008).

Unlike with whole numbers, where students have experiences both counting and working with sets of objects, children cannot work with negative sets of objects (unless we artificially impose a negative value onto objects). Therefore, playing a similar linear board game that includes negative integers may be a helpful way to give children experiences with the order and values of negative numbers. On the one hand, this extension might only make sense to children after they have developed a whole-number mental number line (in first grade). On the other hand, they may be able to learn about negatives while they are simultaneously developing the whole-number mental number line (in kindergarten). Based on these possibilities, we explore the following research question: To what extent can playing a linear board game including negative integers help kindergarteners and first graders develop a linear representation of the integers?

Methods

This study took place over two years. In the first year, we worked with first graders, and in the second year, we replicated the study with kindergarteners.

Setting and Participants

The participants came from an elementary school located in a low-income area in the Midwest with a large proportion of English Language Learners. In the first year, we conducted the study during the first three months of the school year with 50 first graders (26 female; 24 male); however due to two students moving and one not completing the tests, we only present complete data from 47 students. In the second year, we conducted the study during the first three weeks of the school year with 45 kindergarteners (27 female; 18 male).

General Design

Each year, the study involved an experimental design, which included a pretest, stratified random assignment to control or experimental ("game") group, intervention, posttest, and follow-up. We only present data from the pre- and posttest portions of the study. The design and materials replicated those used by Siegler and Ramani (2009) but included some modifications and additions to include a focus on negative integers. For the intervention, each participant worked with a researcher (professor or one of two graduate students) for three, 15-minute sessions. One of the graduate research assistants worked with four kindergarten students who benefitted from Spanish translation. During the first year there were 22 first graders in the game group and 25 first graders in the control group with complete data. During the second year, there were 23 kindergarteners in the game group and 22 kindergarteners in the control group.

Pre-test and Post-test Measures

The pretest and posttest were identical and conducted as individual interviews with the students; we did not provide specific feedback on their performance. Across the sections of the test, the problems used positive integers as well as negative integers with tasks involving counting, ordering integers, determining which integer was closer to or further from 10, and solving addition and subtraction problems involving positive and negative values (for further descriptions see Bofferding & Hoffman, 2014). We describe the two main measures of interest here.

First, on the integer identification task, we presented numerals on isolated pages in random order and asked students to identify integers from -10 to 10. Second, in the final section of the test, students were asked to place integers on number lines. Students completed a packet involving positive integers followed by one involving negative integers. Each page of the packet contained an empty number line 25.5cm long with two integers marked. On the first page of both packets, students were asked to put a pen mark where 0 would go, given the locations of -5 and 5. For the positive packet, the remaining pages contained empty number lines marked with 0 and 10. The placement of zero in the middle, i.e., leaving space for the negative numbers to the left, was an important feature. Students were asked to make a mark where a given integer should go a total of 18 times (1 through 9 in random order, twice). The researchers gave instructions such as, "If here is 0 [point to the middle] and here is 10 [point to the right], then make a mark on this line [motions to whole 25.5 cm line] where 6 should go." The negative number packet worked similarly, only with -10 marked on the left and 0 marked in the middle. Students were told to place the negative integers -1 through -9 on the respective pages.

Control Group

For their three sessions, the control group students rotated through three types of activities with the researcher. The first activity involved counting a collection of 1-10 items and counting backward as far as they could. No feedback was given on correctness. On the second activity, students put a set of six integer cards in order from least to greatest. For example, one set they ordered included the following integers: 2, 1, 0, -5, 10, and -8. After the students ordered the set, they were asked to show the least and the greatest. No feedback was given on the ordering or the identification of the cards. The last activity in the sequence was a game of memory where the goal was to match integers. Corrective feedback was given if students attempted to collect an incorrect match, but they were not told the names of the numbers.

Treatment (Game) Group

During each 15-minute intervention session, the experimental group played a board game against the researcher using a board labeled with the integers -10 to 10 (see Figure 1).

										START											
-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	FINISH

Figure 1: An illustration of the linear, numbered game board.

Players started by placing their tokens at zero, and the first player drew a card from a card deck. In the first version, all but one of the cards was labeled with a 1, 2, or 3. The remaining card contained the text, "All players go back to -10." When this card was drawn, the student had to count backward while moving the tokens back to -10. The researcher always stacked the deck so that this card would come up in the first few turns of the game, ensuring players would advance from -10 to 10 in each round. Players drew a 1, 2, or 3, moved their tokens that number of spaces, and named the numbers on the spaces they passed through. For example, if a player on -7 drew a "2," then she would move her token to -6 and say "negative six," then move her piece to -5 and say "negative five." The game ended when a player crossed 10.

During the third session, the card sending players back to -10 was replaced with a stack of cards marked with -2 or -4. Players began the game by drawing from this stack and counting backwards as they moved to -10. Once a player reached -10, on her next turn she would begin drawing with the deck containing positive numbers. From this point, play continued as normal, with the game ending once a player crossed over 10.Students played an average of 4 games in 15 minutes, and the researchers gave feedback (if needed) to correct the name of the integer that students landed on or correct the number of spaces they moved their game piece. Students had to repeat the correct name or counting sequence before the game play continued.

Analysis

Measurement

All items from the assessments were marked as correct/incorrect, except those in the number line estimation tasks. For the latter, we measured how far away from zero the student made a mark on the empty number line (to the nearest half-millimeter). We also gave the magnitude a sign, positive or negative (because zero was in the center of the line). While students were instructed to make a single, vertical line segment as their mark, some made several segments (by moving the pen rapidly up and down) or drew the numeral instead of a line segment. When measuring in these cases, we took the average of the left and right-most marks.

After one researcher completed the initial measurements for a set (e.g., measured one student's placements of positive numbers on the pretest), another researcher randomly checked five measurements. If there was disagreement on even one measurement, the second researcher checked all the measurements for that set. Lastly, a third researcher took measurements to resolve all disagreements.

Coding

To interpret the measurements, we created a four-tiered coding system. When dealing with only whole numbers as Siegler and Ramani (2009) did, it was sufficient to use two quantitative measures. The R^2 values measured the degree to which the placements were linear, and the slope of the regression line measured whether increases in the numbers to be placed resulted in a proper increase in the placements. When negatives were introduced, a complication was added. Students not only had to space the numbers evenly (high R^2value) and with equal spacing (slope near one), but they also had to know on which part of the number line to make the placements. As an example, consider a student who counted from the left (at -10) when marking positive numbers. The R^2 and slope could be exactly one, but the student would have major errors as 1 would end up at -9, 2 at -8, 3 at -7, and so on. To capture errors such as this, we added a third quantitative measure: numbers placed on the wrong side of zero. For students to show great understanding in their placements, they needed to have high R^2 values, a slope near one, and few numbers on the wrong side of zero.

To make this systematic, we created codes for four levels of understanding. A student with Level 3 understanding had an R^2value ≥ 0.90, a slope of 1 ± 0.3, and at most one value placed on the wrong side of zero. A student with Level 2 understanding did not show Level 3 understanding and had an R^2value ≥ 0.80, a slope of 1 ± 0.8, and at most two values placed on the wrong side of zero. Level 1 understanding meant not fitting into the higher levels and an R^2value ≥ 0.60, a positive slope, and at most four values placed on the wrong side of zero. Finally, Level 0 understanding was for students who did not fall into any of the higher levels. The cutoffs for these levels evolved after familiarizing ourselves with the data, including looking at scatterplots, regression lines, and using qualitative codes.

Comparing Groups

Our primary hypothesis was that the game groups would make significantly more gains in their ability to place integers on an empty number line. We operationalized this using the level-of-understanding codes described above. Specifically, we hypothesized that the mean increase in level of understanding would be significantly higher for the game group both in kindergarten and in first grade, and both with positives and negatives. We were most cautious in our hope with the kindergarteners' performance with negatives, especially if they lacked the ability to correctly identify negatives. In addition to using inferential statistics to test the above hypotheses, we also sought qualitative patterns in the data to motivate fuller explanations and future research.

Results

Identifying Negative Integers

On the pretest none of the kindergarteners in either groups could identifying any of the negative integers; instead they ignored the negative signs and either identified the positive numeral or said random number names. On the posttest, none of the kindergarteners in the control group were able to identify the negative integers. However, six students (26%) in the game group could correctly identify the majority of them. In first grade, three students (12%) in the control group and four students (18%) in the game group were able to identify negative integers on the pretest. By the posttest, eight students in the control group (32%) and 21 (95%) students in the game group did so.

Number Line Estimation

Overall, the kindergarteners showed low levels of proficiency at placing integers on an empty number line. Despite the fact that six students in the game group had success identifying negative integers on the posttest, none of the students showed Level 2 or 3 understanding according to our coding system (see Table 1). Similarly, there was limited success with positives in the game group; only two students achieved Levels 2 or 3. No students in the control group for kindergarten moved above Level 0 for positives or negatives. While several kindergarteners' R^2 values improved, they often had a tendency of placing the numbers on the wrong side of zero (see Table 2 for an example).

Table 1: Students' Levels of Number Line Estimation on Pre- and Posttest

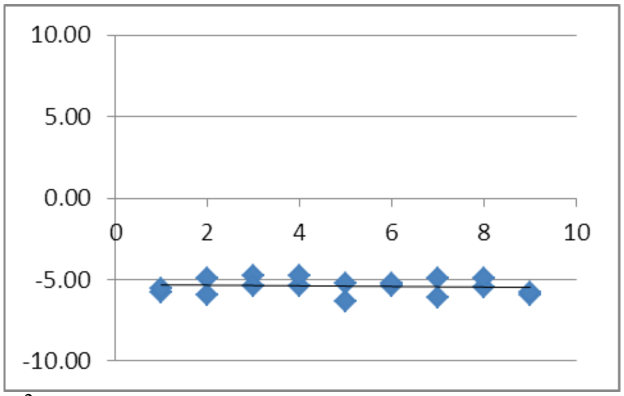
	Kindergarten Control (N=22)		**Kindergarten** Game (N=23)		**1st Grade** Control (N=25)		**1st Grade** Game (N=22)	
Level	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Positive								
3	0	0	2	2	7	6	2	3
2	0	0	0	0	6	4	6	6
1	1	0	1	3	3	2	1	2
0	21	22	20	18	9	13	13	11
Negative								
3	0	0	1	0	4	3	3	7
2	2	0	1	0	6	4	5	6
1	1	0	0	2	2	3	3	1
0	19	22	21	21	13	15	11	8

The first graders performed better than the kindergarteners in every way. There were seven students who achieved Level 3 with negatives on the posttest and six who achieved Level 2. Thus, well over half (59%) showed high levels of proficiency. With the positive integers, nine students achieved Levels 2 or 3 (41%). Even the control group experienced success: ten students achieved Levels 2 or 3 with the positives (40%) and seven with the negatives (28%).

To make the comparison between the groups more rigorous, tests of four a priori hypotheses were conducted using Bonferroni adjusted alpha levels of .0125 per test (.05/4). The four hypotheses consisted of checking for significant differences between the mean change in level of understanding, pretest to posttest, for the positive and negative integers, crossed with the two grade levels. Results

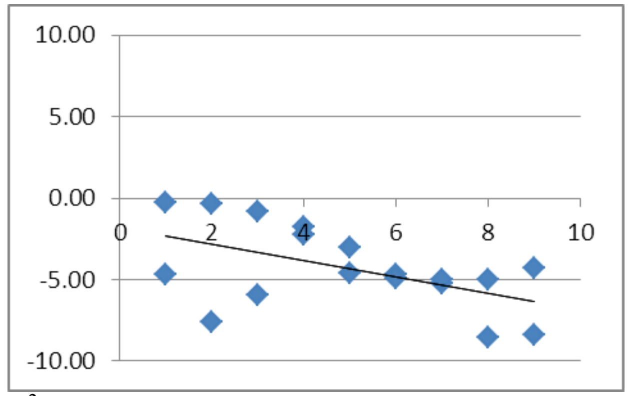
Table 2: A Kindergartener's Number Line Estimation Data (Level 0) on Pre- and Posttest

Student G02, Pretest Positive Values



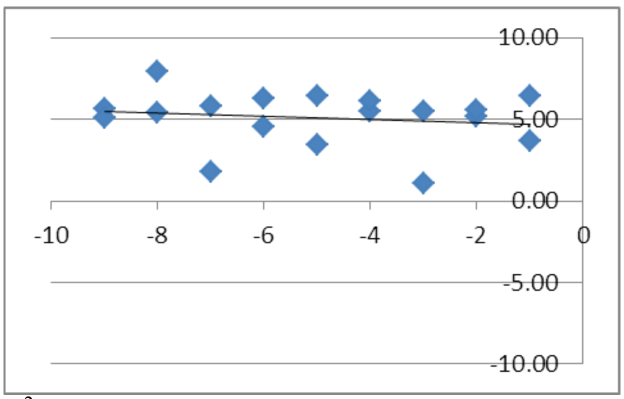
R^2 = .01; Slope = -.02, all with wrong sign

Student G02, Posttest Positive Values



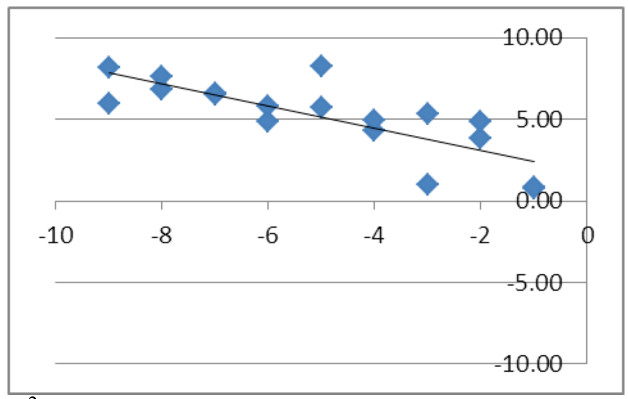
R^2 = .28; Slope = -.50, all with wrong sign

Student G02, Pretest Negative Values



R^2 = .02; Slope = -.10, all with wrong sign

Student G02, Posttest Negative Values



R^2 = .62; Slope = -.68, all with wrong sign

indicated that the mean change in level of understanding was not significantly different with the positives for the kindergarteners between the control group (*M* = -0.05, *SD* = 0.21) and the game group (*M* = 0.09, *SD* = 0.29). Also for the kindergarteners, the mean change was not significantly different with the negatives between the control group (*M* = -0.23, *SD* = 0.61) and the game group (*M* = -0.13, *SD* = 0.46). Likewise, there was no significant difference seen in the first graders with the positives between control (*M* = -0.32, *SD* = 0.95) and game (*M* = 0.18, *SD* = 1.01). However, there was a significant difference seen the mean change with respect to the negatives in first grade between the control (*M* = -0.24, *SD* = 0.72) and the game group (*M* = 0.55, *SD* = 1.18), $t(34) = -2.70$, $p = .011$. Therefore, the intervention, i.e., playing the linear board game significantly impacted participants' ability to place negative integers on an empty number line.

Students who were less successful on the number line estimate task fell into two major groups. One set of students spaced out the numbers along the entire line, ignoring that 0 fell in the middle of the line. Therefore, these students had close to half of their points fall on the wrong side of zero (see Figure 2).

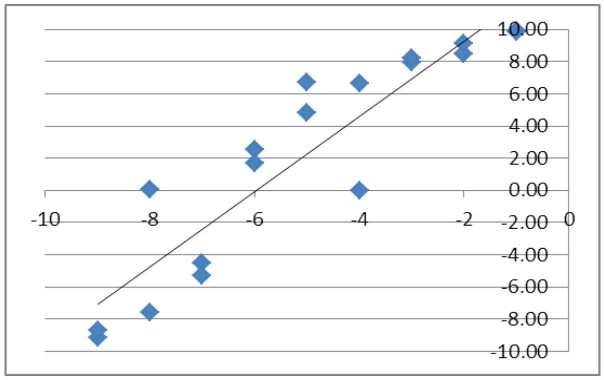


Figure 2: An illustration of data spanning the whole number line when placing -9 to -1.

A second set of students placed numbers in two to three similar locations, regardless of the number shown, as if they split the number line into small and large or small, medium, and large. Therefore, their points formed distinct clusters along the line (see Figure 3). Sometimes, these students started at 0 and counted up to place numbers 1-5 or started at 10 and counted down to place numbers 6-10, which accounted for the clustering.

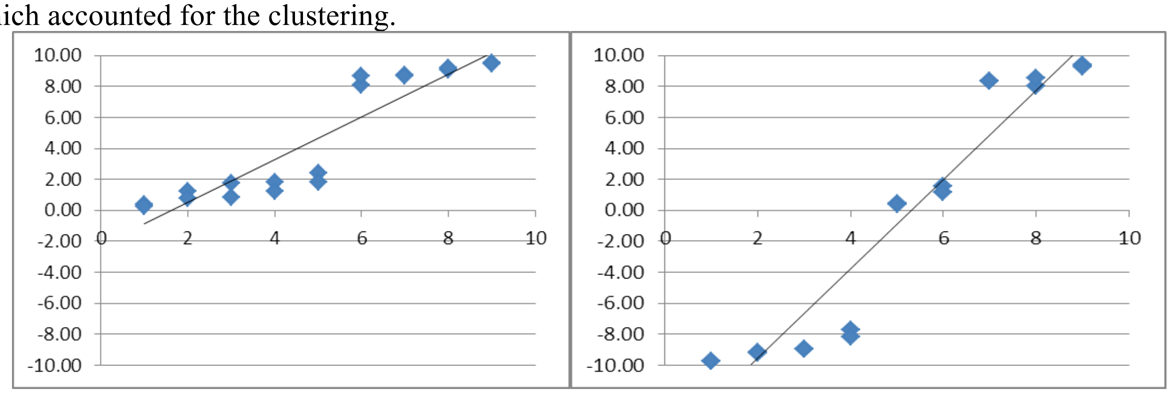


Figure 3: Illustrations of data chunked along the number line when placing 1 to 9.

Conclusions and Implications

Based on the results, we conclude that playing the board game did not help kindergarteners develop a mental number line including negative numbers. Although they started to space out their placement of the integers, they frequently placed numbers on the opposite side of zero. More surprising, they did not improve on placing the positive values on the board as was found in previous studies with preschoolers (Ramani & Siegler, 2008; Siegler & Ramani, 2009). A likely reason for this is that students were given space to mark positive numbers before 0 (as opposed to having zero at the edge of the page). Therefore, they often chose to mark 0 and 1 near the left edge of the paper, to the left of zero, at the beginning of the line. This suggests that as students learn about positive numbers, they need opportunities to see zero in other locations than just at the edge of the paper, and also suggests that Ramani and Siegler's (2008) results may overestimate students' abilities. Because the kindergarteners here learned about positive numbers *and* negative numbers, it is also possible that the kindergarteners had too much to learn compared to children in Ramani and Siegler's study (2008), and the time spent on negatives might have taken away from time needed with positive numbers. Alternatively, providing a longer intervention may lead to a stronger effect for both positive and negative numbers.

On the other hand, the first graders in the game group benefitted from playing the board game. Almost all of the students were able to identify negative numbers on the posttest and a significant number were able to estimate the placement of all integers on the number line fairly well. These results suggest that students are more likely to develop a mental number line that includes negative numbers if they already have a whole number mental number line.

Finally, students' placement of the numbers suggests a few areas to focus on in instruction. Students had an inclination to take up as much space as they were given, spacing out the positive numbers across both negative and positive parts of the number line (and similarly for negative numbers). Further, they often started counting from the very left of the page, rather than attending to the given points. When introducing and using visual aids such as the number line in the classroom, teachers should present numbers in multiple formats (not always starting at the left of the paper) and talk about numbers on either side of key reference points, such as zero. Presenting number lines with different numbers marked and with different scales may help students attend to the relevant features of the number lines and placement of numbers.

Acknowledgements

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ANALYSIS OF STUDENTS' PROPORTIONAL REASONING STRATEGIES

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Proportional reasoning is key to students' acquisition and application of complex mathematics and science topics. Research is needed regarding how students' progress towards and come to demonstrate key developmental understandings within proportional reasoning. To this end we created and administered assessment items to 297 middle grades students. We categorized student solution processes qualitatively, followed by Rasch analysis to examine item difficulty and strategy use in relation to an anticipated trajectory. Our findings indicate that different strategies manifest themselves in a hierarchical manner, providing initial confirmation of categories based on strategy efficiency and emphasizing the importance of teacher (and researcher) analysis of classroom assessments from a student cognition perspective.

Keywords: Learning Trajectories; Measurement; Number Concepts and Operations

Purpose

Proportional reasoning is a lynchpin for future success in mathematics and science (Lesh, Post, & Behr, 1988). Based on a substantial body of proportional reasoning research (e.g., Lamon, 2005; Lobato, Ellis, & Charles, 2010; Tourniaire & Pulos, 1985), there have been several calls for shifting instruction from the typical focus on the cross-multiplication algorithm to students' meaningful understanding and application of ratio related concepts (e.g., National Governors Association & Chief Council of State School Officers, 2011). However, implementing this shift in instruction is difficult. Schools and teachers need resources to support this change and more information is needed on how students' proportional reasoning develops from less efficient to more efficient strategies.

The overall purpose of our research is to develop measures to assess students' flexibility and efficiency in proportional reasoning situations. Our work revolves around: (a) measuring and identifying qualitatively different categories or aspects of student reasoning, and (b) determining whether these categories manifest themselves along a hierarchical progression. The qualitative and quantitative confirmation of different categories or aspects of students' proportional reasoning along a continuum would contribute to a better awareness of how students' progress towards important understandings and assist in designing classroom instruction, curriculum, assessment, and teacher professional development.

More specifically, the present research uses Simon's (2006) KDUs as a theoretical framework for examining student work samples to identify qualitatively different categories or aspects of proportional reasoning in simple missing value contexts. From there we use the structure of Steinthorsdottir's (2009) hypothetical trajectory to create assessment items which measure students' proportional reasoning and enable us to analyze and categorize the resulting student thinking. Lastly, we use Rasch modeling to determine whether our identified categories for student thinking manifest themselves hierarchically, indicating the potential usefulness of the assessment items and qualitative rubric for teachers and researchers in their analysis of students' thinking.

Theoretical Framework

Simon's (2006) articulation of key developmental understandings (KDUs) provides a framework for analyzing students' flexibility and efficiency in proportional reasoning situations. Simon (2006) describes key developmental understandings by stating,

> …I am not claiming that these understandings exist in the student; rather, specifying understandings is a way that observers (researchers, teachers) can impose a coherent and potentially useful organization on their experience of students' actions (including verbalizations) and make distinctions among students' abilities to engage with particular mathematics (p. 360).

We see KDUs as a potential framework for identifying important categories of students' reasoning when analyzing work samples. The following section articulates a KDU important to students' initial development of proportional reasoning.

Research indicates that students' demonstration of flexible and efficient use of the scalar and functional perspectives in proportional reasoning situations may be a KDU (Lobato, Ellis, and Charles 2010; Lamon 2005). A scalar perspective entails recognizing a ratio as a composed unit that can be scaled up or down by multiplying each quantity in the ratio by a constant factor. For example, given the problem "Callie bought 7 cookies for \$3. How many cookies can Callie buy for \$12?" a student recognizes the original 7 cookies to \$3 ratio can be scaled up by multiplying each quantity in the ratio by 4 to generate the 28 cookies for \$12 ratio (see figure 1a). A functional perspective entails recognizing and using the constant multiplicative relationship between the two quantities within the ratio and applying this relationship to create equivalent ratios. For example, given the similar context "Callie bought 6 cookies for \$2. How many cookies can Callie buy for \$13?" a student recognizes the number of cookies to be purchased is three (6 ÷ 2) times the number of dollars paid. This understanding allows the student to quickly realize Callie can purchase 3 x 13 or 39 cookies (see figure 1b).

Figure 1a: Callie bought 7 cookies for \$3. How many cookies can Callie buy for \$12?

Scalar Perspective

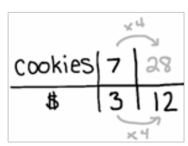


Figure 1b: Callie bought 6 cookies for \$2. How many cookies can Callie buy for \$13?

Functional Perspective

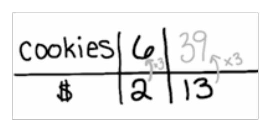


Figure 1. Scalar and functional assessment items and solution perspectives.

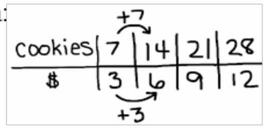
In simple missing value problems, students demonstrate attainment of this initial proportional reasoning KDU by flexibly and efficiently demonstrating knowledge of either the scalar or functional strategies based on the situation or number relationship presented (Steinthorsdottir & Sriraman, 2009). For example, given the situation illustrated in Figure 1a, an efficient and flexible strategy is to scale up by a factor of four. A student who applies the functional multiplier of 2.33 is likely performing a standard procedure without reasoning through the proportional relationships, indicating a possible lack of flexibility in their proportional reasoning.

In addition to examining a students' work for application of the scalar or functional perspective, one must also examine scalar situations for the level of efficiency used in the scaling process (Authors, 2015). For example, in missing value situations with an integral scalar relationship strategies can often be differentiated as additive or multiplicative (see figure 2).

Additive strategies may indicate initial understanding of the scalar relationship but a multiplicative understanding is needed for eventual generalization of the scalar perspective to non-integral relationships. Therefore, student work must also be examined for use of an additive versus multiplicative approach in addition to the scalar and functional perspectives.

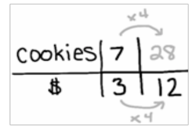
Callie bought 7 cookies for $3. How many cookies can Callie buy for $12?

Figure 2a:



Additive

Figure 2b:



Multiplicative

Figure 2. Additive and multiplicative solution strategies for a scalar problem.

In sum, we observe students' flexible and efficient application of the scalar or functional perspective in simple missing value situations as an initial KDU in a proportional reasoning learning trajectory. In students' application of the scalar perspective, student solution strategies must be examined for an additive versus multiplicative approach to ensure students are able to eventually generalize their strategy to non-integral situations. The next section details a potential developmental trajectory for these ideas, followed by a description of the assessment items designed to capture students' understanding of these concepts.

Steinthorsdottir and Sriraman (2009) articulated a potential progression for students' proportional reasoning. They identified four levels of increasingly sophisticated strategies students used to solve missing value problems. In level one, students incorrectly focus on the difference in quantities either within or between the ratios. In level two, students focus on either additively iterating or multiplicatively scaling the given ratio as a composed unit to reach the missing value in the equivalent ratio (scale-up). Level three involves scaling down a given ratio, and includes two sub-levels, the ability to partition the given ratio as a composed unit to reach the missing value in the equivalent ratio (scale-down) and the ability to combine iteration and partitioning to reach the missing value (scale up and down). Level four involves the flexible use of either the scalar or functional relationship depending upon the ease of calculation with the numbers in the problem.

Based in part on the progression outlined by Steinthorsdottir and Sriraman (2009) and the identified KDUs for scalar and functional perspectives, we developed an assessment with a focus on manipulation of number relationships to examine students' types of reasoning and the level of efficiency in their solution process. For the sake of brevity we focus on presenting three exemplar problems from the assessment and their solution strategy analysis (Table 1).

Table 1: Assessment framework and anticipated solution strategy analysis

Item Construction		Solution Strategy Analysis	
Type of Reasoning	*Integral Number Relationship*	*Less Efficient*	*More Efficient*
1. Scalar	Scale Up	Additive	Multiplicative
2. Scalar	Scale Down	Additive	Multiplicative
3. Functional	Functional	Scale Up and Down	Functional

Methods

Research Questions

1. Can we identify qualitatively different categories of student thinking related to use of an additive or multiplicative solution strategy and/or fluent and flexible use of the scalar or functional relationship depending on the number relationship presented?

2. Do the items manifest themselves as anticipated in relation to a progression from easiest-scale up, moderate-scale down, to functional-hardest?
3. Do the strategy labels and associated scoring codes based on strategy efficiency manifest themselves as anticipated along a continuum? In other words, are less efficient strategies used by less able students and more efficient strategies associated with more able students?

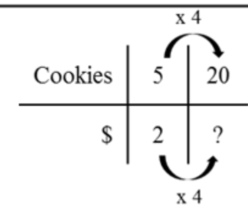
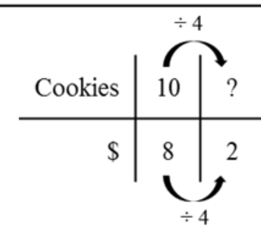
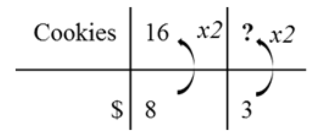
Participants

The respondents represent a convenience sample of 297 students from fourth to ninth grade with the majority of the students coming from 6th and 7th grade (*n*=198).

Measure

As described in Table 1, we constructed a measure based on the Steinthorsdottir progression. We focus on presenting data from three of the assessment items: scale up, scale down, and functional. Table 1 provides the 3 items and the anticipated ‘efficient’ strategy based on the number relationship.

Table 2: Assessment items included in analysis.

Strategy Focus	Context	Anticipated Strategy
Scale Up	Callie bought 5 cookies for \$2. How much will it cost to buy 20 cookies?	
Scale Down	Thomas found a cookie deal with 10 large cookies for \$8. How many cookies can he buy for \$2?	
Functional	Jason found a cookie deal with 16 cookies for \$8. How many cookies can he buy for \$3?	

Timeline and Setting

In order to focus on initial cognitive understanding rather than procedural knowledge, assessment items were administered in the fall prior to formal proportional reasoning instruction. Older students in our sample should have received instruction around proportional reasoning and we would expect more efficient strategies from these students. However, contact with teachers in our study indicated instruction was based primarily on implementation of cross-multiplication, with little or no emphasis on scalar or functional perspectives.

Results

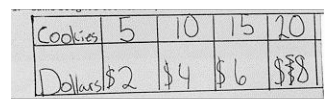
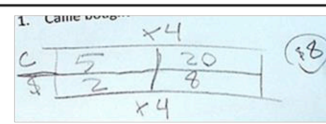
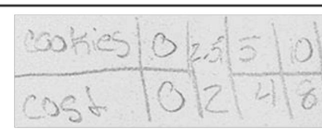
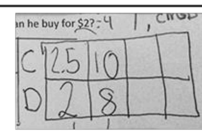
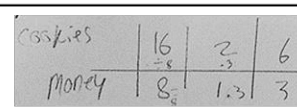
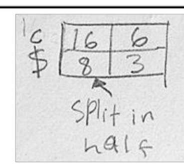
Qualitative analysis of the outcome space

We analyzed the student strategies for each assessment item. We coded each for an additive or multiplicative solution strategy and/or fluent and flexible use of the scalar or functional relationship. Table 3 provides an overview of the coding framework with example student work for each efficiency level. In addition to the qualitative coding of strategy name, description and example, we identified the following scoring categories with respect to strategy efficiency: 0 = incorrect, 1 = correct but inefficient strategy, and 2 = correct and efficient strategy.

Rasch Analysis

We selected Rasch modeling for our quantitative analysis due to its usefulness in (a) identifying the difficulty level of an item in relation to other items, and (b) evaluating the strategy thresholds of our efficiency-based scoring model (Van Wyke & Andrich, 2006). Assessments created to fit the Rasch model consists of items designed to assess a single (unidimensional) construct. Rasch analysis situates test takers' understanding (person ability) and item difficulty along a common equal interval scale, often with a score range between -4 to 4 with 0 as the mean. Therefore, person ability and item difficulty scores can be interpreted in relation to one another through probabilistic language. In situations involving dichotomous scoring (0=incorrect, 1=correct), when person ability and item difficulty are the same, this indicates a 50% probability that the individual would respond correctly (or incorrectly). When a person ability score is higher (e.g., 1) than the item difficulty (e.g., -1) the person is more likely to solve the problem correctly and vice versa. Figure 3 provides an example of a Rasch

Table 3: Strategy coding framework for assessment items with example student work.

Number Relationship	Number Relationships	Rubric Points	Strategy **Name** and Description	Strategy Example
Scale Up	Cookies | 5 | 20 $ | 2 | ?	1	**Scale Up Additive:** Iterate initial ratio additively or double and continue to double resulting ratios until desired number of given component is reached	
		2	**Scale Up Multiplicative:** Determine and use scale factor that scales initial ratio to desired number of given component in one step	
Scale Down	Cookies | 10 | ? $ | 8 | 2	1	**Scale Down Additive:** Halve initial ratio and continue to halve resulting ratios until desired number of given component is reached	
		2	**Scale Down Multiplicative:** Determine scale factor that scales initial ratio down to desired number of given component	
Functional	Cookies | 16 | ? $ | 8 | 3	1	**Scale Up and Down:** Use a combination of scaling up/down to reach desired number of given component	
		2	**Functional:** Determine constant of proportionality and apply to the number of given component	

item scale map based on our analysis. Typically Rasch item maps display person ability on the left side of the scale and item ability on the right. However, for ease of interpretation, we focus on item difficulty and are not displaying person ability.

Analysis 1

In the context of our present work, we first used Rasch analysis to examine item difficulty in relation to the number relationships being manipulated across problems. Our initial analysis examined whether the item difficulties manifested themselves as anticipated in relation to our hypothetical progression; easiest-scale up, moderate-scale down, and functional-hardest?

Figure 3a presents the results of Rasch analysis using the Winsteps program with dichotomous item scoring (0=incorrect, 1=correct). The difficulty scores in the box represent the item difficulty (and standard error) and determine an item's placement on the scale. For example in figure 3a, for the scale up item, -2.99 is the point on the continuum where students with an estimated ability below -2.99 are more likely to get the problem incorrect. Students with an estimated ability score above -2.99 are more likely to get the problem correct. If a student had an estimated ability score of 0, it is likely they would correctly solve the functional and scale up item but incorrectly solve the scale down item.

Findings from Analysis 1

The item order from easiest to most difficult was (1) scale up, (2) functional, and (3) scale down. Our empirical data indicated more students were likely to correctly solve the functional item than the scale down item. This was different than we anticipated. However, we recognized potential issues with this analysis. First, there was the issue with the scale down item not resulting in an integer answer (i.e., 2.5 cookies) and it is highly possible this non-integer result influenced the level of item difficulty.

However, perhaps more importantly, there was also an issue with examining the data from a dichotomous or correct/incorrect perspective instead of investigating students' strategy approach given our qualitative scoring rubric. We knew students used different strategies to correctly solve an item. Our dichotomous scoring model did not allow us to take student strategies into account, nor did it allow us to examine whether students had selected a flexible and efficient approach based on the number relationships presented in the problem. For example, on the scale up problem we wanted to determine whether students who were more likely to use an additive strategy (less efficient) versus a multiplicative strategy (more efficient) differed in ability score.

Analysis 2

Analysis 2 evaluated the qualitative rubric and associated scoring model of less and more efficient strategies. Our specific question was, did our strategy labels and associated scoring codes related to strategy efficiency manifest themselves as anticipated along the interval scale? In other words, are less efficient strategies associated with less difficult threshold scores and more efficient strategies associated with more difficult threshold scores. Thresholds are the point on the continuum where adjacent categories (or strategies) are equally probable. For example, the transition point between correct but inefficient (1) and correct and efficient (2) for the scale- up problem is .48 (see table 4). A student with an ability score of 1 would be more likely to use an efficient strategy, while a student with an ability score of 0 would be more likely to use a correct but inefficient strategy. To conduct this analysis we used a partial credit Rasch model in the Winsteps program. The strategy categories were scored as more efficient (2 pts), less efficient (1 pt), and incorrect (0 pts).

Findings for Analysis 2

Table 4 provides the Rasch item statistics for the three items and the associated category thresholds (other assessment items administered on the test are not included for ease of interpretation). The fit statistics indicate the items are 'fitting' the model. Figure 3b presents findings from analysis 1 and 2 in conjunction with each other. For example, the less efficient scale up additive strategy for the scale up item had a difficulty threshold of -3.85, and the more efficient scale up strategy for the same item had a difficultly threshold of 0.48. This indicates that a student with an ability level below -3.85 would likely get the item incorrect. Those students with ability levels between -3.85 and .48 would likely get the item correct, but use an additive strategy. Those students with an ability level above .48 would be more likely to use a multiplicative strategy.

Table 4: Rasch item statistics for proportional reasoning items

	Strategy Category Threshold	Threshold Difficulty	SE	Infit MNSQ	Outfit MNSQ
Scale Up	0-1	-3.85	0.22	1.01	1.00
	1-2	0.48	0.21	1.06	1.12
Scale Down	0-1	-0.38	0.18	0.89	0.80
	1-2	1.54	0.37	1.20	1.07
Functional	0-1	-0.97	0.18	0.98	0.92
	1-2	1.57	0.37	1.07	1.00

The threshold levels for additive versus multiplicative strategies also held true for the scale down item. While still preliminary, these findings support analysis of students' use of additive or multiplicative strategies in classroom and assessment practices to determine their depth of understanding of the scalar perspectives. This will support later work on related topics, such as geometric scaling where students must be flexible and efficient in applying a scalar multiplicative strategy. In addition, threshold levels for the functional item strategy categories supported the progression articulated by Steinthorsdottir and Sriraman (2009).

In conclusion, we recognize a continuum of ordered strategies related to students' ability is not equivalent to a progression of how students' develop understanding of a KDU. However, Rasch analysis has the potential to support our qualitative findings in ways that would assist in identification of hierarchical relationships between strategy approaches. This can provide important information to inform future research. In addition, the fact that the observed strategy thresholds match the scoring rubric indicate the usefulness of the qualitative rubric in analyzing student work through an efficiency perspective.

Lastly, while not a focus on this investigation, comparison of the two scoring models highlights the importance of analyzing students' strategies in the evaluation of students' understanding of a topic. We cannot assume students have demonstrated knowledge of a key development understanding simply because they provide the correct answer. A student could (and did) correctly solve all three assessment items using a scalar additive or scale up and down strategy. However, these strategies do not demonstrate understanding of the scalar and functional perspectives and will not continue to work as number relationships increase in difficulty.

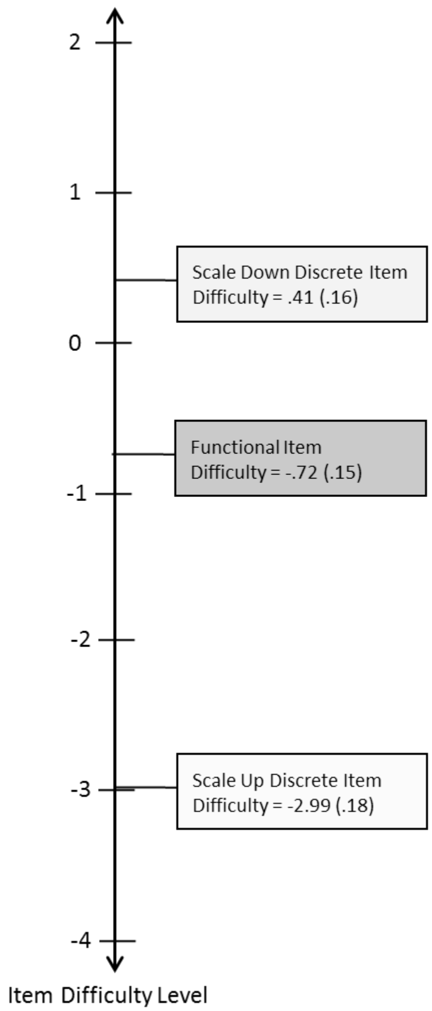


Figure 3a: Analysis 1 with dichotomous scoring

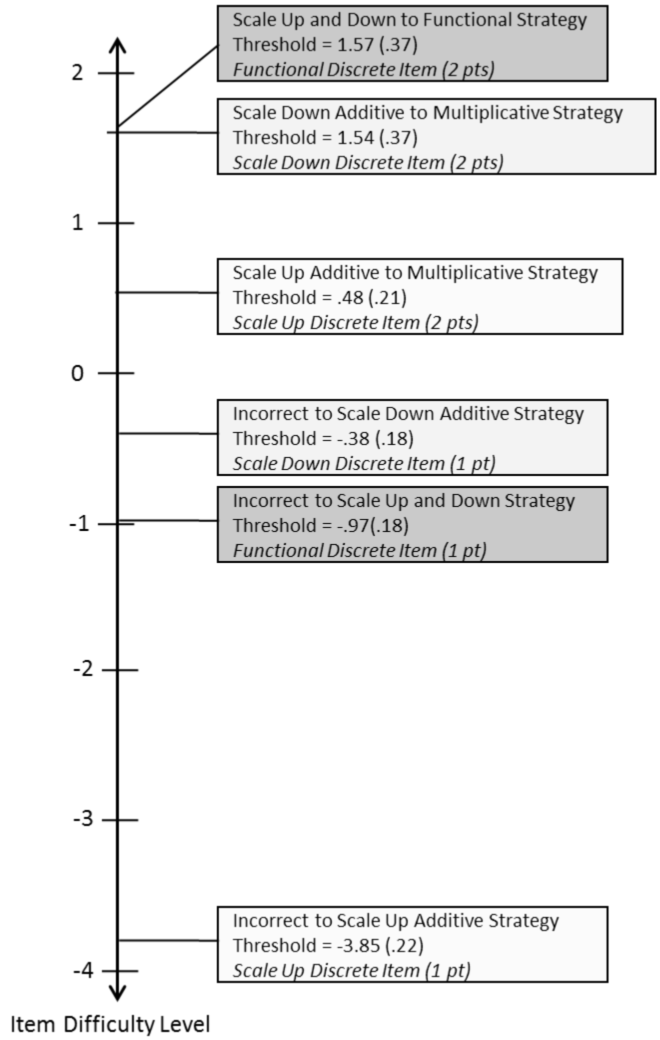


Figure 3b: Analysis 2 with partial credit scoring by strategy

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FRANK'S PERCEPTUAL SUBITIZING ACTIVITY RELATIVE TO NUMBER UNDERSTANDING AND ORIENTATION: A TEACHING EXPERIMENT

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This proposal explores how the activity of subitizing – quickly apprehending the numerosity of a small set of items – changes with the development of number concepts. We describe how varying the orientations of items in teaching experiment sessions promoted one pre-schooler, Frank, to attend to subgroups of items and change his thinking about conjoining two numbers. The results illustrate how game-play oriented subitizing activities may support growth in number understandings.

Keywords: Number Concepts and Operations; Pre-School Education; Learning Trajectories

Introduction

Subitizing is a quick apprehension of the numerosity of a small set of items (Sarama& Clements, 2009). Sarama and Clements (2009) suggested that subitizing processes transition from a reliance upon orientations to a reliance upon number understandings. MacDonald (2013) conducted four concurrent teaching experiments to investigate how subitizing activity changed in relation to understanding of number and perceived space between items. In this study, we focus on how one student's (Frank) understanding of number changed over time to rely on more conceptual processes.

Literature Review

Subitizing

Sarama and Clements's (2009) argue that children rely on either *perceptual subitizing* or *conceptual subitizing* when subitizing. Perceptual subitizing, an innate ability to discriminate different quantities, emerges in infants as young as three to five months of age and is limited to five items. Conceptual subitizing is grounded in a child's number understanding due to a child's ability to subitize groups and then compose the total number of items (Sarama & Clements, 2009). When children in kindergarten through grade two engaged in *Building Blocks,* a computer learning environment, their subitizing activity became more sophisticated and included conceptual processes (Clements & Sarama, 2007).

Number Understanding

Number understandings will be grounded in theories stemming from number conservation (Piaget, 1941/1965). This is characterized as a child's simultaneous coordination of their *serial* (number follows a sequential order) and *algebraic* (number is composed by smaller subgroups) thinking structures (Piaget, 1968/1970). Number understanding in this study will be centered on the following four areas: (a) counting (Steffe, Cobb, & von Glasersfeld, 1988), (b) composition and decomposition of number (Fuson et al., 1997), (c) links between quantity and number words (Krajewski & Schneider, 2009), and (d) perceived dimensionality (Piaget & Inhelder, 1948/1967).

Counting. Counting is described as a child relying primarily on serial thinking structures, as items are empirically pointed to and coordinated with a sequence of words. (Steffe et al., 1988). Multiple sets of research findings emphasize the importance counting has in children's mathematical development (Chan, Au, & Tang, 2014; Jones et al, 1994; Steffe et al., 1988). Steffe et al.'s (1988) research findings essentially describe how counting promotes children's number development through a coordination of units. Van Nes & van Eerde (2010) found that relationships between spatial reasoning and counting exist, as children's counting changed in relation to block arrangements. Thus, spatial orientations of objects promote children to rely on different types of counting and construct sophisticated number understandings.

Composition and decomposition of number. Fuson et al. (1997) found that children able to construct multi-digit number understandings had more sophisticated grouping techniques. Essentially when children compose and decompose number they progress through six stages of development. Jones, Thornton, and Putts' (1997) also suggest that many aspects of number understanding, including composition and decomposition of number, are foundational for students' development of number. Thus, these findings imply that young children build multi-digit number understandings through effective composition and decomposition of numbers less than ten.

Link between quantity and number words. As number is understood in a more abstract manner, number words are said to link to quantities (quantity-number competencies [QNC]) (Krajewski & Schneider, 2009). Krajewski and Schneider found that kindergarten children's QNC explained about 25% of these children's achievement scores four years later. Implications from this study suggested that future research consider how young children's empirical activity with concrete material promotes children's QNC ability prior to entering kindergarten.

Dimensionality. Topological thinking structures involve a child's attention towards the perceived topology of objects and sets of objects (Piaget, 1968/1970). One aspect of topological thinking structures, *dimensionality,* is described by Piaget and Inhelder (1948/1967) as directly promoting the flexible thinking necessary for children's later conceptualizations of formal Euclidean Geometry. Four areas of development characterize dimensionality: (a) proximity (nearbyness), (b) separation (betweeness), (c) continuity (connecting objects in spatial fields), and (d) enclosure of shape (surrounding) (Piaget & Inhelder, 1948/1967).

Purpose

Parallels between children's construction of number and subitizing activity have been suggested in the research literature (e.g. Freeman, 1912; Sarama & Clements, 2009), but a fine-grained analysis of this relationship is absent from the literature. To understand number, children need to engage with empirical items to group, partition, compose, decompose, and count. Linking number to number words would allow for number to be abstracted. Subitizing activity would essentially promote several of these empirical and mental activities to provide a child a vehicle in which to construct number understandings. The purpose for this study was to investigate how one child's understanding of number changed as he engaged in subitizing activity.

Methodology

Teaching Experiment Methodology

This study uses teaching experiment methodology (Steffe & Ulrich, 2014) and is grounded in the *radical constructivist paradigm* suggesting that mathematics understanding is actively constructed (von Glasersfeld, 1995). A teaching experiment includes a teacher-researcher, a witness for each teaching episode, at least one student, and a way to record student actions and words in each teaching session (Steffe & Ulrich, 2014). In this study, the first author was the teacher-researcher, and the second and third authors alternated as the witness. The teacher-researcher and witnesses initially met prior to the start of the experiment to establish a similar theoretical grounding and establish the functional aspects of each of our roles in this study.

Participants

Fifteen students between the ages of three years, 11 months, and five years, five months were initially recruited to participate in a larger study. The 15 students were enrolled in a preschool located near a university campus located in the southeastern portion of the United States. Four students spoke a second language at home. Eleven students were male and four students were female. Following an initial screening, six of these preschool students were selected to participate in a teaching experiment. The selection was based on their ability or lack of ability to conserve number, count, and subitize two to five items. An in-depth analysis of one student, Frank, is the focus of this study.

Frank. Frank is a male student whose family is from China. He was four years and five months old at the onset of this study. He spoke English, and in his home spoke Mandarin. Frank was interviewed two separate times on June 5th to determine if he was able to conserve number and to determine his counting and subitizing abilities. Throughout the interviews, Frank wanted to have the "correct" answer. This disposition promoted Frank to reflect more often on his activity. Frank engaged in 22 teaching experiment sessions.

Procedures

Interviews. Frank's first two interviews were used to determine (a) whether he conserved number, (b) whether he could keep track of items when counting, and (c) whether he perceptually or conceptually subitized. Frank was found to not be able to conserve number, and used perceptual subitizing. When counting, it seemed as if Frank was able to initially "count on" from 12 items. Knowing that Frank was able to count on and not conserve number seemed atypical, as Steffe et al.'s (1988) findings indicated that "counting on" required a more comprehensive understanding of number.

Teaching experiment session tasks. The teaching experiment was comprised of 22 sessions occurring two days per week no more than 20 minutes each. Tasks were designed to either assess or provoke change in Frank's thinking. Item orientation, reliance upon empirical material, and QNC were considered in the formation of the tasks throughout analysis. Tasks required him to subitize a set of items, draw or use counters to show what he remembered, and use words or actions to justify his response. They were refined prior to the teaching experiment sessions, allowing for orientation, quantity, or color of items to change to test and expand the limits of Frank's thinking. The five following tasks were used: (a) Draw what you saw, (b) Camera game, (c) Concentration, (d) Board games, and (e) Hidden Pictures. Below, we focus on the first two.

Draw what you saw. The teacher asked Frank to subitize a set of dots or counters and then to draw or use counters to show what he "saw" or "remembered." This activity was also followed up with, "How do you know you saw _______?" Frank was given material to draw what he remembered or use counters to represent what he remembered.

Camera game. The camera game was adapted from Clements and Sarama's (2007) activities. Clements and Sarama's (2007) camera game used a computer program, but in this study the activity had a series of camera pictures on a three-ring notebook. Frank was shown quickly an image of the viewfinder of a camera with dots arranged. He was asked how many dots were seen, and then he drew what the picture would look like when it came out of the camera.

Analysis

Two forms of analysis, conceptual and retrospective, were used to model and describe Frank's thinking (Steffe & Ulrich, 2014). *Conceptual analysis* regards students' responses between tasks and sessions, and *retrospective analysis* regards changes over a longer period of time. Each session was videotaped with two video cameras. Each session's video footage was reviewed after each session (conceptual analysis) and sections of video footage from the entire study were reviewed six times throughout the study (retrospective analysis).

Results

Conceptual Analysis

Subitizing activity relative to the perceived symmetry of items. Initially, Frank subconsciously relied on symmetry when subitizing, as it seemed the symmetrical orientations of the items afforded Frank the opportunity to build towards four. In Frank's first teaching experiment session he was shown four dots arranged in a square-like orientation. Frank stated that he saw "T…four," but when asked about almost stating two, he responded that he did not remember seeing two. When asked to draw what he remembered, he drew two dots and wrote the numeral two beside them. After seeing the orientation a second time, he stated that he saw four and drew the four dots in the same square-like orientation and wrote the numeral four beside them. His response suggests a subconscious attention to the two by which to build towards four.

Subitizing activity relative to the perceived space between items. In the middle of his fourth teaching experiment session Frank was playing the "Camera Game." He was shown an image with five dots (see Figure 1). The space between the square and the one dot seemed to disrupt Frank's ability to subitize the total group. This is evident when Frank draws four dots and one dot, and then writes the corresponding numerals beside his drawings (see Figure 1). After Frank describes seeing "four and one," he is asked "how many is there altogether?" This question elicits his response of "fourteen." This happened in subsequent sessions when Frank used counters, and it suggests that Frank's QNC was grounded in a procedural understanding for two-digit numbers (Krajewski & Schneider, 2009).

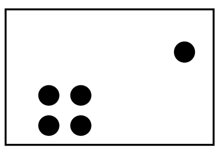
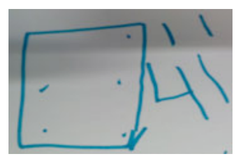

Figure 1: The orientation shown to Frank and the drawing Frank made respectively.

In Frank's earlier sessions, it seemed as if Frank's QNC was primarily procedural. This was evident when he used two-digit numbers to describe what he saw (i.e. "four and one…that makes fourteen"). Space between clustered items played a critical role in Frank's subitizing activity, as it seemed to promote him to attend towards subgroups, but he lacked the ability to compose these two groups of number. Thus, Frank would rely on a procedural QNC by stating a two-digit number. Symmetrical orientations with a regular amount of space between items prompted Frank to build towards the total number of items with one subgroup. This was evident when Frank said, "T…four." This activity does not require Frank to compose groups but to count up after subitizing two, eliciting a serial thinking structure. Thus, it seemed as if he was having difficulty coordinating his serial and algebraic thinking structures.

Changes in Frank's subitizing activity relative to changes in Frank's understanding of number. Throughout the first three sessions, Frank was capable of subitizing four items, but when shown five items, he needed space between clustered groups of two or three to subitize the subgroups of two or three. Frank did not have a conceptual understanding of five at this point because he could not compose the subgroups to name the total group of five. When Frank was shown five items without a space between the clustered items, he either named this as "six," "seven," or rearranged the items to look like the "X" orientation shown on the face of a typical die before describing this total set of items as "five."

Frank continually described two-digit numbers (i.e. fourteen, twenty-three) when attempting to conjoin the two subgroups he subitized, which indicates that his number understanding remained procedural. He may have been taught to name and identify two-digit numbers before understanding

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single digit numbers. To perturb this notion, we asked Frank to count items and name the total number, or covered up portions of an orientation, incrementally building (+1) from a group of three to the total number of items. This task design utilized Frank's counting and subitizing ability, to perturb what he understood number to entail.

In the early portion of Frank's seventh teaching experiment, Frank is describing "two" and "three" items as "twenty-three." To perturb his thinking, the teacher asked him to count the counters he placed on his mat. He counts out three counters and then counts out two counters. After two more attempts, Frank's counting responses remain the same, so the teacher counted the counters in front of her and had him "parrot" her counting. Immediately following this task, Frank is shown items clustered to represent "two," "one," and "two," (see Figure 2) and Frank describes seeing "two plus one plus two makes five." This is the first time Frank begins to construct five by composing groups of numbers, suggesting a change in how Frank understands five.

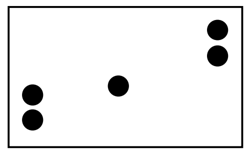

Figure 2: This orientation was shown to Frank in the middle of his seventh session.

Throughout the subsequent sessions, Frank's responses reveal more conceptual understandings of number (i.e. "three and then one is four," "two and two is four," "three and one is what tells me four"). These responses seem to reflect how Frank is perceiving the space between clustered groups of items, as he simply subitizes the total groups of items when no space is evident. Also, it is important to note that each time Frank explains his thinking he needs to rely upon his description of the subgroups to describe the total group. This seems less abstract than if Frank were to state the total group and then reverse his thinking to then describe the subgroups. Thus, it seems as if his number understanding is still developing.

Frank's number understanding in session 17. Prior to session 17, Frank was capable of subitizing four with subgroups "three and one," "two and two," and "one and three," and his QNC was more conceptually grounded with regard to four. However, Frank was not able to describe the total group and then reverse his thinking to describe the subgroups of four. For Frank to describe the total group and then the subgroups would suggest a cognitive reorganization of what Frank knew about four because he would have to compose four and then decompose four. Composing and decomposing a number would require Frank to reflect on his actions and be cognizant of the subgroups he used to compose four.

Additionally, Frank's understanding of five was still limited, and after his seventh session, he was not able to carry his description of five as "two plus one plus two" into subsequent tasks. Often when shown an orientation that promoted "two, two, and one" he would describe seeing four because he saw two and two or state that he saw, "four…five." These responses suggested that Frank was still solidifying what he knew "four" to be and was not able to coordinate the composition of two subgroups to build towards four and then coordinate a third subgroup to build towards five. Thus, we planned to use symmetrical orientations to promote Frank's subitizing of five in session 17.

In the Data Excerpt below, Frank was in his 17th teaching experiment session and his teacher showed him the circular counters in an orientation (see Figure 3), and asked him how many he saw. Frank brought his stuffed mouse to this teaching experiment session, and at times he pretends the mouse is the one responding to the tasks. This task was near the end of the session, and just before this task Frank was shown five counters which he determined were four and one, which then made fourteen. Once Frank counted these five counters, he determined the group was five. So again, Frank was having difficulty with composing five. As Frank described the subgroups which made up the composite groups, it seems evident that symmetry supported this activity, as only one two is mentioned.

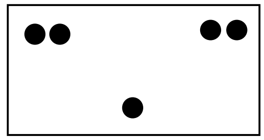

Figure 3: This orientation was shown to Frank near the end of his 17th session.

Data Excerpt.

Teacher: Okay, set mousey aside. Okay, one, two, three. [Teacher-researcher lifts the top piece of cardstock revealing counters arranged so that two counters are on the left-hand portion of the mat, and two counters are on the right-hand portion of the mat. In the middle there is one counter (see Figure 3).]
Frank: Five. [Frank talks in a squeaky "mouse-like" voice.]
Teacher: Five? How did you know that so fast?
Frank: Mousey says that.
Witness: How did mousey know it so fast?
Teacher: Yeah, how did mousey know it that quickly? Do you agree with mousey?
Frank: Yeah.
Teacher: Yeah? Why?
Frank: Because mousey said five [says five again in a squeaky "mouse-like" voice.] But mousey wins.
Teacher: He did win, but why did mousey know it was five? I don't know why that's five.
Frank: But you put it…you put two and down [motions with both his hands to show two on his left hand and right-hand portion on the bottom portion of his mat] and one and up [motions in the middle top portion of his mat.]

This data excerpt illustrates three cognitive changes in how Frank is understanding five. First, the symmetrical aspects of the orientation seemed to scaffold a change in Frank's understanding of five, as he described "two and down and one and up." Second, Frank composed subitized groups to quickly state that he saw five, but then "unpacked" or decomposed five to describe the actions and the groups he saw when subitizing five. This activity is more sophisticated than building up to five, and the symmetrical nature of the orientation seemed to promote this activity. Third, Frank no longer needed to "make" the orientation, but pointed to the imagined areas where the counters are located. This step away from the perceptual material was an important one; it seems as if Frank relied on more abstract actions when subitizing.

Retrospective Analysis

Frank's subitizing activity initially relied on common images or patterns to describe four, and at times, five. Also, Frank's initial QNC was grounded in procedural knowledge related to what he "knew" two-digit numbers "to be" which did not support a conceptual conjoining of number. To press Frank to attend to subgroups, Frank was shown orientations with large amounts of space between small clustered items that Frank was capable of subitizing. To connect what Frank knew about five to his subitizing activity, Frank reflected upon his counting activity and the relationship between subgroups when items were covered up. Frank developed more sophisticated number understandings as a result of subitizing symmetrical orientations. Frank was replacing the visual patterns with actions, as he was able to explain his thinking without having the perceptual material in front of him as evidenced in the Data Excerpt. Frank used prepositional phrases to explain what he saw (i.e. down, up) which suggests he may also be describing his eye movements. Thus, it seemed that Frank developed strategies that promoted changes in his number understanding (i.e. composition, decomposition, QNC). These strategies seemed to result from counting, subitizing, and the orientations (i.e. symmetry and space between items).

Conclusions

The purpose of this teaching experiment was to investigate how a child's understanding of number changed as he engaged in subitizing activity. Item orientation, reliance upon empirical material, and QNC were considered throughout the analysis. The two main findings of the study were related to (a) how Frank composed number and (b) the nuances in his perceptual subitizing activity.

Number Composition and Decomposition

With such a societal push to promote early childhood instruction in mathematics, it is important to understand what is (in) appropriate to teach young children. Entering the teaching experiment, Frank's understanding of conjoining two groups of items was a procedure resulting in a two-digit number word. This understanding of composition changed for situations in which the procedure was linked to a conceptual conjoining of groups to explain five. Though it seems that his procedure for naming two-digit numbers distracted from his construction of five, perhaps developing the procedure helped him attend to subgroups. Future research that focuses on helping children make appropriate connections between procedures and conceptual number understandings would be important for developing early childhood curricula. Findings from this study brought new ideas to light about how subitizing activity can press students to engage in meaningful activities when beginning to understand number.

Nuances in Perceptual Subitizing Activity

Throughout the teaching experiment, Frank's subitizing activity was described as perceptual instead of conceptual because he relied heavily upon perceptual material when discussing subgroups and total groups of items. The identification of sub-stages of perceptual subitizing are useful for further understanding topological, serial, and algebraic thinking prior to number conservation (Piaget, 1941/1965). First, shapes and patterns were initially described when explaining how a number was understood. This reliance upon patterns seemed to provide Frank a template to work from when recreating the orientations. Second, attention to subgroups before describing the total group of items indicated a "building up" of number. Third, near the end of the study, it seemed that Frank could compose items quickly when subitizing and then decompose these groups to explain his thinking. Children making connections between early perceptual activities and conceptual processes gives purpose for particular curriculum choices. Thus, early childhood educators' utilization of findings from this study could inform their pedagogical choices when designing subitizing tasks embedded in game play.

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"NATURAL RESOURCES": TWO CASE STUDIES IN EARLY EXPRESSIONS OF GENERALITY

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In this paper, we analyze individual semi-clinical interviews conducted with one kindergarten and one first-grade student. We build on prior research to offer evidence, via excerpts from these interviews, that children as young as kindergarten have a powerful, intuitive sense of generality and indeed naturally draw upon it to reason through mathematical scenarios. We identify within these children's utterances four features of generalizing for which educators might attend.

Keywords: Classroom Discourse; Algebra and Algebraic Thinking; Elementary School Education

Introduction, Issues, Theoretical Framework

Hyman Bass credits generality as "one of the most important and powerful characteristics of mathematics" (2003, p. 326); John Mason dubs it mathematics' "heartbeat" (1996, p. 65). Concordant with these voices, both the *Common Core State Standards for Mathematics Initiative* (CCSSI) and the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* emphasize generalizing as a key mathematical practice throughout the grades, one that can and should be encouraged early and honed over time (CCSSI, 2010; NCTM, 2000). To truly foster this practice beginning in the early grades, however, educators must first recognize young students' generalizations, a task complicated by the extent to which the language children employ differs from that of adults. The risk of underestimating the robustness of children's understandings or overlooking their insights altogether may be especially pronounced in the domain of mathematics, whose exactness can lend itself to particularly formalized conventions as to how generality should be expressed. For instance, universal qualifiers, be they explicit or implicit — "for all $x \geq 0$, $|x| = x$," "multiples of four are even," or "every integer greater than 1 has a unique prime factorization" — are the standard fare of generalizations in mature mathematical discourse. As students are inducted over time into communities of mathematical discourse, they gradually assimilate these conventions of communication. But there is no reason to believe that children in the earliest grades would have learned these communicative conventions, and there is no simple litmus test for whether children are thinking in generalized terms. These challenges call for further research into the range of ways in which very young children may express generality. By honing our own ability to recognize early inclinations to generalize, we can, in turn, better nurture their development.

Identifying young children's generalizations can be challenging, requiring "the skilled and attentive ear of a teacher who knows how to listen carefully to children" (Kaput, 2000, p. 6). Our goal in this paper is thus twofold: (1) to offer evidence, via excerpts from mathematical interviews with two case students, that children as young as kindergarten have a powerful, intuitive sense of generality and indeed naturally draw upon it to reason through mathematical scenarios, and (2) to identify within these children's utterances several characteristics indicative of generalizing for which educators might, in Kaput's phrase, "listen carefully."

Prior research has yielded numerous examples, largely from teacher documentation or classroom discussions, of young children's generalizing about comparison of quantities (Schifter, Bastable, Russell, Riddle, & Seyferth, 2008a) including non-specified quantities (Dougherty, 2008), commutativity (Schifter, Monk, Russell, & Bastable, 2008b), classes of numbers (Bastable & Schifter, 2008), additive and multiplicative identity properties (Carpenter & Levi, 2000), and use of the equal sign (Carpenter, Franke, & Levi, 2003). As a body, this research demonstrates that it is within the reach of young children to observe mathematical regularities and talk about their discoveries in a variety of ways.

What does it mean, really, to generalize? Rowland (2000), Carraher, Martinez, and Schliemann (2008), and Kaput (2000) each propose perspectives. According to Rowland (2000), generalizations are statements of beliefs about properties of an entire class that have not and indeed cannot be inspected and tested. For Carraher et al. (2008), generalizations involve claims for an infinite number of cases, where "the *scope* of the claim is always larger than the set of individually verified cases" (p. 3, italics in original). Finally, Kaput (2000) highlights that generalizations "deliberately extend … the range of communication beyond the case or cases considered, [to the] patterns, procedures, structures, and the relations across and among them" (p. 6). These three proposals share an emphasis on the generalizer's conviction with respect to an inference that includes many cases simultaneously, a conviction that obtains in the absence of envisioning each of those individual cases.

Mode of Inquiry, Data Sources

In this paper, we analyze individual semi-clinical interviews conducted with one kindergarten and one first-grade student as part of a research project focused on exploring kindergarten through second-grade (K-2) children's understandings of functions. The data are drawn from an eight-week classroom teaching experiment (CTE). Individual interviews with a subset of students in each of the three grades were carried out immediately prior to, halfway through, and at the end of the CTE. The students in the school in which the CTE was conducted are 98.6% minority (non-white), with 89.5% categorized as low SES and 33.9% as ESL.

To facilitate the process of data reduction, a series of steps was taken. First, because the goals of the study were to explore young children's intuitive sense of generality and to provide evidence for what generalizations look like among young students, we focused on the initial interviews (pre-interviews), which took place before any lessons were implemented. By selecting for analysis data collected before the teaching experiment began, we ensured that each child's productions, both verbal and written, would best approximate what might be described as "naturally" present. Second, we focused our selection on kindergarten and first-grade students to explore expressions of generalization among the youngest children in our study.

In our pre-interview protocol, students were first asked how many noses a dog has. They were then asked how many noses there would be altogether among two dogs, three dogs, and so on. We did not doubt that students would answer correctly for all specific cases proposed. Rather, we were interested in students' explanations of how they knew — that is, in how they spoke about their reasoning. Each student was asked at some point in the interview to organize the information in a function table, reason about far values (e.g., the case of one hundred dogs), and reverse the relationship (supply a number of dogs given a number of noses), as well as to respond to a proposed "mismatched" case (e.g., the suggestion of five dogs and six noses).

Students were also asked how they might tell a friend how to know the number of noses for any number of dogs and whether there was a rule for making this determination. They were also invited to generate their own examples rather than simply responding to interviewer-generated values. These questions were intended to create an open-ended space for students to verbalize their understanding of the problem and the functional relationship that governed it.

All interviews were transcribed verbatim from video, and all video and transcripts were reviewed. This review facilitated a progressive selection of the dataset for this paper. Our criterion for selecting the students for this paper's analysis was that they exhibit diverse verbalizations of mathematical ideas. Our aim was not to showcase the most sophisticated thinking in young students, but rather to highlight a range of ways young students' mathematical thinking might find expression. In this way, we selected interviews with Kinetta and Ferdinand, a kindergartener and first-grade student respectively.

In analyzing these interviews for generalizing, we appealed to the common thread among the perspectives on generalizing noted above: conviction in an inference about many simultaneous cases that is independent of envisioning of those individual cases. We reviewed the transcripts line-by-line using the constant comparative method (Glaser & Strauss, 1967). Our research goal was to identify features of this thread within these students' interviews.

Results

Bills (2001) theorizes that qualitative differences in students' language correspond to qualitative differences in their conceptual constructions and that these shifts in language may be markers of progress towards recognizing the generality of procedures. Adopting this premise, we highlight four prevalent features of the two case students' verbal productions that emerged as a result of our analysis. Examples of each of these features will be provided below in Table 1.

- *Definite Articles, Indefinite Quantities*: We observed that when students were asked questions that might easily have been construed as centering on a particular case, they often nonetheless replied with a generalized answer that could accommodate any case. In such instances, the student also indicated that a strict rule would uniquely determine the relevant numerical "output" based on the input, whatever the input might happen to be. We took these instances, referred to here as "definite articles, indefinite quantities," to indicate that the student was spontaneously generalizing, displaying conviction about many simultaneous cases not individually envisioned.
- *Certain Denial*: If a student was prepared to cry foul without hesitation in response to a mismatched (e.g., five dogs and six noses) scenario proposed by the interviewer and to justify and defend his or her position, we took this as an indication of the student's conviction. (These impossible scenarios were akin to Carpenter and Levi's [2000] and Davis' [1964] false number sentences, leveraged as windows into young children's ability to justify generalized properties of whole numbers.) Moreover, if the student justified his or her conviction by giving reasons that appealed to the general logical structure of the problem, as opposed to simply the particular case, we took this as evidence of generalized thinking about cases not individually envisioned. We refer to this feature as "certain denial."
- *Generic Examples*: We adopt this terminology from Balacheff (1988), who lists generic example as the third of four main forms in the cognitive development of proof. Balacheff makes much of the transition from the second form — termed "crucial experiment" — to its successor. While a crucial experiment offers only the outcome of a particular case to support a general conclusion, the case has been chosen deliberately for its perceived particular ability to carry that import.[1] Balacheff maintains that one crosses a "fundamental divide," or undergoes a "radical shift in ... reasoning" in stepping from crucial experiment to generic example, and that in the latter territory, one "establish[es] the *necessary nature of [a] truth* by giving reasons ... by means of operations or transformations on *an object that is not there in its own right, but as a characteristic representative of its class*" (pp. 218-19, emphasis added to correspond to conviction with respect to many simultaneous cases not individually envisioned). Accordingly, in our study, we took students' generic examples to be evidence of

generalizing. Bills (2001) also regards generic examples as steps toward more formal generalizations.

- *Authoritative "You"*: Rowland (1992, 1999, 2000) has paid considerable attention to children's use of the pronoun "you." One use is as a substitute for the more formal "one" and thus as an indicator that a generalized procedure is being described. Rowland notes that this procedural "you" is common even in non-mathematical situations, especially but not only among children — for instance, in explaining how to play a game. He has observed that a shift from "I" to "you" in children's discourse seems to parallel a shift from explaining work done with specific cases towards describing a more generalized procedure. This procedural "you" also appears in excerpts from Schifter et al. (2008a), Bastable and Schifter (2008), and Carpenter and Levi's (2000) research with young children (though these researchers do not highlight it as such in these studies): for instance, "you don't have to pay attention to the 6s," "each time you add a number to a group that can go, you get a group that can't," and "when you put zero with one other number, just one zero with the other number, it equals the other number," respectively. A second use of "you," distinct from the first, is as a pronoun of direct address. Rowland notes that it is considerably less common for students to use "you" in this fashion when speaking to teachers than vice versa, an imbalance he attributes to power relations. Thus we take students' usages of direct-address "you"s to be indicative of their willingness to assume an authoritative position and their usages of procedural "you"s to be indicative of explaining generalized rules or procedures — which in turn we take as indicative of conviction with respect to many simultaneous cases not individually envisioned. We refer to this feature as "authoritative 'you.'"

Owing to space constraints, we give limited excerpts from each case student's interview and highlight these four features within the transcript. We use **boldface** text to foreground particular phrases that, situated in context, exemplify these features. Following the transcripts, we organize these examples in a table (see Table 1) with some discussion.

Excerpt from Initial Interview with Kinetta (Kindergarten)

Interviewer: If there are three dogs?
Kinetta: Three noses.
Interviewer: How do you figure out? How do you know how many noses there are?
Kinetta: You count.
Interviewer: How do you count?
Kinetta: One, two, three.
Interviewer: Mm-hm. And how do you know how to stop — when to stop counting?
Kinetta: When **you** get to **the** number.
Interviewer: So [...] for instance, what if there were ten dogs? How many noses [...]?
Kinetta: Ten.
Interviewer: How do you know that?! I didn't see you count! I didn't see you do any counting.
Kinetta: That's because I counted in my head.
Interviewer: Oh! You went all the way to ten that quickly?
Kinetta: [Nods.]
Interviewer: [...][2] What if there are one million dogs? How many noses are there?
Kinetta: One million.
Interviewer: You did not count that fast. How do you know what number to say?
Kinetta: Because **you** just said **it**.
Interviewer: Oh! What do you mean?

Kinetta: **You** said "one million."
[...]
Interviewer: What if there were twenty-four noses?
Kinetta: Twenty-four dogs.
Interviewer: Was that easy? Yeah? How did you figure that out?
Kinetta: Because **you** just said **it.**
[...]
Interviewer: Can I show you something that some people use? It's called a table. [Draws function table setup.] They do this, and here they put th— how many dogs, and here they'll put how many noses. So for instance, if it's one dog [writes the numeral 1 in the left-hand column of the function table], how many noses?
Kinetta: One.
Interviewer: [Writes the numeral 1 in the right-hand column.] If it's two dogs [writes 2 in the left-hand column]...
Kinetta: Two noses.
Interviewer: And can you put it? Can you show it right there [in the right-hand column]?
Kinetta: [Writes 2 in the right-hand column of the table.] [...]
Interviewer: What if there were three here [in the left-hand column]? How many noses?
Kinetta: [Writes 3s in both columns.]
Interviewer: Oh, great, and what if there were— let's put another number here [in the left-hand column]. Whatever number you want.
Kinetta: **[Writes 100, 100.]**
Interviewer: Oh my goodness! What number is that?
Kinetta: **One hundred.**
[...]
Interviewer: If we put a number here, whatever number here [in the left-hand column], what number do we have to put here [in the right-hand column]?
Kinetta: **The same one.**
[...]
Interviewer: So, you know what someone told me? Someone told me that there were five dogs and there were six noses. What do you think of that?
Kinetta: **[Shakes head no.]** [...]
Interviewer: No? Why not?
Kinetta: **Because it needs to be the same.** [...] Six noses and six dogs. Five noses and five dogs.

Excerpt from Initial Interview with Ferdinand (First Grade)

Interviewer: So, if — what if we put here that there's one dog, okay [writes the numeral 1 in the left-hand column]? How many noses are there gonna be?
Ferdinand: One.
Interviewer: One [writes the numeral 1 in the right-hand column]. What if there are two dogs [writes 2 in the left-hand column]?
Ferdinand: It's two.
Interviewer: Okay, can you show me that?
Ferdinand:[Writes 2 in the right-hand column.]
Interviewer: What if there are three dogs?
Ferdinand: Then three [writes a 3 in each of the two columns].
Interviewer: Do you want to show me some other numbers of dogs?

Ferdinand: We're going like number?[3]
Interviewer: Whatever you want. If you want to do it that way, we can do it that way.
Ferdinand: **I'm gonna do a five [writes 5 in the left-hand column of the table].** [...]
Interviewer: What would you put on the other side? [...]
Ferdinand: Oh, five [writes 5 in the right-hand column]. [...]
Interviewer: How do you know it's five noses?
Ferdinand: 'Cause fiv— it's five 'cause five noses **has to** be the same, **like** — they could play, **if** they didn't have a lot of— **like** they're playing hide and seek **if** they didn't have a lot of people to play—
Interviewer: Mm-hm.
Ferdinand: **So they have to** have five and five.
Interviewer: Mm, okay.
Ferdinand: **Like** teams.[4]
Interviewer: So, so if a friend asked you what number — how you know what number to put here, what would you tell them?

Table 1: Examples of the Four Features

Feature	Key phrases (regarded in context)	Discussion
Definite Articles, Indefinite Pronouns	"When you get to **the** number" (line 8); "Because you just said **it**" (line 19, line 26); "**The same one**" (line 48). "**It** was **the** number that — that you said" (line 82).	Lines 8, 19, 26, and 82: Rather than give answers specific to the particular cases the interviewer has just referenced — of three and five dogs, respectively — as they might reasonably do, both students instead respond in generalized terms even though they have not been "asked" to generalize. Line 48: Kinetta appears to have no trouble responding when posed a question in general terms.
Certain Denial	"**[Shakes head no]** [...] **Because it needs to be the same.** [...] Six noses and six dogs. Five noses and five dogs" (lines 52, 54-55). "You say, 'You **have to** take one more out, 'cause we **have to** have five and five, **'cause** [...] we **have to** play five-five, **'cause**[...]'" (lines 86-88).	Both children appeal to the logical necessity of sameness: it "needs to" or "has to" happen for a reason. Additionally, that Kinetta allows for both possible corrections of the mismatch (five-five and six-six) suggests an understanding that this sameness is not only necessary but also sufficient. She's willing to vary the number of dogs *or* the number of noses, but she insists that those two counts have to match.
Generic Example	"**[Writes 100, 100]** [...] **One hundred**" (lines 42, 44). "**I'm gonna do a five [writes 5 in the left-hand column of the table]**" (line 69); "It's five 'cause five noses **has to** be same, **like** — they could play, **if** they didn't have a lot of— **like** they're playing hide and seek **if** they didn't have a lot of people to play [...] **So they have to** have five and five [...] **Like** teams" (lines 73-75, 77, 79).	While the dogs-and-noses problem centers on an identity function, and thus in some sense masks the "transformations" enacted on objects, we maintain that these examples proposed by Kinetta and Ferdinand are nonetheless "characteristic representative[s] of [their] class," "objects ... not there in [their] own right" (Balacheff, 1988, pp. 218-219). In Kinetta's case, one hundred dogs is fundamentally a case not individually envisioned. While one might precisely visualize one, two, or three dogs, it's reasonable to surmise that it's fairly impossible to hold a quantitatively precise mental image of one hundred dogs. One hundred serves as a representative of the class of theoretically possible, potentially arbitrarily large quantities of dogs. Ferdinand's usage of "if" and "like" (lines 73-75, 79) resonates with Bills' (2001) findings that children often use "if it's like" to introduce generic examples (where adults, Bills posits, might use "for example," "consider for instance," or "suppose"), as well as with the diction that accompanied the use of generic examples in Carpenter and Levi's (2000) study. Furthermore, that Ferdinand says that "five noses **has to** be the same" (line 73) and concludes, "**So they have to** have five and

		five" (line 77) suggests that, as Balacheff describes, what he's arguing about is the "necessary nature of [the] truth."[5]
Authoritative "You"	"When **you** get to the number" (line 8); "Because **you** just said it" (line 19, line 26); "**You** said one million" (line 21). "It was the number that — that **you** said" (line 82).	Line 8: "You" is used impersonally to convey a generalized procedure. Lines 19, 21, 26, 82: "You" is used as a pronoun of direct address.

Ferdinand: "**It** was **the** number that—that **you** said."
Interviewer: Okay. [...]What if you had a friend who said that they counted that [...] with five dogs, there are six noses? [...] Five dogs and six noses [points to these places on function table]. What would you tell your friend?
Ferdinand: You say, "You **have to** take one more out, 'cause we **have to** have five and five, **'cause** [...] we **have to** play five-five, **'cause** if we don't have five-teams, we ha— we have more than five."
Interviewer: Okay.[...] If your friend says that there are ten dogs and he counted twelve noses?
Ferdinand: It'll be — twelve.
Interviewer: Twelve what?
Ferdinand: Dogs.

Significance of the Research

The examples we provide from interviews are far from exhaustive; nonetheless, they build a portion of a catalogue to which educators might look to identify instances in which students as young as kindergarteners are spontaneously generalizing, so that we might better capitalize on opportunities to foster this type of reasoning. Importantly, for our purposes, they contribute to an existence proof that young children do indeed bring natural intuitive powers of generalizing to formal schooling and that, as such, the power of generalizing is "natural, endemic, and ubiquitous" (Mason, 1996, p. 66).

Endnotes

[1]This contrasts with the first of the forms, naïve empiricism, which lacks the deliberate "*this* is my *test* case" element, in that it entails believing a proposition simply because one conceives of *a* case, or a handful of cases, that "works."

[2]Throughout the transcripts, the symbol [...] indicates dialogue removed for irrelevance.

[3]We take Ferdinand to be asking here whether he was expected to increment the number of dogs consecutively.

[4]We take the analogy to be to the necessity that two teams have equal numbers of players.

[5]Kinetta also speaks to the logical necessity of sameness when she says (line 54) "it needs to be the same." However, since this was an instance of her rejecting a proposed "impossible case" as opposed to proposing an example of her own, we have not considered it a generic example.

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HOW STUDENTS' INTEGER ARITHMETIC LEARNING DEPENDS ON WHETHER THEY WALK A PATH OR COLLECT CHIPS

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In light of conceptual metaphor theory, historical mathematicians' and students' difficulty with negative numbers reveals that the collecting objects metaphor may be a cognitive obstacle to those first learning about negative numbers. Moreover, consistency of physical motions with targeted ideas is a factor of cognition. Thus, this pre-post-delayed post study randomly assigned 8 classes of initial learners to a collecting objects integer model (chip model) or a moving-along-a-path metaphor-based model (number line model) to learn integer arithmetic with the four primary operations during an eight-day mini-unit. The study investigated the questions: What do students demonstrate learning with each model and what, if any differences in learning are found between models? Findings did support theory that a motion-aligned model using a moving-along-a-path metaphor would likely support learning better than collecting objects.

Keywords: Number Concepts and Operations; Cognition; Learning Theories

In this paper I share results from a larger study designed to understand the effects of instructional models on students' learning about integers. This study contributes to resolving two enduring challenges in mathematics education: one practical and one theoretical. The first concerns improving the way that classroom-based research can inform teachers' practical decisions about teaching integer arithmetic. The second offers new insights into the theoretical and practical debate about whether and how physical experience supports learning mathematics. The study provides evidence about how students' physical motions or model-movements can support or interfere with mathematics learning.

Prior Research

Integer arithmetic with negative numbers is counterintuitive, yet essential to most mathematics beyond middle school. It has been extensively studied (Gallardo, 2002; Küchermann, 1981; Liebeck, 1990; Linchevski & Williams, 1999; Vlassis, 2008), yet recommendations are contradictory about how to help students adapt their arithmetic concepts to embrace negative numbers (Star & Nurnberger-Haag, 2011). Most integer research has assessed students' learning during or immediately after instruction and focused on addition and subtraction only. Given that students need to understand negative number arithmetic to build on and use in complex ways in formal algebra (Vlassis, 2008), research that investigates a larger set of integer knowledge, including all operations, along with longer-term implications of instructional experiences is crucial.

Küchermann (1981) categorized three types of integer instruction models as (a) cancellation models in which two opposites cancel, (b) number line, or (c) abstract models. Although integer thinking and learning has been extensively studied, investigations of integer learning with particular models, particularly those most accessible in classrooms, have yet to be conducted. In spite of this lack of research about student learning with different models and theoretical critiques of models offered (e.g. Star & Nurnberger-Haag, 2011; Freudenthal, 1973; Vig, Murray, & Star, 2014), multiple integer models are promoted in methods textbooks for prospective teachers as well as school textbooks, particularly cancellation and certain number line models. One study did compare a cancellation model and a number line model (Liebeck, 1990) but it did not involve parallel instruction and post-tests nor did it include a pretest. For educators to make effective instructional decisions, it is important to compare methods (Nunez, 2012) and to understand how different

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methods might offer similar or different learning opportunities. Consequently, the present study experimentally compared students' initial learning of negative number arithmetic using a cancellation model or a number line model.

Theoretical Perspectives

Documentation of historical mathematicians and of modern students has described cognitive obstacles for negative numbers (e.g., Fischbein, 1987; Pierson, et al., 2014). A reading of these works in terms of conceptual metaphor theory reveals that the *object collection metaphor* may be a cognitive obstacle to those first learning about negative numbers (hereafter called *initial learners)*. The conceptual metaphors identified by Lakoff and Nunez (2000) applicable to integer arithmetic include the *object collection*, *motion along a path*, and *measuring stick* metaphors. Flexible use of more than one metaphor may be necessary for expert understanding of negative numbers (Chiu, 2001). Yet, research of integer arithmetic via conceptual metaphors is in its infancy and has focused on what metaphors people used while thinking (Nurnberger-Haag, 2013; Chiu, 2001; Kilhamn, 2011), rather than the impact of metaphor-based physical motions on initial learning.

Conceptual metaphor theory offers educational researchers valuable insights to identify the ways that different models invite students to conceive of numbers. *Cancellation models* treat numbers as objects (using an object collection metaphor for negative number concepts). In schools, the cancellation model most commonly used is an integer model that uses chips of opposite colors to represent opposite numbers. In research additional models have also been promoted and studied that draw on an object collection metaphor (Nurnberger-Haag, 2013; Kilhamn, 2011). At first glance Küchermann's (1981) characterization of cancellation versus number line models might offer a sufficient framework. Or one might suspect that Fischbein (1987) and Freudenthal's (1973) explanations that thinking of numbers as objects renders use of conceptual metaphor theory moot, because this neatly aligns with the object collection metaphor. Using conceptual metaphor theory to investigate integer arithmetic learning, however, affords considering the differences between the mathematical representations and the ways we think about these representations. For example, *number line models* all use a commonly accepted representation of a number line, but there is not a single number line model. A number line representation can be thought of using a measuring stick metaphor as Descartes did historically (Berlinghoff, & Gouvêa, 2002; Lakoff & Nunez, 2000) in which numbers are found at the end of a positive length in a particular direction. Number line representations can also be thought of with a motion-along-a-path metaphor (Lakoff & Nunez, 2000; Nurnberger-Haag, 2007; Kilhamn, 2011), or as a combination of both of these metaphors to be discussed in later studies. Moreover, considering conceptual metaphor theory aides recognition that the differences of model metaphors might matter because of how these ways of thinking fit with the ways humans think more generally. In other words, conceptual metaphor theory assists with explaining and unifying mathematical thinking to show that and in what ways mathematical thinking is part of the varied abstract thinking humans do (Lakoff & Nunez, 2000).

Lakoff and Nunez treat conceptual metaphor as an object of thought that results from physical experiences, which grounds how we think about abstract ideas. Whereas other research has referred to conceptual metaphors similarly using nouns (Chiu, 2001; Kilhamn, 2011), I use the verb forms (*collecting objects*, *moving-along-a-path, and measuring*) to emphasize and transform the original claims to consider the metaphorical mechanism as grounding the patterns of interacting with the world as part of an on-going dynamic system (Nurnberger-Haag, 2014).

Furthermore, when promoting one metaphor with students, such as the moving-along-a-path metaphor on a number line, we must realize that this metaphor does not prescribe how students move on a number line. The typical number line model promoted in curricular resources directs students to move backwards or forwards depending on the sign of a number and to move in a particular direction depending on the operation. In other words, these models inform students *which* direction to move. The walk-it-off model, in contrast, was designed to promote the opposite operator meaning of the negative sign necessary for algebra, but not afforded by other models (Nurnberger-Haag, 2007). Rather than written symbols indicating *which* direction students travel, the written symbols in the walk-it-off model emphasize to change direction by turning the *opposite direction* or not to change direction for addition or positive values.

Purpose of Study

Although broadly, hands-on learning or manipulatives have been extensively studied in mathematics education, this research has not attended to the ways the physical model-movements may represent or misrepresent the mathematics students learn. Yet, research from psychology has shown that consistency of physical motions with targeted ideas is a factor of metaphor comprehension (Glenberg & Kaschak, 2002). Thus, this study examined how two different conceptual metaphors and students' physical motions and related explanations affect initial learning of integer arithmetic in classrooms. With a goal of practical impact, I chose to compare models with which students physically represent integer arithmetic that could be or are easily implemented in schools and what I thought would be the best case of each metaphor. In order to capture the complexity of students' initial learning of integer arithmetic on all four basic operations and assess longer-term learning, this pre-post-delayed posttest study used multiple methods to address the questions: After using either a chip model, or a number line model that emphasizes opposites and magnitude, what do students demonstrate knowing about integers and what, if any, differences in learning are found between students who used each model?

Method

The most common *collecting objects* metaphor-based model (a chip model) or a *moving-along-a-path* metaphor-based model (a number line model see Nurnberger-Haag, 2007) was randomly assigned to eight intact classes of initial learners (four classes per model). Here I report findings from written pre/post/delayed posttests.

Settings, Participants, & Research Personnel

A power analysis indicated that including at least 50 students per integer model should detect a medium effect. To study initial learning, district sites that met the following criteria were recruited: curriculum had not yet addressed integer operations in the recruited grade and all students in this grade attended the same school with the same mathematics teacher. Two public rural districts in a Midwest state participated. According to the grade level data on the state website, 45% of the students I instructed had free or reduced lunch and were primarily European American. After removing students from analysis due to absences, 78 chips and 76 walk-it-off students remained in the analysis. The study instruction occurred in the grade prior to when integer arithmetic is typically taught in the district (School A first semester sixth grade, School B second semester fifth grade). As the researcher-teacher, with approximately 20 years of experience teaching mathematics (including integer arithmetic to K-16+ students), I taught all students in the targeted grade. The classroom teacher remained in the classroom to ensure safety of students, but not to teach. Only those students who themselves assented and whose guardians consented participated in the research by giving their written work and assessments to the researcher-teacher for an incentive the equivalent of a university folder and pencil.

Instruction and Measures

Each class experienced parallel instruction with the same tasks and activities, differing only by the integer model used (two colors of chips or ten-foot long empty number lines, the language about how to use those representations and model-movements). During each lesson students worked on tasks and played games in assigned trios or pairs and had the opportunity to participate in whole-class discussions. I planned and implemented eight approximately 50-minute lessons about negative numbers, ordering numbers, and operations with negative integers (addition, subtraction, multiplication, and division) including sums of additive inverses, hereafter called opposite sums. The written measures assessed these constructs as well as opposite operators (transfer problems) that were not taught during the lessons (e.g., -(-4) and – (6-8)). The data reported here are from a 46-item open response skill-based test *Integer Arithmetic Test* (IAT) and one of the items from a seven-item *Explain and Draw Test* (EDT). These measures were developed through several phases of piloting and analysis including factor analysis to remove items that did not perform as expected. Students could only use pencils when completing these written tests (neither chips nor number lines were provided).

Data Analysis

The IAT data reported here included item accuracy and qualitative assessment of student reasoning on the EDT Opposite Sums Item. The IAT total test score was scaled to a 100-point test by weighting the subtotals of the following constructs: ordering numbers (20%), addition and subtraction (35%), multiplication and division (35%), and opposite operations (10%).

Multiple methods were used to determine each student's level of opposite sums knowledge at pre, post, and delayed posttest. Two types of IAT questions (calculation problems such as -19 + 19= and generative problems ___ + ___ = 0) , were each separately subtotaled for accuracy 0 to 2. An example of the EDT Opposite Sums Item follows:

> Trina and Jaleesa are students in your grade at another school.
> Trina said that $^{-}8 + (^{-}7 + 7)$ does <u>not</u> give the same answer as $^{-}8 + (^{-}5 + 5)$.
> Jaleesa said they will. Circle who is right: Trina or Jaleesa.
> Draw and write an explanation in words to convince a friend that this student is right.

I coded student explanations with a qualitative coding scheme using a constant comparative approach (Glaser, 1965). A second trained coder assessed 20% of the randomly selected tests with 92.7% agreement. A K-cluster analysis informed determination of Leveled Opposite Sums knowledge profiles (no/low, moderate, or strong) using the three ways of demonstrating opposite sum knowledge (calculation problems, generative problems, and EDT Opposite Sums Item).

Statistical controls were built in to the study design and analysis including a pretest, whole number fact test, gender, and preconceptions of negative number multiplication and division. No significant differences were found between the eight classes or between integer models at pretest. Although I planned to include district and class as statistical controls, the model would not run with both and a statistical model that accounted for either accounted for 48.4% of the variance, so district, which had significant differences at pretest was included in spite of it not being a significant predictor of the statistical model. Scaled data (overall test and subtest scores) were analyzed using multivariate analysis of covariance (MANCOVA). I used an ordinal regression to analyze leveled data.

Results and Analysis

Which, If Either, Model Supports Better Overall Learning?

Multivariate analysis of covariance (MANCOVA) was conducted on the IAT post and delayed posttest total scores to compare student learning with the walk-it-off model to learning with a chip model controlling for pretest IAT, whole number fact test, gender, and preconceptions of integer multiplication and division. Both integer models supported statistically significant integer learning from pre to posttest, indicating that both models were reasonable models for integer learning. Statistically significant integer model differences were found between the overall learning of students using walk-it-off compared to chip models: $F(2, 146)=11.414$, $p<0.001$, $\eta\rho^2=.14$. On average in the short-term students using the walk-it-off integer model for initial instruction scored 11 points higher on this 100-point test than students using chips (posttest $\beta=-10.8$, 95% CI [-15.7, -5.9]) and 13 points higher in the longer-term (delayed posttest $\beta=-12.7$, 95% CI [-18.3, -7.0]).

Does the Way Model-Movements Represent Mathematics Impact Learning?

The IAT problems that I argue require chips students to move in ways that are inconsistent with the targeted integer operations, were grouped into Inconsistent Model-Movement Problems and those consistent into Consistent Model-Movement Problems. Inconsistent Model-Movements for example $^{-}4 \times {}^{-}3$ and $^{-}2 - {}^{-}5$ require students to put in enough chips to represent zero with sufficient numbers of chips to be able to remove 4 groups of 3 negatives or 5 negatives, respectively. Multivariate analysis of covariance (MANCOVA) was conducted on the IAT operation post and delayed post Consistent Model-Movement and Inconsistent Model-Movement problems (22 and 14 respectively) controlling for these scores at pretest and the rest of the controls used in the total test analysis. The differences were again statistically significant, but with more than twice the practical effect, ($\eta\rho^2=.34$): $F(4, 145)=18.358$, $p<0.001$. When the chip model movements were consistent with the mathematics of integer arithmetic operations, no significant differences were found between the integer model groups' test performance. Statistically significant differences were found, however, for those problems for which I argue that the chip model required inconsistent movements ($p<.001$).

Opposite Sums Knowledge

Ordinal regression analysis on the students' levels of opposite sums knowledge showed that the chips group did demonstrate greater learning at posttest than the walk-it-off model ($p=0.002$), but this difference was not maintained five weeks later. No significant differences on opposite sums knowledge were found between students who learned with the chip model or walk-it-off model on the delayed posttest ($p=0.090$).

To test if students who did not have strong opposite sums knowledge prior to instruction had more difficulty learning with this chip model than the walk-it-off model, multivariate analysis of covariance (MANCOVA) was conducted on the post and delayed post IAT scores only for the 118 participants whose opposite sums knowledge was not strong at pretest. These students assigned to the chip model did significantly worse overall on the IAT than students assigned to the walk-it-off model: $F(2, 110)= 13.35$, $p<0.001$, ($\eta\rho^2=.20$). Students using chips scored lower on the posttest -12.7, 95% CI [-18.4, -7.0] and delayed posttest -15.8, 95% CI [-22.1, -9.6].

Discussion & Conclusions

These overall results fit the theoretical reasons drawn from cognitive science, analysis of mathematical processes, and practical classroom experience. These issues converge to predict that the walk-it-off model should better support student learning, which the data confirmed. Several reasons for these results were identified: the walk-it-off model uses model-movements consistent with mathematical ideas (whereas the chip model is often inconsistent) and the chip model visually violates the meaning of zero students would expect when thinking of numbers using a collecting objects metaphor.

Consistency with Mathematics Matters: Model-Movements

Some theoretical work referred to the mathematical alignment of integer models as breaking or requiring model-rules that differ from the mathematics (Star & Nurnberger-Haag, 2011; Vig, Murray, & Star, 2014). This study offers evidence to support these theoretical arguments that incongruent mathematical alignment does impact students' learning outcomes. Moreover, this study offers reasons related to human cognition why these breaks are likely problematic. One predicted reason was that students move differently in order to interact with the representations with these models. If this difference and the consistency of the ways model-movements represent mathematics did not matter for student learning, then there should be no significant differences in performance on the Consistent and Inconsistent Model-Movement problems. This analysis, however, did show that when the chip model required model-movements that contradicted or were extraneous to the mathematical processes and ideas, this interfered with learning. In contrast, the walk-it-off model-movements consistently represent the mathematical ideas, so I classify this model as a Motion-Aligned-Model. When approaching integer arithmetic problems with model-movements consistent with the mathematics, the results demonstrated that either model could be equally effective. This lack of significant differences on Consistent Model-Movement problems further supports the claim that model-movement alignment with mathematics could be a factor in students' learning with models.

Consistency with Mathematics Matters: Conceptual Metaphor

In order to calculate almost every integer problem with chips, students need to represent the idea of cancelling opposite values (e.g., -9 and 9). Thus, due to repetition of the underlying metaphor of cancelling opposite things, one might suspect that a benefit of the chip model might be to better support student learning opposite sums. The results did fit this prediction, but the benefits of using a cancellation model for this purpose were only in the short-term significantly different from the number line model used in this study. Longer-term on average students performed equally well after learning with either model.

Opposite sums knowledge may actually be required in order to learn integer arithmetic with the chip model, because this model requires students to sum opposite values to calculate almost every integer problem. The findings seem to support this, because students without strong cancellation knowledge at pretest who used this collecting objects metaphor model, scored the equivalent of about 1.5 grades lower than students who used this particular moving-along-a-path metaphor model. A more global reason for these initial learners' challenges with the chip model may be because it visually violates a central feature of applying the collecting objects metaphor to numbers. Rotman (1993) articulated, that when thinking of numbers using a collecting objects metaphor, zero should be visually represented as nothing or "no thing." Yet, chip models require students to use multiple things to represent no thing.

Implications

This study that tested several aspects of integer learning, including all four basic operations, suggests that the walk-it-off model may be a parsimonious model for initial integer instruction. It also reveals for which aspects of integer knowledge a collecting objects metaphor (in the form of a chip model) might add richness to student thinking. The delayed posttest results reflect longer-term learning, which although rare in educational experiments, is crucial to make claims about educational impact that matters in students' lives.

Practical Implications

Given that the walk-it-off model was more effective overall and improved learning even more for those who have less integer knowledge prior to instruction, this model is likely the most parsimonious model with which teachers might begin instruction that meets the diverse range of learning needs in real classrooms. The sample was economically diverse (45% free and reduced lunch), but were primarily European American fifth and sixth grade students in a rural district, so the study should also be replicated with other populations to ensure that these results appropriately inform instruction for all students. Anecdotal evidence suggests the walk-it-off model, which a teacher developed and has shared with hundreds of other teachers, is a feasible model for teachers to implement (Nurnberger-Haag, 2007). Nevertheless, this study was conducted by an experienced researcher-teacher, so future investigations should confirm that students using these models with typical classroom teachers experience similar results.

Research could investigate beginning integer instruction with a moving-on-a-path metaphor in the form of the walk-it-off model, which works for every integer problem, and then integrating other conceptual metaphors in the real-life contexts in which these metaphors make sense (as well as have students assess in which contexts these metaphors make sense). The study reported here used a number line model designed to encourage students to move in ways that represent opposite operators, which differs from other approaches, so the findings of this study should not be generalized to other number line models. To further consider how learning with a model affords and constrains integer learning, other collecting objects, measuring, and moving-along-a-path metaphor-based models could be experimentally compared.

Theoretical Implications

Humans are always moving. Research in cognitive science shows that these movements influence what and how we think (Antle, 2013; Glenberg & Kaschak, 2002). Important work about moving to learn mathematics has begun (e.g., Gerofsky, 2012; Roth & Thom, 2009). Yet, more is needed, and it is crucial that mathematics education research attend to the *ways* that students move due to instructional models, instead of *whether* they move during instruction. This study contributed to this theoretical goal specific to integer arithmetic and findings suggest further investigating if motion-aligned models are more parsimonious instructional models across mathematics topics.

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MENTAL MATHEMATICS AND ENACTMENT OF SPECIFIC STRATEGIES: THE CASE OF SYSTEMS OF LINEAR EQUATIONS

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This study on the mental solving of systems of linear equations is part of a larger research program, aimed to gain a better understanding of the potential of mental mathematics activities with topics/objects other than numbers. Through outlining the study details and the activities engaged with, the paper reports on the variety of strategies developed for mentally solving systems of linear equations. The analysis pays attention to specificities and particularities of these strategies, focusing on their economical, tailored, and spontaneous nature to solve the tasks. In reflecting on the context of mental mathematics, the paper closes with discussions of the potential and richness of these strategies for exploring systems of linear equations.

Keywords: Algebra and Algebraic Thinking; Problem Solving; Instructional Activities and Practices

Context of the Study

To highlight the importance of teaching mental calculations, Thompson (1999) raises the following points: most calculations in adult life are made mentally; mental work develops insights into number system/number sense; mental work develops problem-solving skills; and mental work promotes success in later written calculations. These aspects stress the nonlocal character of doing mental mathematics with numbers, where the skills being developed extend to wider mathematical abilities and understandings. Indeed, diverse studies show the significant effect of mental mathematics practices with numbers: on students' problem solving skills (Butlen & Pézard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Murphy, 2004; Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Pézard, 1992) and on their estimation strategies (Heirdsfield & Cooper, 2004; Schoen & Zweng, 1986). There is thus an overall agreement, and across contexts, that the practice of mental mathematics with numbers enriches students' learning and mathematical written work about calculations and numbers: studies e.g. conducted in US (Schoen & Zweng, 1986), France (Butlen & Pézard, 1992), Japan (Reys & Nohda, 1994), and UK (Murphy, 2004; Thompson, 1999). However, there is much more. For Butlen and Pézard (1992), the practice of mental calculations can enable students to develop new and economical ways of solving arithmetic problems that traditional paper-and-pencil contexts rarely affords because these are often focused on techniques that are too time-consuming in a mental mathematics context. In a similar vein, Poirier (1990), in reviewing historical curricular documents where the practice of mental calculations was salient, underlines the fact that mental calculations have their own processes, which differ from regular written calculations. Others, like Murphy (2004) or Threlfall (2002, 2009), focus on the aliveness and on-the-spot nature of mental calculations, where strategies are developed and tailored for the problem at hand and often differ from the writing processes that are usually focused on in paper- and-pencil contexts: something I have also discussed at PME-NA-35 (see Proulx, 2013a).

This being so, as Rezat (2011) explains, most if not all research studies on mental mathematics have focused on numbers/arithmetic. However, mathematics involve much more than numbers, and are predominantly studied through paper-and-pencil activities. This rouses interest in studying (1) what doing mental mathematics with mathematical topics/objects other than numbers (e.g., algebra, functions, trigonometry) might contribute to mathematical reasoning and understanding of these topics/objects, as well as studying (2) the kinds of strategies and solving processes engaged in to solve mental mathematics tasks related to other mathematical topics/objects than numbers. This paper focuses on this latter interest in relation to systems of linear equations, that is, studying the nature and specificities of the strategies developed through solving systems of linear equations tasks in a mental mathematics environment.

Defining Mental Mathematics

Because most work on mental mathematics is on numbers (often referred to as mental *arithmetic* or *calculations*), no definition of mental mathematics appears in the literature. For Thompson (2009), mental calculations represent a *subset* of mental mathematics; however he does not offer any definition of mental mathematics. This said, even if thought of in terms of numbers and mental arithmetic, definitions about mental calculations can be adapted to other mathematical objects/topics to help define mental mathematics. Building on Hazekamp (1986), who offers a definition that summarizes what is generally considered by mental calculations, one tentative definition is: Mental mathematics is the solving of mathematical tasks through mental processes without paper and pencil or other computational (material) aids.

In order to help develop a finer understanding of what is meant by mental mathematics, various dimensions about mental strategies with numbers are found in the literature (e.g. Butlen & Pézard, 1992, 2000; Kahane, 2003) and are adaptable to other objects/topics than numbers. One of these dimensions concerns *reasoned computations*, implying the elaboration of personal strategies, often nonstandard and adapted to the problem, versus *automatized computations*, which implies access to an immediate result through the use of known facts or memorized procedures. An example of this could be, for area, between using the formula $\frac{D\times d}{2}$ to find the area of the rhombus versus cutting the figure into triangles to find or compare the area. A second set of dimensions concerns *approximate computations*, based on estimation and approximation to gain an order of magnitude for the answer, versus the mental application of an algorithm or a fact to obtain an exact answer. An example for trigonometry could be between using the fact that $sin30° = \frac{1}{2}$ versus establishing a visual order of magnitude that the opposite side of a 30° angle enters approximately twice in the hypotenuse. A third dimension concerns *rapid computations*, which require quick execution to find the answer. Often criticized because it is perceived as a speed exercise detrimental to sense-making, it can also be seen as helping to develop new solving methods because it forces the solver, in trying to be economical, to abandon methods that may be slower (e.g. standard procedures) or less efficient for completing the task (e.g. one-on- one counting). In the case of algebra, an example could be the development of a *global reading* of an equation like $x + \frac{x}{4} = \frac{4}{x} + 6$ giving $x = 6$, avoiding numerous algebraic manipulations in order to isolate *x* (Bednarz & Janvier, 1992). These dimensions illustrate possible entries for solving mental mathematics tasks. In the case of systems of linear equations, what these dimensions represent is something probed into in this paper. In addition to having value for refining what mental mathematics can mean, these dimensions have value for data analysis and are reinvested in the subsequent analysis of strategies developed by solvers for systems of linear equations.

Methodological Considerations

This study is part of a larger research program that focuses on studying the nature of the mathematical activity in which solvers engage with through working on mental mathematics with objects/topics other than numbers. This is probed through (multiple) case studies that take place in educative contexts designed for the study (classroom settings/in-service education activities), where participants are asked to solve a variety of tasks. The reported study is one of these case studies, from a day-long session with 12 secondary-level teachers (Grade 9 to 12). The general organization took the following structure: (1) a task is offered to the group in writing on the board; (2) participants have approximately 15 seconds to solve the task; (3) at the signal, participants are asked to write their answers; (4) strategies are shared in plenary. The data comes from the strategies orally explained by participants, recorded in note form by two research assistants, who collected and compared their notes to produce more substantial information about the strategies developed. The session was also video-recorded, which allowed the research team to return to the tapes to enlarge on the notes and analyse the strategies in depth.

In this study, teachers are regarded as problem solvers; as would any participant solving the given tasks in this project. The decision to work with teachers is methodologically important. Indeed, these teachers are not novice solvers of systems of linear equations and thus are not in a new solving context or familiarization with the topic. This enables them to "enter" into tasks and attempt to solve them, giving access to their strategies and mathematical activity. This could be otherwise with participants who are newcomers to the topic, as they could experience significant difficulties with systems of linear equations and possibly would not be able to "enter" into the tasks for solving them. In that sense, the intention in this study is not to offer prescriptions for practice or to show how an approach through mental mathematics is "better" for learning about systems of linear equations than another focused on paper and pencil. The intention is neither to trace parallels between what these teachers do and what could happen with secondary-level *students* in a regular classroom. The analysis is focused, along the lines explained by Douady (1994), on the nature, meaning and functionality of the strategies developed in this mental mathematics context in order to analyse the mathematical activity engaged with and study the specificities of the strategies deployed for solving systems of linear equations tasks.

Findings

The tasks focused on for this paper consisted mainly in finding the solution, the intersection point, of a system of two linear equations (given algebraically), and then drawing the coordinate point on a sheet of paper displaying a Cartesian graph (with $y = x$ drawn as a reference line). Without being too caricatural, in a paper-and-pencil context the participants (as they explained during the session) would have resorted to algebraic manipulations and known algebraic strategies (comparison, elimination, substitution methods) to find these intersection points and then draw them on the graph. While solving the tasks, some participants opted for these algebraic methods, but in most cases they could not do this as the burden of algebraically manipulating without paper-and-pencil support became too important, and they had to develop alternative ways of solving. Below, I report on these alternative strategies to give a sense of their nature and analyse the mathematical activity involved. I first focus on the strategies given for solving the system of linear equations: "$y = x$ and $y = {}^{-}x + 2$". This is followed by other strategies developed for similar tasks, focusing again on their specificities for solving.

Strategies developed for Task 1 ("$y = x$ and $y = {}^{-}x + 2$")

Strategy 1: The role/influence of parameters. To solve this system, the participant focused on the fact that both lines would normally cross at (0,0). However, because the second equation was not "$y = {}^{-}x$" and had a y-intercept of 2, the answer was elevated by 2 on the y-axis, giving (0,2) as an answer. The focus on the parameter (y-intercept) played an important role in determining what and where the answer would be in the graph. This said, the answer (0,2) was not seen as the y-intercept, but mainly as an elevation of the point (0,0) toward (0,2); it ends up being the same, but at the time it was not thought of in these terms, as the participant expressed.

Strategy 2: Finding a "line" of possible solutions. The participant drew a vertical line at x=1, explaining that he did not have enough time to find the exact value of y, but that the solution was on that line because x=1 gave the same answer for both equations. Of interest is that substituting x=1 also gives the value for y. But in his algebraic manipulations for finding the value that gave the same answer for both equations, his focus was on finding a common x that gave the same answer for both equations ($x = ?$ and $^{-}x + 2 = ?$) and not on finding the value of *y*, even if it is the same. Both ventures were seen as separate. The first venture (focus on *x*) gave an infinite number of solutions within a restricted domain of *x*=1. He did not have enough time to look for *y*.

Strategy 3: Visualizing in the graph. The participant mentioned having found approximately where the point would be, in the 1st quadrant, by visualising the lines as one that crosses the first quadrant in the middle (y=x) and the other going through the 1st quadrant as well ($y = {}^{-}x + 2$, with a negative slope and starting from 2 on the y-axis), both intersecting on $y = x$ and in the 1st quadrant. Visualising the lines played an important role to position where the intersection point would be.

Strategy 4: Visualizing the lines with objects. Having pencils and pens on her table (for drawing the solution afterwards), the participant imagined them as lines in the graph and saw, as did the participant in Strategy 3, that the intersection point was in the 1st quadrant. It is through "seeing" the graph that the solution was developed, as the participant explained.

Strategy 5: Finding the right quadrant. For this participant, the first step was to realize that the solution would be in the 1st quadrant, because of the equations of the lines: one that splits the 1st quadrant in half and the other that goes through it (without being precise in exactly how). The point was not placed precisely in the graph (close to the *x*-axis, a little to the right of the origin), and the participant knew this, but the focus was on finding the right quadrant for the intersection.

Strategy 6: The y-intercept as a focus. Similar to Strategy 1, the *y*-intercept played a role in determining the intersection point. This time, with the difference that the *y*-intercept did not influence a previously obtained answer (as in Strategy 1 where the point (0,0) was elevated by 2), but influenced the fact that the solution had to be "in that area" of the *y*-intercept because it played a role in the position of the line $y = {}^{-}x + 2$ in the graph. Hence, the solution was placed at (0,2), mainly because of time constraints and because an answer had to be given. Here again, even if it is the same coordinate points, (0,2) is not thought of as the *y*-intercept for the participant (if it had been, it would have been discarded, because the participant knew that the other line did not have 2 as *y*-intercept). It was seen as a possible intersection point where both lines would cross each other.

Strategy 7: Visualizing the lines with gestures. Similar to Strategy 4, a participant imagined placing and crossing hands to represent the slopes of the lines, which is a common teaching practice. This gave an idea of where the intersection point would be. He considered the point to be in the 1st quadrant and saw that the intersection point could be in the "middle" of the crossing hands (not necessarily realizing that the point was on *y*=*x*, with the same value for *x* and *y*). Here too, the participant explained that it is through seeing the lines that the solution was developed.

Strategy 8: Trial and error. The participant attempted some numerical solutions (however not of the form *x*=*y*), but because it led nowhere he was unable to place any point of intersection (albeit knowing that there was one because both lines did not share the same slope).

In the following section, the above strategies are grouped and discussed in relation to their similarities, drawing out their specificities related to the context of mental mathematics. Since a strategy's attributes can be related to diverse groups, it can be placed in more than one group.

Discussion of strategies developed for Task 1 ("$y = x$ and $y = {}^{-}x + 2$")

Order of magnitude strategies: Strategies 2, 3, 4, 5 and 7. For these solutions, the focus is on having a good idea of where the solution is in the graph, of what is happening in the system of equations. These solutions can thus be related to the *approximate calculation* dimension. The focus is not on finding an exact answer, because time i s an issue in the mental mathematics context. This forces an analysis of the system to obtain an order of magnitude, an approximation, of where the solution would be: whether by focusing on the value of *x* as in Strategy 2, on the fact that the solution is on the line *y*=*x* as in Strategy 3, on gaining a visual idea of where the lines intersect as in Strategies 4 and 7, or of knowing in which quadrant the solution is as in Strategy 5. The need to develop an order of magnitude appears to derive from the mental mathematics context, a strategy quite removed from the algebraic manipulations that aim to find the exact solution to the system. The specificity of these solutions underlines important aspects, mainly visualizing the system and understanding what is happening. The focus on x=1 in Strategy 2 and on x=y in Strategy 3 are exam ples of how this focus on the order of magnitude offers significant information about the system, because, even if it may seem obvious, the value of x and the value of y need to be the same for finding a solution that satisfies both equations, one of them requesting that x=y! The same is true for the more visual strategies where lines are positioned, because one can directly see that there is an intersection point (something important to know about) and that the links between both algebraic and graphical representations are salient. Hence, one gets an idea of the system, how it functions, and what its possibilities are. It is in this sense that these strategies appear as specific for mental mathematics: not that they are better or worse, but simply different, provoked by the context of solving and offering another way into solving the system that a paper-and-pencil context does not necessarily afford.

Study of the equation: Strategies 1 and 6. Even if they give an answer that is mathematically inadequate, these strategies have something interesting to offer and can be related to the *rapid computation* dimension for a *global reading* of the equations. In particular, they are quite far removed from the usual algebraic manipulations that one would normally plunge into to solve the system. When solving algebraically in a paper-and-pencil context, the presence of the 2 in the second equation, the y-intercept, is not given much consideration or seen as important as it is simply a “2”, the number “2”, that is manipulated to find the value of x or y for solving the system. In algebraic manipulations, this “2” is not related to a “2” as the y-intercept, and the answer obtained is not considered as being influenced by this “2”; any more than would the negative sign affecting the x in the second equation (in algebraic manipulations, this negative sign affecting an x does not signify a “negative” slope). But, in Strategies 1 and 6, this “2” *is* the y-intercept and not simply a number to manipulate. Hence, consideration of the “2” is not related to the mechanics of manipulating algebraically, but for reflecting and thinking of the answer, the point of intersection of both lines. Thus, in these two strategies, one finds that the equation is studied, its attributes are considered and evaluated for finding something in it, for reading it so that it speaks differently than just algebraic symbols that need to be mingled. In this sense, these strategies call attention to the “-” sign affecting the x, to the “2” in the second equation, that makes the equation different from one that does not have the “2” (e.g. y=^{-}x). These strategies offer a focus on differences in equations that make a difference for determining the solution.

Because of time constraints, the effect of the “2” made the participants decide on an intersection that was about that “2”, but the specificity of these strategies is not about the answer obtained or the consideration of the “2”, but rather about the effect that this “2” had on the answer: the fact that it influenced and played a role in the answer (e.g. 2 more, 2 higher). The specificity of Strategies 1 and 6 concerns the focus on the *effect* of the parameters, here the y-intercept, on the solution to the system: something of lesser interest in algebraic paper-and-pencil manipulations.

Focus on numerical points: Strategies 2 and 8. These strategies focus on exact values for the solution by attempting to substitute possible answers to satisfy the system: being a hybrid between automatized and the opposite of *approximate computations*. It can be said that these strategies are elementary, as they are only an attempt to try out possible answers in a trial-and- error, unsystematic venture. However, their interest lies not in what they are not, but mainly in what they can be and can offer: that the solution needs to satisfy both equations simultaneously to be a solution to the system. Indeed, if there were only one equation to satisfy, the numbers attempted would have given a solution, and this would be it (e.g. x=2 in y=^{-}x+2 would give y=0 and this would be satisfactory for an answer to this equation). But, in the case of a system of equations, the solution needs to satisfy both equations simultaneously to be its solution. Hence both x and y need to be solution of both equations.

The x=1 answer in Strategy 2 illustrates this, as x=1 is explained as a solution for both equations, meaning that it gives the same answer in each equation. (Again, here the focus of the participant was on finding x first, and time did not allow for finding y, even if the "answer" and the "y" were the same). The next step would have been to find the value of y that was the same for both equations, which would satisfy both simultaneously. The same is true for Strategy 8, as the solutions attempted (here coordinate points) was intended to test a value for x and for y at the same time in both equations and to see if both satisfied the equations. The specificity of these strategies lies in the fact that the mental mathematics context provoked a need to find an answer, an x and a y, that satisfied both equations. In an algebraic manipulation context, this intention to find values that satisfy both equations simultaneously is often hidden behind the mechanical manipulations, and if it emerges it does so at the end of the process when establishing the values of x and of y that are solution to the system. It is in this sense that the trial and error, focused on numerical points, appears as specific in this mental mathematics context.

Discussion of a variety of other strategies developed for other tasks

In what follows, I discuss two other strategies developed to solve similar tasks. In detailing these strategies, I explore their specificities and what they focus on for solving.

Visualizing the lines in the graph through studying the equation. When solving the system "y=3x+1 and y=7x", one strategy was to analyse where the lines would be in the graph and then consider their intersection; related to a *global reading* dimension of the system. Thus the participant explained that y=7x passed by (0,0) and is quite inclined, whereas y=3x+1 "starts at 1", is less inclined, and thus crosses y=7x at a y greater than 1 and on the right of the y-axis (note that the analysis is made only in the 1st quadrant, as this participant knew where the lines would intersect from visualizing both lines, whereas an algebra-equation analysis would not offer the same information about the 1st quadrant). Another similar strategy was also related to the inclination of the lines, where the y=7x was seen much more inclined than the y=3x+1, leading to a value in x being between 0 and 1. Also, the value of y was seen as greater than that in x because one of the two lines, the y=3x+1, had a y-intercept of 1 (and a positive slope). In both cases, the analysis of the line is precise, focusing on aspects of the equation that gives information about the lines in order to visualize them for subsequently finding the solution. Understanding that the solution is in the 1st quadrant because of the inclinations of the lines, that the value of x is between 0 and 1, that the value of y is higher than 1, and so forth, is not a necessary part of the algebraic manipulating process, as these facts have little influence on the manipulations needed to obtain the solution. But in this case, in the mental mathematics context, these specific aspects are provoked in the strategies. And similar visualizations of the line in the graph, through studying and analyzing the equation, were made for other tasks. E.g., with the line "$y = x + 10$", one participant said that it was parallel to "$y = x$" but higher because of its y-intercept. Another example is with the system "$y = \sqrt{8}x + 5$ and $-\sqrt{18}x + 3y = 9$", where one participant approximated the value of the slope of $\sqrt{8}$ as being close to $\sqrt{9}$ and thus close to 3 with 5 as the y-intercept. For the second one, a mistake was made in relation to the sign of the slope, which was seen as positive rather than negative, but the participant considered the y-intercept as being of 3 (from dividing 9 by 3), and thus the solution in y as being between 3 and 5 in the 2nd quadrant. This analysis is quite impressive. For this participant, the intersection could not be lower than 3 in y, because it would be in the 1st quadrant, which is impossible because in that quadrant the other equation "starts" at 5 in y (its y-intercept being at 5). Thus by being a negative slope that has a y-intercept of 3, it had to intersect in the 2nd quadrant and between 3 and 5. These strategies represent specific ways to manage solving the system, performing a fine analysis of the equation combined with a visualization of the lines for solving it. It offers a specific way into the system, visualizing it, but mostly understanding how it works and where and how the solution can be. The specific analysis of the equation renders some "information" in the equation as significant to solve the system: something un-usual in a paper-and-pencil algebraic manipulations context.

Approximate algebraic manipulations. When asked to solve the system "$4x + y = 10$ and $x - 2y = {}^{-}6$", some participants opted for the elimination method by quadrupling the second equation, transforming it from $x - 2y = {}^{-}6$ to $4x + 8y = {}^{-}24$, and then subtracting it from the first (and obtaining$9y = 24$). Because of time constraints, the value obtained for y was said to be around 5; which can be related to the *approximate computation* dimension. Then, substituting the value of *y* in the first equation led to a value for *x* of around 1, for a coordinate point being about (1,5). Obviously, the degree of errors in regard to the solution is significant, because there is a first approximation for y, and then one for *x* based on that y. But of interest in this strategy is the approximation of the values in an algebraic system. This offers an order of magnitude for both values even if they were obtained through what is often seen as a precise strategy, that is, the algebraic route of elimination. Whereas earlier solutions focused on gaining an order of magnitude for situating the coordinate points in the graph, here the order of magnitude is in relation to algebraic manipulations: quite different from what is usually seen in an algebraic manipulation paper-and-pencil context.

Final remarks

These strategies reported illustrate some specificities of ways of engaging with the task in this context of mental mathematics. They are no doubt provoked by time constraints and because notes cannot be taken in writing, but this is the context of mental mathematics and it promotes these kinds of specific entries. It creates a need to grab something, to draw out aspects of significance for solving the problem. In addition to their specificities, these strategies have important potential for understanding or developing meanings about solving systems of linear equations, away from mechanical treatment and in relation e.g. to considerations of what is going on in the equations themselves, and how they behave. The reported "analyses/studies of the equations" represent such possibilities/potential for understanding systems of linear equations. Even if these strategies are not always optimal or do not render correct answers (precise or not), they offer ways of understanding what a system of equations is, ways that are different from what is usually done with algebraic manipulation strategies. These strategies are specific in the sense that they offer a way into the systems, an analysis of it, through drawing out particular aspects. Whether this is through analyzing the equations to gather significant information on the system, through having an order of magnitude of the solution or the system itself, through attempting to satisfy simultaneously both equations of the system, all this offers a specific way of engaging with solving systems of linear equations.

These results are promising. What comes out of this current work, and that on solving algebra equations (e.g. Proulx, 2013b) and operations on functions (e.g. Proulx, 2013a) in mental mathematics contexts, is the emergence of unusual ways of tackling and working on these mathematical topics, bringing forth varieties of strategies that focus on aspects not usually engaged in or focused on (here e.g. the *y*-intercept, the order of magnitude, the boundaries where the value in *x* and *y* could be). These strategies are creative and exploratory, and in this sense they suggest *extending* what can be done with these topics: not in terms of the tasks offered, but in terms of the meaning to be given to the topic itself. These strategies bring us elsewhere, focusing on different aspects and on other ways of solving. It is in this sense that they are specific, as is argued similarly for mental calculation on numbers, where they differ from the usual ways of solving and enable a focus on other aspects for solving. Clearly, more is to be studied and researched, but the focus on extending the scope of solving for these topics promise great value for mathematics teaching and learning.

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JAKE'S CONCEPTUAL OPERATIONS IN MULTIPLICATIVE TASKS: FOCUS ON NUMBER CHOICE

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This case study examined how a teacher's choice of numbers used in tasks designed to foster students' construction of a scheme for reasoning in multiplicative situations may afford or constrain their progression. This scheme, multiplicative double counting (mDC) is considered a significant conceptual leap from reasoning additively with units of one (1s) and composite units. A researcher-teacher's work with Jake allowed us to center on his gradual cognitive advance as different numbers chosen for the unit rate in problems (e.g., 5 cubes-per-tower) were used in the context of the Please Go and Bring for Me platform task. Our findings show that a child's use of an evolving scheme may initially depend on the numbers used in the task. We discuss the key recognitions that (a) a new way of operating does not evolve in a "once-and-for-all" way for all numbers and (b) the support our study provides for Pirie and Kieren's core notion of folding-back.

Keywords: Number Concepts and Operations; Elementary School Education; Teacher Knowledge

Introduction

In recent years, a growing body of research has been focused on various aspects involved in children's transition from additive to multiplicative reasoning (Empson& Turner, 2006; Hackenberg & Tillema, 2009; Sherin & Fuson, 2005; Tzur et al., 2013; Verschaffel, Greer, & DeCorte, 2007). However, unlike in other mathematical areas, such as numbers chosen for addition and subtraction tasks (Fuson, 1992), how a teacher may consider the use of numbers in tasks designed to help students overcome the conceptual leaps involved in progressing from additive to multiplicative reasoning has received little attention. To embark upon this lacuna, our study addressed the problem: How may specific numbers a teacher uses in tasks for promoting students' advance from additive to multiplicative ways of operating with/on different types of units (1s, composite), afford or constrain a child's conceptual progression when solving multiplicative problem situations? In particular, this paper examines such affordances and constraints as a child begins the transition from additive reasoning to the first scheme in which she or he coordinates operations on two types of composite units—the multiplicative double counting (mDC) scheme.

Conceptual Framework

A constructivist perspective on knowing and learning (Piaget, 1985) underlies this study. Specifically, we drew on von Glasersfeld's (1995) construct of scheme as a three-part mental structure: (a) situation, a recognition template into which a learner assimilates a problem situation (or task), that triggers her or his goal; (b) a mental activity the mind carries out to accomplish that goal; and (c) a result the learner expects to follow from the activity. The situation part includes recognition of and bringing forth of objects, such as numbers, upon which the mental activity operates.

Working within such a perspective, Steffe's (1992) seminal work contributed to distinguishing between additive and multiplicative schemes. He proposed to focus on the units on which one is operating mentally (1s or composite units) and the operations a learner uses that underlie her or his performance when solving tasks. He thus distinguished additive from multiplicative reasoning not on the basis of observable behaviors ("strategies") the child uses in and of themselves, but on the basis of inferences into what mental operations on units operations could give rise to those behaviors. Specifically, he asserted that multiplicative schemes involve a coordination of composite units in which the child distributes the items of one composite unit (e.g., towers made of 5 cubes each) over items of another composite unit (e.g., a compilation of 3 towers). Such a coordination may be manifested in the use of some figural items in place of the objects alluded to in the task (e.g., a finger standing for one tower) and later in the abstract, double counting of two sequences of composite units (e.g., first tower is 5 cubes, second-is 10, third-is-15). Tzur et al. (2013) recently proposed a developmental pathway of six schemes through which children's multiplicative reasoning may progress; this study focuses on the use of numbers to promote the transition from additive reasoning to the first scheme in the progression.

This study also drew on Pirie & Kieren's (1994) constructivist stance on the non-linear growth in learners' mathematical understandings, particularly the key construct of folding-back. Rooted in studies of children in other areas (e.g., fractions), they showed that an ordinary path to higher-level understanding (outer layers in their model) might include frequent 'regresses' to the use of lower-level, previously constructed understandings. Our tasks were designed to promote students' transition to the first of six schemes, multiplicative double counting (mDC), while allowing the research team to analyze how numbers chosen for these tasks would possibly bring about folding-back and upward shifts in the units/operations a child may use.

Methodology

The case study reported in this paper was part of a larger constructivist teaching experiment (Cobb & Steffe, 1983) we have conducted with four 4th graders identified by their western US school as requiring intervention in mathematics based on state assessments and classroom teacher recommendations. The two first authors conducted the video recorded teaching episodes with each child individually. This paper analyzes data from their work with one student, Jake (pseudonym), twice a week, from October through December of 2014, around 30-45 minutes each episode. The second author (Nina, pseudonym) served as the researcher-teacher in those episodes, as Jake was accustomed to working with her as the school intervention teacher.

All teaching episodes engaged students in playing the task-generating game of Please Go and Bring for Me (PGBM), which Tzur et al. (2013) described in detail. In a nutshell, PGBM is a turn-taking game played in pairs, with one's peers and/or the teacher. Each turn, a "Sender" asks a "Bringer" to build and bring back from a box containing individual cubes a compilation of same-size towers, one tower at a time (e.g., 3 towers, 5 cubes in each). Once all towers were brought to the Sender's satisfaction, she or he asks the bringer four questions (in our work – those are written on a poster to promote students' use of full sentences and explicit mention of units): (a) How many towers did you bring (emphasizes number of composite units)? (b) How many cubes are in each tower (emphasizes unit rate – number of 1s in each composite unit)? (c) How many cubes are in all the towers? (d) How did you figure this [total of 1s] out? (Last two questions emphasize operations child used to figure out number of 1s in the entire compilation of composite units.) Similarly, the poster included 'answer-starters' that enabled the bringer to express her or his answers as full sentences (e.g., "I brought __ towers"). Initially, the teacher constrains the game so children can only use particular numbers of cubes per tower (e.g., 2 or 5) and of towers in all (e.g., up to 6), while also asking children to use different numbers for each kind (e.g., a sender cannot ask to bring 5 towers of 5 cubes each).

Our line-by-line retrospective analysis of video records, transcripts, and researcher field notes taken during each episode focuses on the first two teaching episodes with Jake. Focusing on his initial transition to multiplicative reasoning serves the purpose of 'zooming in' on the interplay between his ways of operating and the numbers chosen in each task. The two first authors conducted ongoing analysis following each episode; the entire team of authors then conducted the line-by-line analysis of the four segments presented in the next section.

Results

To study how numbers chosen for tasks may afford and constrain a child's operations with different types of units as the teacher promotes Jake's transition to the mDC scheme, this section includes analysis of four data Excerpts. In the first episode (Excerpts 1 & 2), Jake worked with a peer on PGBM tasks constrained to unit rate of 2 or 5 and a number of composite units up to 5. In the later episodes (Excerpts 3 & 4), Jake played a bringer role with Nina as sender, with tasks allowing both unit rates and number of composite units to be 2, 3, 4, 5, or 6.

Starting Point: Less than Five Composite Units, Unit Rates of 2 or 5

Excerpt 1 shows data from Jake's first turn as a bringer, after he had produced (from single cubes), brought 4 towers of 5 cubes each, and properly responded to the first two questions (SS stands for Student-Sender).

Excerpt 1

SS: How many cubes did you bring in all?

Jake: (Glances at the towers for 1 second, then says) I brought … (uses his left hand to tap five times on the palm of his right hand); I brought 20 cubes altogether.

SS: How did you figure this out?

Jake: I figured this out by counting.

Nina (Teacher): How did you count?

Jake: I counted by 5s[1].

Nina: How did you know to stop counting?

Jake: Cause if you don't, cause if you can't … (reaches with his left hand and brushes over the towers that the SS is holding)

Nina: I can count by 5s too: 5, 10,15, 20, 25,30 ... So how did you know to stop at 20?

Jake: (Looks at the towers that the student sender is holding) It's … because that … (turns his head away from the towers for five seconds, then turns his gaze back on the towers). It's because I only brought 4 towers.

Nina: So you knew to stop because ... ? How did you know you had counted the four towers, Jake? I agree that you only brought 4 towers. How did you know that 20 was 4 towers.

Jake: I counted on my fingers.

Nina: Can you show me?

Jake: (Holds up his right hand, then folds his index finger while stating) 5; (folds middle finger) 10; (folds ring finger) 15; (folds pinkie finger) 20! (Body indicates, "I am done").

Excerpt 1 provides a glimpse into Jake's mental operations while solving a problem with 'easy number' unit rate 5. Jake first reestablished the number of 1s that constituted each composite unit (five palm taps). That is, he seemed to have created a figural mental template of the size of every composite unit. Because counting by 5s was within his capacity, his count of the accrual—which he demonstrated to the teacher, indicated a purposeful method of keeping track of the number of composite units so he would know to stop at four towers. His ability to perform such a purposeful action did not yet seem to support expressing how he did it. Rather, when prompted to explain how he knew to stop at 20, he initiated a shift to a different figural re-presentation, by using the fingers of his right hand to represent the composite units in the situation (towers) and his number sequence (by five) to re-present the accruing 1s (cubes). This seemed to assist his growing anticipation of the link between coordinated actions taken to figure out a progressive total and the effect of stopping the count of 1s when reaching the number of composite units (which fits his statement, "I brought 4 towers"). Jake, when operating mentally on a unit rate that is a known counting sequence (5s), could both initiate and complete the coordinated, goal-directed mental activity involved in multiplicative double counting.

It is important to note the purpose of the teacher's interventions (e.g., "how did you count"). A child may initiate and carry out goal-directed actions while not being aware of his own actions, let alone the steps he took to monitor those actions. By asking Jake to explain, she thus attempted to orient his reflection on and awareness of his own purposeful actions (e.g., he did keep track of the composite units). To explain his strategy, Jake used fingers, which the teacher intended as a means to promote two critical reflections in constructing the mDC scheme. First, she focused Jake's attention on a specific aspect of his coordinated counting—monitoring accrual of the composite units. Second, she simultaneously focused his attention on the key in his stoppage monitoring.

In the following PGBM task in which Jake played the bringer, he had to figure out and explain how many cubes are in 5 towers with 2 cubes each. Again, Nina pressed for his explanation of his solution, as his response (10) came after quietly nodding his head five times but not using his hands/fingers. Excerpt 2 shows he re-used the previous way of explaining.

Excerpt 2: Finding the total number of cubes in 5 towers of 2 cubes each

Nina: Can you show me?
Jake: Like this … (holds up right hand, folds down his index finger while saying) 2; (folds his middle finger) 4; (folds his ring finger) 6; (folds his pinkie) 8; (folds his thumb and concludes) 10.
Nina: So, again, each one of your fingers … this [seems] similar to something else you did. Each one of your fingers, you put them down: 2, 4, 6, 8, 10 (paraphrases J's motions and utterances); so you stopped here (wiggles her thumb); Why?
Jake: Its because I only brought 5 towers.

Excerpt 2 provides further evidence to the evolving regularity in Jake's ways of operating as well as the teacher's involvement in that process. Freed from a mental focus on the accrual of 1s in the sequence of multiples of 2, he initiated and completed a coordinated count of figural composite units(5 towers) through nodding his head and later through using fingers. He seemed to anticipate the need to coordinate two accruing number sequences: composite units (towers) with unit rate (cubes distributed over each of the towers). Similarly, when prompted to explain his thinking, Jake used the fingers of his right hand to re-present each composite unit in the sequence of 5 towers while keeping track of accrual of 1s via his number sequence (by 2s) to ten. Considering Excerpts 1 & 2 combined, Jake seemed to have established at least an enactive anticipation of multiplicative double counting as a means to accomplish his goal.

Starting Point: Composite Units of Less Than 5, Unit Rates of 4 and 6

Excerpts 3 & 4 present Jake's work two days later, with Nina serving as the sender. Based on his facility with double counting in tasks with unit rates of 5 or 2, she chose to send him to bring 3 towers of 4 cubes each—slightly harder numbers (for him) that he could still work out by using each hand separately. Excerpt 3 starts after Nina asked Jake how many cubes he brought in all.

Excerpt 3

Jake: I brought 11 cubes altogether.
Nina: (Lays the towers down on the table closer to Jake) Can you double check?
Jake: (Takes apart each tower into its individual cubes while counting them out loud but without keeping them as distinct groups) 1, 2, 3, 4,5,6,7,8,9,10,11,12!
Nina: So how many [cubes] did you bring altogether?

Jake: 12.
Nina: I want to know how you figured that out (she takes 4 cubes and reassemble them into a tower.
Jake: (Reassembles another tower, albeit from 5 cubes.)
Nina: (Reassembles another tower from the remaining 3 cubes, and asks him to give her a cube so she can make a tower of 4.) Jake, when you counted the first time, I saw you use your fingers (she folds her left hand's fingers to emulate his motions). Can you tell me what you were doing?
Jake: (Folds down 4 fingers on his left hand, then folds down a finger on his right hand; repeats the process for the second and third towers, while left-hand fingers seem to stand for 1s and right-hand fingers for composite units/towers.)
Nina: And why did you do this (replicates Jake's motions)?
Jake: Each finger was a tower.
Nina: Each finger was a tower? So can you show me again?
Jake: (Holds up his right hand with the palm face up) Each finger was a tower. (Folds down three fingers on his right hand, one at a time, while saying) That's the first tower, that's the second tower, and that's the third tower. (He then adjusts his hand motion and folds down four fingers on his left hand, one at a time, while saying)If we add them all up we go 1(folds down a finger),2 (folds down a second finger),3 (folds down a third finger), 4 (folds down a fourth finger),5 (looks at his hand as if having an 'oops' experience). (After 2 seconds, he opens his hand, says "No," then raises three fingers again on his right hand and says)No; this is three towers and you [have to] count by fours. He uses his left hand to put down one of the fingers on his right hand and says "Four." Then he raises his left hand, counts four fingers, and says "5-6-7-8." He then uses his left hand to fold another finger down on his right hand, then opens his left hand again and says "9-10-11-12."(Then, folds the third finger on his right hand.)
Nina: And you knew to stop at 12 because these were like the towers (touches the fourth finger on *his* right hand and this was like the 12th …
Jake: (Completes her sentence) … Tower
Nina: (Inquiring about the unit type) The 12th what?
Jake: Tower
Nina: (Places her hand on the real towers and says) How many towers do you have here?
Jake: (Corrects himself in response to her prompt) That was the last, third tower.
Nina: It was the last, third tower; but it was the 12th what?
Jake: Cube

Excerpt 3 provides further evidence of the role numbers in task played in his evolving way of operating, as well as the teacher's involvement in that process. Challenging him to solve a problem with 4 as the unit rate, Jake's mental system was no longer freed from focusing on the accrual of 1s. Thus, he folded back to a focus on 1s (single cubes) that composed each unit. Taking apart all 3 towers without any allusion to their grouping indicated his need to individualize units—no longer reasoning multiplicatively. Yet, when prompted to explain, Jake made several attempts to coordinate the use of his fingers to re-present, on each hand, a different type of unit (cubes on the left hand, towers on the right). In his first, unsuccessful attempt, he identified the right fingers as towers, but began counting them as cubes. In an "oops" moment, he realized that counting the figural tower re-presentations on his right hand was not moving toward his goal and started over. On his second attempt, Jake was able to accurately use his fingers to keep track of the unit rate of 4. He began his count not from 1 (cube) but from the first multiple of 4, indicating the coordination of a first composite unit with the numerical value of the unit rate. Due to lack of facility with the next two multiples of 4,heturned to using the left hand four counting accrual of 1s while keeping track of the composite units with fingers on the right hand. Indeed, Jake's solutions to this more challenging (for him) task differed, and folded back, from his coordinated actions to solve the previous tasks (with unit rates of 5 and 2). This suggested that Jake's goal-directed coordination of double counting operations on composite units was still evolving and thus dependent on the numbers used in the task (in the sense of his facility with the multiples of the unit rate).

To further demonstrate the interplay between numbers chosen for a task and a child's ways of operating, we present his solution to the task that followed (Excerpt 4). Here, the teacher decided to keep the number of towers at 3, but to increase the unit rate (6 cubes per tower) so it exceeds the number of fingers on one hand. This choice intended to explore how Jake would cope with the challenge of keeping track of composite units while using fingers on both hands to keep track of a sequence of multiples with which he was not facile.

Excerpt 4

Nina: How many cubes did you bring altogether?

Jake: (Silently touches each cube in the first tower, perhaps re-counting them all. He then holds up and stares at his two hands for 1 second, puts his hands down, reaches for a pen, looks back at the cubes, points to the cubes for 1 second, and writes on a small white board the number 6 first and then the letter "T" above it.)

T
6

He looks back at the towers on the table and writes on his white board another set:

T T
6 6

He looks back a third time at the towers, and completes his writing to correspond with the numbers given in the task:

TTT
6 6 6

Seemingly not knowing how to proceed, he turns back to the real towers; taps on each of 8 cubes individually (all cubes in the first tower and two from the second), stops, and erases the white board completely. At this point, he begins counting while pointing with his finger to each individual cube in the first tower, shifting to counting by 2s to expedite the process, finally saying): I brought 18 cubes altogether.

Nina: (Not indicating if he was correct or not) How did you figure it out?

Jake: (Picks up all of the towers and then puts them back on the table) I figured it out a different way; I just counted it this time. I used the cubes (picks up one tower). I pretended I broke it up like … (breaks off individual cubes from each tower); like 1-2-3-4-5-6;that is a tower I counted.

Nina (a little later): Did it get a bit harder today when the numbers got harder?

Jake: It was hard for me because I wanted to use the way I did last time (holds up both hands and shows the hand motions that he made when using his fingers to re-present a coordinate, double-count with 4s); but I don't have as much fingers.

Nina: So you couldn't, you couldn't, you don't have as much fingers so does that mean you couldn't? What do you mean by that (shows holding up of both hands)?
Jake: I had to do it a different way.
Nina: What do you mean you don't have as much fingers?
Jake: I only have 10 fingers and … (rebuilds the 3 towers of 6 using the cubes on the table) Since …So if there are towers of 6 (holds up both hands) and I went like that (begins to count out six on his right hand);So I wouldn't have enough fingers.

Excerpt 4 provided further evidence how the numbers Jake operates on make a difference in his goal-directed activity. When presented with a unit rate of 6, for which he had neither a mental number sequence nor enough fingers to re-present the items, he seemed unable to complete a coordinated count the way he did for unit rates of 2 or 5. Jake's first, spontaneous attempt to re-present the towers was to write a "T"(for tower) and the number"6" under each "T" to indicate how many 1s constituted that composite unit. While insightful and resourceful, this symbolized re-presentation did not proceed to a double-counting activity. Seemingly having no other recourse, Jake abandoned this initiative (e.g., erasing the white board) and instead folded back to counting each of the tangible cubes (albeit shifting from counting by 1s to counting by 2s to expedite the process). Jake's explicit utterance, about not having enough fingers, indicated he was acutely aware of the need to but unable to operate in a coordinated way when the number in one, unfamiliar composite unit precluded using each hand for a different component of the coordinated count. This was evident in his show of two hands, used to this end for 3 towers of only 4 cubes each, and the statements that followed ("I had to do it a different way"; I only have 10 fingers").

Discussion

This paper focused on a key consideration for teaching specific mathematical ideas to students, namely, the choice of numbers used in tasks. Particularly, our study focused on this consideration when using tasks to foster students' construction of the multiplicative double counting (mDC) scheme. Our study can contribute to the field in two important ways. First, it provides further support to the stance that construction of a particular scheme (e.g., mDC) is not a "once-and-for-all" event. Rather, when a shift in the child's way of operating is to be promoted, such as the cognitive leap from additive to multiplicative reasoning (Steffe, 1992), initial emphasis in task design and implementation needs to be placed on orienting the child's mental powers onto the novel coordination of operations on units. To this end, choosing 'easy' numbers seems highly productive, because the child can bring forth available knowledge of numerical calculations with which she or he seems facile (e.g., multiples of 2s and 5s). Initially solving tasks that involve 'easy numbers' enables the child to construct the intended goal-directed activities (a new scheme), which can then become an invariant way of operating she or he could apply to solving tasks with numbers that require engaging more complex mental capacities. (The scope of this paper did not allow us to provide data on how that change was successfully fostered in Jake after what we have seen in Excerpt 4.)

Second, our findings provide further support to the core construct of folding-back in Pirie & Kieren's (1994) model of growth in understanding. Specifically, we showed how Jake was able to use his newly constructed understanding of the coordinated count of composite units and begin operating in multiplicative situations when using 'easy numbers' (Excerpts 1 & 2). Nevertheless, he folded back to operating on 1s, while also conflating unit rate with the number of composite units (Excerpt 3), and was initially unable to solve similar (from an adult's point of view) tasks presented with numbers for which he had no facility with the multiples. He further folded back, foregoing use of double counting when the unit rate (6) exceeded what he could signify with one hand's fingers. As Tzur and Simon (2004) noted, folding-back may be a good behavioral indication for a stage in constructing a new scheme at which the child's evolving scheme (here, multiplicative coordination/distribution of composite units) is yet to become independent and spontaneous. This seemed to be Jake's case, when anticipating he would not have enough fingers to account for all 1s in a tower of 6 cubes, and thus producing the abstract diagram to create a tool for keeping track that seemed not yet readily available for him conceptually. Folding-back can thus also be a good indicator of the need to pay close attention to the child's conceptualization, in the following, threefold sense: (a) not simply attributing to the child too high a level of conceptual growth due to successful performance on 'easy-number' tasks; (b) not attributing to the child outright failure to conceptualize the intended math due to unsuccessful performance on 'harder-number' tasks; and (c) designing tasks and using tools that can gradually bring the intended mathematics to be within the child's mental reach.

Endnote

[1]To improve readability, the authors have elected to use numerals in the transcriptions when speakers make number references.

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STUDENTS' GENERALIZATIONS IN THE DEVELOPMENT OF NON-LINEAR MEANINGS OF MULTIPLICATION AND NON-LINEAR GROWTH

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The study reported on in this paper is an interview study conducted with 20 7th and 8th grade students whose purpose was to understand the generalizations they could make about non-linear meanings of multiplication (NLMM) and non-linear growth (NLG) in the context of solving combinatorics problems. The paper identifies productive challenges for the students, and thus fruitful areas where the students could generalize their reasoning about NLMM and NLG.

Keywords: Cognition; Number Concepts and Operations; Middle School Education; Algebra and Algebraic Thinking

As students progress into the middle grades they are expected to begin to understand situations that involve NLMM and NLG. Despite this curricular structure, studies in this area have highlighted the difficulties that students have in reasoning about a broad range of contexts that can involve NLMM and NLG (Van Dooren, De Bock, Janssens & Verschaffel, 2008). These difficulties include a tendency for students to generalize linear meanings of multiplication (LMM) and linear growth (LG) to situations that involve NLMM and NLG. For example, students frequently conclude that scaling the side lengths of a square by a factor of k produces a change in the area of the square by a factor of k rather than k^2 (DeBock, Van Dooren, Jansenns, & Verschaffel, 2007; Vlahovic-Stetic, Pavlin-Bernardic, & Rajter, 2010), and treat non-linear functions, for example quadratic functions, as if they have similar properties as linear functions (Chazan, 2006; Ellis & Grinstead, 2008; Zaslavsky, 1999). Thus, an important issue for researchers to investigate is how to support middle grades students to begin to establish NLMM and NLG, and further, to identify productive generalizations students can make in the context of establishing NLMM and NLG (Ellis, 2011). The purpose of this study, which is currently underway, was to address these two issues.

We addressed these issues by presenting 20 7th and 8th grade students with combinatorics problems whose solution could be represented with a two-dimensional array. Thus students had the potential to establish NLMM and NLG, and generalizations about NLMM and NLG as a result of reasoning about relationships among one and two-dimensional quantities. Here we report results from the first of two interviews. The purpose of the first interview was twofold: (a) to establish which of three qualitatively distinct multiplicative concepts students were using (Author, 2009; Steffe, 1994); and (b) to identify productive challenges in the domain of NLMM and NLG for students that were using each multiplicative concept. Once productive challenges were identified for students using each of the three multiplicative concepts this information was used to design tasks for the second interview whose purpose was to examine the generalizations that students using the three different multiplicative concepts could make.

Perspectives and Theoretical Framework

A Quantitative Approach to Non-Linear Meanings of Multiplication and Growth

We developed NLMM and NLG using combinatorics problems like the *Digit Problem*.

Digit Problem: You have a deck of cards with the digits one through seven written in black on them. You draw a card, replace it, and draw a second card. How many possible coordinate points (e.g., (1,7) is one coordinate point) could you make using this process?

We specifically selected combinatorics problems to investigate these issues because of their potential to support students to establish one and two-dimensional quantities, and relationships between them.

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That is, combinatorics problems can involve *ordering* the digits in the deck of cards (a first digit, second digit, etc.), and *ordering* the draws from the deck of cards (a first draw, and a second draw) (Author, 2013). We considered this to be a potential basis for spatially structuring the units on the axes, and the axes themselves, which in turn could create a spatial structuring for the coordinate points in the array. Further, we considered that these problems could involve *pairing* a digit from the first draw with a digit from the second draw (Author, 2013), and thus give students an operative way to create what we considered two dimensional units (i.e., coordinate pairs) from one dimensional units (i.e., digits).

We considered that the solution of the digit problem could involve students in establishing a meaning for 7^2, an NLMM (Figure 1a). Once students solved this initial problem we were interested in introducing NLG by having students consider how changing the number of digits would change the number of coordinate points, which we considered to involve whole number co-variation[1]. We wanted to investigate this issue through two types of problems: (a) having students consider how adding additional digits to the deck of cards (e.g., adding the digit 8, then adding the digit 9) would change the total number of coordinate points by considering how many *new* coordinate points there would be after a given change in digits; and (b) having students consider how doubling, tripling, etc. the number of digits would change the number of coordinate points. The goal of the first task was for students to have an experience of NLG by investigating how *equal* changes in the number of digits (e.g., changing the number of digits from 7 to 8, and then from 8 to 9) yielded growth in the number of coordinate points that was *not constant* (e.g., 15 new coordinate points when the number of digits increased from 7 to 8, and 17 new coordinate points when the number of digits increased from 8 to 9) (Figure 1b).

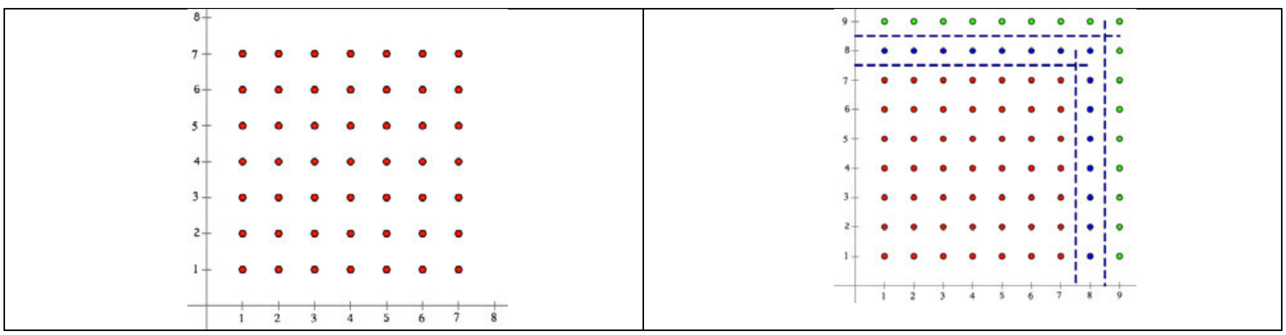

Figure 1a (left) & 1b (right): Arrays for the Digit Problem

The goal of the second task was for students to have an experience of NLG by investigating how multiplicatively increasing the number of digits (e.g., doubling) increased the number of coordinate points by the square of the increase in the number of digits (e.g., quadrupled) (Figure 2). Both of these problems we considered important to students' initial understandings of NLG.

Discussion of Students' Multiplicative Concepts

Prior research has identified three qualitatively distinct multiplicative concepts, all of which are rooted in students' units-coordinating activity (Author, 2009; Steffe, 1992, 1994). We outline the second and third of these multiplicative concepts because we report on data related to students using each concept. We use the Candy Problem to outline the concepts:

> *Candy Problem:* Brandy has 3 packages of candy each containing 6 candies. How many candies does she have in all?

A student using the second multiplicative concept (MC2) has interiorized two levels of units, which enables her to strategically reason with sixes in solving the Candy Problem. For example, to solve the candy problem a student might reason that six and four is ten, and that two more is twelve

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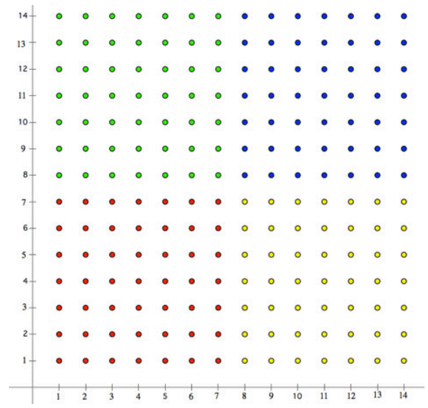

Figure 2: Array for the Digit Problem

and then finish the solution by reasoning that six more than twelve is eighteen. The ability to operate on each six by breaking it into, for example, four and two and strategically combining the parts is a hallmark of MC2 students because it is one indication that they are able to operate on a unit of six units. MC2 students are also able to produce three levels of units in activity. That is, once they have solved the Candy Problem they may regard the 18 candies as a unit of three units of six units, but they cannot use this three level of unit structure in further operating. For example, if these students were told that Brandy received 8 more packages, they could determine that this was 48 more candies, producing 48 as a unit of 8 units of 6 units. They could then unite 18 candies and 48 candies to determine there were 66 candies in all. However, determining that 66 candies was 11 packages would be a separate problem for an MC2 students because they would not retain the 18 candies as a unit of 3 units of 6 units and the 48 candies as a unit of 8 units of 6 units. This means that MC2 students are able to create three levels of units in activity, but are not able to operate on this third level of unit.

In contrast, students using the third multiplicative concept (MC3) have *interiorized* three levels of units. This means that an MC3 student would be able to determine that the 66 candies constituted 11 packages because the 18 candies would retain there status as a unit of 3 units of 6 units, and the 48 candies would retain there status as a unit of 8 units of 6 units. So when they united 18 and 48 they would establish the 66 candies as a unit of 11 units of 6 units.

Methods

The data for this research was drawn from clinical interviews (Clement, 2000) with 20 7th and 8th grade students who were attending an urban public school in a large Midwestern city during the 2014-2015 academic year. The school population, of which the study population mirrored, is approximately 69% African American/Black, 16% Hispanic/Latino, 13% White, and 2% multiracial, with slightly over 85% of students qualifying for free or reduced lunch. The study consisted of three hour-long interviews. The first interview was a selection interview and helped identify the multiplicative concept students were using as well as identified potential productive challenges in the domain of NLMM and NLG. Using the information gathered in the first interview, the second and third interview explored generalizations that students using each of the three different multiplicative concepts made. There were seven tasks in the selection interview protocol, however, tasks were altered based on the interactions with individual students in order to gain authentic insight into students' thinking processes (Clement, 2000). All interviews were video recorded and conducted in the presence of two researchers. Following the interviews, written notes were taken and discussion within the research team served as a form of data triangulation.

Results

We present two data excerpts from the first interview in order to illustrate how we identified what kind of generalizations would be productive for us to work on with MC2 and MC3 students

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during the second interview of the study. The first data excerpt is from an MC2 student, Keon, who was presented with the digit problem, which he solved and then created a seven by seven array. He was then asked how many new coordinate points there would be if he was given an eighth digit. The interviewer asked Keon to determine a solution to this problem without initially drawing the coordinate points on the array. This condition was presented by the interviewer in order to test the extent to which Keon could monitor the number of new coordinate points he created as he created them, and to see how he structured his creation of the new coordinate points. Moreover, the goal was to avoid having him put the points on the array and then count the number of points he had put in the array because such a solution had the potential to significantly simplify the problem because it would be less clear in this case if the points retained there status as two-dimensional coordinate points or if they were simply enumerated as units of one.

Excerpt 1: Keon's Solution to Adding a Single Digit

I: Would you say what all of them (new coordinate points) are so you can figure out how many there would be?
K: Eight, one; eight, two; eight, three; eight, four; eight, five; eight, six; eight, seven. Yeah. It's eight times seven, right?
….
I: Would there be any others?
K: Oh, and eight, eight.
I: Eight, eight. Yeah that is good. So how many total is that?
K: Just the new ones? It would be eight, right?
I: Okay. So you said eight, one; eight, two; eight, three...Are there any others you could get?
K: [sits in concentration for 5 seconds]: Oh, one, eight.
I: Mm-hmm.
K: Two, eight; three, eight; four, eight; five, eight; six, eight; seven, eight; and eight, eight.
I: Mm-hmm. So how many total new ones would there be?
K: It'd be sixteen, right? No. It'd be sixteen, right? Cause it'd be eight, one; eight, two; eight, three; eight, four; eight, five; eight, six; eight, seven; eight, eight [puts up each finger on his right hand and his thumb, index finger, and middle finger on his left hand as he says the pairs]. One, eight; two, eight; three, eight; four, eight; five, eight; six, eight; seven, eight; eight, eight [puts up his ring finger and pinky on his left hand, and then reuses the fingers on his right hand and the thumb on his left hand as he says each pair]. So sixteen.
I: Sixteen. Did you count any of them twice?
K: Huh?
I: Did you count any of them twice? Check and just see.
K: Count any of them twice?
I: Mm-hmm. Say them again, and just think about if you counted any of them twice.
K: Eight, one; eight, two; eight, three; eight, four; eight, five; eight, six; eight, seven; eight, eight.
I: Mm-hmm and what were the other ones you counted?
K: One, eight; two, eight; three, eight; four, eight; five, eight; six, eight; seven, eight. Oh, and then eight, eight, I can't say that one no more cause I already said it.
I: Yeah that is right.
K: So it'd be fifteen.

Keon's initial response in which he said aloud seven new coordinate points (eight, one through eight, seven), and then said, "It's eight times seven, right?" indicated that he anticipated that the problem would involve linear growth. That is, he anticipated that if he added one new digit this would produce seven new coordinate points because when he had created his seven by seven array each digit produced seven coordinate points. Upon questioning, he realized that eight could also go with itself, and that this meant he produced eight new coordinate points. Moreover, after he produced the coordinate point eight, eight, he had the insight (with the support of further questioning) that eight could also be the second digit in a coordinate point, and was able to state the rest of the coordinate points that he could create.

After he had stated all of the coordinate points, he thought that there would be sixteen coordinate points although was not positive, “It'd be sixteen, right? No.”, and so restated all of them again counting each coordinate point using his fingers as he said them, and concluded there would be a total of sixteen coordinate points. This portion of the data excerpt is of interest for two interrelated reasons: first, Keon was not totally certain about how many new coordinate points he had created and so stated them again in order to count them; and second, he double counted the eight, eight coordinate point, an issue that has been reported frequently in prior research on students’ two-dimensional reasoning (Battista, 2007). He was subsequently able to conclude that he had double counted the eight, eight coordinate point after questioning on the part of the interviewer, and re-stating all of the coordinate points again, to check to see which, if any, of them he had counted twice.

We account for these features of his solution by appealing to the levels of units coordination that are likely involved for a student to immediately conclude that, for example, he had created sixteen new pairs, and that this number of new pairs contained the coordinate point eight, eight twice. That is, our inference is that Keon created the first eight coordinate points as a unit of eight pairs in activity, and that the creation of such a unit structure is equivalent to establishing a three level of unit structure in activity because each of the coordinate points can be considered equivalent to a unit that contains two units (and so a unit of eight pairs is like a unit of eight units of two units). Keon could create a unit of eight pairs in activity but could not operate further with this unit structure; had he been able to operate further with this unit structure our inference is that he would have simply combined the first unit of eight pairs he created with the second unit of eight pairs he created to determine the total number of new coordinate points to be a unit of sixteen pairs without re-creating and counting each of the pairs. Additionally, our inference is that he would have identified, without re-creating all of the pairs, that each unit of eight pairs contained the coordinate point eight, eight.

Nonetheless, this data excerpt suggested to us that for Keon, and students like him, an appropriate and productive challenge would be considering the relationship between a change in the number of digits and the number of coordinate points as a single digit was added. Moreover, we identified that a productive challenge would be structuring the new coordinate points as seven new coordinate points that contained eight as the first digit *only*, seven new coordinate points that contained eight as the second digit *only*, and one new coordinate point that contained eight as both the first and second digit.

The second data excerpt is from an MC3 student, Armando, who was also presented with the digit problem, which he solved and then created a seven by seven array. He was then asked how many times the number of coordinate points he could make if he had twice as many digits. The goal of the interviewer was to have Armando use his array to quantitatively establish the relationship between the number of old and new coordinate points. Once Armando had accomplished this goal, then the interviewer wanted to determine whether Armando could symbolize the situation as 14 x 14 = (7 x 2) x (7 x 2) = 7^2 x 2^2 so that he could also use his symbolic statement to see why there were four times as many coordinate points.

Excerpt 2: Armando’s Solution for Doubling the Number of Digits

I: [A extends the axes of the array, and puts in the digits eight through fourteen on each axes] How many times more numbers are you going to have if you fill those all in?

A: Um...can I just write it out?
I: Well what are you going to do?
A: Multiply fourteen times fourteen.
I: Yeah okay so I want you to try to use your picture to figure it out.
A: Hmm, okay. How can you use the picture?
I: That is a good question. So like this would be like one time, right? [Points to the seven by seven array that is filled in].
A: Mm-hmm.
I: And how many were in here [circles the 7 x 7 part of the array]?
A: Forty-nine.
I: Are there going to be any equivalent sections that are going to be this size in your picture?
A: No ish. Wait yes.
I: Okay so make another one that is going to be the same size as this.
A: Hmm, I could do...hmm. I just got this small picture in my head in some way. So since there are seven xs over here [points to the digits eight through fourteen on the x-axis of the array] and going up is seven [points to the digits one through seven on the y-axis] all this area over here is actually also going to be forty-nine[draws the box in the lower right corner].
I: Okay good.
A: And the same thing with up here [puts a square in the upper right], and also over here [puts a square in the upper left]. So there is forty-nine four times.
…
I: What were you going to multiply when I said to use the picture?
A: Oh, I was going to multiply fourteen by fourteen.
I: Okay so will you write horizontally fourteen times fourteen. And that is equal to? How many times did you have seven in fourteen?
A: Once, er, what do you mean? Oh! Twice.
I: Twice here. So can you write fourteen as seven times. Fourteen is seven times what.
A: Equals seven times two [writes 7 x 2].
I: And what about on this side.
A: Um, same thing [writes 7 x 2 x 7 x 2, and then (7 x 2) x (7 x 2) so that he has written on his paper that 14 x 14 = (7 x 2) x (7 x 2)]
I: Do you see any number squared in your picture?
A: Ah, seven squared.
I: And then do you see some other number squared in your picture?
A: The other forty nines. Those would also be seven squared [writes $7^2 + 7^2 + 7^2 + 7^2$, and now has written on his paper that $14 \times 14 = (7 \times 2) \times (7 \times 2) = 7^2 + 7^2 + 7^2 + 7^2$].
…
I: And how many seven squareds did you say you have?
A: Four.
I: So rather than adding them. What could you do?
A: Seven squared times four [writes 4×7^2 and sets it equal to what he already has on his paper].
I: The four is that any number squared?
A: Two.
I: Do you see two squared somewhere in your picture, and if so where?
A: Would it be that [points to the digits one through four on the horizontal axis]?
I: Okay so say a little bit more.
A: Like four is two squared. Oh, since, um, eh. Since this right here would be four squared, I mean two squared [circles the four coordinate points in the bottom left of his array].

Our interpretation of this excerpt is that Armando established multiple three level of unit structures in his solution of the problem: he established the fourteen digits on each side of the array as a unit of 2 units of 7 units, and each of the four regions contained in the array as a unit of 7 units of 7 pairs. We make this interpretation because of how he operated to determine that the 14 by 14 array would contain forty-nine coordinate points four times, and the way he subsequently symbolized his reasoning. He first established that there would be a second region that would contain forty-nine coordinate points by identifying that the digits eight through fourteen on the x-axis could be paired with the digits one through seven on the y-axis, which he envisioned could create another region in the array that contained forty nine coordinate points. He then envisioned two other similar 7 by 7 regions. The fact that he could establish these regions without actually having to create any of the coordinate points provided initial indication that he could take these regions as something to operate with, and so was treating the forty-nine coordinate points as a unit of 7 units of 7 pairs.

The assertion that he was operating with three level of unit structures is also supported by how he symbolized the problem: he was able to see 14 x 14 as a product of 7 x 2 and 7 x 2. In particular, once the interviewer supported him to consider that 14 was equal to 7 times 2, he independently contributed writing 14 x 14 as equal to the product of 7 x 2 and 7 x 2. This way of symbolizing the problem provides indication that after he established fourteen as containing two sevens he could envision operating further with the two sevens by taking the product of two sevens and two sevens. It is important that he considered this to be a product, and not a sum, because considering it a product was essential for him to see that the product was equivalent to $7^2 + 7^2 + 7^2 + 7^2$: each of seven digits could be paired with seven other digits to produce a region that contained 7^2 coordinate points. Our contention is that to see two sevens and two sevens as a product required maintaining 14 as a unit of 2 units of 7 units because it entailed envisioning operating further with each seven, namely each of seven digits could be paired with another seven digits to make coordinate points without actually having to carry out these operations.

An interesting feature of Armando's solution is that he did not constitute the product as 2^2 x 7^2. The interviewer attempted to find out whether he saw the product in this way at first with an indirect question: "I: Do you see any number squared in your picture? A: Ah, seven squared. I: And then do you see some other number squared in your picture? A: The other forty nines. Those would also be seven squared." The interviewer intended the second question to be about whether Armando saw 2^2 in his picture, but Armando answered that what he saw were the other regions in the array that were 7^2. The interviewer then tried to ask Armando more directly about this issue once Armando had written that the array was equal to 4 x 7^2: "I: The four is that any number squared? A: Two." Computationally Armando was able to state that two squared was four, but when asked to identify where this would be in his picture he initially marked the digits from one to four on the x-axis, and then circled the four coordinate points in the lower left corner of his array—indicating that he did not see the four 7^2 sections of the array as 2^2.

Discussion

The initial interviews in the study have helped us to identify what kind of work is challenging for MC2 and MC3 students related to NLMM and NLG, and thus the kind of problems that are likely to produce interesting generalizations for these students. For MC2 students, the data suggests that problems in which students are asked to consider how the number of coordinate points changes as the number of digits is increased by one are likely to be challenging, but solvable. We think that the potential exists for these students to consider that, for example, $8^2 = 7^2 + 7 + 7 + 1$, and $9^2 = 8^2 + 8 + 8 + 1$, to understand these symbolic statements show that the new number of coordinate points is equal to the old number of coordinate points plus the change in coordinate points where the change in coordinate points is structured as 7, 7, and 1 based on classifying them according to which position the new digit appears in the new coordinate point, and to generalize this understanding.

For MC3 students, the data suggests that problems in which they are asked to consider how multiplicatively changing the number of digits changes the number of coordinate points are likely to be challenging, but solvable. In particular, the challenge appears to lie in seeing that, for example, a 14 by 14 array can be structured as 7^2 x 2^2. We conjecture that a problem like the following may support them to establish this way of seeing an array.

Two Color Digit Problem: You have a set of cards where the numerals 1 through 7 are written in blue. You have a second set of cards where the numerals 1 through 7 are written in orange. You select a card, replace it, and select a second card. What different color combinations could you get? What coordinate points could you get in each color combination?

We make this conjecture based on the fact that the problem includes two levels of pairing—pairing colors and pairing digits within a particular color combination. We make this conjecture because pairing digits seemed integral to Armando establishing the 49 coordinate points as 7^2. We expect that from this type of problem students will be able to generalize that given n colors and x digits in each color that $(nx)^2 = n^2x^2$. We will report on the generalizations that students made in these contexts as part of the presentation.

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UNDERSTANDING OF PLACE VALUE EXPLORED THROUGH NUMERICAL COMPARISON

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We present part of an investigation related to place value subject with first grade elementary school students (6 years old). Through this research, we look to analyze students' understanding of place value and difficulties evidenced during an experimental teaching that incorporates briefly other numerical bases and the use of manipulative material in a way that reflects the structure of our number system. For that purpose, we conducted individual interviews with students before and after experimental teaching. The results obtained in one of the tasks included in the interviews regarding numerical comparison show an improvement in understanding place value as a result of the experimental teaching.

Keywords: Number Concepts and Operations; Elementary School Education; Instructional Activities and Practices

Introduction

Place value is a fundamental mathematical idea. Several studies have demonstrated that there is a relationship between understanding place value and the arithmetic performance of students (among others, Moeller, Pixner, Zuber, Kaufmann, and Nuerk, 2011). However, it "is a highly sophisticated concept that is not really understood by many children even at the end of their primary schooling" (Thompson, 2000, p.292).

Based on the above, we are developing a research in which we approach the place value from the perspective of teaching and learning. In this article we focus on one of the objectives of this study: to describe what is the students' understanding of place value at the beginning and the end of an experimental teaching, for which we analyzed the results of one of the tasks of the interviews applied to students about numerical comparison.

Theoretical Framework

There are diverse factors associated to the difficulties of students of the place value understanding; one of them is the language. Miura, Okamoto, Kim, Steere and Fayol (1993) consider that understanding place value and mathematical performance of children may be influenced by the characteristics of their numerical language, in particular, by the level of correspondence between written and oral number system.

Another issue related to difficulties in understanding place value is the teaching itself. For example, Fuson (1990) observes that in usual teaching the multi-digit numbers are treated as linked single digits. So, "for most of the children, a number is a digit alignment" (Bednarz and Janvier, 1988, p. 300).

Considering the issues associated to difficulties presented in understanding place value, some of which we have mentioned before, several studies have proposed alternative ways for its teaching. In some of these studies, it has been taught the explicit number names (e.g., 1-ten 1, 1-ten 2, 1-ten 3 … 2-ten, Cotter, 2000). In others, it has turned to the game (e.g., Association of Independent Schools of South Australia, 2004) and to manipulative materials for teaching of the decimal number system and the place value.

Furthermore, in the teaching and learning of number system the introduction of different number bases, particularly small bases, represents diverse advantages according to Vergnaud (1991): the formation of groupings of second and third order does not represent a difficulty in comparison to the base 10 since the number of objects to handle is smaller and the number system rules are essentially the same in all bases, and so they will be built by the student regardless the number base.

Method

This study was conducted with a first grade group of twelve students (6 years old), in a public elementary school in Mexico City. In this group, we implemented an experimental teaching, consisted in 20 sessions of approximately 45 min each one, in order to benefit the understanding of decimal number system and place value.

The teaching sessions scene was an adaptation of *The Base Ten Game* presented by Pengelly (1991), consisting in throwing two dice, add the points, take the same number of wood sticks and place them on a board that contains place value columns. The game rule is that there can be no more than nine items in any column. So, once there are more than nine sticks in the unit column, ten sticks must be grouped and placed in the ten column (Association of Independent Schools of South Australia, 2004; Pengelly, 1991). The adaptation of this game consisted of a brief introduction of different number bases, as other investigators have done, by the advantages that this represents (Vergnaud, 1991).

Our main contribution to the game was the introduction of an abacus of own design with an unique peculiarity of two columns for the units order and two for the tens order (with ten beads each), to facilitate the exchange: ten for one or one for ten depending on the case; the hundreds order is limited to one column because it was expected that students understand numerals up to three digits. Furthermore, the first five beads of each column of this abacus have different color in order to facilitate the recognition of quantities (Cotter, 2000). This abacus was used by students in *The Base Ten Game*. Thus, when they had more than nine beads in the unit column, they had to trade ten (units) for one (ten).

Students who participated in this study were interviewed individually in order to explore their understanding of place value before and after the experimental teaching. In both interviews, the following tasks were presented: a) Cognitive representation of number (Miura et al, 1993), b) Digit-correspondence (Ross, 1986), c) Positional knowledge (Ross, 1986), d) Additions and subtractions and e) Numerical comparison.

Through numerical comparison, task that we focus in this report, we pretend to know if students consider the value of the digit according to its position when they compare numbers. In the initial interview, there were presented the next pair of numerals to the students on a sheet: 32 and 28, they were asked which was the biggest and why or how they knew it. This allowed us to know the strategy they used and their place value understanding. The same task was performed with the numerals 71 and 59, this time asking what was the smallest. In the final interview the following numerals 139 and 151, 198 and 231 were include, since it was expected that as a result of the experimental teaching students could transit from two-digit numerals to three-digit numerals.

Results

In the initial interviews, a third part of the students considered that 28 was bigger than 32 and more than half said that 71 was smaller than 59. In contrast, in the final interviews, almost all students (except two of them) offered correct answers when comparing the four pairs of numerals. The latter two children considered that 151 was smaller than 139, this comparison was the most difficult for children since hundreds digits were the same and they needed to pay attention to tens digits.

As mentioned before, in the interviews we also found out the reason of their answers, which allowed us to explore their understanding of place value before and after the experimental teaching.

First we approached the lack of arguments, which did not allow us to know if their answer was a random decision or what was the strategy employed.

In the initial interviews, almost half of the children did not provide any argument about why the numeral selected was the biggest in the case of 32 and 28. In the comparison of 71 and 59, a third part of the students did not offer argument about their choice of the smallest numeral. In these cases, when students were asked about why or how they knew it, some children did not respond and others simply said the word or number words associated with the numerals ("Because this is the 32 and this 28"). In contrast, in the final interviews only 2 of the 12 children interviewed gave no argument only in one of the four tasks.

The strategies used by children in the numerical comparison task, explicit through their arguments were organized according to the strategies proposed by Sinclair and Scheuer (1991): a) Ordinality-cardinality, b) Face value and c) Place value (Table 1).

a) Ordinality-cardinality. Within this category the students refer “to the number relation of the two numbers (cardinality) ... or to the position of these numbers in the counting sequence (ordinality)" according to Sinclair and Scheuer (1991, p. 208). Some of the arguments presented by the children in this category were: "Because 32 is farther away than 28"; "Because ... 71 is a lot and 59 is a little"; 3 of 24 responses corresponded to this category in the initial interview and 5 of 48 responses in the final interview.

b) Face value. Children grouped within this category generally focused only in one of the digits of numerals compared regardless of its position and all their answers were incorrect. Examples of the arguments offered by the children in this category were: "Because it has 1" (71 <59), "3 is less than 8" (28> 32), "This (151) is the smallest because this (139) has no another one" (151 <139). In initial interviews we found 8 answers within this category while in the final interviews we found only 1 answer.

Table 1: Numerical comparison strategies used by students

Strategy	32 & 28		71 & 59		139 & 151	198 & 231
	Initial	Final	Initial	Final	Final	Final
Without argument	5	0	4	0	2	0
Ordinality-cardinality	1	3	2	0	1	2
Face value	4	0	4	0	1	0
Place value	2	9	2	12	8	10

Note: Initial = Initial interview, Final = Final interview, Total students = 12.

c) Place value. In this category, there are children who considered the value of the digits according to its position in the numerical comparison. Generally, they focused in one of the positions and all their answers were correct. In this category is where we appreciate the most important changes resulting from the experimental teaching. In the initial interview, only 2 of the 12 children interviewed considered the value of the digits according to its position when they compared numbers. In contrast, in the final interview most of the children interviewed offered an argument that showed an understanding of place value (39 of 48 responses).

Some of the arguments offered for 32 and 28 (the biggest) were: "Because 30 is bigger than 20"; "... And it does not matter that this (8 of 28) is bigger but the first is larger (3 of 32) that this (2 of 28)"; “Because from 20 … it follows 30".

When comparing 71 and 59 (the smallest): "Because 5 ... ... is smaller than 7"; "Because this (71) takes 7 and this (59) takes 5”; “Because it has 1 (the child points 1 of 71) does not mean that it is smaller, this (59) although it has 9 does not mean it is big"; “And it doesn't matter that this (1 of 71) is. If only was 1 (the child covers 7 of 71 with his finger) then this (1) would be".

For 139 and 151 (the smallest), some of the arguments were: "... because it's 10, 20, 30 (139) and this (151) is 10, 20, 30, 40, 50, and they would be missing other two to be like this (151)”; "The two would be equal, wouldn´t they?, but here it changes ... because here is like they were just these (39 y 51, the child hides with her fingers the hundreds of both numerals) and this (39) is smaller".

In the last comparison, 198 and 231 (the biggest), some of the reasons given by the children were: "Because this takes 200 ... and this only takes 100". "And if these two were not here (98 of 198 and 31 of 231), who would win?” As we can see, within this category children's arguments reflect an understanding of place value, understanding that will favor their arithmetic performance as evidenced in previous studies (e.g., Moeller et al., 2011).

Conclusions

Based on the results of the numerical comparison task presented above, it is clear that the experimental teaching implemented benefit the students' understanding of place value. This research shows the benefits of implementing alternative approaches in teaching place value, including the use of manipulative materials in a way that reflects the structure of our decimal number system. It remains to analyze the rest of the tasks that were part of the interviews applied to students, which will allow us to comprehend better the development of understanding of place value and the difficulties presented by students.

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A LEARNING PROGRESSIONS APPROACH TO EARLY ALGEBRA RESEARCH AND PRACTICE

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We detail a learning progressions approach to early algebra research and how existing work around learning progressions and trajectories in mathematics and science education has informed our development of a four-component theoretical framework consisting of: a curricular progression of learning goals across big algebraic ideas; an instructional sequence of tasks based on objectives concerning content and algebraic thinking practices; assessments; and posited levels of sophistication in children's reasoning about algebraic concepts within big ideas of early algebra. This research balances the goals of longitudinal research on supporting students' preparedness for algebra while attending to the practical goals of establishing connections among curriculum, instruction, and student learning.

Keywords: Learning Trajectories; Curriculum; Algebra and Algebraic Thinking

Learning progressions and trajectories are currently receiving much attention in mathematics and science education, especially in advancing recommendations for standards, curriculum, assessment, and instruction (Daro, Mosher, & Corcoran, 2011). Some key issues within this domain of research include the use and meaning of terminology, methods of assessing sophistication in student thinking, and connections among curriculum, instruction, and student reasoning (Barrett & Battista, 2014; Ellis, Weber, & Lockwood, 2014). This paper addresses these issues, with particular attention to the ambiguous use of the term *learning progression*—"sometimes indicating developmental progressions, and at other times suggesting a sequence of instructional activities" (Clements & Sarama, 2014, p. 2). We take the stance that a learning progression includes both.

This research is situated within the Learning through an Early Algebra Progression (LEAP) project, which is grounded in a research agenda concerned with a fundamental question of how to prepare students in the elementary grades for success in middle grades algebra and beyond (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015). The LEAP project builds on Kaput's (2008) framework for early algebra in documenting changes in students' learning of both algebraic content and algebraic thinking practices over time. Our purpose is to elaborate a theoretical framework for an *Early Algebra Learning Progression* (EALP), making progress in a program of research whose aim is to support an integrated system of curriculum, instruction, and student learning in early algebra.

Theoretical Framework

The EALP advanced in this research includes four components: (1) a curricular progression of learning goals across five big ideas and corresponding core concepts, (2) a sequence of instructional tasks based on objectives for content and algebraic thinking practices across big ideas, (3) assessments and coding schemes for analyzing student strategies, and (4) levels of sophistication in children's thinking about core concepts in early algebra. See Table 1.

Table 1: A Theoretical Framework for an Early Algebra Learning Progression (EALP)

EALP	Description and Sub-Components
Curricular Progression	The foundation from which a sequence of instruction and hypothesized paths of student thinking were built and investigated over time: • Big ideas and algebraic thinking practices (cf. Kaput, 2008) • Core concepts (Battista, 2004) or constructs (cf. Shin, Stevens, Short, and Krajcik, 2009) • Learning goals (cf. Clements & Sarama, 2004) or claims (cf. Shin et al., 2009)
Instructional Sequence	A sequence of instructional materials and tasks designed to guide activity around the Curricular Progression: • Lessons and lesson objectives • Tasks (jumpstarts, problem-solving tasks)
Assessment Items	The primary means to measure students' understanding of concepts within big ideas (cf. Battista, 2004) of early algebra. All items have multiple entry points; "anchor items" appear in multiple grades to track growth over multiple years.
Levels of Sophistication in Children's Thinking	Levels represent qualitatively distinct ways of thinking, capturing patterns in students' thinking and reasoning. For a core concept a level includes a description of a way of understanding (which could be a misconception), evidence (i.e., responses to assessment items), and an assessment item or subset of items.

This work builds on several related perspectives across "progressions" research in both mathematics and science education, elaborated next across each dimension of the framework.

Curricular Progression

The multi-year scope of the LEAP project warrants attention to a continuum of levels of specificity in content across grades and within grades and lessons. Thus we define our *curricular progression* to encompass various grain sizes (from largest to smallest): big ideas, core concepts, and learning goals (or claims). The curricular progression establishes a foundation of targeted learning goals from which instruction, assessments, and levels of student thinking are based.

Big ideas are "key ideas that underlie numerous concepts and procedures across topics" (Baroody, Cibulskis, Lai, & Li, 2004, p. 24). Drawing from early algebra the big ideas of the EALP are: (a) equivalence, expressions, equations, and inequalities (EEEI), (b) generalized arithmetic, (c) functional thinking, (d) variable, and (e) proportional reasoning. The multi-year curricular progression is organized around these content strands and the *algebraic thinking practices* of generalizing, representing, justifying, and reasoning with mathematical relationships (Blanton et al., 2015; Kaput, 2008). A *core concept* is an idea critical to understanding a big idea. For the big idea of EEEI, a core concept is "The equal sign is used to represent the equivalence of two quantities or mathematical expressions." We take a *learning goal* (Clements & Sarama, 2014) to include claims about the nature of understandings or skills (Shin et al., 2009) expected of students regarding a concept. For example, in our work a learning goal for the big idea of EEEI is to "understand the equal sign as a relational (rather than operational) symbol," evidence of which is seen through students' actions in interpreting true/false and open equations.

Instructional Sequence

The EALP *instructional sequence* is defined to include a sequence of lessons that entail lesson objectives, jumpstarts, and problem-solving tasks designed to address both concepts and algebraic thinking practices. *Lessons* are defined as guides for an instructional intervention session (typically one 60-minute class period). *Lesson objectives* are defined as statements of targeted performances; they are derived from the curricular progression's learning goals and offer a systematic framework for designing or adapting tasks and allow for the revisiting and extending of algebraic ideas across the grades. Each lesson begins with a *jumpstart* designed to engage students in revisiting and strengthening their understanding of core concepts and algebraic thinking practices addressed in previous lessons. New concepts are introduced through *problem solving tasks*, or structured opportunities for students to build and extend understandings and practices around a goal-driven assignment. As an example for EEEI, students are asked to engage in tasks adapted from research (e.g., Carpenter, Franke, and Levi, 2003) that have been successful in supporting students' relational understanding of the equal sign.

Shin and colleagues (2009) note that learning progressions do not set forth a single, linear path to understanding, but a web of interconnected constructs within a big idea. We likewise acknowledge that our instructional sequence represents one possible path for supporting the development of algebra understanding, and different productive paths certainly exist.

Assessments

Written assessments for each of grades 3-7 were designed to elicit student reasoning across the big ideas and algebraic thinking practices. Assessment items were often adapted from those that had performed well in previous research (e.g., for equivalence items see Knuth, Stephens, McNeil, & Alibali, 2006) and were piloted and revised prior to administration. The assessments include several "anchor items" that appear in multiple grades to allow us to measure growth on the same item over multiple years. Assessment items offer multiple points of entry so that students at the very beginning of the progression as well as those more experienced in early algebra can demonstrate what they know regarding algebraic content and thinking practices. For example, the *True/False* task 57 + 22 = 58 + 21is posed across grades 3-5 to elicit understandings of the equal sign and equation structure and can be solved in multiple ways.

Levels of Sophistication in Children's Thinking

The final piece of our approach to learning progressions in early algebra research concerns the documentation of changes in students' learning over time. Levels of sophistication are "benchmarks of complex growth that represent distinct ways of thinking" (Clements & Sarama, 2014, p. 14). We initially conjectured levels of sophistication in student thinking based on extant empirical research on student conceptions, misconceptions, and difficulties. We then refined these after analyzing students' responses to assessment items (i.e., student strategies)across grades to discern patterns in children's thinking. Each level of sophistication represents a level of understanding as evidenced in their responses to one or more assessment task(s).

For example, the levels of sophistication we conjectured and observed for students' developing understanding of the equal sign range from Level 1's "Student has operational view of the equal sign and inflexible view of equation structure" to Level 5's "Student has advanced relational-structural understanding of the equal sign and flexible view of equation structure and can consider relationships across equations." We view the levels of sophistication identified in our work as dependent on the learning goals and sequence of tasks that drive the intervention.

Conclusion

The large-scale nature of the LEAP project and our desire to speak to both research and practitioner audiences led to practical decisions about our theoretical frame. This included clearly stated objectives and assessment tasks that integrate algebraic thinking across several grades. We also integrate several perspectives across learning progressions and learning trajectories. Science education literature on learning progressions (e.g., Shin et al., 2009) provided an initial frame for coordinating disciplinary and research-based perspectives on student thinking. It also led to our organizing the content of our EALP according to big ideas, core concepts, and claims. In mathematics education, our EALP parallels Battista's (2004) emphasis on connections among core concepts, assessment items, and levels of sophistication. We also emphasize that the observed levels of sophistication in student strategies are inseparable from the curricular and instructional context in which the learning was supported, yet given the large scale scope of the LEAP project, these connections are not as tightly linked as in some learning trajectories research (cf. Clements & Sarama, 2004).

Our continued research on a comprehensive approach to curriculum, instruction, and student learning is important work to share with the research community towards the goal of coordinating efforts to promote effective early algebra education and identifying important milestones in students' thinking. We also feel it is important that this work be available in a practical form for teachers (e.g., lesson plans and professional development) as they engage in the day-to-day and year-to-year work of developing students' algebraic reasoning. A feasible future direction of this work is to more closely examine paths of students' thinking across grades and in turn, to posit tighter links between tasks and instructional strategies that could be productive in supporting students' engagement in more sophisticated ways of reasoning.

Acknowledgments

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STU'S INITIAL AND EVOLVING CONCEPTIONS OF UNIT FRACTIONS

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We illustrate conceptions of fractional quantity, evidenced through problem solving strategies and observable operations, of a fifth grade child with learning disabilities (LD) before, during, and after seven instructional sessions situated in equal sharing. The child's work documents levels of conception reflective of a trajectory. Implications for future research are discussed.

Keywords: Cognition; Rational Numbers; Learning Trajectories; Equity and Diversity

Much of the intervention literature in special education reports children's responses to strategies or representations imposed through the teacher's instruction as opposed conceptual development within children's thinking (Hiebert & Grouws, 2007). Because conceptions cannot simply be imposed onto children (Baroody, Cibulskis, Lai, & Li, 2004), a depiction of the informal notions of fractions children with learning disabilities (LD) *do* possess along with examples of how varying idiosyncratic advances in conceptions may occur is needed to conceptualize possible teaching and learning environments. Trajectories (Daro, Mosher, & Corcoran, 2011; Simon, 1995) model critical transitions in how a child conceives of a mathematical idea along a series of dynamic (i.e., changeable) tasks to elicit cognitive dissonance. Documenting trajectories of children with LD is an important initial step to address the needs of a historically marginalized population in school mathematics. Accordingly, we explored the following research question: What conceptions of fractional quantity, observable through problem solving strategies and operations does a fifth grade child with LD display before, during, and after tutoring situated in equal sharing?

Conceptual Framework

Problem solving strategies, language, and observable mental operations within equal sharing tasks provide evidence of a child's conceptions. We synthesized existing research (e.g., Empson & Levi, 2011; Steffe & Olive, 2010; Tzur, 1999) on children without LD and used it as a framework for the study. Initially, children may not view the whole as divisible. A child sharing, for instance, three sticks of clay among four people may add to the quantity to be shared (i.e., adding a fourth stick) or create unequal shares (e.g., one of the four people does not receive a share). Once the child accepts the divisible whole, they employ skip counting or knowledge of halving to create enough shares for each person. Yet, the child does not yet determine the number of parts needed to exhaust the wholes before their activity because they do not relate their partitioning plan with the number of sharers. Through their activity, the child begins to develop a notion of equality and size of the parts relative to each other but does not yet unitize one whole when they conceive of the size of the quantity they create (Steffe & Olive, 2010). The child views the unit fractions as a number of pieces whose magnitude, to the child, may or may not be equivalent to each other but not yet be different from that of the whole. The exception is the case of one-half, for which the child has strong informal notions.

As children's conceptions of fractional quantity advance, they begin to use the number of sharers as an a priori plan to create a predetermined number of parts. Their partitioning activity is no longer counting based partitioning in activity but a mentally planned relation supported by an interiorized unitization of the whole. The child now considers one whole as a quantifying unit (i.e., Part whole scheme) and knows that the parts they create must exhaust each whole. A flexible understanding of composite units (e.g., four is four units of one or one unit of four) may aid in the child's a priori partitioning plan (Olive, 2001; Steffe & Olive, 2010). A partitive unit fraction conception and use of an iterating mechanism to prove parts equal and related to the whole is surmised to underscore developing multiplicative notions. Over time, children's mental records concerning the equal sharing division becomes distributive (3 clay sticks shared by 4 people is now, to the child, "3 ÷ 4", or (1 ÷ 4) + (1 ÷ 4) + (1 ÷ 4)) and independent of modeled activity. Anticipatory strategies reflect the child's mental recall of the relationship between a value of fraction (e.g. three-fourths of a whole unit) and dividing the numerator by the denominator (e.g., 3 divided by 4).

Method

The data used in the current study focuses on the case of a fifth grader who we refer to as Stu. Stu (age = 12 years) attended elementary school in the Northwestern United States. He was purposively chosen to participate in the teaching experiment because he was identified by his school system as LD in mathematics and by his teacher as needing additional instruction in fraction concepts and operations. Stu was diagnosed as LD by subtracting his score on an academic achievement test from his standard intelligence quotient. Stu has documented, sustained low achievement in mathematics and a processing disorder documented by the school.

Teaching Experiment

Data collection was facilitated through a teaching experiment (Steffe & Thompson, 2000). We worked with Stu one-on-one in seven 40-minute tutoring sessions between mid February and late April. Sessions were held during school hours and were in addition to the child's regular mathematics class time. Researchers collected three sources of data in the teaching experiment: a) transcribed video-recordings, b) the child's written work, and c) observation field notes. The first and second researcher attended all tutoring sessions and collaborated throughout the ongoing analysis of teaching episodes. The first author was the researcher-teacher. The second author observed the interaction between the researcher-teacher to provide an outsider's prospective during on-going analysis and deter ineffective interactions from continuing.

Task trajectory. We designed an initial sequence of problem tasks and possible teaching moves (i.e., questions) based on the theoretical framework of how children with LD might advance their strategies, language, and operations. Tasks were planned to be dynamic (i.e., adaptable to the child's current conceptions), situated in equal sharing (e.g., 2 people share 5 items; 7 people share 4 items), and presented to Stu in realistic contexts that we changed according to his preference. In each task, the number of sharers ranged from two to ten and the number of objects shared ranged from three to 13. The problem solving tasks were designed so that Stu could use a variety of strategies and representations to reason about the mathematics.

Analysis

Ongoing analysis of events that appeared critical (Powell, Francisco, & Maher, 2003) for the child's thinking and learning were noted and discussed before and after each session. The focus was on generating (and documenting) initial hypotheses as to what conceptions could underlie the child's apparent problem solving strategies during these critical events. These hypotheses led to (a) planning the following teaching episode. We used retrospective analysis after all data were collected to delineate Stu's informal conceptions of fractional quantities, how his strategies and language shifted during each tutoring session, and what his conceptions were during the final session. Finally, identified possible indicators of Stu's conceptual growth using the constant comparison approach

(Leech & Onwuegbuzie, 2007). The first and second author read through the initial clinical interview and the first two tutoring sessions as a team. We coded the child's evidenced problem solving strategy by task and coded any associated apparent operations and language employed by the child within problem solving strategy by task. We compared each set of codes generated for each problem solution and task with previously generated coded data to ensure consistency. This led to the creation of an initial codebook. We then independently coded two more sessions using the initial codebook. Codes were then compared using peer debriefing and collaborative work. Researchers used the amended codebook to independently code all remaining sessions (Intercoder reliability = 95%).

Selected Results

Initial Conceptions: Level 1

In the initial clinical interview, the child had just solved a task involving 12 cookies and four people by directly modeling 12 cookies with unifix cubes, dealing out each cookie into four piles until he arrived at a solution of three cookies. The excerpt below depicts an extension of the situation to involve a 13th cookie ("I" stands for the researcher- teacher instructor, "S" for Stu, and italicized text in parentheses indicates actions or gestures).

S: There would be one left over. One remaining (smiles).

I: What if we didn't want it to remain? Suppose they wanted to eat that one, too.

S: All of them? (Long pause) OK. (Deals out 3 unifix cubes, one at a time, into four piles) So we can take this little cookie (grabs a 13th unifix cube) and this equals… four of them? (Grabs four unit cubes, stacks them, and compares the size to the Unifix cube) But oh…no no. It doesn't work like that. We need…another four?

I: Another four?

S: To make it the same size!

I: Oh, I see. That's pretty clever.

S: So we did three to this one, three to this one, three to this one, three to this one. And they want more (deals out the smaller unit cubes, one at a time, to each of the four piles until none remain). Two small pieces of cookie with two big pieces for each one. And it's called three and a half two.

In Level 1, Stu (a) creates fractions in his activity yet does so reluctantly and (c) does not yet employ a plan for creating a known number of total pieces across the wholes. Interestingly, Stu seems to use a ratio like correspondence (i.e., "trade in" the larger Unifix cube for some number of smaller unit cubes as opposed to a halving strategy, such as partitioning through repeated halving) to share the leftover cookie. Yet, Stu uses the correspondence in a manner that reflects his focus on the size of the objects versus the creation of an anticipated number of parts. He also evidences a guess and check conception rather then one supported by a planned number of parts in the whole.

Level 4 Conceptions

In later sessions, we exclusively utilized tasks that result in a non-unit fractional quantity less than one. Stu's thinking in previous tasks showed his negotiation of the total parts within one whole through his partitioning plan (as opposed to across wholes) along with his increased attention on the necessity of the parts being equal with respect to each other. Stu also quantified the equal share in terms of one whole with language and was beginning to attach this language to symbolic representations that he may have gleaned from previous school based experiences. The following excerpt shows Stu's first thinking in two tasks from the final sessions involving five sandwiches and six sharers.

S: You just start with one and break it into six (draws a rectangular shaped bar. Draws a part within a bar; looks at the size of the piece he creates. Pauses, and then creates next piece methodically. Erases; redraws the part until he is satisfied they are equal. Repeats process until he has six parts exhausting the first whole) There.
I: How many more bars will you draw?
S: Imagine there are more. From each, they would get…one, one, one, one, and one (puts a dot on the board for each "sandwich"). Six-sixths in each!
I: Ok. If I was one of the sharers, what I would get?
S: Well, if there was one sandwich you'd get one- sixth.
I: Oh ok. So we can write one-sixth over here for one sandwich?
S: Yep (writes "1/6"…process continues to "5/6").
I: OK, so how much does he get all together?
S: Five. Five parts of six.

In Level 4, Stu (a) comes into the sharing situation with a definitive plan for how each item will be shared before enacting activity, (b) seems to base his plan in sharing one item and then projects his plan across the rest of the items to be shared, and (c) verbalizes previous directly modeled thinking. Stu's increased need to create equal pieces through his correspondence-based partitioning activity seems to fall away it is not observable to us now because he is relying on verbal actions to engage in the activity.

Discussion

It is noteworthy that Stu's use of correspondence seemed to be instrumental to his ability to conceive of a number of parts within one whole prior to activity. It is possible that children with LD rely on correspondence as a mechanism to through which to build and solidify notions of unit fractions as opposed a halving mechanism. More research is needed to test this assertion. The case of Stu also provides preliminary evidence children with LD evidence similar understandings of children without disabilities as they grow in their conceptions of fractional quantity (Empson & Levi, 2011; Steffe & Olive, 2010). We hypothesize such pathways may show again in future research with more children labeled as LD and advocate for further inquiries in intervention research.

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UNDERSTANDING ISSUES OF QUANTITY THROUGH COMPARISONS: MATH LEARNING DISABILITIES AND FRACTIONS

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Although many students struggle with fractions, students with math learning disabilities (MLDs) experience pervasive difficulties because of neurological differences in how they process numerical information. When comparing fractional quantities, students with MLDs make errors on the easiest fraction comparison problems and these errors persist over the years. To investigate the origin of these difficulties I conducted a detailed analysis of videotaped tutoring data with two students with an MLD. Their explanations revealed that both students relied upon atypical understandings of fractional quantities, which may help explain the unique and persistent error patterns identified in students with MLDs.

Keywords: Rational Numbers; Cognition

Many students struggle with fractions, but not all students struggle for the same reasons. For students with math learning disabilities (MLDs) their difficulties stem from a cognitive origin, specifically difficulties processing quantitative information (Butterworth, 2010). Because of these cognitive differences, students with MLDs make qualitatively different kinds of errors than their peers. For example, Mazzocco Myers, Lewis, Hanich, and Murphy (2013) determined that students with MLDs make errors on the *easiest* fraction comparison problems (i.e., comparisons of fractions with the same denominators and comparisons involving the fraction 1/2). Although these error patterns differentiate the performance of students with MLDs from their peers, it remains unclear what the origin of these error patterns are and *why* these qualitative differences emerge and persist over time.

To begin exploring what underlies these unique patterns of error, I conducted a secondary analysis of two students with MLDs. The case studies were originally conducted as part of larger study that investigated MLDs during students' attempt to learn basic fraction concepts during one-on-one videotaped tutoring sessions. This study addresses the following research questions:

Do the case study students demonstrate errors similar to those documented in Mazzocco et al., (2013), specifically difficulties comparing fractions with the same denominator and comparing fractions with 1/2?

What do the students' explanations reveal about their reasoning and how does this relate to their pattern of errors on comparison and non-comparison problems?

Theoretical Framework

In this study, I depart from the predominant deficit approach used to conceptualize MLDs (e.g., Geary, 2010) and rely upon a cognitive *difference* model proposed by Vygotsky. From this perspective, a child with a disability "is not simply a child less developed than his peers, but is a child who has developed differently" (Vygotsky 1929/1993, p. 30). Students with disabilities develop differently because sociocultural (i.e., mediational) tools that have evolved over the course of human history may be incompatible with the student's biological development. For example, for students with MLDs, standard mathematical mediational tools (e.g., numerals, drawings, manipulatives), which support the development of typically developing students, may be inaccessible due to incompatibilities with how students with MLDs cognitively process numerical information.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

These incompatibilities may result in a student developing atypical understandings of standard mediational tools.

Methods

Data collected for this study were used for two purposes: determination of the student's MLD status and case study analysis (see Figure 1 for an overview of the design of the study). Out of the 11 students with potential MLDs who were initially recruited, only two students, "Emily" (a White 18-year-old recent high school graduate) and "Lisa" (a White, 19-year-old community college student), met all the qualifications for having an MLD. Both students demonstrated (1) persistent low mathematics achievement, (2) no social or environmental factors that could explain their low mathematics achievement, and (3) a lack of response to a tutoring protocol that had been effective for typically achieving students (see Lewis, 2014 for more details).

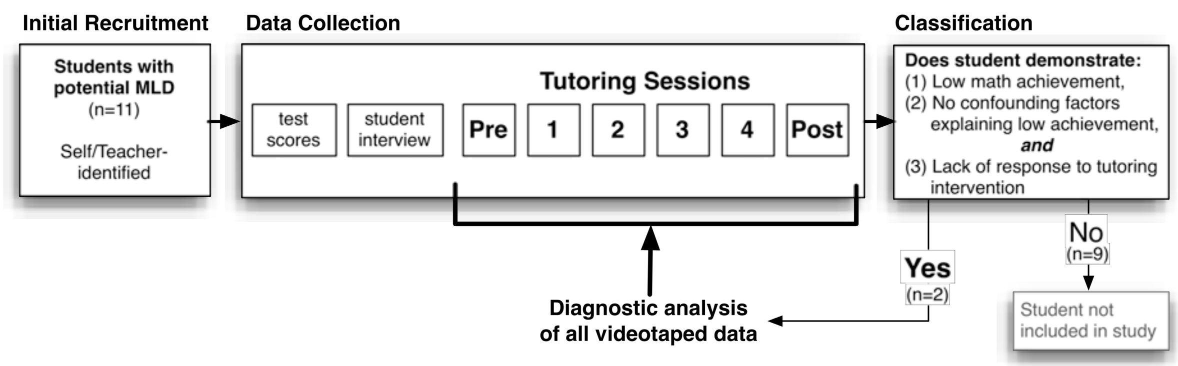


Figure 1: Schematic overview of methods used in this study

Analytic Approach

The videotaped tutoring sessions were transcribed, and parsed into individual problem instances. In addition to evaluating the errors on comparison problems throughout the tutoring sessions, a systematic grounded analysis (Glaser & Strauss, 1967) of the data was conducted, which revealed persistent atypical understandings of fractional quantity (Lewis, 2014). These atypical understandings provide insight into the nature of the errors made on comparison problems and will be discussed in the results section.

Results

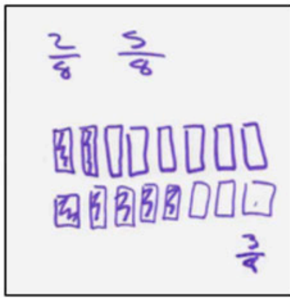
The results of this study are presented in two parts. First, the students' performance on comparison problems are considered to evaluate whether Emily and Lisa experienced the same error patterns noted in the Mazzocco et al. (2013) study. Second, exemplar excerpts are presented for each kind of problem to highlight the ways in which the students incorrectly reasoned about (1) comparisons of fractions with the same denominators and (2) comparisons involving the fraction one-half. For each of these problem types, I consider how these kinds of atypical understanding were supported by the systematic analysis of the data.

Error Analysis

An evaluation of comparison problems was used to establish that Emily and Lisa demonstrated a similar pattern of errors to those documented in the Mazzocco et al., 2013 study. Emily answered 63% of all same denominator problems incorrectly and 37% of all problems involving 1/2 incorrectly. Similarly, Lisa answered 14% of comparison problems with the same denominator incorrectly and 76% of problems involving 1/2 incorrectly. At the time of the post-test both students answered same denominator comparison problems and one-half comparison problems incorrectly. Given the high percentage of errors made on these problems and the continued evidence of errors during the posttest, Emily and Lisa's performance was judged to be consistent with the findings of the Mazzocco et al. (2013) study. To understand why the students answered these questions incorrectly I consider the student's solution process and explanation when solving comparison problems. The examples presented highlight the atypical understandings that the students relied upon when reasoning about fractional quantities both during comparison and non-comparison problems.

Same Denominator Comparison Problems

During the pretest Emily was asked to compare the fractions 2/8 and 5/8. To solve the problem she correctly drew a representation of both fractions. However, once she drew these two representations, she began interpreting 5/8 as 3/8 and attending to the *non-shaded* pieces (relevant transcript bolded in Figure 2).



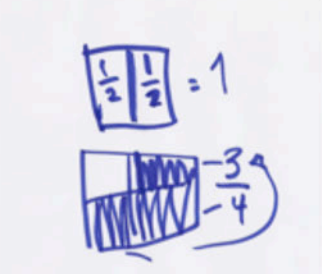
Tutor: What if we had the problem, two-eighths and five-eighths, which one would be bigger there?
Emily: *(writes 2/8 and 5/8. Draws 8 rectangles, shades in two. Draws 8 rectangles, shades in 5)* So, this is... *(points to the 5 shaded pieces in drawing of 5/8. Points to the* ***3 non-shaded*** *pieces of drawing of 5/8)* Um. *(writes* ***3/8.*** *Points to the* ***6 non-shaded*** *pieces in the drawing of 2/8.)* I don't know.

Figure 2: Scanned artifact of Emily's solution to the comparison of 2/8 and 5/8

Although Emily correctly represented both fractions, these drawings did not support her comparison of the fractional amounts. Once drawn, Emily shifted from attending to the 5 shaded pieces to attending to the 3 non-shaded pieces for 5/8 and interpreted her drawn representation as 3/8 – the fractional complement. In addition, Emily also attended to the six non-shaded pieces of 2/8 by pointing to each of the pieces in turn, before determining she did not know how to answer the question. What this excerpt reveals is not that Emily had difficulty comparing 2/8 and 5/8, but that when presented with the fraction 2/8 and 5/8, she was unsure whether she should be comparing the shaded pieces (representing the fractional quantity) or the non-shaded pieces (representing the fractional complement). As in the above example, the tendency to attend to the *fractional complement* was evident in both Emily's and Lisa's sessions in comparison and *non*-comparison problems. The students' attention to the fractional complement provides a potential explanation for why both students would make errors on fraction comparison problems involving the same denominator.

One-Half Comparison Problems

During the first tutoring session Lisa was asked to compare the fractions 1/2 and 3/4. She incorrectly determined that 1/2 was larger than 3/4, and justified her answer by explaining that 1/2 had larger pieces than 3/4 (see Figure 3).



Tutor: So if we were comparing three-fourths and one-half. Which one of those is going to be bigger?
Lisa: The half.
Tutor: Ok, and how do you know?
Lisa: Because it's larger, because it's closer to a whole. You have to split this (pointing to 3/4) up four ways to create a whole. Whereas the half only needs to get split into two.

Figure 3: Transcript and written work for Lisa's solution to the comparison of 1/2 and 3/4

Lisa justified her determination that 1/2 was larger than 3/4, focusing exclusively on the number of partitions. As in this example, the fraction 1/2 was not understood as a quantity but as the splitting into two pieces (i.e., "halving"). Both Lisa and Emily understood the fraction 1/2 as the act of splitting something in two, rather than the quantity 1/2. This "halving" understanding was evident at the time of the posttest for both students. The persistence of the halving understanding for both Emily and Lisa suggests why these students would make errors on fraction comparison problems involving the fraction 1/2.

Conclusion

Emily and Lisa experienced unusual difficulties comparing fractions with the same denominators and comparing fractions to 1/2, which were consistent with the errors documented in Mazzocco et al. (2013). The students' explanations revealed that they understood standard representations of fractions in atypical ways. These patterns of reasoning persisted throughout the tutoring sessions and were problematic for non-comparison problems as well. As in Emily and Lisa's cases, it may be an atypical understanding of fractional quantity may underlie the fraction comparisons errors documented in students with MLDs. Therefore, investigations of students' atypical understandings of quantity may be a productive avenue to consider in future studies of MLDs.

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FLIPPING NUMBERS AND TURNING ARRAYS: STUDENTS' JUSTIFICATIONS AND CONCEPTIONS OF THE COMMUTATIVE PROPERTY OF MULTIPLICATION

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This interview study explores how 19 students from grades 4 through 12 attempt to justify the commutative property of multiplication (CPM). Harel and Sowder's (1998) taxonomy of proof schemes is used as a general framework for interpreting students' justifications. Students showed evidence of symbolic, authoritative, empirical, quasi-transformational, and transformational proof schemes. An important relationship was noted between students' ability to articulate why it makes sense to multiply to enumerate the number of objects in an array, and the production of a transformational justification of the commutative property. Two types of conceptions of the commutative property emerged from students' justifications: syntactic and structural conceptions.

Keywords: Number Concepts and Operations; Reasoning and Proof

Introduction

According to the Common Core State Standards for Mathematics (CCSS-M), beginning in the third grade students should "understand the properties of multiplication" including the commutative property (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010, p. 23). It is relatively simple to describe the commutative property of multiplication (CPM): $a \times b = b \times a$. It is less clear, however, what it means to "understand" it. The CCSS-M, in fact, do not specify what it means to understand the commutative property, nor how that understanding is to be attained, though the writers note that "one hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity *why* a particular mathematical statement is true or where a mathematical rule comes from" (p. 4). The purpose of this study, then, is to explore how students justify the commutative property of multiplication. Identifying the ways in which students justify the commutative property can contribute to our knowledge of how children use and understand multiplication.

This study draws from research into several aspects of children's understanding of multiplication, including the classification and relative difficulty of different problem types (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Vergnaud, 1983), the strategies children use to solve multiplication problems (Ambrose, Baek, & Carpenter, 2003; Carpenter et al., 1999), and the schemes of action and operation that underlie children's strategies (Confrey, 1994; Steffe & Cobb, 1998; Steffe, 1988, 1994).

Several previous studies specifically investigated children's use and understanding of the CPM. Two studies found that populations who do not study multiplication in school (e.g., indigenous groups in Africa and Brazilian street sellers) are unlikely to apply the CPM, while those who do study multiplication in school routinely apply it (Petitto & Ginsburg, 1982; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998). Various studies found that young children generally do not apply the CPM when solving contextual problems in which the multiplier and multiplicand are clearly defined, though they may apply it to number-only problems (Ambrose et al., 2003; Baek, 2007; Carpenter et al., 1999; Nunes & Bryant, 1995; Vergnaud, 1983). Some authors suggest that children may find it more intuitive to apply the CPM to problems involving an array context (Ambrose et al., 2003; Carpenter et al., 1999; Nunes & Bryant, 1995). Battista and colleagues, however, caution that children do not initially see arrays as multiplicative structures; this is something they must construct over time (Battista, Clements, Arnoff, Battista, & Van Auken Borrow,

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

1998). Finally, two studies describe some of the justifications of the CPM produced by children in grades 3 and 6. Children justified the property by citing a rule, generating examples, and using array-based justifications (Bastable & Schifter, 2008; Valentine, Carpenter, & Pligge, 2005).

Method

Nineteen students spanning grades 4 through 12 from a rural public school district were interviewed in order to learn more about how students justify the CPM. The interviews were semi-structured, (Bernard, 1988). During the first two segments of the interview, students solved a range of contextual and number-only multiplication problems and described their strategies. Contextual problems were of two main types: (a) problems in which the multiplier and multiplicand were clearly defined, and (b) problems involving rectangular arrays and area. Number-only problems varied in difficulty, but all were intended to be solvable mentally. During the final segment of the interview, students were asked to justify the CPM in reference to the times they used the property while solving problems from earlier interview segments. The interviews were video recorded and students' written work was collected. The videos were transcribed, and transcripts were analyzed to identify how students justified the CPM. Harel and Sowder's (1998) taxonomy of proof schemes was used as an initial framework for categorizing students' justifications of the CPM. A process of open coding was used to identify subcategories, although this process was also informed by the research literature (e.g., Bastable & Schifter, 2008; Nunes & Bryant, 1995; Valentine et al., 2005).

Results

Students articulated numerous justifications of the CPM that can be broadly categorized as authoritarian, symbolic, and empirical using Harel and Sowder's (1998) taxonomy of proof schemes. Students' most interesting attempts to justify the commutative property, though, are—or come close to being—examples of what Harel and Sowder (1998) call transformational proof schemes. Transformational proof schemes are characterized by goal-oriented operations on objects and anticipations of the operations' results. These schemes are also characterized by considerations of generality. Some of the justifications for the CPM were classified as "quasi-transformational" because they (a) involve reasoning that goes beyond external and empirical proof schemes, and (b) share some of the characteristics of transformational proof schemes, such as transformations of mental objects, and considerations of generality; yet, they fall short of demonstrating the logical necessity of the CPM. The quasi-transformational and transformational justifications are the most interesting because they reveal the most about how children's conceptions of multiplication and commutativity.

Quasi-transformational justifications included (a) a balancing conception (e.g., 5 groups of 7 is the same as 7 groups of 5 because with 5 groups, there are fewer groups and more in a group, while with 7 groups, there are more groups, but fewer in each group); (b) a justification based on equivalent partial products (e.g., $8\times13=13\times8$ because both are 80+24); and (c) a partition model (e.g., $9\times6=6\times9$ because 54 can be partitioned into 6 segments of 9, or into 9 segments of 6). Transformational justifications included array-based justifications and justifications based on a rectangular area model. A significant relationship was noted between students' ability to articulate why it makes sense to multiply to enumerate the number of objects in an array, and the production of an array-based justification of the CPM. If students were not able to see the rows and columns of the array as iterable, composite units, they were not able to articulate how an array could represent both $a\times b$ and $b\times a$.

Discussion

Based on students' justifications of the CPM, there appear to be two types of conceptions of the CPM reflected in students' uses and justifications of the CPM: *syntactic* and *structural*. Students holding a syntactic conception of the CPM offered authoritarian and empirical justifications of the CPM. For these students, the property is a rule dictating how the symbols in an expression can be manipulated while still maintaining equivalence. There are two parts to the syntax: you can switch the order of the factors, and you get the same answer either way. This conception can sound like just restating the CPM in students' own words. But for some students, it is an explanation as well. This may be a symptom of the students' beliefs about the nature of mathematics. They may see mathematics simply as a collection of rules to be followed, which tell them what they can and cannot do when manipulating mathematical objects.

Structural conceptions of the CPM draw on mental imagery connected to students' conceptions of multiplication. Additionally, structural conceptions of the property incorporate the "logical necessity" (Simon, 2011) of the numerical equivalence of the two products. Students who produced array-based justifications of the commutative property clearly hold a structural conception of the property. Justifications based on an area model are also evidence of a structural understanding of the property, provided that the relationship between multiplication and area is well understood. Though no students in this study articulated a rearrangement justification of the property, this would be another example of a structural conception of the property, based in this case on equal groups imagery. Ethan (6), who used a partition model to justify the commutative property, seems to have a somewhat structural conception of the property. His justification draws on imagery connected to his conception of multiplication, but does not include the logical necessity that two products are equivalent. Justifications based on a balancing idea, though not entirely structural because of their qualitative nature, may be initial steps toward a structural understanding of the property, as they are based on equal groups imagery. With appropriate experiences, students who produced a balancing justification of the commutative property might be able to construct a rearrangement conception of the property.

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A COGNITIVE SCHEME THAT EMERGED FROM AN ALGEBRA CLASSROOM TEACHING EXPERIMENT

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This study investigated the realized learning trajectory that emerged as sixth-grade students participated in an algebra whole class teaching experiment. A learning trajectory evolved based on the students' changing interpretations of a variable. Cognitive schemes emerged as students' understanding of expressions and equations became more sophisticated. The first phase of the learning trajectory that transpired is explored. This phase, involving a Variable-as-Label Scheme, is described. Specifically, students' initial misconception of a variable is discussed, including conditions, the focus of the classroom activity, forms of reasoning, key mechanisms that shifted student thinking, and type of thinking.

Keywords: Algebra and Algebraic Thinking; Design Experiments; Learning Trajectories

Theoretical Framework

A hypothetical learning trajectory (HLT) (Simon, 1995) is a model of student learning that consists of the goal for students' learning, the tasks that will be used to promote students' learning, and hypotheses about the process of this learning. Reconceptualizing a mathematics concept to lead to student learning and understanding is challenging (Simon & Tzur, 2004). For instance, development of an HLT requires the teacher to have a solid understanding of the mathematics content and the current knowledge of the students in order to make hypotheses about the process of student learning and to select learning tasks based on these hypotheses.

An HLT must be developed, tested, and refined to produce a realized learning trajectory (RLT) (Lamberg & Middleton, 2009). The RLT results from the HLT and shows the actual learning/thinking that took place in the classroom as the lessons were implemented. The RLT explains what happened, in other words, describes the sense making that took place. In addition, it documents "why and how" it happened. Once an RLT is established, it can then become the HLT and be retested and redeveloped to find a new and improved RLT.

The actual path that students take might differ from the HLT and can be described by schemes used to designate students' cognitive structures (Norton & McCloskey, 2008; Steffe, 2004; Steffe, Cobb, & von Glasersfeld, 1988). These schemes are constructed to explain and document *mechanisms* for supporting shifts in student thinking (Lamberg & Middleton, 2009). This paper reports on the first phase of the RLT that emerged.

Learning Algebra

Research on algebra shows that students have difficulty interpreting letters as variables and studies have focused on how students learn to represent values using variables (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; NRC, 2001). Students learning algebra must move from arithmetic problems to symbolic representations of relationships with variables (Moseley & Brenner, 2009), requiring students to adapt their prior knowledge to new experiences. This requires the integration of symbols used in arithmetic (e.g., +, -, x, •, ÷, and =) to be used in the transition to algebra with variable expressions. Once students learn to work with variables without thinking about the numbers that the variable might represent, they have achieved manipulation of "opaque formalisms" (Kaput, 1995, p. 8).

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Variables can represent different situations and therefore can be interpreted by students in different ways. For example, a variable can be interpreted as a *changing quantity* such as varying prices of a specific item in different stores. Or, it can be interpreted as an *exact value* such as the price of a specific item in one store. With experience, it becomes clear to algebra learners that these two interpretations are actually representing the standard definition of a variable as an unknown quantity that can change; however, we found that this is not initially the case. Clearly, learning the meaning behind symbols and variables is essential for students to become proficient in algebra. Kaput (1995) found that many students view algebra as "little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test" (p. 4). Thus, students must find meaning in algebra not only to understand why they are solving algebraic equations, but also to understand what situations these equations represent. Therefore, further research on how to support student learning in terms of using variables in algebra is needed. We investigated how students respond to algebra tasks and the cognitive schemes that emerged as part of a whole class teaching experiment (Lamberg & Middleton, 2009). The purpose in using a design research approach was to understand the means of supporting and organizing student learning of algebra through tasks presented in a teaching experiment. In this paper, we describe one cognitive scheme that emerged as part of an RLT as students solved problems involving adding different types of quantities.

Methods

Students from a sixth-grade classroom in an urban elementary school in a western state participated in the study. The sample included a total of 22 predominately Latino(a) students, ages 11 to 12. There were 11 female students and 11 male students. The majority of the students were from lower to middle socioeconomic backgrounds. A whole class teaching experiment using design research (Lamberg & Middleton, 2009) was conducted (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The framework of design research allows researchers to observe as well as intervene throughout the study process. It involves engineering learning environments, systematically studying what takes place, and making adjustments to the curriculum (Cobb et al., 2003; Collins, Joseph, & Bielaczyc, 2004; Kelly, 2003). Design researchers develop theories about the learning process, as well as techniques designed to support learning (Cobb, et al., 2003). We used a variety of data sources in order to analyze this teaching experiment using design research. Any source that related to the broader phenomena being studied in the experiment was collected (Cobb et al., 2003). Multiple sources of data allowed for retrospective analysis once the study was complete, as well as iterative analysis throughout the study. Data sources included field notes (Maxwell, 2005), video recordings, and documentation of anything that occurred in the classroom such as students' work and researchers' reflections. We had regular and ongoing discussions with the teacher about the in-class interventions and interpretations of the data. These discussions were important because they enhanced the quality of the study and research process. Data collection and analysis was a parallel process with prospective analysis occurring throughout the teaching experiment and retrospective analysis occurring after the teaching experiment.

Results

The whole class realized learning trajectory emerged around schemes documenting the meaning of a variable in various contexts of expressions, equations, and functions. Within this sequence of three phases, students' mathematical thinking revealed five schemes: *Variable-as- Label, Variable-as-Changing Quantity, Variable-as-Known value, Variable-as-Unknown value, and Independent and Dependent variable.* This paper presents the first scheme in this learning trajectory: *Variable-as-Label.*

Variable-as-Label Scheme

Students developed a *Variable-as-Label* scheme when they encountered a problem where they had to keep a record of adding multiple quantities. This scheme progressed as students encountered two different units and represented the problem context as an expression. Students were exploring the context scenario of going to soccer game. The class concluded that each family was unique and a different combination of people from each family would go to the game. The teacher asked students to figure out how many boys and girls would go to the game so that they could determine how many colored t-shirts to order.

Students reasoned that a variable *B* represented boys and therefore, the *B* was a shortened form of writing the word "boys". Similarly *G* was a shortened form for writing the word "girls". Therefore, students reasoned about variables as representing a label for a category. This reasoning became further evident when they began using variables to label a category and not the quantity. Specifically, students were asked to find the total of two groups, boys and girls, by writing an algebraic expression or equation. For example, a student labeled the category of boys with *B* and the category of girls with *G.* A variable was used as a label for the specific category of boys and a different variable was used as a label for the specific category of girls.

In the next case, students used a variable as a label to keep track of their counting. Students in a small group were given M&Ms and were asked to figure out the total red M&Ms given to their small group. Students recorded each person's amount in the expression and were asked to write an equation that represented the total M&Ms. The purpose in doing this activity was to have students combine like terms. Students added the quantities together and indicated the total. When students added, they were indicating the total M&Ms and using the variable as a label for the color of M&M. Students added the quantities independently and then indicated a total. At this point, they were treating the variable *r* as a label for a category of "red M&M" and the 16 represented the total. They were not thinking about the variable *r* as representing *one* and, thus, were thinking additively as opposed to multiplicatively ($7r$ did not mean $7 \cdot r$). The misconception of variables in the previous examples is to label colors or certain shapes as opposed to quantities. For instance, $7r$ means "7 red" in which the variable *r* labels the color *red*. In the second case, the variable *T* is labeling the word "total" – $68T$ is 68 Total. In another activity, students recorded that a soccer ball has 20 hexagons, and wrote an *h* to label and replace the word *hexagon*.

The teacher challenged the children to think about the meaning of the numbers. For instance, $1h$ represents 1 hexagon and $20h$ represents 20 hexagons. When the students added the number of hexagons, they were thinking additively. In other words "16 plus 4 equals 20". At this point, they were combining like terms by adding quantities together. However, they were not thinking multiplicatively until the teacher challenged them to think about what 20 hexagons represents – that there are 20 times the amount of hexagons. These examples demonstrate that students were thinking both additively by finding a total, and also multiplicatively because they were beginning to grasp the concept of $20h$ as $20 \cdot h$ where *h* is one hexagon. The variable represented a label to refer to the hexagon.

Discussion

Students' initial understanding of variable was a letter that labeled a specific category. However, their thinking evolved to thinking about a variable as representing a single quantity and thinking of a coefficient as multiple units of that item, such as $5r$ represented five red M&Ms. Their thinking also shifted to thinking multiplicatively where the *5* represented five times as many M&Ms. The scheme of Variable-as-Label emerged as students abbreviated the name of a group with the first letter of the word or used a letter to represent one object or one group. This scheme developed out of a need to keep track of objects that represent a unit when counting.

Prior research identifies the importance of learning about the different meaning of variables (Kaput, 1995; Blanton, 2008; Carpenter, Franke, & Levi, 2003). However, the schemes in which students acquire different meanings and misconceptions for variables and the relationship between these schemes are not adequately addressed. This research provides a starting point for teachers and researchers to develop learning trajectories to promote student learning, keeping in mind that developing problem types and carefully sequencing them based on variables should be considered to support student learning of expressions and equations.

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UNDERSTANDING STUDENTS' CHALLENGES WITH INTEGER ADDITION AND SUBTRACTION THROUGH ANALYSIS OF REPRESENTATIONS

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This study investigates middle school students understanding of integer operations through basic computation problems and in their ability to generate story problems for certain number sentences. Consistent with other research, this study finds that students struggle more with certain types of subtraction problems than with others. Additionally, this study finds students conceptions of integers may make certain problem types more difficult to navigate. Student reasoning is compared to the typical classroom presentation of integer operations. Implications for instruction on integer operations are made.

Keywords: Middle School Education; Number Concepts and Operations

Student difficulties solving problems involving addition and subtraction of positive and negative values are well known to mathematics teachers and mathematics education researchers. Many studies have explored whether and how concrete models and/or real world contexts may be helpful in building student understanding about integers and integer operations (Ball, 1993; Stephan & Akyuz, 2012). Helpful models include using different colored chips, and number lines (Hayes & Stacey, 1998; Stephan & Akyuz, 2012; Tillema, 2012). Helpful contexts have included assets and debts, elevators going above and below ground, and stories about temperature changes, among others (Ball, 1993; Hayes & Stacey, 1998; Stephan & Akyuz, 2012). These studies show that while these models and contexts support student reasoning about integer operations, students consistently find subtraction more difficult than addition, even after instruction (Ball, 1993; Hayes & Stacey, 1998; Stephan & Akyuz, 2012). Recent work focusing on the ways students reason about integer addition and subtraction has found that integer addition is not uniformly easier than integer subtraction, but rather that performance varies by problem type and prior knowledge (Bishop et al., 2014; Bofferding, 2014; Whitacre et al., 2012).

In partnership with three sixth grade teachers, we have worked over three years to develop, test, and refine an instructional sequence meant to support students' informal reasoning about integer addition and subtraction. As part of this work, we observed that while integer addition and some integer subtraction problems are fairly easy for students, other subtraction problems remain challenging even after instruction. Thus we sought to examine both performance and reasoning strategies when students were asked to provide a context for a given subtraction problem in order to improve the instructional sequence. The research questions guiding this study were: (1) Are there patterns of errors within simple integer computation problems?; (2) In what ways do students represent integers and subtraction?; and (3) How does student choice of representation of number and operation relate to success in computation?

Theoretical Framework

The ability to represent mathematical concepts in a variety of representations along with the ability to translate between those representations plays an important role in the development of mathematical understanding (NCTM, 2000). With that in mind, this study uses the Lesh Translation Model (Lesh & Doerr, 2003) as a framework for representational fluency. According to this model, conceptual understanding is marked by an ability to translate between and within modes of representation, and asking students to represent concepts in multiple ways and to translate between

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those representations promotes conceptual understanding. For example, asking students to translate an equation (a symbolic representation) to a story problem (a real world representation) can both promote and reveal their understanding.

A key feature of the Lesh Translation Model is that the different representational forms are but one component to building student understanding. Students must also be able to make connections between the different representations and understand how the different representations each embody the deeper mathematical concept being taught. When students fail to see the connections between the different representations, they also fail to comprehend the deeper mathematical concept embodied by this connection (Cramer, 2003). Thus, when employing multiple representations for the purpose of deepening student understanding, we should intentionally incorporate opportunities for students to translate within and between the representational forms. This was emphasized in this study by asking students to model an equation with chips or to write a story that would match an equation, thus providing them with opportunities to translate between modes of representation.

Method

This study was conducted during the 2012-2013 and 2013-2014 school years with the approximately 300 sixth grade students at a suburban, Midwestern middle school where 24% of students are on free or reduced lunch. This study is part of a larger project that is following a design-based research approach (Clements, 2007), with the ultimate goal of designing an instructional sequence that develops and supports student conceptual understanding of integer addition and subtraction appropriate for use with middle school students. Based on site restrictions, the unit was very short in duration, typically about one week in length, and included introducing the concepts of negative integers and integer addition and subtraction.

This study took place during the third iteration of the design-based research cycle and included a game called *Floats and Anchors* that had shown promise in supporting student reasoning about integer addition and subtraction. In the game, students model integer addition and subtraction by moving a ship up and down a vertical number line (i.e. above or below sea level) by adding or removing floats or anchors from the ship. This iteration of the study was designed to investigate the particular features of integer addition and subtraction problems that students found most challenging so that the game and follow-up activities could be redesigned to better support student reasoning for the most difficult problem types.

In addition to playing the Floats and Anchors game, instruction in the classes included moving forward and backwards along a number line, real-world contexts including money and good/bad guys, and mnemonic devices for remembering rules. The researchers made recommendations for and against some of these models, but the classroom teachers ultimately made the final decisions on instruction. All three teachers introduced a variety of models, but classroom observations conducted by the researchers at the time of the intervention indicated that one teacher placed comparatively more emphasis on mnemonic devices than the others while another placed more emphasis on the number line model. The unit, including the intervention, lasted between five and six class periods for all teachers.

Study Design

In this study, a concurrent explanatory mixed methods design was used to explore the relationship between student understanding of number and operations and student performance on various forms of integer addition and subtraction problems both before and after instruction. This design involved collecting and analyzing both quantitative and qualitative data then using the qualitative data to explain the quantitative results

In the quantitative component of the study, pre- and post-test data were collected to assess student performance on nine different forms of integer addition and subtraction problems to measure students' basic computational abilities with integers. The pretest was administered prior to any instruction and the post-test was given the class period immediately following the completion of the unit. Time between tests was approximately one week.

Pre- and post-test items were matched across the two tests, with numbers differing but the structure of the problem remaining the same across time. All problems involved only integers between $^{-}9$ and 9, excluding 0. Twelve different categories of problem types were created based upon the possible combinations of the sign of the subtrahend, minuend, and answer (+ or -) and the operation (addition or subtraction). Magnitude of the numbers was not considered except as necessary to generate an answer of an appropriate sign, i.e. $^{-}3 – 5$ and $^{-}5 – 3$ were considered the same problem type while $^{-}3 – ^{-}5$ and $^{-}5 – ^{-}3$ were not because of the differing sign of the answer. Of the twelve possible problem types, three were not included, specifically (+) + (+) = (+), (+) + -) = (+), and (+) - (+) = (+). The problem types not tested were excluded because nearly all students in a pilot study answered these problems correctly on both the pre- and post-test.

For the qualitative component of the study, students were asked to translate an open number sentence of the form (+) – (-) = (+) into a story problem at two different times during the unit. This problem type was chosen in part because it was a subtraction problem type that was most difficult for students on the pretest. The student generated stories were coded to determine the way they represented both positive and negative numbers and the operation of subtraction. In particular, we were interested in the ways students tried to make connections between real world contexts and symbolic representations.

Results and Discussion

Similar to previous work, integer subtraction was more difficult for students than addition. However, when we looked at student responses based upon the type of subtraction problem, we found that students performed only slightly worse on subtraction problems positive minuend and subtrahend (e.g. 3 - 5) than they did on the addition problems. Prior to instruction, students performed far worse on subtraction problems with a negative subtrahend or in which the signs differed between the subtrahend and minuend. Of these, problems like 3 - (-5) were the most difficult on the pretest, with fewer than 5% of students answering these problems correctly, while after instruction, student performance on this problem type improved to nearly the same success rate as addition problems. However, most of the other subtraction problems remained very difficult for students even after instruction. For example, problems like 5 – -2 = 7, -3 – (-5)=2 and (-3) – 5 = -8 had a very low success rate on both the pre- and post-test.

One result from the quantitative component of this study is that integer subtraction is not uniformly difficult for students but rather varies in difficulty depending on other features of the problem. This validates other research that has suggested that the sign and magnitude of the subtrahend and minuend both influence student success in solving such problems (Bofferding, 2014). Moreover, the students showed no or very little gain in their ability to solve these most difficult problem types even after instruction. Only problems such as 3 - (-5) showed considerable improvement, suggesting that the instructional sequence as implemented was not doing enough to support student reasoning about these difficult problems. Additionally, students were highly successful on addition problems of any type before and after instruction.

Qualitative analysis of the stories students generated for the open number sentences showed that nearly all the students modeled subtraction as take-away and used contexts that involved either discrete objects or money on both the pre- and post-tests. Students who used money as a context typically referred to "dollar bills." For both dollar bills and discrete objects, negative numbers were typically represented as those dollars/object being "borrowed" or "owed." Such representations were supportive for reasoning about subtraction problems involving a positive minuend and subtrahend but negative difference (e.g. 3-5), but were not supportive for cases involving a negative subtrahend (e.g. 3 - (-5)). Students who tried to create stories that involved taking-away owed objects were unilaterally unsuccessful in creating meaningful stories.

Together, these results suggest that instructional sequences must include explicit opportunities for students to learn how to model subtraction involving both positive and negative quantities. Instruction that works with discrete models and subtraction as take-away will connect with students' prior knowledge, but attention must be paid to helping students understand the need to use contexts that involve opposites (such as floats and anchors or credits and debts) as they are unlikely to note the importance of this on their own. Finally, a significant portion of instructional time should be spent developing students' understanding of integer subtraction when the subtrahend and/or minuend are negative, while relatively less time needs to be spent developing integer addition concepts.

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STUDENT SUCCESS AND STRATEGY USE ON MISSING-VALUE PROPORTION PROBLEMS WITH DIFFERENT NUMBER STRUCTURES

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As students develop proportional reasoning, they employ a variety of solution strategies with problems of different complexity. The solution strategy they choose gives insight to their thinking. In this report, we vary three characteristics of the number structure of missing-value proportion problems and compare both the success rate and types of strategies employed by middle school students. Findings indicate student thinking is more influenced by the character of the scale factor than the unit rate. Also, a hierarchy of difficulty in problem types is suggested.

Keywords: Rational Numbers; Middle School Education

Purpose of the Study

In school settings, the common way to develop students' proportional reasoning is to engage them in solving proportion-related problems. Since these problems are an important part of the curriculum, we need to understand how grappling with them advances the development of proportional reasoning. In our research, we focus on missing-value proportion problems and vary number structure characteristics. We seek to identify milestones of understanding, that is, essential understandings students require to reason through increasingly more difficult problems. Which variations in number structure provoke changes in students' success rate and strategy use? Which aspects of proportional reasoning are required for students to solve subsequent problems? This report is based on a preliminary analysis of eight problems in which three number structure characteristics vary. We compare student success rates and strategy use to establish a hierarchy of problem difficulty and aim to identify number structure characteristics that are most influential on student thinking.

Theoretical Framework and Methods

Our research is based on the idea there is a sequence of recognizable developmental stages achieved by students as their proportional reasoning matures. We use a learning progression for missing-value proportion problems proposed by Carpenter et al. (1999) and refined by Steinthorsdottir and Sriraman (2009) as our starting point. Associated with this learning progression is an implied hierarchy of problems. Students who have attained a particular developmental stage are predicted to be able to solve certain problems, but fail on problems that require more advanced reasoning. A premise of our investigation is we can infer the developmental stage of a student based on his/her work on a set of problems. Research, such as that involving cognitively guided instruction, has used this approach (Carpenter, Fennema, Franke, Levi, & Empson, 1999). In this study, instead of following individual students over several years, we are investigating the written work of many students of different ages for patterns of success and strategy use on a range of problems. As problems increase in difficulty, we expect to see both lower rates of success as well as decreased use of efficient strategies. We aim to identify what additional knowledge students employ at different levels of problem difficulty.

Prior research indicates many task features affect students' ability to solve missing-value proportion problems. Key among these are semantic type and number structure (Heller, Post, Behr & Lesh, 1990; Lamon, 1993; Tourniaire & Pulos, 1985; Vergnaud, 1983), though many others have been identified (Harel& Behr, 1989). We designed our instrument paying most attention to features related to number structure since a comparison of the impact of semantic type and the presence or absence of integer ratios indicated students were more influenced by the number structures of problems (Steinthorsdottir, 2006). We considered tasks in which the within measure space ratio was either an integer or non-integer, the between measure space ratio was either an integer or non-integer, and the given proportion was either to be enlarged or decreased in the target. The term *within measure space ratio* refers to the ratio of the given quantities with identical units; it is the scale factor needed to transform the given into the target quantity. *Between measure space ratio* refers to the ratio between the given quantities with different units. This ratio is a unit rate.

For our study, a pencil and paper instrument was administered to 409 middle school students (5th – 8th grades) in a small Midwestern city. Since the instrument is long (26 questions), three class periods were used over a two week period at the beginning of the school year. There was no instruction given in conjunction with data collection, therefore we consider the students' work to represent their natural approach, built on both informal and formal prior experiences. Students showed their work and did not have access to calculators.

The problems in this study, categorized by their number structure, are presented in Table 1. Each was given as a story problem, such as Q18: "Kris works at the stable and it is feeding time. If 24 horses eat 40 bales of hay each day, how many bales would 6 horses eat?" Prior research suggests that problems with an integer scale factor (Cells A and B) are easier than those with a non-integer scale factor (Cells C and D). Further, the ability to scale down by an integer, as in a Cell B shrink problem, is predicted to prepare students to succeed on problems in which the scale factor is a non-integer (Carpenter et al., 1999; Steinthorsdottir & Sriraman, 2009).

Table 1: Number Structures of Problems Analyzed

	Integer Scale Factor	Non-integer Scale Factor
Integer Unit Rate	Cell A Q23 4:16 = ?:48	Cell C Q19 60:20 = ?:45 Q03 3:9 = 11:?
Non-integer Unit Rate	Cell B Q02 5:7 = 40:? Q18 24:40 = 6:? (shrink) Q06 36:24 = ?:8 (shrink)	Cell D Q11 18:16 = 45:? Q12 42:35 = ?:10 (shrink)

Many researchers discuss common student strategies, both correct and error (Bezuk, 1988; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985). We used a detailed rubric to code strategies. Broad categories of error strategies included haphazard calculations, additive comparisons, and incomplete multiplicative comparisons. Broad categories of correct strategies included build-up and multiplicative proportional reasoning strategies. We also coded a variety of errors students made after beginning a problem using proportional reasoning. These partially correct strategies had errors that included either calculation or conceptual errors. In all categories, we captured details of the students' work. Each strategy was coded by two people, and any disagreements were discussed and resolved. For this report, codes were then collapsed into the following categories. *Essentially correct* solutions are those with answers achieved using any proportional reasoning strategy with at most a minor calculation error. *Partially correct* are those that begin with proportional reasoning but finish with incorrect reasoning or contain a major (conceptual) calculation error. These typically arise from mishandling non-integer factors. *Incorrect* solutions are those in which students do not apply proportional reasoning.

Essentially correct solutions are further categorized as representing *efficient* or *less efficient* strategies. An efficient strategy is multiplicative in nature and uses two calculations; for example, a factor (either the scale factor or unit rate) is computed, then applied. Less efficient strategies are build up strategies (e. g. 5:7 = 10:14 = 20:28 = 40:?, answer 56) which take more than two steps. This labeling is subjective but indicates the sophistication of the strategy chosen by the student.

Results and Discussion

Results are presented in Table 2 with questions arranged from easiest to hardest based on the number of essentially correct solutions observed.

Table 2: Categorization of Strategies, Percent of 409 Students

		Q23 Cell A	Q02 Cell B	Q18 Cell B shrink	Q06 Cell B shrink	Q19 Cell C	Q03 Cell C	Q11 Cell D	Q12 Cell D shrink
Essentially Correct Strategies	Efficient	52%	44%	47%	40%	23%	24%	21%	24%
	Less Efficient	16%	21%	9%	8%	19%	17%	11%	2%
Essentially Correct		67%	65%	56%	48%	41%	41%	32%	26%
Partially Correct		1%	2%	3%	2%	19%	20%	19%	10%
Incorrect		32%	33%	41%	49%	39%	39%	49%	64%

McNemar's test is appropriate for comparing two paired proportions. Using this test, the proportion of essentially correct responses in all but two question pairs (Q23 and Q02; Q19 and Q03) were significantly different at the 5% level. As predicted by the learning progression, the enlarge problems with integer scale factors were the easiest, followed by the shrink problems with integer scale factors. The enlarge problems with a non-integer scale factor but integer unit rate are next, followed by the enlarge problem with two non-integer factors. The shrink problem with two non-integer factors is the hardest problem.

The amount of change between problems in different cells strongly suggests the complexity of the scale factor plays a larger role in student thinking than the complexity of the unit rate. Compare the enlarge problems from cells A and B in Table 2, in which the unit rate is integer and non-integer, respectively. The difference in the proportion of essentially correct solutions for Q23 and Q02 is negligible. On the other hand, when the scale factor changes from integer to non-integer, Cell A to Cell C, the success rate drops from 67% to 41% (Table 2, Q23, Q19 and Q03). We hypothesize students more easily notice the multiplicative relationship for quantities with identical units rather than for those with different units.

The ability to correctly partition a given ratio appears to represent a developmental milestone. Shrink problems have lower success rates than enlarge problems with a similar number structure. Compare results in Table 2 for Q02, Q18 and Q06, all cell B questions. The striking difference is the drop in less efficient strategies. This drop also occurs in Cell D, Q11 (enlarge) and Q12 (shrink). Most students who correctly solve a shrink problem do so with an efficient strategy. The ratio of efficient to inefficient solutions is about 2:1 on the enlarge problem and about 5:1 on the shrink problems in Cell B. Perhaps less advanced students have more entry points into an enlarge problem, allowing them to figure out a correct answer. Not seeing an entry point in a shrink problem, these students have a greater tendency to immediately use non-proportional reasoning. In fact, if our questions are ordered by increasing percentage of incorrect responses, the Cell C enlarge problems precede the Cell B shrink problems.

The ability to scale up and down by an integer should enable students to solve problems with non-integer scale factors (Cell C and D problems) equally well. We observed, however, students were more successful on Cell C problems (Q19 and Q03) than on Cell D problems (Q11 and Q12). We conclude the presence of an integer unit rate is beneficial, even though the explicit use of the unit rate strategy does not explain the differences in success. Rather, the integer unit rate appears to enable students to correctly finish a build-up strategy. It also may be that the "nice" unit rate provides an entry to the problem and so prevents students from turning immediately to non-proportional strategies.

Data from additional questions and deeper analysis should clarify our findings regarding the relative importance of number structure task characteristics.

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ALGEBRA NOTATION FOR FUNCTIONS IN GRADES 5 THROUGH 9

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We examine how 1343 students in grades 5 to 9 solved an arithmetic problem and attempted to provide a general representation for a function described in the problem. We found that use of letters appears from grade 5 and continuously increases through grade 9 and that students who solved the arithmetic part of the problem were more likely to use letters as variables.

Keywords: Algebra and Algebraic Thinking; Assessment and Evaluation

Research shows that middle and high school students often fail to solve algebra problems (Knuth et al, 2011), have trouble generating equations from word problems (Kieran, 2007), and do not use letters to solve them (Booth, 1984; Küchemann, 1981). Difficulties with algebra have been attributed to the inherent abstractness of algebra (Collis, 1975, Kuchemann, 1981) or to teaching that emphasizes the "meaningless manipulation" of variables (Chazan, 2000).

More recently, a different approach to high school algebra (see Schwartz and Yerushalmy, 1992), which emphasizes the relationship between quantities and focuses on functions and their multiple representations (e.g., tables, symbolic expressions, and Cartesian graphs) have shown to promote a deeper understanding of algebra (Chazan, 2000). In fact, even in elementary school, a functions approach to algebra can promote students' understanding of algebraic principles and representations and use of variables to represent verbal statements (Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2008). These results support the proposal for a different curriculum, aimed at integrating algebraic reasoning across all grades (Kaput, 1998). To design such curriculum, we need to better understand students' intuitive and conventional ways of attempting to use letters as variables across grades and to identify their typical mistakes. This study contributes to this goal by examining the evolving use of letters as a variable by students in grades 5 to 9, as they attempted to represent a function described in a verbal problem, after they had solved a specific instance of the problem using arithmetic.

Method

A total of 1343 students from New England (U.S.A.) completed a mathematics written assessment at the start of the 2012-2013 school year; 233 students were in grade 5 (17.3%), 465 in grade 6 (34.6%), 378 in grade 7 (28.1%), 131 in grade 8 (9.8%), and 136 in grade 9 (10.1%). We analyzed students' answers to parts (a) and (b) of a five-part problem taken from a grade 10 State Assessment test. Part (a) relates to grade 6 Common Core State Standards (on expressions and equations) and part (b) relates to grade 8 standards (on functions). The problem situation and the two questions examined were:

> Liam and Tobet are going to walk in a fund-raising event to raise money for their school. Liam's mother promised to donate to the school $4 per mile that Liam walks, plus an additional $30. Tobet's father promised to donate to the school $6 per mile that Tobet walks, plus an additional $20. (a) If Liam walks 15 miles during the event, what is the total amount of money his mother will donate? Show or explain how you got your answer. (b) Write an equation that represents y, the total amount of money Liam's mother will donate if Liam walks x miles during the event.

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Results

Missing or incorrect answers were scored as 0 and correct answers as 1. Then, part (a) answers were coded into no response (blank answer or I don't know), incorrect answer, and correct answer; and part (b) into no response, do not use letters (e.g., 90, $60+$30=$90; $50=5 mi; 60, 15*4 = 60 or 20*4 = 80, 1120; 30 by 7.50 miles; if he walks 40 miles then his mom will donate $110), and use letters. Answers to part (b) that used letters were also coded into the six categories in Figure 1.

Use of Letters

a. **Conventional Correct Notation**: (e.g., y=4x +30)

b. **Different Correct Notation**: uses different letters to represent x (number of miles walked) and y (total donated) (e.g., d=4m+30 or 4y+30=x)

c. **Reverse Notation**: reverses order of number and letter
   - **i. Correct Answers** (e.g., x4+30 = y)
   - **ii. Incorrect Answers** (e.g., y = x6+20)

d. **Incomplete Notation**: provides an incomplete equation
   - **i. Correct Answers** (e.g., 4x+30)
   - **ii. Incorrect Answers** (e.g., y = x + 30 or y=4x)

e. **Wrong Information** (uses wrong number information in a complete, incomplete, or incorrect set-up; e.g., 30 + (x*5)=y; y = 4x + 40; y = 15x + 20; y = 6x, y = x + 20; y *x = 450; y=90)

f. **Unidentified Use of Letters** (Randomly combines letters given/not given in problem; e.g., x + y = m; (x*y)+20=90; y/x+30=60; 16xy; 15xy=y; 4x=y+30=y; y=x)

Figure 1: Categories for Use of Letters as a Variable in Part (b) of the Problem

Representing and solving the arithmetic part of the problem seems to have been easier than providing a general algebraic representation of the situation. Here, the percentage of correct arithmetic answers (part a) ranged from 38.2% in grade 5 to 77.9% in grade 9, with no significant differences (analyzed with an ANOVA) between grades 7 to 9, but significant differences between the other grades. Correct algebra representations (part b) ranged from 4.7% in grade 5 to 70.6% in grade 9; differences across all grades were significant.

The arithmetic part of the problem was significantly more difficult for 5th graders, with 38.2% correctly solving the problem, than for 7th through 9th graders, where more than 70% did so.

The algebraic representation of the problem was difficult for 5th, 6th, and 7th graders, with from 4.7% to 36.5% of the students correctly using letters. These percentages increased to 57.3 in grade 8 (the grade where variables are introduced) and to 70.6 in grade 9.

Table 1 shows, for students in each grade, how the percentage of no response, incorrect, or correct responses to part (a) relates to no response, did not use letters, and used letters in part (b). Percentages in the cells for each grade level add up to 100%.

The last column in the table shows the total number of responses to part (a), for each category and grade level. Here, 5th graders show almost equal percentages for each of the three categories. Among 6th graders, there was a slight improvement in results. For the three higher grades over 70% of students provided a correct response to part (a).

Overall, in response to part (b), 33.6% of students in all grades gave no response, 5.3% did not use letters, and 61.1% used letters as a variable.

Even though a high percentage (33.9%) of 5th graders provided no response for both parts (a) and (b), as many as 21.9% correctly solved part (a) and used variable(s) in their solution to part (b). From grades 6th through 9th, the percentage of students who correctly solved part (a) and used variables in part (b) increased from 42.4% to 75%.

As expected, across all grades, a high percentage of students who provided no response to part (a) also provided no response to part (b), while those who correctly solved part (a) had a higher percentage of use of variables in part (b). This suggests that students who understand the arithmetic relations in the problem are more likely to use variables in part (b), at any grade level. Also, the

Table 1: Percentage of Each response by Grade

Grade	Part (a) Response	Part (b) Response			Total
		No Response	Didn't use letters	Used Letters	
5 (N=233)	No Response	33.9%	0.9%	0	34.8%
	Incorrect	10.7%	9.0%	7.3%	27.0%
	Correct	12.0%	4.3%	21.9%	38.2%
6 (N=465)	No Response	21.7%	0	0.9%	22.6%
	Incorrect	9.5%	2.4%	9.5%	21.4%
	Correct	10.8%	3.0%	42.4%	56.2%
7 (N=378)	No Response	12.2%	0	0	12.2%
	Incorrect	3.7%	0.8%	13.0%	17.5%
	Correct	7.9%	1.6%	60.8%	70.3%
8 (N=131)	No Response	10.7%	0	0.8%	11.5%
	Incorrect	0.8%	0	14.5%	15.3%
	Correct	0.8%	2.3%	70.2%	73.3%
9 (N=136)	No Response	10.3%	0	0.7%	11.0%
	Incorrect	0	0.7%	10.3%	11.0%
	Correct	2.9%	0	75.0%	77.9%
All Grades (N=1343)	No Response	18.9%	0.1%	0.4%	19.4%
	Incorrect	6.3%	2.7%	10.6%	19.6%
	Correct	8.4%	2.5%	50.0%	60.9%

percentage of students who correctly answered part (a) and used variables in part (b) positively correlates with grade level.

A total of 672 (50%) of the 1343 students were able to correctly solve part (a) and then use letters as a variable in part (b). These were 51 students in grade 5 (21.9% of 5th graders), 197 in grade 6 (42.4% of 6th graders), 230 in grade 7 (60.8% of 7th graders), 92 in grade 8 (70.2% of 8th graders), and 102 in grade 9 (75.0% of 9th graders).

The percentage of students correctly using letters as a variable increases with grade level as does the percentage of those using the conventional notation. Accordingly, the percentages of most types of incomplete, wrong, and unidentified use of letters decreased with increasing grade levels but was still found among 27.3% and 12.8% of 8th and 9th graders, respectively. Only 9th graders used the correct conventional notation in more than 50% of responses, with 71.6% of them in this category. However, 5th to 8th graders would also use correct notation but reversing the order of variables and constants. Total percentages for correct use of variables, including cases of inversion and incomplete answers, show that 17.6% of the 5th graders, 49.2% of the 6th graders, and 56.5% of the 7th graders who used variables did so correctly, even though this is an 8th grade standard.

Discussion

In keeping with previous research on high school students' difficulties with algebra, our results show that generalized use of letters as a variable, although more frequent in later grades, is still problematic for more than half of 7th graders, more than 40% of 8th graders, and nearly 30% of 9th graders. However, among students who could solve the arithmetic part of the problem and used variables in its algebraic part, use of letters in the correct conventional notation or reverse correct notation start to appear from grade 5 and continuously increase through grade 9.

In terms of solving the arithmetic part of the problem, even though 7th to 9th graders results were better than those for earlier grades, as many as 38.2% of 5th graders and 56.1% of 6th graders understood what the problem was asking them to do. This is noteworthy because this study's data were collected at the beginning of the school year, before 5th and 6th graders were taught the content of part (a).

Our results show that even 5th graders are capable of using letters to represent variables described in verbal statements and suggest that instruction about variables can start much earlier in the K-12 curriculum. The results also suggest that the ability to understand and solve arithmetic problems is a basis for considering variables and using letters for representing them.

Concerning students mistakes, reversing the conventional order for representing variables and constants was present at all grade levels. This could be easily addressed with practice. Other mistakes such as incomplete, wrong, and unidentified use of letters, present even in 8th and 9th grades, call for introducing algebra and functions through activities that promote a clear understanding of variables and of how algebra notation relates to verbal statements.

This study's results and previous findings on young students' use of letters as a variable, suggest that the integrated teaching and learning of arithmetic and algebra may successfully start in elementary school to promote a deep understanding of mathematics across the curriculum.

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LOST IN TRANSITION: DIFFICULTIES IN ADAPTING RELATIONAL VIEW OF EQUALS SIGN

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The concept of equivalence is one of the first algebraic notions introduced to students in early childhood. The future success of students in mathematics, and related subjects, is believed to rely significantly on their understanding of this basic algebraic concept. However studies have found that the operational view of the equals sign is a major obstacle in students' proper understanding of equivalence, and hinders their mathematical abilities. In this paper we report findings from part of a teaching experiment involving second-graders where students were observed developing their conceptions of equivalence. We describe the overlap of the operational view of equivalence and the relational view in two students.

Keywords: Algebra and Algebraic Thinking; Classroom Discourse; Reasoning and Proof

Background and Objectives

Numerous research studies have found that students' misconception of equivalence surfaces in the form of an operator conception the of equals sign (Carpenter et al., 2003; McNeil & Alibali, 2005). Students with such an operational view treat the equals sign as an operator to produce the result of the operation performed on the numbers on one side of the sign. Thus, equals is conveyed as "makes" or "produces" instead of "the same as." These effects have been observed to be long lasting and hinder students' success in their mathematics learning in later grades (McNeil & Alibali, 2005). Research spanning over 40 years has reported various reasons for this misconception and suggest that many students retain this misconception in some form or other even in higher grade levels (Kieran, 1981; Knuth et al. 2006). The present study extends this large body of literature by focusing on how children's conceptions of equivalence changed and varied across several weeks. Therefore, the purpose of this study is to examine the interaction of children's conception of the equals sign as an operator with their ability to grasp of a relational view.

Theoretical Framework

Children with an operational view of the equals sign generally give significance only to the number immediately following the equals sign and usually ignore any operations thereafter. However, with a relational view, students consider expressions appearing on the either side of equals sign as equivalent/same (Baroody & Ginsburg, 1983). The particular focus of research on students' conception of equivalence has been on the nature of children's misconceptions. Such research has found that beyond the conventional arithmetic problems (e.g., $2+1=\square$), children demonstrate difficulty with alternate forms. These include equations in reverse-order such as $a=b-c$ (McNeil et al., 2006), equivalent expressions such as $a=a$ (Baroody & Ginsburg, 1983), or missing numbers in equivalent expressions such as $a+b= \square+d$ (Rittle-Johnson & Matthews 2011).

Our primary interest in this study is to look for any nonstandard difficulties faced by students in realizing the relational view of equals sign. We used the construct map for mathematical equivalence knowledge, suggested by (Rittle-Johnson & Matthews 2011) to determine students' demonstrated level in terms of an operational or relational view of equals sign.

Methods

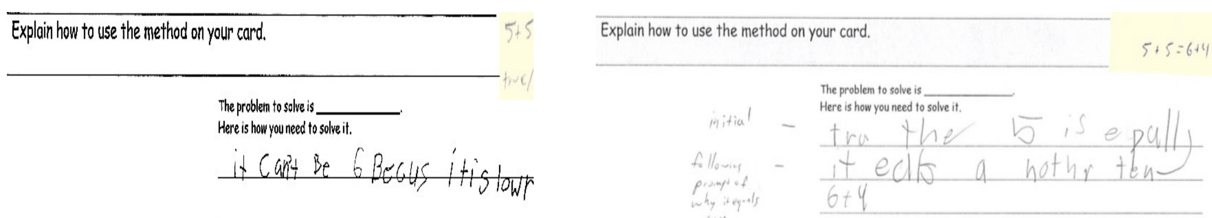
The data used in this paper focuses on the work of two students, Julia and Jacob, who were a part of a year-long teaching experiment with four second-grade students from a public school in a Midwestern U.S. state. For the sake of simplicity and space, we focus on six of these sessions (session 8 to 13). Based on Steffe & Thompson's (2000) description of teaching experiment, the first author served as the observer and the second author as the teacher-researcher. During the sessions of focus here, children were asked to consider different number sentences and their properties, including their association on number balances. The number sentences ranged from single expressions with single digit numbers (e.g., 5 + 7) to two expressions with multi-digit numbers in the equation (e.g., 15 + 10 = 18 + 7). All the sessions were video recorded and transcribed and written work of participants was digitally scanned. Field notes collected from the first author were also included.

Analysis

After each session, both authors used video, students' writing, and observer notes to examine students' understanding of equivalence. Transcripts were used along with other data following all sessions for further analysis. We used the construct map provided by Rittle-Johnson & Matthews (2011) to code students' demonstrated understanding. The concept map proposes four levels of mathematical equivalence knowledge; *rigid operational* ($a+b=c$); *flexible operational* ($c=a+b$, $a=a$); *basic relational* ($a+b=c+d$, $a+b–c=d+e$); and *comparative relational* (e.g., knowing that 85 + 14 = 80 + 19 without needing to find the sum of each expression). The findings were refined through continuous review of student data. After careful observation and independent analysis of data, both authors reached agreement on levels of the construct map for each student across different sessions.

Results and Discussions

In early sessions Julia demonstrates an operational view of the equal sign, as opposed to Jacob who consistently demonstrated the relational view. However, it should be noted that the very first problems involved equivalent expressions containing single digit number only (Figure. 1).



Julia: "It can't be 6 because it is lower" **Jacob:** "True the 5's equal...it equals another ten, 6+4"

***Figure 1*. Julia and Jacob's responses to whether 5+5=6+4 and why.**

Notice how, in Figure 1, Julia identifies only the 6 to the right of the equals sign, and does not identify the + 4 afterwards. Jacob, however, describes the whole equation in framing his response. Despite her rigid operational view in the beginning, Julia started to shift how she began describing the equals sign in using language resembling "same as" descriptions, but never reached a point where her descriptions of the arithmetic aligned with a *flexible operational* stage (Rittle-Johnson & Matthews 2011). Jacob's performance demonstrated a unique misconception hindering his advancement to the *basic relational* stage. For the sake of clarity, we discuss Julia and Jacob's performance separately for the remainder of the paper.

Julia's work

In view of (Rittle-Johnson & Matthews 2011) construct map, Julia would, at times, seemingly demonstrate a transition from the rigid operational view of equals sign to a flexible operational view, but this transition was not consistent. During one session, Julia responded to the question "Explain why 38 + 16 = 40 + 14" by stating "They did not show the same numbers but they are the same answers. So it is a different equation but it is the same answer.*"* Thus, Julia performs the operation of addition irrespective of the appearance of expression on the left or right of the equals sign, but does not demonstrate the equivalence of the expression. Rather, she talks about the answer of the addition operation separately on either side. We infer from this that she is, at this time, considering the two expressions either as $a+b=c$ or $c=a+b$.

Jacob's work

Jacob's responses for the equivalent expressions using single digit numbers suggest that his level of equivalence knowledge is beyond the *rigid operational* level (see figure. 1). When asked to consider the equivalent expressions involving two digit numbers, Jacob did not demonstrate an understanding of equivalence (see the Figure 2 below).

40+14=38+16: *"Because they are not the same patterns and the same numbers and the same equal"*
10+15=17+8: *"It doesn't work because it is not equal".*

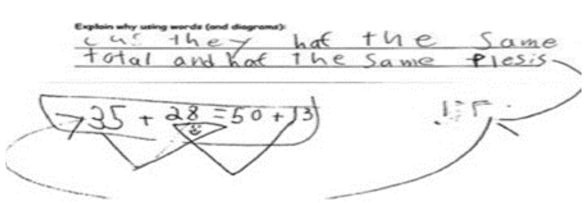
***Figure 2*. Jacob's failure to realize equivalence of expressions.**

Based on his work with equations with single-digit numbers, we initially considered that Jacob had an operational view, but we found disconfirmatory evidence in this regard. Jacob correctly identifies the equivalence in some other problems involving two digit numbers. For example consider his responses for the questions in Figure 3.This evidence seemingly contrasts an assumption of an operational view. To understand this discrepancy in Jacob's responses, we considered his work in response to various problems such as "*35+28=50+13*", as shown in Figure 3.

35+28=50+13: *"Because they have the same total and have the same places"*
21+36=50+7: *"Because they have the same total"*

***Figure 3*. Jacob's realization of equivalence of expressions.**

On careful observation, we found that Jacob's strategy with equivalent expressions involving two digit numbers relied primarily on adding the numbers in ones place on the left hand side of equals sign. He then looked for the same number (the sum of numbers in the ones place) on the right hand side of the equals sign. If the equation had the same number on the right hand side of equals sign, Jacob would conclude that the expressions are equivalent, and are not equivalent otherwise.



35+28=50+13; *"Because they have the same total and have the same places"*

***Figure 4*. Jacob's responses to equivalent expressions with two digit numbers.**

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Confirmatory evidence for Jacob's adding strategy is presented in Figure 4. Jacob demonstrates adding the numbers in the ones place on left hand side of equals sign, but then looks for the resulting number on the right hand side. Thus, he adds 8+5 to get 13. He then checks to see 13 on the right hand side of equals. Finally he adds the numbers in tens place on the left hand side of the equals sign to get 30+20. He then verifies 50 to appear on the left hand side again and concludes that the expressions are equal. He does the same for 21+36 = 50 +7 and declares it to be correct (see Figure 3). Using the same method, he concludes that the two examples in Figure 2 are not equivalent. Based on the discussion above, we conclude that Jacob's level of equivalence knowledge can be placed somewhere between flexible operational and basic relational. His exhibition of a "pseudo-relational" view of the equals sign might be considered as his preliminary advent to the basic relational level of equivalence knowledge. Rather, at times he is able to demonstrate what appears to be a relational view, and provide what appear to be relational definitions, but disconfirmatory evidence suggests that such a view is too ill defined by Jacob for him to show consistency in his responses.

Conclusion

In this report, we have shown that children's relational and operational views of the equals sign may be mutually exclusive, but there is not as clear a line between these views as much prior research has suggested (Rittle-Johnson & Matthews 2011). We observed that Jacob's pseudo-relational understanding of the equals sign prevented him from consistently demonstrating a basic relational understanding. Such a view may signal false indicators of a basic relational view and thus be misleading for teachers (and also researchers).

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PRESERVICE AND INSERVICE TEACHERS' CONCEPTIONS OF NUMBER AND OPERATIONS CONCEPTS

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Across the professional spectrum, teachers' mathematical knowledge for teaching impacts their teaching practice and students' learning opportunities. This brief research report details an ongoing study focused on the conceptions of place value and whole number operations among preservice (n=26) and inservice (n=8) teachers engaged in a multi-week teaching intervention involving an unfamiliar number system in base six. Using mixed methods, analysis of pre- and post-Assessment of Place Value data, participant reflections, and pre- and post-surveys of productive and unproductive beliefs about teaching and learning mathematics, yields insights about teachers' conceptions of place value and whole number operations, specifically noting differences across preservice and inservice teachers. Supporting literature and pilot data indicate the potential for positive impact on participants' mathematical knowledge for teaching.

Keywords: Number Concepts and Operations; Teacher Education-Preservice; Teacher Education-Inservice (Professional Development); Teacher Knowledge

Number and operations are a consistent component of the K-12 mathematics curriculum, and supporting teaching and learning of these concepts remains an enduring challenge in mathematics education. In U.S. states that have adopted the *Common Core State Standards for Mathematics* (*CCSSM*), teachers are expected to build upon concepts of number, operations, and place value, applying those concepts with increasing sophistication from natural and whole numbers, to fractions, integers, rational, irrational, and complex numbers. Similar progressions can be found in alternative curriculum standards. Here we report on an ongoing study of preservice and inservice teachers' conceptions of place value and whole number operations throughout an intervention designed to deepen teachers' understanding of these foundational concepts.

Purpose of Study

Preservice and inservice teachers' conceptual understanding of numeration systems, place value, and mathematical properties can influence how they teach the concepts and help students make meaning of them (Ma, 1999). Teachers must possess a deep conceptual understanding of numeration and place value in order to select the appropriate order of tasks to engage students in knowledge construction, choose appropriate manipulatives and representations to support those constructions (Morin & Samelson, 2015), analyze students' reasoning, facilitate productive mathematical discourse, and anticipate possible alternative solutions and strategies. The purpose of this study is to explore teachers' conceptions of place value and whole number operations through an intervention involving a base-six number system and alternative numerals. We also explore howthese conceptions differ among and between preservice and inservice teachers.

Theoretical Perspective

Building on Shulman's (1987) concept of pedagogical content knowledge, *mathematical knowledge for teaching* (Hill, Rowan, & Ball, 2005; Ball, Thames, & Phelps, 2008) is now a well-established construct in mathematics education. We draw from this construct as we consider conceptions of place value and operations among elementary preservice teachers (PSTs) enrolled in an undergraduate mathematics methods course and inservice teachers enrolled in a graduate mathematics education course. The study described in this paper engages teachers as mathematical learners of place value and operations concepts in an unfamiliar number system, providing them with opportunities to construct deeper understanding of mathematics and pedagogy through exploration and productive struggle with cognitively demanding mathematical tasks in base-six (Stein, Smith, Henningsen, & Silver, 2009).

There exists a robust research literature relating to teaching and learning elementary number concepts, with much attention directed toward progressions and characteristics of teaching and learning about place value and multi-digit addition and subtraction concepts (see, for instance, Fuson, 1990; Baroody, 1990; Fuson et al., 1997; Clements, et al., 2011). In order to facilitate student learning that focuses on such concepts, teachers must possess deep conceptual understanding (CBMS, 2001; Sowder, Philipp, Armstrong, & Schappelle, 1998). Yet, deep conceptual understanding cannot simply be assumed. For example, Thanheiser (2009) examined elementary pre-service teachers' (PSTs') conceptions of multi-digit whole numbers in a variety of contexts and found that the majority of PSTs experienced some gaps in their conceptions of whole numbers and place value. Variation in teachers' understanding of place value concepts has also been documented among practicing teachers (Tanase, 2011; Tanase & Wang, 2013).

One approach to developing teachers' knowledge and conceptual understanding of place value and operations has been to engage teachers as learners in a non-base ten number system (Andreason, 2006; McClain, 2003; Roy, 2008). Hopkins and Cady (2007) describe how learning experiences with the Orpda system, a base five number system that used symbols other than Hindu-Arabic numerals, provided preservice and inservice teachers with insight about student thinking and manipulative use, while developing a deeper understanding of place value concepts. In her dissertation study involving the Orpda system, Price (2011) also documents positive changes in teachers' conceptual understanding of place value and operations. Likewise, dissertation studies by Rusch (1997) and Radin (2007) emphasized a variety of number bases in coursework for pre-service elementary teachers. Positioning preservice and inservice teachers as learners of familiar mathematics in unfamiliar number systems can facilitate the construction of conceptual understanding, while providing authentic experiences with pedagogical approaches, potentially impacting teachers' mathematical knowledge for teaching.

Methods

The specific teaching intervention used in this study engages participants in place value and whole number operations tasks that parallel the progression of Number and Operations in Base Ten from the *CCSSM* (CCSSI, 2010). Throughout the multi-week unit, participants create, represent, and solve tasks in a base-six numeration system, termed the Sloth System, that uses the symbols **O**, •, ♥, ▽, □, ★, to represent the Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, respectively. As instructors for these two courses, the authors pose tasks that include bundling objects in groups of six; representing and comparing quantities; manipulatives and representations of base-six blocks and six frames; composing and decomposing numbers; and adding, subtracting, multiplying, and dividing, all in the Sloth System. The intervention models many of the base ten concepts across elementary mathematics using the Sloth System with the goal of improving participants' mathematical knowledge for teaching place value and operations.

A mixed-methods approach is used throughout the study involving 26 preservice teachers in two sections of an undergraduate elementary mathematics methods course and eight inservice teachers in a graduate course on Numbers & Operations. Each participant is asked to complete a pre- and post-Assessment of Place Value, adapted from Rusch (1997), at the beginning and end of the semester. Additional data sources include teachers' written reflections on their learning related to the teaching intervention, and results of a four-point Likert-type beliefs survey adapted from productive and unproductive teaching beliefs as described in *Principles to Action: Ensuring Mathematical Success for All* (National Council of Teachers of Mathematics, 2014), administered at the beginning and end of the semester.

Using the constant comparative method (Strauss & Corbin, 1998), participants' reflections will be coded and examined for themes. The unit of analysis is the individual teacher. Additionally, reflections and the responses to open-ended items on the pre- and post-assessments will be analyzed using the *conceptions of multi-digit whole numbers framework* (Thanheiser, 2009). Results from qualitative analyses will be mapped to the findings from the quantitative results for each participant. Analysis of data from the subset of preservice participants will be compared with the subset of inservice participants, as well as among individuals in each group.

Preliminary Findings

We report on preliminary findings from this study in progress. Two prior iterations of the base-six teaching intervention yielded pilot data from elementary preservice teachers' reflections. Three themes emerged from the qualitative pilot data: preservice teachers reported increased understanding of student thinking and misconceptions about place value and operations; acknowledged that discussion of multiple strategies fostered their own understanding of place value and operations; and, gained appreciation for the manipulatives and representations. Preservice teachers described their initial frustration with representing and operating on numbers in an alternative numeration system, but were cognizant that their struggle was likely similar to that of many of their future students learning the same concepts in base ten. Because the teaching intervention in this study builds upon prior iterations, these findings suggest that teachers' understanding of both mathematical content and pedagogy, or perceptions thereof, may be impacted. Yet the extent to which that impact differs among students and between preservice and inservice teachers cannot yet be determined. Additionally, preliminary analysis of preservice teachers' pre-tests for this study indicates lack of familiarity with non-decimal number bases prior to the teaching intervention. Thus, participants' facility with alternative number bases, builds from a rudimentary baseline understanding of place value and operations that are not in base ten.

Discussion

Prior research has indicated positive results when preservice teachers' learn in non-base-ten numeration systems (Rusch, 1997; McClain, 2003; Andreason, 2006; Hopkins & Cady, 2007; Radin, 2007; Roy, 2008; Price, 2011). To this body of literature, the study described here will add results about inservice teachers' conceptions of number and operation concepts. Increased understanding of preservice and inservice teachers' conceptions of place value and whole number operations will support researchers and mathematics teacher educators to better target teachers' preparation and professional development needs, in turn potentially impacting teachers' mathematical knowledge for teaching (Ball et al., 2008) and students' learning of foundational number and operations concepts (Hill et al., 2005).

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ALICE'S DRAWINGS FOR INTEGER ADDITION AND SUBTRACTION OPEN NUMBER SENTENCES

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Alice, a fifth grader who participated in twelve weeks of a teaching experiment on integer addition and subtraction, produced drawings as part of her strategy for solving integer addition and subtraction open number sentences. The drawings she created during the twelve weeks of the teaching experiment were analyzed and grouped into the following categories: Single Set of Objects, Double Set of Objects, Number Paths & Number Lines, and Number Sentences. These drawings provide insight into how children may directly model or count when solving integer addition and subtraction problems.

Keywords: Number Concepts and Operation; Elementary School Education; Cognition

For solving addition and subtraction problem with positive integers we know that children often use strategies that incorporate drawings that include direct modeling, counting, or derived facts (Carpenter, Fennema, Franke, Levi, & Empson, 2015). Children often use drawings paired with direct modeling or counting strategies as they begin to invent strategies for solving addition and subtractions problems. We also know that children often draw upon direct modeling strategies, which may incorporate drawings, when the number size changes. However, as a field, we know little about the drawings that children employ as they transition from using positive integers to negative integers. Bofferding (2010) demonstrated that children often use a number path when solving integer problems. Other researchers have shown that children will use a variety of ways to reason about the integers which include order-based or number line reasoning (e.g., Bishop et al., 2014). Despite the insurgence of research on the ways that student think about integers (e.g., Bofferding, 2014; Bishop et al., 2014; Wessman-Enzinger & Mooney, 2014), we need to know more about the ways that children reason about integers in relationship to they ways that children employ direct modeling or counting. One way to identify more of these direct modeling and counting strategies from children is to look at the drawings that they produce and create. Vig, Murray, Star (2014) highlight that understanding the productive aspects of models, as well as their breaking points, is an important component to integer addition and subtraction. Understanding ways that children use these drawings productively and unproductively could help provide insight into affordances and hindrances of models.

Theoretical Perspective

Word use, visual mediators, narratives, and routines are the central tenets of discourse in commognitve theory (Sfard, 2008). Although all of the tenets of commognitive theory work together synergistically, the visual mediators are the focus of this paper. Visual mediators include recognizing artifacts such as gestures or drawings as part of a students' discourse. Drawings that children produce while solving integer addition and subtraction open number sentences represent a component of their discourse that is just as important as their verbal reasoning. For students, drawings can be as communicative as their verbal expressions and investigations into their drawings for negative integers can be illuminative. During the time with the Grade 5 students, they often used drawings to help them make sense of the negative integers. This research brief highlights one of these three students, Alice, and her drawings. Specifically, this research brief addresses the research question:

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

What types of drawings does Alice produce as she solves integer addition and subtraction open number sentences?

Methodology

Three Grade 5 students from a rural Midwest school participated in a 12-week teaching experiment (Steffe& Thompson, 2000) centered on integer addition and subtraction, using both contextual problems and open number sentences. The students met in both individual and group sessions during the teaching experiment and all sessions were videotaped. The students primarily solved problems in contexts during both the individual and group sessions; however, there were four individual sessions where students solved open number sentences.

Integer addition and subtraction open number sentences were solved during four individual sessions across the 12-weeks. During these sessions the open number sentences were provided on paper, with no manipulatives and only a box of markers available. The students were asked to explain their reasoning for solving the open number sentences. Alice was chosen as the participant to report on in this research brief because of the three participants Alice used drawings the most. Alice's drawings from the individual sessions with open number sentences represent the unit of analysis. A grounded theory approach (Glaser & Strauss, 1967) was utilized for categorizing the different types of Alice's drawings. Both the verbal interactions from Alice and the teacher-researcher, as well as, the process of her drawings were transcribed. Each of the drawings, paired with these descriptions and transcripts, was examined and sorted for common themes.

Results & Discussion

Single Set of Objects

Alice often drew a Single Set of Objects to solve the open number sentences. The Single Set of Objects were utilized in two different ways, by either crossing off objects or adding objects on. For crossing off a Single Set of Objects, Alice began by drawing an initial set of objects (e.g., boxes or tallies), which represented either a positive or negative integer. She then crossed some of the objects off (see Figure 1). The objects crossed off represented either the addition or subtraction of a positive or negative integer. Alice's objects that she drew included either boxes or tallies for the Single Set of Objects.

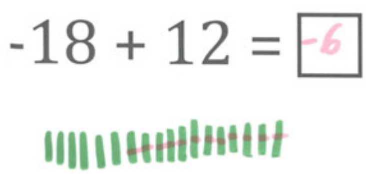

Figure 1: Single Set of Objects for Solving -18 + 12 = □

In Figure 1, Alice used 18 tallies to represent the negative integer, -18. Then, Alice crossed off 12 tallies, representing the positive integer being added.

Double Set of Objects

Other drawings that Alice produced frequently included two layers or two separated groups of objects. The drawings that included these layered or separated objects were considered a Double Set of Objects (see Figure 2). For example, Alice used the Double Set of Objects drawings with when solving □+ -4 = 13. In Figure 2, the pink tallies represent negative four and the green tallies represent 17. Alice added tallies until the leftover tallies totaled 13. Then, she counted all of the green tallies to determine the solution of 17.

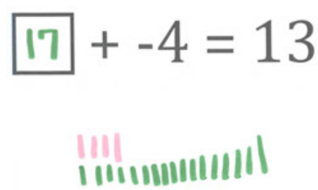

Figure 2: Double Set of Objects for Solving□ + -4 = 13

Although Alice would use objects that were layered on top of each other, Alice also represented the addition and subtraction with segregated layers. For example, in Figure 3, Alice represented the -4 with boxes and then segregated the second set of boxes, but this layer of boxes was not stacked on top of the other boxes like in Figure 2. She then added up all of the boxes to get 14. Alice described her drawing, "I did four for negative four (motions across four boxes), then I did how many I was adding a box for how many it would take to get me up to ten." Although this type of drawing is reminiscent of a Number Path (see, e.g., Bofferding, 2010; Wessman-Enzinger & Bofferding, 2014), Alice did not recognize these boxes as orderings of numbers. Instead, she described the quantities -4 and 10 without order.

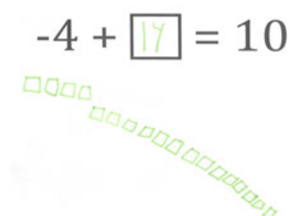

Figure 3: Double Set of Objects for Solving -4 + □= 10

Number Sentences

Alice often drew horizontal or vertical number sentences to solve the open number sentences. Sometimes the horizontal or vertical number sentences used only positive integers, while sometimes the horizontal or vertical number sentences incorporated negative integers. For example, to solve -12 – -11 = □, Alice drew a vertical number sentence involving negative integers. She vertically wrote, -11 – -12. Yet, Alice still obtained -1 as a solution. This is consistent to findings that children often incorrectly apply the commutative property when subtracting negative integers (Bofferding, 2010).

Number Path & Number Lines

Alice only drew a Number Path with negative integers once during all of the individual sessions (see Figure 4). Alice did not draw a conventional Number Line; rather, she drew a Number Path (see, e.g., Bofferding, 2010; Wessman-Enzinger & Bofferding, 2014).

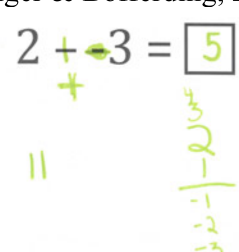

Figure 4: Number Path for Solving 2 – -3 = □

Her drawing in Figure 4 included the ordering of a Number Line and has a close relationship to the formal Number Line, yet is not a Number Line. She drew this Number Path after solving the open number sentences 2 – -3 = □ during the last individual session of the twelve weeks. To solve 2 – -3 =

□, Alice first solved it by recalling a rule that the student developed during the group sessions, “Because it's just like the last one. You do plus (changes minus sign to plus sign) and take that off (scratches off the negative symbol of -3). And, it just be like two plus three.” When asked why it worked, she drew both a Single Set of Objects (e.g., tallies) and then a Number Path to try to justify her reasoning. Although she drew a Number Path, she didn’t utilize it to solve or justify her solution. In fact, Alice shared that she didn’t know how to use either of her drawings (Single Set of Objects or Number Path) to explain her answer.

Conclusion

Alice used a variety of drawings that were productive for solving integer addition and subtraction number sentences. Despite the different types of drawings she drew, she did not draw very many Number Paths or any Number Lines during the individual sessions. Instead, Alice drew the quantities of objects that seem to be related to direct modeling strategies (Carpenter et al., 2015). Towards the end of the teaching experiment Alice began to utilize a Number Path, which may highlight that the development of drawing Number Lines takes extended time for some children. This may point that the development of using Number Paths and Number Lines takes significant for some students. These different types of drawings (e.g., Single Set of Objects, Double Set of Objects) provide further insight into the ways we understand student thinking about addition and subtraction with integers.

Acknowledgements

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COORDINATING EQUATIONS AND GRAPHS OF POLYNOMIALS: WHAT DO PATTERNS IN STUDENTS' SOLUTION STRATEGIES REVEAL?

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What strategies can secondary precalculus and calculus students use to identify whether representations correspond to the same function, and how do the coordination strategies a student uses correspond with his/her success in identifying matching representations? We analyzed 64 think aloud protocols in which high school students were asked to identify whether a given pair of representations (graph, table, equation) corresponded to the same function to answer these questions. Results indicated that two clusters of students using distinct strategies were identifiable on tasks that required matching equations and graphs and equations and tables. However, we were not able to identify clusters by strategy for graph-table representations. We discuss how task demands might explain this pattern of results and consider implications for teaching that are responsive to students' conceptions of mathematics.

Keywords: Algebra and Algebraic Thinking; High School Education; Cognition

Objectives

A critical ongoing challenge for mathematics educators is helping more students achieve success in advanced mathematics courses such as calculus where flexibly using multiple representations is a critical skill. This study seeks to extend the field's understanding of how secondary calculus and precalculus students identify whether pairs of equations, graphs, and tables corresponded to the same function. In secondary mathematics and especially in reform approaches to calculus (e.g. Hughes-Hallet et al., 2012), students are often required to identify whether two graphs, tables, or equations are equivalent (Leinhardt, Zaslavsky, & Stein, 1990). This identification is part of a more complex set of skills related to coordinating multiple representations (CMR). This study builds on our past work (Zahner, 2012, Chang, Cromley, & Tran, in press) through incorporating mixed methods to more fully describe how 64 secondary students approached CMR tasks.

Consider the CMR identification task in Figure 1. A student has multiple options when deciding whether these two representations correspond to the same function. Since the y-intercept and constant term match, these two representations might correspond to the same function. How could a student check? She could use the equation to evaluate $f(x)$ for a set of inputs (e.g., let $x = \{1, 2, 3\}$). This point-testing strategy would solve this mismatch problem (e.g. since $f(3) = 7$ but (3,7) is not on the graph, the graph and equation are not the same function). However, with some background knowledge another strategy is possible. Knowing that degree two (quadratic) functions have a U-shaped graph, a student could coordinate the shape of the graph and the equation and conclude these cannot be the same function. These two hypothetical solutions represent different strategies for solving CMR problems.

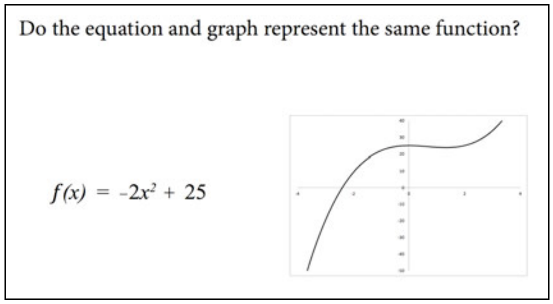

Figure 1: Sample Equation Graph Item

This study was designed to investigate the strategies used by precalculus and calculus students to coordinate representations of polynomials. Our main research questions were 1) What are the strategies students use to solve CMR identification tasks such as this? 2) Is it possible to identify distinct groups of students based on strategies used? 3) If it is possible to identify distinct groups, are there data to show that one group is more or less successful than the other?

Theoretical Perspectives

We draw on theoretical models and analytical methods from Siegler's overlapping waves theory (Siegler, 2005). This theory describes children's development and use of problem solving strategies as probabilistic. Rather than developing expertise in strategy use along a linear progression, problem solvers often mix more and less sophisticated strategies, and the problem prompt is an important mediator of strategy choice. This latter insight from Siegler's cognitive work brings to mind Wertsch's (1998) cultural historical and semiotic analysis of the affordances of representational tools that make particular problem solving strategies more or less viable. Thus in our discussion we tie together these two distinct branches of research.

Methods & Analysis

The participants were 64 secondary students enrolled in precalculus and calculus in suburban high schools in the Northeast. The sample consisted of 60% female and 79% White students. As a proxy for socio-economic status, we measured maximum parental education; 80% of families had at least one parent with a Bachelor's degree or above.

The students were presented with a set of 12 questions similar to the question in Figure 1 and they were asked to follow a think-aloud protocol (Ericsson & Simon, 1984) while solving the problems. The questions were presented on a computer screen that included eye-tracking tools. The 12 prompts included four items with each of the possible combinations of equation ↔ graph, equation ↔ table, and graph ↔ table. The functions were distributed among linear, quadratic, and cubic functions with integer coefficients. Half of the items showed a match.

The think aloud data were transcribed and divided into utterances roughly corresponding to a phrase containing a subject and a verb such as "the negative x squared matches the parabola." Codes were created to identify CMR strategies including matching ordered pairs (MOP for non-intercepts and MOPX or MOPY for matching an intercept without naming the intercept as such), explicitly matching the x- and y- intercepts (MINTX and MINTY), evaluating the degree of a function (DEG), and evaluating the direction (DIR) or magnitude of the leading coefficient (MAGC). A total of 2889 utterances were coded. After the initial coding by strategy, two overarching strategies were identified: evaluating global characteristics of the function (DEG, DIR, and MAGC), and evaluating on a point-by-point basis (MOP, MOPX, MOPY, MINTX, and MINTY).

In the process of coding we also noted that although some students tended to use global strategies while others used "point-by-point" comparisons, both groups of students who used different strategies had similar levels of accuracy on the CMR tasks (Zahner & Cromley, 2014). To further explore students' strategy-use profiles by representation pair, we employed a cluster analysis (Aldenderfer & Blashfield, 1984) on strategy use by representation pair. Cluster analysis is a person-centered approach that uncovers homogeneous groups underlying a set of data (DiStefano & Kamphaus, 2006). We used a hierarchical cluster analysis procedure based on squared Euclidian distances with the Ward's method on standardized ratios of strategy use (i.e., Z scores of [occurrences of a strategy/occurrences of all strategies per student]; Milligan, 1996). We then evaluated the validity of the obtained cluster solutions by comparing the subgroups with respect to accuracy in answering the CMR items, time used in answering the CMR items, and total number of strategies used.

Results

For the equation ↔ graph (EG) items, cluster analysis results indicated two distinct groups of students by strategy use: EG group 1 (n = 49) used significantly more EVAL and fewer MOP strategies, whereas EG group 2 (n = 16) applied significantly more MOP but fewer EVAL strategies. Comparing the two groups in accuracy and average time used, we found both groups were similar in accuracy but EG group 1 spent significantly less time than EG group 2 to complete the equation-graph items.

Table 1: Summary of cluster analysis for EG items

	Accuracy Rate	Time Used**	No. of Strategies Used
Group 1 (n=49)	85.0%	25.1	9.4
Group 2 (n=16)	87.5%	36.8	9.3

**p<.01

For the equation ↔ table (ET) items, cluster analysis results also indicated two distinct groups of students. However, the groups were only marginally different with respect to their use of EVAL. Instead the ET group 1 (n = 12) explicitly matched the y-intercept (MOPY) more often while ET group 2 (n = 52) matched order pairs without explicitly referencing the y-intercept as a salient point. Comparing these two groups in accuracy and time used, we found members of ET group 1 were significantly quicker but less accurate in responding to the equation-table items.

Table 2: Summary of cluster analysis for ET items

	Accuracy Rate**	Time Used***	EVAL~	MOP***	MOPY***
Group 1 (n=52)	86%	51.6	1.4%	68.7%	22.0%
Group 2 (n=12)	67%	23.2	9.6%	22.5%	66.3%

Note: ~ $p< .10$; * $p< .05$; ** $p< .01$; *** $p< .001$.

Finally, the graph ↔ table (GT) cluster analysis did not yield a solution we consider meaningful in practice due to the size difference in clusters.

Discussion & Conclusions

In summary, the cluster analysis confirmed one of the hypothesized relationships. For the items that presented an equation-graph pair of representations, one group of students tended to use strategies from the superordinate category "Evaluate" while the other group tended to use strategies under the superordinate category "Match Ordered Pairs." Interestingly there was no significant difference between the two groups in terms of accuracy on the task, but the students who used evaluate tended to complete the task faster than those who matched ordered pairs. It is notable that the accuracy of the respondents' answers was not significantly related to cluster identification. This indicates that there are multiple possible solution pathways on CMR identification items like the ones used in this study.

Looking ahead, we are currently testing whether modifications such as adding arrows and boxes to highlight particular features of the prompt will be associated with a shift in the strategy that students use while solving the problem.

Second, the cluster analysis revealed that clusters of students by overarching strategy were not readily identifiable on the table ↔ graph items. We relate this to the affordances of a table, which makes point-by-point comparison easy and global comparisons relatively difficult. In future research where we attempt interventions, it may be worthwhile to build on Lobato's focusing perspective (Lobato et al., 2003) and investigate how modifications to table items (e.g. adding a difference or ration column) can prompt students to use more sophisticated strategies.

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EXPLORING A ‘NOT-SO-COMMON’ COMMON FRACTION REPRESENTATION

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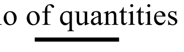
This study examines undergraduate responses to a fraction task designed to investigate how different individuals leverage a perceptually based “sense of proportion” with a continuous, nonsymbolic representation of fraction values. Participants were asked to double two parallel line segments used to represent the fraction 3/5, write an equation, and give explanations to show what they did mathematically. The representation elicited a diversity of symbolic operations and explanations of reasoning among participants. This may provide further insight into the complex psychological foundations involved with learning fractions.

Keywords: Rational Numbers; Elementary School Education; Reasoning and Proof

Developing a strong understanding of fractions is one of the main goals of K-8 mathematics education (e.g., CCSSM, 2010).Yet children often experience considerable difficulties developing an understanding of fractions. Indeed, the National Math Advisory Panel (2008) declared fraction knowledge to be the most important foundational skill that is not presently developed in the school-aged population. Additionally, fractions are also considered a topic not adequately developed in primary education (NMAP, 2008)

One potential source of these difficulties may be located in foundational assumptions about how fractions are learned. For example, fraction knowledge is often approached from the assumption that fractions knowledge must be rooted somehow in whole number knowledge (e.g., Steffe & Olive, 2010). However, some research suggests that we should re-evaluate this widely held assumption. Notably, evidence strongly suggests that humans (and even non-human primates) easily compare visuo-spatial ratios in multiple types of representations such as line segments, dots, and areas (e.g., Jacob, Vallentin & Nieder, 2012; Lewis, Matthews, Hubbard, 2014; Matthews & Chesney, 2015). This perspective accords well with Carraher’s (1996) distinction between a *ratio of quantities* and a *ratio of numbers*. A ratio of quantities concerns two magnitudes such as the ratio of two lengths (e.g., one half instantiated as , or even the ratio between a circle’s circumference and its diameter). If we take seriously the proposition that number knowledge is “developed through acting and reflecting upon physical quantities” (Carraher, p. 241), then we should pay due attention to research suggesting fractions qua ratios might be accessed by the perceptual system – even in approximate form. The implication is that this type of representation may provide a valuable foundation for understanding fraction magnitudes in ways that may be elusive to current research paradigms.

Although some research does suggest that humans have an early ability to identify and reason with visuo-spatial ratios, little research has investigated how such nonsymbolic ratio abilities interact explicitly with mathematical reasoning. With this research, we sought to investigate this question using a sample of undergraduate students from a selective Midwestern university. This study was undertaken as part of a larger study of undergraduates’ understanding of fractions which has shown a) that fractions knowledge is often fragile - even at a selective university, and b) that undergraduates’ nonsymbolic ratio processing abilities are predictive of standardized math achievement test scores (Lewis, Matthews & Hubbard, 2014).

Theoretical Question

Ratios as Units?

It is not clear that research in mathematics education has sufficiently explored whether a continuous visuospatial ratio can be taken as a mathematical unit. To clarify, can the relationship between two line segments deemed to be *perceptual items* by Steffe & Olive (2010) in the plural sense potentially be conceived alternately as a *perceptual item* in the singular? Stated alternately, can perception of an *intensive* ratio between what are commonly understood as two *separate* items itself be taken as a mathematical unit? We explore this possibility in the discussion section by considering the construct of *unitizing*- which we take to mean "the mental act of forming unit items out of sensory experience (Olive & Lobato, 2008, p.5). For example, 3 equal slices of a pizza divided into 5 equal slices can be *unitized* as a 3/5-unit (such that two of these 3/5-units can then be understood to make 6/5). *Units-coordination* is necessary to *simultaneously* understand, for example, that 3/5 is three 1/5's, which is in turn 3/5 of a unit whole (e.g., Steffe& Olive, 2010). Our question examines how the unit may be identified in a continuous representation that is in some sense equivalent to 3:5 such as .

Methods

Participants and Task

27 undergraduate participants were given a pencil-and-paper assessment of aspects of their fractions knowledge that included a task adapted from Carraher (1996) shown below (Fig. 1).

5. The fraction shown by the line segments below represents $\frac{3}{5}$. If you double the length of **both** line segments, what did you do mathematically? (Write an equation and try to solve the problem. Then explain.)

a. Equation:

b. Explanation:

Figure 1: Selected task

Data Analysis

First, we considered any nonsymbolic drawings or markings made by participants on the representation. We categorized markings as one of five types: partitioning each segment separately, partitioning both segments, extending both segments, extending only one segment, and no markings/drawings. Second, we categorized equations as either showing the fraction as a ratio remained the same ("correct") or did not ("incorrect"). Subcategories were created for equations that were procedurally correct and those that were not, those that included idiosyncrasies such as the presence of algebraic symbols, and those treating the problem mathematically as either two separate equations or one equation. Finally, explanations were classified as either congruent with the procedure, in conflict with the procedure, or not present.

Results

Results were consistent with the extensive body of research demonstrating learner difficulties coordinating numerical and non-symbolic representations of fractions (e.g., Moss & Case, 1999).

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Here, we focus on examples that help communicate the diversity of responses generated. The figures below show examples of a "correct" result with a procedurally correct equation where the verbal response is in agreement (Fig. 2), a "correct" result with algebraic symbols and an incorrect procedure that is in conflict with the explanation (Fig. 3), as well as examples of an "incorrect" result with an incorrect procedure (Fig. 4), and a "correct" result with a correct procedure that is approached in two separate equations (Fig. 5). It is of note that 15 participants verbally explained the result as an equivalent fraction – not including those participants who found the result to be 6/10 or 3/5. Finally, two participants *explicitly* identified the two segments as a single "whole" or unit (Fig. 2 & Fig. 3).

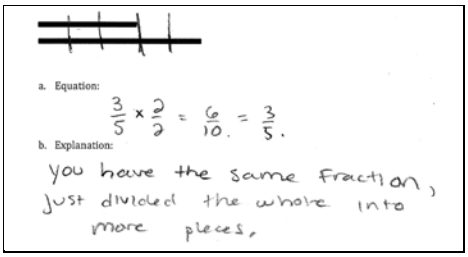


Figure 2: Partitioned "whole"

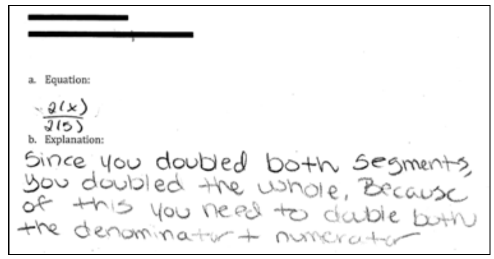


Figure 3: Algebraic symbol use with unit "whole"

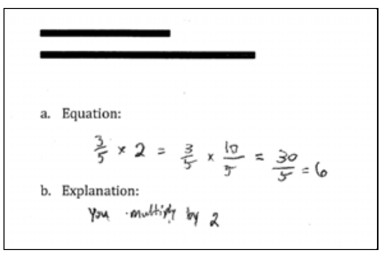


Figure 4: "Incorrect" response

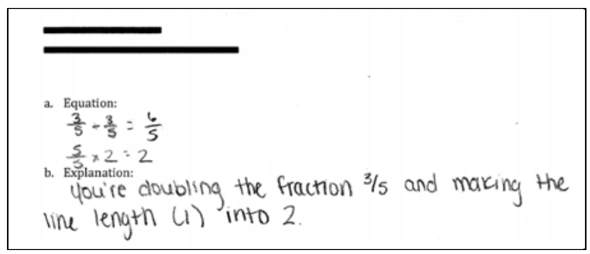


Figure 5: Separate equations (and operations)

Discussion

Although this study is clearly limited in scope (indeed, it only considers a singe task), we feel the wide variety of procedures brought to bear on identifying the "same fraction" as well as the explicit identification of two segments as a unit whole are notable. They demonstrate a marked need to continue the project of disentangling numerical, nonsymbolic, and verbal understandings of what might constitute a unit and under what circumstances. For example, it may simply have been the case that a numeric unit of 3/5 was identified and that the procedural knowledge necessary to multiply the numerator and the denominator was sufficient to recognize the resulting equivalent fraction. However, participants' drawings, explanations, and inscriptions of the nonsymbolic fraction as single or separate equations suggests different ways of interpreting the representation - from potentially a single perceptual item that can be "doubled" (Fig. 2 & Fig. 3) to separate unit items that are not integrated (Fig. 5). Similarly, it is difficult to determine what additional mathematical processes support idiosyncratic symbolic responses that recognize the preserved ratio (Fig. 3) or those that appear not to rely on perceptual supports at all (Fig. 4). Thus, further research is needed to delimit alternative ways in which units come to be understood– particularly insofar as basic perception may support certain intuitive conceptions which may help ground more formal knowledge. Ultimately, if unit ratios can in fact be both perceptually *and* mathematically identified, this may have implications for early fractions pedagogy by leveraging a more intuitive “sense” of proportional and multiplicative reasoning.

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DYLAN'S COORDINATING UNITS ACROSS CONTEXTS

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Keywords: Rational Numbers; Learning Trajectories; Instructional Activities and Practices

Introduction/Purpose/Methods

Researchers have identified a shared cognitive necessity for understanding fractions as numbers and for conceptualizing multiplication (and division) – an ability to *coordinate multiple levels of units* (Hackenberg & Tillema, 2009; Steffe, 1992). We describe how a teacher's alternating tasks with fractions and tasks with whole numbers in multiple contexts contributed to growth in a sixth-grade student's (Dylan) ability to coordinate units during an 11-session constructivist teaching experiment (Steffe & Thompson, 2000).

Results/Conclusions/Implications

Dylan had interiorized two levels of (whole number) units and could coordinate a third level of whole number unit in activity at the onset of the teaching experiment. Introducing tasks where the units coordination demands varied helped Dylan begin to anticipate coordination of three levels of units in activity with fractions. Dylan thus made progress toward interiorizing three levels of units, a developmental precursor to understanding improper fractions as numbers (Steffe & Olive, 2010).

Within-session task sequencing generally involved whole numbers tasks preceding fractions, but the random sequencing of tasks in some contexts, including engagement with the iOS app *CandyDepot* (LTRG, 2013)meant that any task could potentially involve improper fractions or whole numbers. This seemed to help prolong productive interaction in Dylan's zone of potential construction (Norton & d'Ambrosio, 2008). Future plans include further investigations of this conjecture and its generalizability.

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THE IMPACT OF AN EARLY ALGEBRA PROFESSIONAL AND CURRICULUM DEVELOPMENT PROJECT ON STUDENTS AND TEACHERS

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Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Elementary School Education; Teacher Knowledge

This poster reports the evaluation findings of a sustained early algebra professional and curriculum development project; Conceptual Algebra Readiness for Everyone, (CARE) in fourth and fifth grade for both students and teachers. CARE is designed to prepare students conceptually for algebra-conceptual algebra readiness. Through a state Mathematics Science Partnership grant with a local school system we developed weekly problem solving activities for students in grades 3-7 to improve their conceptual algebra readiness. Our guiding rationale is that helping students understand certain underlying, foundational concepts of algebra will prepare them for future success in formal algebra.

Three parts of the evaluation are presented. First, treatment/control test results of algebra readiness test given to fourth and fifth grade students are provided. Second, graphs of the mathematics standardized test results over a five year period for grades 4 and 5 are given. Finally, results from the Diagnostic Teacher Assessments in Mathematics and Science (DTAMS) Exam from the University of Louisville was utilized to determine if professional development impacted teachers' mathematical content knowledge. The methodology and participants differed for each part of the evaluation. The project and evaluation occurred in a Midwestern, high-needs, urban school (free and reduced lunch rate of 75% and a minority population of 51%). The results provided were collected in different years and involve different cohorts of teachers and students.

Results from the evaluation indicate that children using the CARE materials are engaged in deep mathematical thinking. Not only are they problem solving, but they are able to model mathematical situations and describe their own thinking in mathematical language. In almost every case, there are indicators that they are engaged in mathematical practices consistent with the Common Core State Standards for Mathematical Practice. Classroom observations of students engaged in these activities indicated that students enjoy solving the activities and that many teachers are developing problem based learning approaches to instruction. Students are frequently engaged in rich mathematical discussions. Many of the teachers and students are realizing the type of instructional practice advocated by the CCSS. These results indicate that students are developing algebraic reasoning through their participation in the project.

Our study supports current research that early algebra interventions can impact algebra readiness (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015) and provides evidence that teachers can successfully provide algebra readiness interventions. Teachers, when provided with the appropriate professional development, can be agents of change for algebra readiness.

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OPPORTUNITIES TO LEARN ALGEBRA IN SECONDARY TEACHER EDUCATION PROGRAMS

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The National Research Council (2010) review noted the dearth of detailed studies of mathematics teacher education programs. As algebra is a gatekeeper for advanced mathematics and science studies and career opportunities (Kilpatrick & Izsàk, 2008), it is important to investigate the opportunities that preservice teachers (PSTs) have to prepare to teach algebra to diverse student populations at a program level. The Conference Board of the Mathematical Sciences (2012) recommends that PSTs should have opportunities to learn advanced algebraic knowledge and connections between advanced algebra and school algebra in five dimensions: (1) applications, (2) algebraic structures, (3) connections between geometry and algebra, (4) reasoning and proofs, and (5) technology. This poster will describe the nature of opportunities provided by three secondary mathematics teacher education programs (Beta, Gamma, and Kappa Universities) according to these five dimensions.

For this poster we draw on case study data from a larger project, but we only analyzed interviews with instructors of Linear Algebra and Abstract Algebra. Here we preview results from Linear Algebra, but the poster will also include results from Abstract Algebra.

Instructors of Linear Algebra at Beta reported opportunities to learn [OTL] advanced algebra across all five dimensions, whereas Gamma offered OTL across the four dimensions other than technology, and Kappa offered OTL across the four dimensions other than applications. Across the three programs instructors reported fewer opportunities to learn connections between advanced algebra and school algebra than opportunities to learn advanced algebraic ideas. Beta stood out by offering opportunities to learn connections across all dimensions. Gamma provided opportunities to learn connections only in the dimensions of applications and reasoning and proof. Kappa offered no opportunities to learn connections at all. In addition to quantitative differences, universities provided qualitatively different experiences to learn in each dimension. For instance, for the dimension of technology, Kappa only used TI-84 calculators, whereas Beta used Maple.

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PIECES OF THE PUZZLE: LEARNING FROM DIFFERENCES IN STUDENTS DRAWING, NOTATING AND EXPLAINING FRACTIONAL RELATIONSHIPS

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Keywords: Algebra and Algebraic Thinking; Middle School Education; Equity and Diversity

An enduring challenge in today's middle schools is the increasing need to understand students' diverse cognitive needs and organize mathematics instruction accordingly. Students enter middle school at different levels of reasoning about multiplicative relationships (Hackenberg & Tillema, 2009). These levels affect the way they understand quantitative relationships, thus affecting the way they work with and talk about fractions. One way schools have traditionally dealt with cognitive difference is to track students into different classes. Another way, which is relatively new, is differentiating instruction by creating tasks that are both accessible and challenging for different groups of students (Tomlinson, 2014). In order to differentiate instruction for cognitively different students in mathematics, it is necessary to create models of student thinking (Steffe, 1994). This requires attention not just to the products a student creates, but also to the meaning of those products.

Our study is part of a larger study that investigates how to differentiate mathematics instruction at middle school. The focus of our poster study is on models of middle school students' reasoning with quantitative relationships and the potential use of those models in differentiating instruction. In our analysis we use video/audio records of individual and group work of middle school students in addition to their written and computer works. Specifically, we look at their drawings, algebraic notation and explanations of fractional relationships between two unknown quantities.

Our preliminary analysis shows interesting differences in the models of thinking we have created for two students who are at the same level of multiplicative reasoning. For example, one student anchored his explanations of quantitative relationships in drawings and seemed to create equations based on those. The other student appeared to anchor her explanations in equations, struggling with visual representations of quantitative relationships. Given the persistence of this level of reasoning for students (Steffe, 2007), we would like to learn more about the models of reasoning for these students by researching the diversity of thinking within the group. We believe this will guide us in responding to our students' diverse cognitive needs through differentiated mathematics instruction.

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SECONDARY PRE-SERVICE TEACHERS' OPPORTUNITIES TO LEARN ABOUT MODELING IN ALGEBRA

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Keywords: Algebra and Algebraic Thinking; Modeling; Teacher Education-Preservice; Teacher Knowledge

The U. S. Department of Education (2009) noted that teaching is becoming increasingly challenging as teachers seek to meet changing needs and requirements: Diversity of student populations is growing and state graduation requirements, especially with respect to algebra, are evolving. Algebra is a foundation for advanced mathematics and a gatekeeper for high school students to enter a college or university (e.g., Kilpatrick & Izsák, 2008). Given this context, teacher education programs play a vital role in supporting pre-service teachers (PSTs) to overcome these challenges and prepare their students to learn about algebra.

Modeling is an area that has been emphasized since *Common Core State Standards for Mathematics (CCSSM)* included modeling as a mathematical practice for all K-12 students (NGA & CCSSO, 2010). The processes of modeling described in *CCSSM* (e.g., identifying and selecting variables) is related to algebra learning because students choose variables and develop algebraic representations (e.g., graphs, equations) to solve realistic problems. The new expectations from *CCSSM* related to modeling raise questions about whether teachers are being prepared to teach algebraic modeling in secondary school mathematics classrooms.

As part of a larger project, *Preparing to Teach Algebra*, we gathered data at five universities to investigate PSTs' opportunities to learn about algebra. We interviewed instructors of required mathematics and mathematics education courses for PSTs and collected corresponding instructional materials. We also interviewed two groups of PSTs from each institution. With these data sets, we aim to answer the question, "What opportunities do secondary mathematics teacher preparation programs provide to learn about algebra related to the modeling standards described in *CCSSM*? For data analysis, we coded our data according to each modeling process described in *CCSSM* and compared different types of opportunities.

We will present concentrations of opportunities provided to PSTs to learn about and learn to teach algebra related to each modeling process within and across mathematics and mathematics education courses in five institutions. For example, at Gamma University, both instructors and PSTs describe diverse opportunities to learn about creating algebraic representations and few opportunities to reason behind their solutions to modeling problems. Teacher educators can look across opportunities provided from these exemplar teacher education programs to find gaps in their programs and refine their instructions about algebraic modeling.

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MATHEMATICAL EQUIVALENCE AND ALGEBRA: FUNCTIONS, VARIABLES, AND EXPRESSIONS

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Mathematics educators and researchers agree that algebra is pivotal in students' mathematical development (National Research Council [NRC], 1998; RAND Mathematics Study Panel, 2003). However, children encounter many obstacles as they acquire algebra (National Council of Teachers of Mathematics, 1988). Because of its importance and the challenges children have learning it, experts have recommended introducing algebra as early as kindergarten (NRC, 1998; RAND, 2003). This recommendation raises questions about how algebraic thinking develops over the longer term, and what prerequisite knowledge is needed for success. The current study investigated students' understanding of mathematical equivalence as it relates to performance on a range of algebraic tasks.

Some students have an operational understanding of the equal sign and view it as a signal to supply an answer or apply the operation. Those students fare worse in algebra than those with a relational view of equivalence in which they view the equal sign as representing a balance between two quantities (Knuth, Stephens, Mcneil, & Alibali, 2006). This research, however, has focused solely on the effects of equivalence understanding on solving algebraic equations. Thus, it is unclear whether equivalence concepts also impact learning in other areas of algebra.

There is reason to think equivalence understanding is implicated throughout this system. For functions, students have to understand that any manipulations that they make to x will result in a particular value of y. The relational view of mathematical equivalence might help students perceive the functional relations among symbols rather than focusing on specific quantities. Similarly, variables are symbols that represent unknown quantities in an equation or expression. A relational understanding of equivalence might help students comprehend their many-to-one mapping to referents.

In the present study, 242 7th and 8th grade students completed a 22-item written assessment that included items about equivalence, equations and equation solving, expressions, functions, and variables. Students' level of equivalence understanding was coded according to the guidelines provided in Knuth et al. (2006). In a series of regressions, with level of algebra instruction controlled, there were significant relations between equivalence understanding and three algebra subskills (variables, functions, and equations). This finding suggests that differences in equivalence understanding can have far-reaching effects on algebra learning.

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USING FRACTION MODELS WITH DEVELOPMENTAL ALGEBRA STUDENTS

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Keywords: Post-Secondary Education; Rational Numbers; Instructional Activities and Practices

The National Center for Education Statistics (NCES) reported that almost 30% of incoming freshman students entering a postsecondary institution in the fall of 2000 needed developmental coursework due to a lack of preparedness for standard credit-earning courses (Wirt, Choy, Rooney, Provasnik, Sen, & Tobin, 2004). The majority of these developmental courses were in the area of mathematics. Despite this remediation, developmental mathematics courses usually have the highest failure and incompletion rates among all developmental subjects (Bonham & Boylan, 2011). According to data from the National Educational Longitudinal Study, only 30% of students pass all of the developmental mathematics courses in which they enroll, as compared to 68% in writing and 71% in reading (Attewell, Lavin, Domina, & Levey, 2006). These developmental mathematics courses become a barrier to students' degree completion. Therefore it is necessary to examine the concepts with which students in these courses typically struggle. Fractions have been studied extensively across the middle and high school grades and have repeatedly been cited as a difficult concept for students of all ages (Moss & Case, 1999). Yet, fraction understanding is essential for further learning of mathematics. To date, little research has been done on adult developmental students' understandings and conceptions of fractions.

The purpose of this study was to examine adult developmental algebra students' knowledge of fraction addition with unlike denominators. Prior research has shown that two typical errors are made by students when adding fractions with unlike denominators: representing fractions with unequal whole units and not using a common denominator (Hui-Chuan, 2014). The present study conducted clinical interviews with two adult developmental algebra students. The first set of interviews revealed errors consistent with the aforementioned research, as well as the tendency to try to remember rules learned in prior mathematics courses. Additionally, participants' responses on fraction addition tasks did not rely upon a model (e.g., pie chart) to represent the fractions or upon rational number sense (e.g., magnitude and/or estimation of the fractions). A teaching activity using paper folding was designed and implemented during the second set of interviews. The activity helped both participants to understand *why* equal whole units and common denominators are necessary in fraction addition.

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THE ROLE OF SLOPE IN CONCEPTUALIZING THE LINE OF BEST FIT

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Keywords: Algebra and Algebraic Thinking; Data Analysis and Statistics; Middle School Education

How students conceptualize slope and the extent to which they are able to consider functions as relations of covarying quantities impacts their understanding of linear functions as well as advanced topics that require a foundational understanding of linearity. In this case study, we consider how one student, Claire, leveraged her notions of slope when conceptualizing and placing the line of best fit (LOBF). In particular, we call on previous research to investigate the conceptualizations of slope (Moore-Russo, Connor, & Rugg, 2011) and the stages of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) Claire used on four interview-based tasks which asked her to place a straight piece of wire to represent the LOBF on a scatterplot (see Casey, 2015).

The first two tasks involved scatter plots representing the relationship between the drop height and corresponding bounce height of a golf ball. Claire struggled with the notion of placing a straight line to represent non-collinear data points. Her responses showed that she was neither reasoning about slope nor the covariational relationship between drop height and bounce height. She placed inaccurate lines that were meant to show the range of values represented by the points. The second two tasks provided a context (ticket price vs. movie attendance) that seemed to resonate with Claire. She immediately described the relationship between the covarying quantities by describing the *Directional Change* (*L2*) between the quantities. This helped her place an accurate line using the *Behavior Indicator* conceptualization of slope. Despite her success on these tasks, Claire was not able to explain how she determined the line's steepness (*Physical Property*) using covariational language (i.e., *Amount of Change – L3*).

The findings suggest that covariational reasoning and the notion of slope are closely connected ideas and suggest the need to more carefully link the two bodies of research. Instruction that focuses on slope as indicating the increasing and decreasing behavior of a line is supported by, and in turn supports, students' ability to think covariationally about the direction of change of two quantities. More clearly describing how covariational reasoning can be developed when studying slope, one of students' earliest interactions with covarying quantities, could greatly improve the learning of slope and strengthen students' understanding of advanced mathematics and statistics.

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BUILDING ALGEBRA CONNECTIONS IN TEACHER EDUCATION

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Given the move toward algebra-for-all across the U.S., most mathematics teachers find themselves teaching algebra in their first position upon completion of their teacher preparation program; in addition, they are expected to teach algebra to a more diverse population of students than ever before (Stein, Kaufman, Sherman, & Hillen, 2011). Given this movement and algebra's role as a gatekeeper to college and career opportunities (e.g., Kilpatrik & Izsák, 2008; Moses & Cobb, 2001), it is imperative that we study how future teachers are prepared to teach algebra.

When future teachers learn algebra and learn to teach algebra, they need to make connections among mathematical topics and in relation to others, as they consider mathematics as a "whole fabric" (National Board for Professional Teaching Standards, 2010). In this poster, we report the findings from an investigation of future teachers' opportunities to learn about such connections. Specifically, our research question is "What opportunities do secondary mathematics teacher preparation programs provide to learn about connections related to algebra and algebra teaching? The algebraic connections of interest here include: (1) connections within algebra, (2) connections between algebra and other mathematical fields, (3) connections between algebra and non-mathematical fields, and (4) connections between college-level algebra and school algebra.

This study is being conducted as part of a larger mixed methods study (i.e., a national survey and five case studies of secondary teacher education programs) that investigated opportunities to learn about algebra, algebra teaching, equity issues related to algebra, and the algebra described in the *Common Core State Standards for Mathematics*. Data were collected from semi-structured instructor interviews, focus group interviews with pre-service teachers (PSTs), and course materials. To answer the research question of focus here, two members of the research team coded interview data for opportunities to learn related to the four types of algebra connections described above; connections that did not fit into these types were also noted.

Analysis of this data is ongoing; however, preliminary findings show that different types of opportunities to learn about algebra connections were found at all case study sites. Instructors and PSTs alike at Gamma University reported more attention to connections within algebra and between algebra and other mathematical field than to the other two types of connections. PSTs provided examples of such connections, along with how technology helped them connect algebra with geometry in the courses designed to link school and college mathematics.

It is only by investigating how PSTs are prepared to teach algebra that we can improve teacher education and effect the enduring challenge of students who are unprepared for the algebra required by college and career opportunities. This study is a step in that direction.

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UNDERGRADUATE STUDENTS' INVERSE STRATEGIES AND MEANINGS

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Inverse function, and more generally reversibility, is important for success in mathematics (Hackenberg, 2010). Researchers (Oehrtman, Carlson, & Thompson, 2008) have argued that college students, pre-service teachers, and in-service teachers construct unproductive inverse function meanings. For instance, Vidakovic (1997) identified that despite having success in determining analytical and graphical representations of inverse functions, students did not relate these representations to critical ideas of inverse function (e.g., function composition).

We used semi-structured task-based clinical interviews (Goldin, 2000) for the purpose of further understanding students' inverse meanings. Specifically, we interviewed 25 pre-service teachers using an interview protocol that consisted of decontextualized and contextualized tasks that included analytical (e.g., equation) or graphical representations. We characterize students' actions based on our perspective of the consistency of their strategies used within and among the various task types. Compatible with previous researchers' findings, we argue that students' meanings are often restricted to carrying out specific actions in analytic or graphing situations. We also illustrate that the students held compartmentalized meanings for inverse function dependent on the function and/or representation. For instance, although each of the 25 students used a consistent technique in analytic, decontextualized tasks, only five of these students were able to determine a viable interpretation of the inverse function they defined analytically in a contextual (e.g., temperature conversion) situation. An important finding is the extent that students focused on carrying out a particular activity (e.g., switching x and y); to most students, carrying out a particular activity dependent on representation and context *was* their meaning for inverse. We did not interpret these students to hold meanings for inverse function that enabled them to conceive connections across all task types. Future researchers may be interested in exploring ways that would support students in constructing productive meanings for function and inverse function in contextualized and decontextualized situations.

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ARITHMETIC PROPERTIES AS A ROUTE INTO ALGEBRAIC REASONING

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For many students, algebra continues to be a gatekeeper to future academic and employment opportunities. As a result, it is now widely accepted that algebra should be treated as a grades K–12 strand of thinking. In response, our project aims to examine the effectiveness of a longitudinal early algebra intervention on grade 3-5 students' algebra learning and algebra-readiness for middle school. The study described here compares understandings of the Commutative Property of Addition demonstrated by students who participated in an early algebra intervention in grades 3 and 4 to those of students who experienced more traditional elementary grades instruction.

Arithmetic properties provide students an opportunity to look at arithmetic expressions "in terms of their form rather than their value when computed," and can serve as a route into algebra (Kaput, 2008, p. 12). Based on the results of our study, we argue that traditional arithmetic approaches to properties do not provide sufficient opportunities for students to engage in the important algebraic thinking practices of generalizing, representing generalizations, justifying generalizations, and reasoning with generalizations (Kaput, 2008).

Participants included approximately 100 intervention students and 60 comparison students from two elementary schools in the same district in the Northeast. Written assessments administered at the beginning of grade 3, end of grade 3, and end of grade 4, were coded based on correctness as well as strategy use. On the grade 3 pre-test, there were no significant differences between the performances of intervention and comparison students. At the grade 3 and grade 4 post-tests, there were no significant differences between the proportions of students who invoked the Commutative Property of Addition when justifying the correctness of a specific numerical example ($23 + 15 = 15 + 23$). However, when students were asked to represent and justify this generalization, the groups' results differed significantly. Students who participated in the intervention were more successful representing the property using variables (e.g., $a + b = b + a$) and justifying why the property holds true for all numbers. Our findings suggest that arithmetic properties can serve as useful contexts to engage students in developing, representing, and justifying generalizations.

The inclusion of field properties in early grade mathematics instruction may provide teachers with the opportunity to use them as a springboard for engaging students in algebraic reasoning. The development of teachers' "algebra eyes and ears" and the use of supplemental "algebrafied" instructional materials are critical to teachers' ability to encourage such algebraic activity amongst elementary students (Blanton & Kaput, 2003).

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RELATING ADDITIVE AND MULTIPLICATIVE REASONING: A TEACHING EXPERIMENT WITH SIXTH-GRADE STUDENTS

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At the end of her chapter in the *Second Handbook of Research on Mathematical Teaching and Learning*, Susan Lamon (2007) lists outstanding questions for future research. The first of these is, "What are the links between additive and multiplicative structures?" (p. 662). Understanding the psychological operations that link additive and multiplicative reasoning remains relatively mysterious for researchers today. From December 2013 until May 2014 we conducted a constructivist teaching experiment (Steffe & Ulrich, 2014) with three pairs of sixth-grade students in order to investigate how the complexity of a student's additive reasoning was related to the complexity of their multiplicative reasoning. As part of this analysis, we compared the students' zones of potential construction (ZPCs) (Norton & D'Ambrosio, 2008) in coordinating multiple additive comparisons and their ZPCs in coordinating multiple multiplicative comparisons.

Using tasks and findings from previous research (Steffe & Olive, 2010), we selected students whose ZPCs included constructing multiplicative comparisons in activity but not using them to assimilate situations (Pair 1), students for whom a single multiplicative comparison could be assimilatory (Pair 2), and students who could coordinate multiple multiplicative comparisons in their assimilatory structures. These ZPCs were confirmed in various contexts throughout the teaching experiment. Multiple additive comparisons were engendered using tasks involving a second difference (a comparison of differences between quantities), such as the following:

> Team B beat Team A and Team D beat Team C. Team D won their game by 18 more points than Team B won their game by. Find the missing score. Team A: 89. Team B: ___. Team C: 54. Team D: 77.

We found that the additive complexity within each pair's ZPCs were either comparable to or greater than the multiplicative complexity in their ZPCs. In particular, these problems were not in the ZPC of Pair 1. Solving these problems was in the ZPC of Pair 2. However, constructing an assimilatory structure for the problem was surprisingly difficult: It took six teaching sessions for Pair 2 to stop regularly conflating quantities and eight teaching sessions until the underlying additive structure of these problems appeared to be assimilatory for both students in Pair 2. Finally, this problem was within the ZPC of Pair 3. One potential implication is that complex additive situations can sometimes be used to challenge a student to work with increasingly complex quantitative relationships when corresponding multiplicative complexity is outside the student's current ZPC.

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LINEAR REPRESENTATIONS OF TWO-DIGIT NUMBERS PROMOTE FIRST GRADERS' ESTIMATION

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Introduction. Previous studies indicated that it is not until about second grade that numerical estimations of two-digit number magnitudes show a linear correspondence between estimated and actual magnitudes (Siegler & Booth, 2004), consistent with the developmental progression depicted by Case and Okamoto (1996). The objective of the present study is to examine the effectiveness of three instructional strategies to promote first graders' acquisition of numerical magnitudes of two-digit numbers. We predicted that teaching children magnitudes of decades (e.g., 10, 20, and 30) in a linear fashion would be more effective than the alternative strategies.

Procedure. Thirty-one first graders participated in the study. The experimenter met with the participants individually for four 15-minute sessions (pretest, two training sessions, and posttest) over a 4-week period. The number estimation task (Siegler & Booth, 2004) was used as a pre and posttest measure. A blank number line (25 cm long) was drawn on a sheet of paper with "0" and "100" appearing below the left end and the right end of the number line, respectively. Each child worked on 26 estimation problems in a random order.

Training. The students were randomly assigned to one of the three training groups. In all three interventions, base-10 blocks were used. The interventions differed in the terms of which blocks were used: (1) multiple 10 blocks and unit blocks (M10U), (2) single-10 block and unit blocks (S10U), and (3) unit-blocks only (U). For the M10U group, the experimenter demonstrated how to construct two-digit numbers by using the precise combination of 10 and unit blocks. For example, 35 was constructed by placing three 10 blocks and five unit-blocks in a linear fashion. For the S10U group, the experimenter constructed 35 by placing one 10 block and 25 unit blocks in a linear fashion. For the U group, the experimenter constructed 35 by placing 35 unit blocks in a linear fashion.

Results. The students in the M10U group improved their PAE from pretest (41%) to posttest (16%) whereas the other two groups did not. All three groups at pretest showed estimation patterns best described as logarithmic but only the M10U group's estimation patterns changed to linear at the posttest ($R_{lin}^2 = 98\%$).

Discussion. The current study showed that use of base-10 blocks to show chunks of 10s in a linear fashion was effective in promoting first-graders' understanding of two-digit number magnitudes. This study showed that unstructured use or non-canonical base-10 use does not lead to improved understanding. It is the activity of placing 10 blocks in a linear fashion that helps students to build canonical base-10 representations.

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CHILDREN'S REASONING WITH FRACTION REPRESENTATION SYSTEMS

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Research suggests teachers should know how to selectively use multiple representations of fractions to teach for understanding (e.g., Cramer & Wyberg, 2009; National Mathematics Advisory Panel, 2008). Yet little is known about how children reason within and between elements of fraction representation systems. Drawing from Behr, Lesh, Post, & Silver's (1983) *representation systems* (i.e., pictures, manipulatives, spoken and written symbols, and "real-world" situations), we investigate this problem with the following research question: *How is children's reasoning related with their use of fraction representations?*

Part of a larger study, the data considered for this report focus on clinical interviews with four children (at the ends of 2nd, 3rd, or 4th grades) from a diverse group of 24 children in the Southeastern U.S. Children were given fraction tasks designed to probe their fraction understanding and encourage use of multiple fraction representations. Results from cycles of open coding that highlight nuances of reasoning types are presented in Table 1.

Table 1. Student Reasoning Types

Reasoning Type	Brief Description	Examples
Informal/Intuitive	"Real-world" prior knowledge is used to extend reasoning beyond the original problem context.	- Sadie introduces brownies that can be partitioned instead of candies in a discrete collection in order to reason about fraction sizes. - Ariel recognizes that 2 cakes can be cut and shared equally with 3 people after saying it was impossible to equally share 2 cookies with 3 people.
Formal and Disconnected	Knowledge within one representation is not extended to another.	- Kevin argues that 2/3 is larger than 5/8 when using an area model to compare fractions but that 5/8 comes after 2/3 on a number line.
Formal and Connected	Knowledge is extended from one representation to another.	- Paul identifies a unit fraction from a bar described as 2/5 and is able to find the length of the whole. He generalizes his knowledge when given 13/15 as a numeric symbol only.

Results both extend previous research (e.g., Cramer & Wyberg, 2009) and suggest a more nuanced approach is needed to understand student thinking with representation systems. Importantly, existing analytical frameworks for understanding how and when knowledge is connected (or not) with different representation types may be insufficient for both researchers and teachers to assess and support student thinking at fine-grained level of analysis.

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Chapter 5

Geometry and Measurement

Research Reports

Brief Research Reports

Poster Presentations

TEACHERS' EXPECTATION ABOUT GEOMETRIC CALCULATIONS IN HIGH SCHOOL GEOMETRY

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This paper reports on a study of the instructional situation in high school Geometry that Hsu (2010) called Geometric Calculation in Algebra (GCA). In particular, we conducted a virtual breaching experiment in order to examine the extent to which high school teachers recognized breaches of two norms that we conjectured to describe geometry teachers' expectations of this work context. The results of our analysis of the data (using z-tests and mixed effect regression models) provide evidence that, in the situation of GCA, (1) teachers appear not to take issue with giving students tasks that require them to set-up and solve equations whose solutions have no geometric meaning (e.g., the length of a side of the figure is zero), and (2) teachers do not appear to expect students to document the geometric theorem or property that justify the setup of those equations (highlighting the contrast between the situation GCA and that of doing proofs).

Keywords: Algebra and Algebraic Thinking; Geometry; Instructional Activities and Practices; Measurement

In an effort to describe the practice of teaching mathematics in schools, some mathematics educators have argued for the importance of understanding the factors that influence mathematics teachers' instructional decisions. Some researchers have described these decisions as motivated by the teacher's individual goals, beliefs, and knowledge (Schoenfeld, 2010). However, others have sought to complement this perspective by suggesting that decisions might also be understood as influenced by customary norms of the practice of teaching. In their account of the practical rationality of mathematics teaching, Herbst and Chazan (2012) suggest that students and teachers recognize some patterns of interaction as defaults for recurring classroom situations (e.g., when doing proofs in high school geometry class).

This study investigates norms of another instructional situation in high school geometry instruction: what Hsu (2010) has referred to as Geometric Calculation in Algebra. We use Herbst's (2006) definition of instructional situation to conceptualize geometric calculations in algebra (GCA): GCA enables an exchange between the work students do posing and solving an equation and the claim their teacher can make that they know properties about a geometric figure (to which the equation refers). As in the case of other instructional situations, we expect there are norms for what the teacher and the student are expected to do to enable them to do such work and operate such exchange. Figure 1 shows an example of a GCA task.

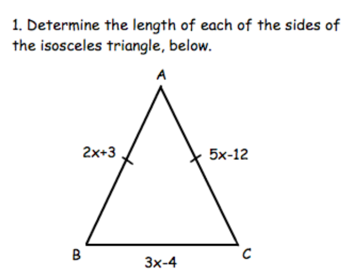

Figure 1: Sample GCA task

The example in Figure 1 illustrates some norms of GCA: Students are informed (often, through a diagram) that certain dimensions of a given geometric figure can be represented by given algebraic expressions and they are expected to use their knowledge of the properties of that figure to set up and solve algebraic equations using the given expressions in order to find one or more of the figure's dimensions. Their success in this work counts toward their understanding of the geometric properties of the figure as well as their retention of algebraic skill.

The work of investigating the norms of this instructional situation may be of interest to mathematics educators (both practitioners and researchers) for several reasons. For one, in terms of efforts to understand and describe the ways that mathematics is actually taught and learned in schools, norms provide interesting insights because they represent both expected actions and a rationale for teacher's instructional actions. Norms also provide a baseline in the process of improving instruction: Efforts to change instruction need to oppose existing norms and propose justifiable breaches of such norms. Along those lines, while this study explores GCA in particular, its methods are equally applicable to the study of norms of other situations in other courses of study beyond high school geometry. Second, for those particularly interested in geometry instruction, the situation of GCA is a common instructional situations in American high school geometry courses, and one that provides students with opportunities to engage in practices that have generally been supported by mathematics educators, such as engaging in algebraic and geometric reasoning as well as connecting multiple representations (NCTM, 2014). Further, Hsu (2010) argued that geometric calculation tasks offer students inroads to the types of reasoning needed for understanding and writing proofs in geometry.

Based on our observations of American high school geometry classrooms and informal analysis of geometry textbooks, we hypothesized that the following were norms of the instructional situation of GCA:

- When a GCA task is given to students the algebraic expressions associated to the dimensions of the figure are such that when an equation is set-up on the basis of one or more true geometric properties of the figure the numerical measures obtained from the solution of such equation will have interpretable geometric meanings (e.g., side lengths and angle measures will be positive).
- Although students may be asked to state orally the geometric property that they use to set-up one or more equations when solving GCA problems at the board, they are not expected to write that property.

For sake of brevity, we will refer to the first of these two norms as the GCA Figure (GCAF) norm and to the second as the GCA Theorem (GCAT) norm.

Similar to other mathematics educators who have endeavored to investigate instructional norms (e.g.,Dimmel, 2015; Herbst, Kosko, & Dimmel, 2013), we adopted a variation of a breaching experiment (Garfinkel, 1963) to determine whether two norms that we conjectured to exist actually describe how high school geometry teachers expect work on GCA problems will unfold, by examining the extent to which high school mathematics teachers recognize breaches of them. Accordingly, we posed the following two research questions:

- Do the GCAF and GCAT norms exist (i.e., represent how high school geometry teachers expect work on GCA problems will unfold)?
- How do participants react to breaches of the GCAF and GCAT norms?

Further, aware that in any instance of an instructional situation more than one norm might be breached, we also sought to investigate how breaches of a given norm at one point in a lesson might influence teacher's reactions to breaches of norms of that situation that occur later in the lesson. We therefore also posed a third research question:

- Is participants' recognition of breaches of the GCAT norm affected by whether the GCAF norm is breached or complied with?

In order to measure teachers' recognition of breaches of these hypothesized norms, we designed and implemented a research instrument, which we describe in the next section.

Methods

Data Collection

The present study represents a first attempt at answering our three research questions – by designing a research instrument and using it with a convenience sample of 40 high school mathematics teachers from a Midwestern state. The instrument designed sought to elicit mathematics teachers' reactions to storyboard representations of classroom scenarios, through online multimedia questionnaires, in order to determine whether hypothesized norms of the instructional situation of GCA exist, by measuring the extent to which participants recognize the breaches of those hypothesized norms (see Herbst, Kosko, and Dimmel, 2013) that occurred in some of those scenarios.

The instrument was comprised of twelve item sets. Each item set consisted of various questions referring to one storyboard. The scenario represented by each storyboard can be described as following one of three experimental conditions, which we refer to as the CFCT, the CFBT and the BFBT conditions (to denote which of the two norms are breached or complied with). The four CFCT scenarios were conjectured to be completely normative (both norms were complied with). The four CFBT scenarios breach the GCAT norm, but not the GCAF norm. The four BTBF scenarios breach both norms. We represented four scenarios of each condition to increase the "construct validity" (Shadish, Cook, & Campbell, 2002) of the instrument – only having one scenario of each condition would threaten the validity of our claim that participants' responses to the scenarios represent whether and how participants might react to breaches or compliance with the two norms (rather than the specific task or other incidental aspects of the scenario). One way of distinguishing the four scenarios in each experimental condition is by describing them as following one of four general storylines (i.e., plots), which differ in terms of how the task and student who solves it are selected, as well as how the correctness of the students' solution is discussed. The scenarios also differ in terms of the figure in the task – a feature that we use to title the storylines: the similar-triangles storyline, the trapezoid storyline, the isosceles-triangle storyline and the parallelogram storyline.

Each storyboard consists of 12 frames. During the first three frames, the class selects a GCA task to work on and the teacher either asks for a volunteer or selects a student to solve it, at the board. This occurs in one of four ways (depending on which storyline the given scenario follows), all of which were conjectured to be normative (e.g., the teacher in the scenario may accept a student's request to review a problem from the homework given the day before or may choose a problem that they conjecture might challenge the students; they may request that a particular student share their solution or ask for a volunteer). In the following three frames of each storyboard, the teacher puts the problem on the board and asks for the selected student to present their solution. In the CFCT and CFBT scenarios, the task complies with the GCAF norm, while in the BFBT scenarios, it breaches the GCAF norm by involving algebraic expressions that imply that the length of one of the sides of the figure is less than or equal to zero (and, therefore, that the figure does not exist). In the following three frames, the selected student writes a correct equation on the board, after which the teacher asks the student what theorem or property they used to set-up that equation. In all cases, the student identifies a correct theorem but, whereas in the CFCT and CFBT scenarios the teacher then affirms the student, in the BFBT scenarios the teacher breaches the GCAT norm by saying something that conveys that they expected the student to write the theorem or property on the board (e.g., the teacher says "Why was that not written on the board? Please always write down the properties you use to justify your work."). In the last three frames, the student finishes correctly solving the problem and asks the teacher to help them determine whether their solution is correct. This occurs in one of two ways, both of which were conjectured to be normative - the teacher either asks the class whether they think the solution is correct (which occurs in two of the storylines) or asks a specific student the same question (which occurs in the other two storylines). In the CFBT and BTBF scenarios, the theorem or property used to set up the first equation is also written on the board, in response to the teacher's request[1].

For each item set, and hence for each scenario, the questionnaire contained seven open-response and five closed-response questions. After being shown the first six frames of a storyboard, participants were asked, "what did you see happening in this first segment of the scenario?" and provided with an open box in which they could type their response. Asking this type of question (a prompt for participants to describe what they notice about a given scenario) permits one to observe what participants tacitly expect will occur in those situations. In line with the notion of breaching experiments, we would expect that most participants who saw a scenario where GCAF had been breached would remark that breach. After that open-ended question, participants were asked to rate the appropriateness ofthe teacher's actions in the first three frames of the storyboard, using a 6-point Likert scale, and asked to explain their rating in an open-response field. The same appropriateness questions were asked about the second group of three frames of the storyboard. From the explanations of their ratings, we also expected to see evidence of participants' recognition of the breaches of the GCAF norm as well as to learn why some teachers might disagree with certain breaches, while others might deem them justifiable.

Participants then saw the second half of the scenario (the third and fourth segments) and were asked the same three open-response and two closed-response questions about those segments. Last, they were asked to rate the appropriateness of the teacher's facilitation of the work on the problem throughout the scenario (again, using a 6-point Likert scale) and to explain their answer (in an open response field). This last question was posed, in particular, to provide participants with an opportunity to remark on breaches of the GCAF norm, in the chance that they had not realized that the task was not normative (if it was not) when it was first put on the board, but realized it once the student finished solving it.

Each participant was randomly assigned to one of three groups, each of which was assigned four item sets (two of one condition and two of another), as follows:

- Group 1 was assignedtwo CFCT item sets, one that followed the similar-triangles storyline and one that followed the trapezoid storyline, as well as two CFBT item sets, one that followed the isosceles-triangle storyline and one that followed the parallelogram storyline.
- Group 2 was assignedtwo CFBT item sets, one that followed the similar-triangles storyline and one that followed the trapezoid storyline, as well as two BFBT item sets, one that followed the isosceles-triangle storyline and one that followed the parallelogram storyline.
- Group 3 was assignedtwo BFBT item sets, one that followed the similar-triangles storyline and one that followed the trapezoid storyline, as well as two CTCF item sets, one that followed the isosceles-triangle storyline and one that followed the parallelogram storyline.

Data analysis

As each participant was assigned two item sets of the same condition, we used mixed effects regression models (Agresti & Finlay, 2009) to analyze the closed-response data, using MemberID (a variable used to keep track of which responses were associated with which participant) as the random effect. The outcome variable in each model was the set of responses to one of the five Likert-scale items that asked participants to rate the appropriateness of the teacher's action (in each of the four segments of the scenario, then overall). To be able to compare ratings of scenarios of different experimental conditions and scenarios of different storylines, we created two variables - Condition (with values CFCT, CFBT, BFBT) and Storyline (with values similar-triangles, trapezoid, isosceles-triangle, parallelogram) – applied these to code each of the closed-responses and, after dichotomizing each, used those dichotomous variables as dependent variables in each model. The CFCT condition was used as the reference group for the CFBT and BFBT conditions, because the CFCT scenarios were designed as control scenarios (i.e., neither norm was breached in them). The choice to have the similar-triangles storyline as the reference group was arbitrary, as all scenarios were designed to be normative except for whether or not they complied with either the GCAF or GCAT norms.

We hypothesized that, when controlling for the Condition variable, there would be no significant differences between the mean ratings of the scenarios of different storylines, in any of the models, as we conjectured that the teachers' actions in each were equally appropriate (as they were designed to be normative, outside of the moments when each of the norms were at issue). Similarly, we also conjectured that there would be no significant differences between the mean rating of segments 1 and 4 of the scenarios, when controlling for the Storyline variable. The only significant differences we expected to observe were in the mean ratings of segments 2 and 3 of the scenarios, because they each relate to part of the text where one of the norms was breached in some items, but not in others. Specifically, since the GCAF norm was only breached in segment 2 of the BFBT scenarios, we expected that it would be rated significantly lower than segment 2 in CFCT (the reference group) scenarios, on average. As the GCAT norm was breached in segment 3 of both the CFBT and BFBT scenarios, we expected that it would be rated significantly lower than segment 3 in the CFCT scenarios, on average.

Last, to evaluate our hypothesis that teachers would recognize the breaches of our two hypothesized norms, we created two dichotomous codes – one representing that there was evidence of recognition (versus non-recognition) of the breach of the GCAF norm and one representing that there was evidence of recognition (versus non-recognition) of the breach of the GCAT norm – and applied each to all open-response items. The following is an example of a response coded for recognition of a breach of the GCAT norm: "Kids solves it but doesn't write justification. Teacher tells kid (with different word choice) to write justifications. Kid does it and we move on."

Each participants was then given two scores – one indicating whether there was evidence of recognition of a breach the GCAF norm in any of their open responses and another indicating whether there was evidence of recognition of a breach of the GCAF norm in any of their open responses. A series of z-tests were then conducted to determine whether most participants who were assigned each item that contained a breach of one or both norms recognized those breaches.

Results

Results related to research question 1

In terms of the results of the twelve z-tests of the proportion of participants who recognized the breach of the GCAT norm in each open-response item (against the null hypothesis of 50% recognition), as predicted, no participants recognized a breach in any of the CFCT-condition items, as the norm was not breached in those scenarios. The proportion of participants who recognized the breach in each of the CFBT-condition and BFBT-condition items, except one, ranged from 0.45 to 0.73 (but none of those values were statistically significant at the level of 0.05).

Further, when coding the open-response data for recognition of the breaches of the GCAT norm, we noticed that there were several participants who noted that the teacher requested that the student state the theorem or property that they used to set-up the equation. Consequently, we also conducted a z-test of the proportion of participants that recognized this request in relation to each item. The results indicate more than 50% recognition in all the CFBT-condition and BFBT-condition items (in some cases, significantly more). In fact, even the proportion of participants that recognized this request in each of the four CFCT-condition items was 0.82, 0.43, 0.47 and 0.33.

One surprising result was that, although all BFBT scenarios contained a breach of the GCAF norm, there was only evidence that one participant recognized one of those breaches. Further, this evidence was in their response to the rating of the third segment of the scenario, when they wrote: "the teacher is allowing the student to discover that there is no solution to the problem". Therefore, there was no recognition of the breach of the GCAF norm when the problem was being written on the board (despite this being done over three frames). As we discuss in the next section, this result suggests the need for deeper exploration of the GCAF norm.

Results related to research questions 2 and 3

The results of the five mixed-effect regression analyses (one for each of the ratings of the four segments of the scenario and one for the rating of the teacher's facilitation of the work on the problem throughout the scenario) are summarized in table 1 and generally confirm most of our hypotheses.

As indicated in table 1, when controlling for the Condition variable, we see that all but two of the coefficients are not significant. This supports our earlier claim that, outside of the breach or compliance with one or both of the norms, participants rated the teachers' actions in the scenarios as being similarly appropriate. On the other hand, the two significant Storyline coefficients suggest that this might not be the case. We will discuss this point further in the next section.

When controlling for the Storyline variable, the coefficients for the Segment-3 ratings and the Overall ratings of the CFBT scenarios and BFBT scenarios are negative and significant. This indicates, as hypothesized, that ratings of the segment of the scenarios in which the GCAT norm was breached would be rated lower, on average, than the equivalent segments of scenarios in which that norm was not breached (CFCT scenarios), and that the same would consequently be true for the overall rating of the scenarios. However, neither the rating of segment 3 or the overall rating associated with BFBT scenarios was significantly different than those of the CFBT scenarios. Although this does not provide us with evidence to believe that participants' reactions to breaches of the GCAT norm are influenced by the reactions to breaches of the GCAF norm, as we also discuss in

Table 1: Summary of Mixed Effect Regression Analyses for Variables Predicting Ratings of Each Segment of the Scenarios and the Overall Rating

	Seg-1 rating B(SE)	Seg-2 rating B(SE)	Seg-3 rating B(SE)	Seg-4 rating B(SE)	Overall rating B(SE)
Fixed effects					
Trap storyline	-0.55(0.28)*	0.14(0.21)	0.02(0.18)	0.06(0.24)	0.11(0.16)
Iso-tri storyline	-0.22(0.24)	0.04(0.25)	0.03(0.23)	-0.26(0.21)	0.27(0.17)
Parall storyline	-0.04(0.23)	0.27(0.21)	-0.02(0.27)	-0.03(0.22)	0.45(0.22) *
CFBT cond	0.46(0.24)*	0.60(0.20)**	-0.95(0.23)***	-0.11(0.18)	-0.63(0.24)**
BFBT cond	0.21(0.22)	0.14(0.20)	-0.98(0.29)***	-0.23(0.18)	-0.56(0.20)**
Constant	4.19(0.23)***	3.72(0.21)***	5.00(0.20)***	4.78(0.21)***	4.58(0.22)***
Random effects					
Constant	0.41(0.23)	-0.19(0.14)	0.24(0.16)	0.32(0.13)*	0.23(0.15)
Residual	0.08(0.09)	0.10(0.07)	0.02(0.09)	0.01(0.10)	0.21(0.08)
N	158	158	158	158	158

Standard errors in parentheses; $^{*}p< 0.05$, $^{**}p< 0.01$, $^{***}p< 0.001$

the next section, lack of recognition of the GCAF norm could be the explanation for this.

Also in line with our hypotheses, the mean ratings of segment 4 in all scenarios were similar, which was expected, as no norm was breached in that segment of any of the scenarios. In contrast, however, segment 1 and segment 2 of the CFBT scenarios were rated significantly higher, on average, when controlling for the Storyline variable, than those segments in the CFCT scenarios, which contrasted with our hypothesis that they would be rated similarly (as both of those segments in all of those scenarios were also designed to represent normative instruction).

Discussion: Potential Revisions to the Instrument

Although the results of the study are mixed, in the sense that the instrument provided us with evidence that the GCAT norm is, in fact, a norm of the instructional situation of GCA, but did not allow us to conclude the same about the GCAF norm, nor to detect any relationship between the way that participants reacted to the two norms, the results do provide us with ideas for future research. For one, we would argue that the general design of the instrument is promising. The "what did you see happening…" questions provided us with evidence that one of the hypothesized norms exists. Another affordance of the open-response questions was that they not only allowed us to collect evidence in support of our hypothesized norms, but also allowed us to consider whether and how to revise these hypotheses. Although the proportion of participants that recognized breaches of the GCAT norm in some of the scenarios was as high as we expected, there were scenarios for which the proportion was lower. Of course, this could be a consequence of the small sample size or the representativeness of the sample, but the proportions of participants that recognized the teachers request that the student state the theorem or property at least suggests that the norms might have been slightly different than we first conjectured. For example, it could be that the norm is in fact that students are not expected to write or state the theorem or property that they used to set-up their equation(s). This alternative is supported by the fact that many of the participants' attention were drawn to the request to state the theorem of property, even in the CFCT scenarios.

In terms of the lack of recognition of the GCAF norm, although it could indicate that the norm does not exist, we argue that this is more likely a consequence of at least one of the following two issues with the scenarios. The first is that, in order to detect that the norm was breached, a participant would have likely had to work through the problem, which they might not ordinarily have to, if they assumed that the task was normative, as it had been put on the board by the teacher in the scenarios. Alternatively, there is also evidence in the open-response data that there were more distracting aspects of the scenarios than whether or not the norm was breached. In particular, many participants commented that the teacher took too long (3 frames) to write the problem on the board and that they should have instead used a document camera or have the student write the problem on the board, while the teacher circulated. We found the suggestion of having the student put the problem on the board to be especially helpful, as we conjecture that the teacher would also more likely analyze a problem if the student was the one putting it on the board, and are considering revising the items to integrate this change.

Last, as we are more likely to detect a relationship between the two norms if participants recognize the breaches of the first norm, we are considering adding a question, after they evaluate the first half of the scenario but before they evaluate the second half, that will ask participants to rate the appropriateness of the task in the scenario, expecting that this will also require them to analyze the GCA task (if they did not do so when it was put on the board). Similar to our other rating questions, their rating of the task and their explanation of that rating could also provide us with some evidence that teachers recognize the GCAF norm, even if breaches are not remarked in the first three open-responses.

Endnote

[1]A table describing the four storylines in more detail is included in an extended version of this paper, available on Deep Blue.

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INVESTIGATING SECONDARY MATHEMATICS TEACHERS' ATTITUDES TOWARD ALTERNATIVE COMMUNICATION PRACTICES WHILE DOING PROOFS IN GEOMETRY

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We used multimedia surveys to investigate secondary mathematics teachers' reactions to storyboards that represented episodes of instruction. Participants were asked open-ended questions about the storyboards. We analyzed the responses to the open-ended questions for evidence of the attitudes *(Martin & White, 2005) that participants conveyed about the episodes. We found that, when presented with storyboards that depart from what is hypothesized to be routine instruction, participants' open responses included significantly more negative than positive linguistic markers of attitude. At the same time, when participants were shown storyboards that represented what routinely happens in classrooms, markers of positive and negative linguistic markers of attitude occurred with equal frequency.*

Keywords: Instructional Activities and Practices; Geometry

Introduction

For as long as mathematics has been taught in US public schools, there have been initiatives that have attempted to improve the quality of mathematics teaching in classrooms. A fundamental challenge for such initiatives is the paradox of *change without difference*: reform efforts that, in principle, could bring about fundamental shifts in classrooms emerge, in practice, as "shadows of their original intent" (Woodbury & Gess-Newsome, 2002, p. 763). One reason for this paradox is that the patterns of classroom interaction that practicing teachers have honed through years of experience are robust. Initiatives that aim to effect change in the way that mathematics is taught thus need to contend with the realities of the already established practice of mathematics teachers (Cobb, Zhao, & Dean, 2009). To understand how reform efforts might contend with such realities we need to raise a natural question: When teachers encounter reasonable departures from routine instruction, how do they relate to such actions?

To answer this question, we conducted a study that used multimedia surveys to investigate secondary mathematics teachers' reactions to storyboards that represented episodes of instruction. Participants were asked open-ended questions about the storyboards. We analyzed the responses to the open-ended questions for evidence of the *attitudes* (Martin & White, 2005) that participants conveyed about the episodes. We found that, when presented with storyboard representations of reasonable departures from what we hypothesized to be routine instruction, participants provided open responses that contained more negative than positive linguistic markers of attitude. At the same time, when participants were shown storyboards that represented what routinely happens in classrooms, positive and negative markers of attitude occurred with equal frequency.

The analysis reported below is part of a larger study whose objective was to investigate instructional routines that pertain to discipline-specific communication practices. The communication skills used by disciplinary experts have traditionally been thought to be gradually and tacitly developed by novices as the novices are apprenticed into a field (Lemke, 2013; Thurston, 1994). But recent work in analyzing mathematical communication suggests that discipline-specific ways of communicating are practices that can be described and taught (Fang, 2012; O'Halloran, 2011; Yore, Pimm & Tuan, 2007). *Doing proofs* is one classroom activity during which students could develop discipline-specific communication practices. As the geometry classroom has historically been the principal instructional setting in which students are introduced to mathematical proof (Knuth, 2002), the instructional situation of doing proofs in geometry (Herbst & Brach, 2006) was the focus of this study.

Theoretical Framework

Classroom activity can be modeled as a social system in which an agent playing the role of teacher and other agents playing the roles of students act in accordance with tacit but mutually held norms (Herbst & Chazan, 2012). Such norms help characterize instructional situations: stable segments of classroom activity in which students' work is exchanged for claims that they have acquired items of knowledge (Herbst, 2006; Herbst & Brach, 2006). Though the instructional activity of doing proofs in geometry has been criticized by mathematics educators for being a misrepresentation of the work of proving in mathematics (Schoenfeld, 1988; Martin & Harel, 1989; Lockhart, 2009), it endures as an instructional setting where students are introduced to the notion that there is such a thing as mathematical proof. The goal of this work was to describe norms of the instructional situation of doing proofs that pertain to how student proofs are presented and checked in geometry classrooms.

We use *norm* to refer to those aspects of social situations that not only regularly happen but also that participants (in social situations) *expect* to happen (Garfinkel, 1963). In social situations, when people confront departures from what they expect, they can react with anxiety, bewilderment, or anger (Mehan & Wood, 1975). Such negative reactions are ways in which people mark that a norm has been breached. The work reported here used the notion of a *breaching experiment* (Garfinkel, 1963) to investigate secondary teachers' reactions to episodes of instruction in which hypothesized norms[1] of instructional situations were breached.

Method

Our method of inquiry combined a planned-comparison study with a virtual breaching experiment (Herbst, Aaron, Dimmel, & Erickson, 2013) in an instrument that we call a *virtual breaching experiment with control* (Dimmel & Herbst, 2014). The instrument was a multimedia survey that used storyboards to represent episodes of geometry instruction that were inspired by video records of actual geometry classrooms. Our use of storyboards to probe for recognition of norms is analogous to how scripted classroom videos have been used to probe teacher professional knowledge (Kaiser, 2014) and is an application of the cyclical use of records of practice (Jacobs, Kawanaka, & Stigler, 1999). Each participant[2] in our study viewed two sets of parallel storyboards: one set of parallel storyboards represented departures from hypothesized norms (i.e., *breach* storyboards), and the other set of parallel storyboards represented instances of instruction that were hypothesized to be routine[3] (i.e., *control* storyboards). The storyboards that were designed to represent routine instruction (i.e., the control storyboards) were based on video records of actual geometry classrooms, hence our claim that these storyboards represent the instruction that might typically occur in geometry classrooms. The storyboards in a set were parallel in the sense that they targeted the same hypothesized norm.

After viewing each storyboard, participants were given four opportunities to provide open-response data. The first question that participants were asked is: "What did you see happening in this scenario?" The purpose of prompting participants with this broad, open-ended question was to capture participants' overall reactions to the instances of doing proofs (hereafter: situation instances) that were represented by the different storyboards. This general open response question has been used in previous virtual breaching experiments (Herbst, Aaron, Dimmel, & Erickson, 2013) as a means to capture participants' reactions to storyboards. Participants had three other opportunities to provide open responses, following their review of each storyboard. These open response fields followed episode—how appropriate was the teacher's review of the proof in this scenario?—and segment-specific—how appropriate were the teacher's actions in this segment of the storyboard?—appropriateness rating questions.[4] Following each rating question, participants were prompted to "Please explain your rating."

Responses to the four open response questions were coded using a scheme derived from the *attitude* system of the appraisal framework (Martin & White, 2005). Coding for attitude is a means to capture participants' ways of feeling (Martin & White, 2005) toward the situation instances represented by the storyboards. The attitude system differentiates statements of affect, judgment, and appreciation. Statements of *affect* convey personal feelings through linguistic markers of emotion, such as "sad", "happy" or "angry" (Read & Carroll, 2012). Statements of judgment convey evaluations of people and their deeds, such as "he is a good teacher." Statements of appreciation convey aesthetic evaluations of non-person things in the world (goods and services), such as "that is a clear proof" (Read & Carroll, 2012). The scheme we developed coded each response for all instances of attitude. Attitudes were classified by type—is it judgment, affect, or appreciation?—target—e.g., the teacher, the proof—and polarity—positive or negative.

An example of a response that conveys a positive judgment of the teacher is: "teacher is guiding students effectively." In this response, the teacher is the target of the attitude and "effectively" is a positive evaluation of the action—guiding—that the teacher is described as doing in the scenario. Since the response is about the quality of how a person (i.e., the teacher) performs an action, it is coded as a positive judgment. An example response that contains a negative judgment of the teacher is: "This teacher is being a bit ridiculous." These examples were coded as judgments because, in each case, the targets of the appraisals are people and their deeds. An example response that contains an appreciation is: "the math proof was not accurate." In this example, a mathematical proof is the target of the appraisal. It was coded as a negative appraisal because the response states that the proof is " not accurate."

Reliability of the Attitude Coding Scheme

The *attitude* scheme was tested for reliability by comparing coded responses of two independent coders[5]. The coders applied each scheme to 100 randomly selected texts in the corpus—25 of each of the 4 response types, roughly 10% of the total number of responses. Before each text was coded, it was blinded with respect to whether the response was provided for a storyboard in which the norm was breached or a storyboard in which the norm was not breached. The purpose of blinding the data was to minimize bias. The *kappa statistics* for the *attitude* coding for which there were sufficient instances of the codes to warrant the statistics are .79 for *negative judgments* of the teacher; .49 for *positive judgments* of the teacher; .77 for *negative mathematical* appreciations; .49 for *positive mathematical* appreciations. These *kappa* scores indicate moderate (.49), high (.76, .77, .79), and very high (.89) agreement between the coders.

Data

Data was gathered from 73 secondary mathematics teachers located within a 60-mile radius of Midwestern University. Participants completed the instrument during in-person and online data collection periods that occurred during the 2013-2014 academic year. The multimedia survey (described above) that contained the four storyboards was one of several instruments participants completed during a day-long data collection event.

Results

Each response in the corpus was coded for judgments, appreciations, or statements of affect, and each instance of an attitudinal appraisal was coded in each response. This means that it was possible for a response to contain multiple instances of the same kind of statement of attitude (e.g., there could have been several judgments of a teacher), as well as instances of different kinds of statements of attitude (e.g., a judgment of the students, an appreciation of the proof), and statements of attitude that had different polarities (e.g., a response could contain both positive and negative judgments of the teacher).

We hypothesized that, in the case of storyboards that represent a breach of a norm, participants would react negatively. These negative reactions would be evident by higher numbers of negative statements of attitude. By contrast, based on the premise that the control storyboards represent routine teaching, we hypothesized that open responses associated with control storyboards would contain roughly equal numbers of positive and negative statements of attitude. Such a distribution of positive and negative statements of attitude could be explained on the basis of individual differences (among participants) alone.

Table 1 shows the total number of positive and negative statements of attitude throughout the open responses in the corpus. The results are reported according to storyboard condition. The entire corpus contained 1168 open responses to the 4 different open response questions, with equal numbers of responses to the breach and control conditions (584 responses per condition). These were divided equally (146 per question) among the 4 open response questions that were described above. The results reported in Table 1 are for the entire corpus across all 4 question types. Furthermore, the results reported in Table 1 are simple counts of the number of statements of positive or negative attitude that were coded in the open responses to the storyboards in the breach and control conditions.

Table 1: Counts of statements of attitude, tallied by polarity and storyboard condition.

Storyboard Condition	Statements of Positive Attitude	Statements of Negative Attitude	Total
Breach Storyboards	211	473	684
Control Storyboards	309	310	619

The results reported in Table 1 are consistent with the hypotheses stated above. The open responses to the storyboards in which a hypothesized norm was breached had more statements of negative attitude than statements of positive attitude. In the case of the control storyboards, there were nearly equal numbers of positive and negative statements of appraisal. A chi-square test indicates that there is a significant association between storyboard condition and the number of positive or negative statements of attitude ($\chi^2 = 48.49$, $p < .001$).

The unit of analysis for the results reported in Table 1 is a statement of attitude. This means that each statement of attitude in a response was included in the totals. To further investigate the relationship between attitude polarity and storyboard condition, we recoded the data to eliminate multiples, by polarity, within each response. Thus, if a response contained 3 positive statements of attitude and 2 negative statements of attitude, it was recoded as (1) for *positive attitude* and (1) for *negative attitude*. Table 2 reports tallies of statements of attitude after applying this reduction. The unit of analysis for the results reported in Table 2 is an open response ($n = 584$ for each storyboard condition).

The results reported in Table 2 are consistent with those reported above. Across the corpus for the control storyboards, there were 248 responses that contained at least one statement of positive

Table 2: Counts of open responses that contain positive or negative statements of attitude, by storyboard condition.

Storyboard Condition	Statements of Positive Attitude	Statements of Negative Attitude	Total
Breach Storyboards	176	352	528
Control Storyboards	248	245	493

attitude and 245 responses that contained at least one statement of negative attitude. By contrast, across the corpus for the breach storyboards, there were 176 responses that contained at least one statement of positive attitude compared to 352 responses that contained at least one statement of negative attitude. A chi-square test of association indicates that there is a significant relationship between storyboard condition and attitude polarity ($\chi^2 = 29.54$, $p < .001$).

The results reported in Table 2 are a refinement of the results reported in Table 1 because multiples have been eliminated. The results reported in Table 3 (below) refine these results further by distinguishing 4 categories of response: those that contain only positive statements of attitude, those that contain only negative statements of attitude, those that contain both positive and negative statements of attitude, and those that contain no statements of attitude.

Table 3: Counts of open responses that contain only positive attitude, only negative attitude, both, or neither.

Storyboard Condition	Statements of Positive Attitude	Statements of Negative Attitude	Both Positive and Negative Attitude	None	Total
Breach Storyboards	78	254	98	154	584
Control Storyboards	166	163	82	173	584

The results reported in Table 3 are consistent with those reported in Table 1 and Table 2. The breach storyboards contained more responses that contained only negative statements of attitude than responses that contained only positive statements of attitude. By contrast, the control storyboards contained nearly equal numbers of responses that contained only positive statements of attitude and responses that contained only negative statements of attitude. A chi-square test of association indicates a significant relationship between storyboard condition and the categories of attitude in Table 3 ($\chi^2 = 54.12$, $p < .001$)

The results reported above provide evidence to support our hypotheses: Throughout the corpus, the responses to the breach versions of the storyboards yielded more negative statements of attitude than positive statements of attitude. That responses to the breach versions of the storyboards produced more negative statements of attitude is consistent with the results of the breaching experiments conducted by Garfinkel (1963) and the virtual breaching experiments conducted by others (Herbst, Aaron, Dimmel, & Erickson, 2013). In contrast with the breach storyboards, responses to the control storyboards contained roughly equal numbers of positive and negative statements of attitude. Because the breach and control storyboards were the same except during those frames where the teacher is shown breaching (or complying with) a hypothesized norm, it follows that the teacher's breach of the norm is what prompted participants to react negatively to the storyboard.

Conclusion

We began with the question: When teachers encounter reasonable departures from the routine, how do they react? The results reported above provide evidence that secondary mathematics teachers react more negatively to episodes of instruction that represent breaches of hypothesized classroom norms than to episodes of instruction that represent instruction that is hypothesized to be routine. We provide context for these findings here by describing the nature of the breaches.

The storyboards representing instances of *doing proofs* were scripted to investigate communication practices expected by secondary teachers when proofs are presented (by students) and checked in geometry classrooms. An example of a routine practice for presenting proofs is that of a student going to the board and creating a mark-for-mark reproduction of an already completed proof—an act we call *proof transcription*. Such transcriptions of proofs by students do not match the practices used by disciplinary experts, where a proof is described verbally (with accompanying gestures) as it is generated at a blackboard (Artemeva & Fox, 2011; Greiffenhagen, 2014; Núñez, 2009).

In our study, one set of breach storyboards depicted teachers interfering with student transcriptions of proofs, for example, by requiring students to provide labels on a diagram before using those labels in a proof. The teacher's interference could be defended as reasonable on the grounds that the teacher is steering the student presenter toward staging a discovery—as opposed to a reproduction—of the proof that shows how the student engaged with the material artifact of the proof (Livingston, 1999). Such a move on the part of the teacher could be seen as an effort to bring the student's proof presentation practices more in line with disciplinary practices for presenting proofs (Greiffenhagen, 2014). In fact, some participants in the study remarked on the positive instructional value of the teacher's interference in such storyboards. Yet on the whole, the attitudes that participants expressed toward the teachers that interfered with the student transcriptions tended to be negative. What are we to make of these findings?

One implication is that it is possible that teachers could recognize, in the abstract, the value of an instructional alternative yet prefer, in actuality, the routines they have developed. The tendency toward routine is not evidence of a deficiency in teachers but is rather a fact of social life (Garfinkel, 1963). We see teachers' preference for routines as a resource that could be used to design instructional alternatives that are likely to have greater uptake by practicing teachers. For the work of managing student presentations of proofs in geometry classrooms, the expectation that students create mark-for-mark reproductions of proofs could be the basis for alternatives that would help students develop discipline-specific communication practices. An example of such an alternative could involve asking students to present proofs in pairs, where one student is responsible for generating the transcription and the other student explains the proof as the transcription is being completed. Such an alternative practice would recognize the value in the existing routine—i.e., that the proof that is displayed on the board is an accurate record of the work the student completed—while at the same time provide a scaffold for students to develop the proof presentation skills that are used by mathematical experts.

Endnotes

[1]The target norms were: (1) hypotheses about the details students are expected to include in a proof, and (2) hypotheses about how students are expected to present proofs during class.

[2]Participants were randomly assigned to one of five different treatment groups.

[3]The design of the study was described in a prior report. See (Dimmel & Herbst, 2014) for details.

[4]Analysis of the closed-ended responses to the rating questions were reported in a prior study (Dimmel & Herbst, 2014).

[5]We acknowledge the support of Nicolas Boileau for assisting with the reliability study.

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WHEN "HALF AN HOUR" IS NOT "THIRTY MINUTES": ELEMENTARY STUDENTS SOLVING ELAPSED TIME PROBLEMS

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This paper presents assessment study results addressing the question: Do students treat elapsed time problems differently if phrased as "half an hour" versus "thirty minutes"? A paper-and-pencil assessment was given to second (n=292) and fourth (n=205) grade students in six New England elementary schools. I compare responses on tasks presented in hour units and minute units. Results indicate that children respond differently to elapsed time questions as a function of the units provided in the question (half hour or thirty minutes) depending on the provided starting time (e.g., on the half hour versus on the second half of the clock).

Keywords: Measurement; Elementary School Education; Problem Solving

Objective

The teaching and learning of STEM topics has been identified as a key to our country's innovation as well as a gateway to individuals' job opportunities. Uniting the various STEM domains, time underlies many scientific explorations of *how things work*, from determinations of speed and impact in physical phenomena to the Earth's orbit and rotation to graphs of functions over time. At the same time, our understanding of how children interpret time in standard units is minimal. While a staple of early elementary mathematics instruction (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010), time has been repeatedly referred to as one of the *least* studied mathematical symbol systems (Blume, Galindo, & Walcotte, 2007; Burny, Valcke & Desoete, 2009; Kamii & Russell, 2012), with little known about how children problem solve using conventional notation and units for time. This is of particular concern given the prevalent role of time underlying the mathematics of change in middle and high school (see Yerushalmy & Shternberg, 2000). Although our culture's pervasive use of digital clocks may imply a collective mastery of time, such tools mask the rich and complex mathematics that underlies the unitizing of time and in determining elapsed time.

Theoretical Framework

My theoretical framework coordinates two areas: time is an area of measurement situated within children's developing *theory of measure* (Lehrer, Jaslow, & Curtis, 2003); and cultural tools and representations mediate our thought and communication (Cole, 1996; Sfard, 2007, 2008).

First, children develop a theory of measure through everyday examinations of the attributes of objects or events. Ideally, instruction provides opportunities for children to coordinate sensorimotor actions and everyday experiences with principles of measure (such as unit iteration, the need for equal units, or tiling; see Lehrer, 2003). Such principles cohere across different measures, including length, weight, volume, and time. Like length measure, time is a measureable quantity for which humans have developed standard units. While developmentally we know how students develop an understanding of length measure and also when notions of sequencing and duration develop (Piaget, 1969), we have little understanding about how children draw upon notation for time as related to their theory of measure.

To support the development of children's theory of measure, research has emphasized the role of units (see Lehrer et al., 2003; Stephans & Clements, 2003). Unitizing is a core concept of measuring, with many measurement principles focusing on the role of unit or resulting entailments, such as the need for a zero-point. With standard units of time measure, a long social history has led to our current system of notation in which units of time are in groups of 12 or 24 (hours) and 60 (minutes). The groupings of these hierarchical units greatly contrast with base-10 system of numeration underlying much of the mathematics content in elementary school, suggesting that drawing upon the proportional relation between hour and minute units may pose challenges. However, little research exists that explores how children may apply and coordinate standard units for time in their problem solving. The present study seeks examines how children solve elapsed time problems in order to document whether such challenges involving unit exist and, if so, how to characterize them.

Second, I consider thinking and learning to be inextricably linked to culture (Cole, 1996; Earnest, 2015; Sfard, 2007), with tools (i.e., a digital or analog clock) and conventional notation serving mediating roles in thought and communication. This may include cultural and mathematical referents such as "half an hour" and "thirty minutes." Analog and digital clocks represent time and its properties in different ways, with the analog clock's intervals of time translating duration into spatial distance (Lakoff & Nuñez, 2000; Williams, 2012). Digital time provides a precise time to the minute without reflecting hour-to-minute relations. The digital time 5:10, for example, provides a quick and precise numeric realization of time. Contrasting with this, the analog clock indicates time through a length based representational context. If we consider the hour hand for 5:10, for example, one may interpret its position as not just showing the "5" as with digital notation, but its displacement from 5:00 to 5:10 as well as the length corresponding to the 50 minutes remaining in the hour from 5:10 to 6:00. Tapping into children's developing theory of measure, the analog clock is a cultural tool that builds on principles of measure reflected on a number line. A premise of this work is that, in children's problem solving involving time, prior experiences with notation and tools serve mediating roles, even when solving problems involving digital notation alone.

Mathematically, elapsed times are equivalent whether provided in hour-units (e.g., half an hour) or minute-units (e.g., 30 minutes). However, children may interpret such units in different ways, though prior research has not considered this. Various articles have provided important examples about how children may treat time notation in terms of base ten (e.g., Breyfogle & Williams, 2008; Kamii & Russell, 2012), such as adding 30 minutes to the time 4:40 to reach 4:70. An underlying complexity underlying standard time notation is that hierarchical units of hours and minutes are grouped by 12 (hours) and 60 (minutes and seconds), a stark contrast to the base-10 groupings underlying place value and standard algorithms in elementary mathematics. The present study seeks to contribute systematic research to support whether such issues are pervasive in children's problem solving.

Research Questions

The present study investigates the research questions: Do children perform differently on elapsed time tasks as a function of the units of elapsed time? Do such differences depend on the starting time provided in the task? Based on these questions, I investigate patterns of responses across the focal tasks.

Methods

Participants included 292 Grade 2 students and 205 Grade 4 students drawn from six elementary schools in urban, rural, and suburban contexts in New England. Grade 2 students were selected because standards indicate children in this grade have already mastered time to the hour and half hour and are currently working on time at the 5 minutes (NGA Center & CCSSO, 2010). Grade 4 students were selected because, according to standards, time concepts including elapsed time have been mastered in prior grades, and their performances therefore illuminate any persisting differences in performance on problems involving time.

The assessment featured 31 items, the design of which was informed by classroom observations and informal interviews with students in second and fourth grades over 1.5 years. This paper focuses on a subset of six assessment items: three items featured minute-units for elapsed time ("30 minutes") and three analogous items involving hour-units ("half an hour") (see Figure 1). For each problem type, one problem had a starting time at x:30, another corresponding to the first half of the clock (x:10), and the final to the second half of the clock (x:40).

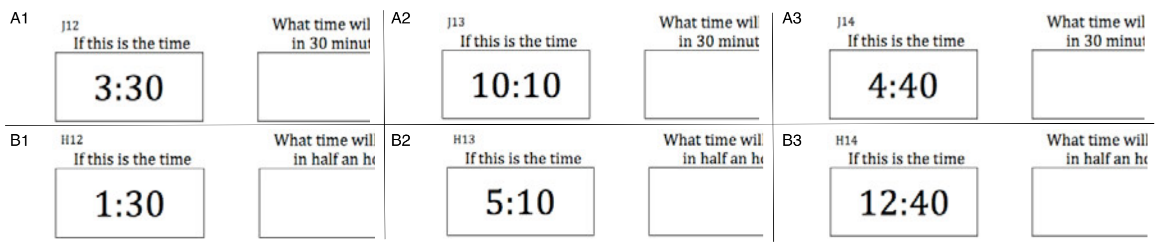

Figure 1: Assessment tasks by Problem Type (rows A and B) and Starting Time (columns 1, 2, or 3).

Teachers administered the assessments in December 2014 and January 2015 (dates were staggered due to weather-related school cancellations). Students had 25 minutes to complete the assessment; students that did not finish the assessment were not included in this analysis. The researcher was present to oversee each administration. All assessments were collected and scanned. Though not reported in the current paper, a subset of grade 2 (n = 72) and grade 4 (n = 72) students participated in problem solving interviews with analog or digital clocks.

Analysis and Results

The analysis is presented in two parts. I first present quantitative results that reveal students perform differently on "30 minutes" versus "half an hour" questions depending on the starting time. Following this, I present an analysis of responses on tasks in order to demonstrate that particular responses arise for one unit type but not for the other.

Performance by Problem Type and Starting Time

Six problems on the assessment addressed elapsed time in digital notation (see Figure 1). Each student was given one point for each correct response; incorrect responses were assigned 0 points. Means and standard deviations are represented in Figure 2. To determine whether there was a difference in performance, I conducted a Two (Problem Type) x Three (Starting Time) x Two (Grade) repeated measures analysis of variance (ANOVA) for performance.

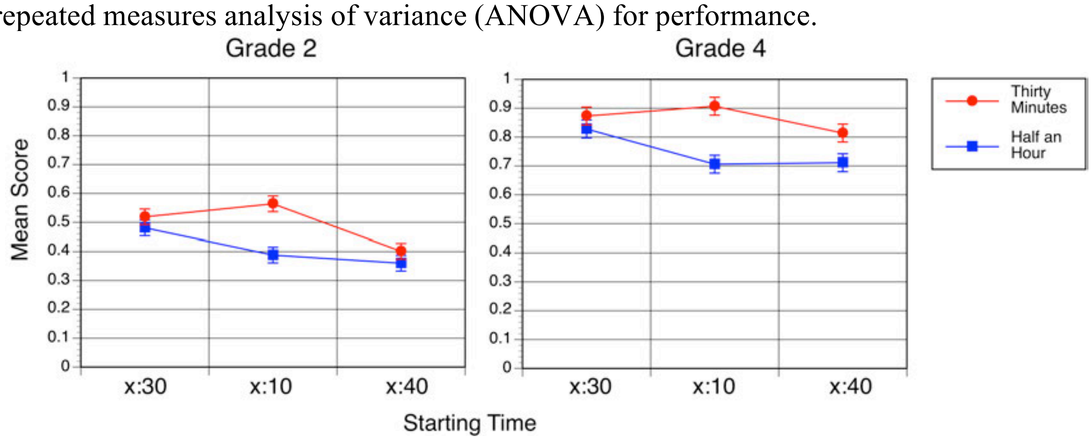

Figure 2: Mean scores with standard deviations on focal tasks.

I first report three-way and two-way interactions in the data. While there was not a statistically significant three-way interaction between Grade, Problem Type, and Starting Time, $p = .558$, there was a statistically significant two-way interaction between Problem Type and Starting Time, $F(2, 982) = 17.798$, $p < .0001$. There were no significant interactions between Problem Type and Grade ($p = .204$) or Starting Time and Grade ($p = .204$), suggesting a similar performance profile across problems independent of grade. Given the significant two-way interaction between Problem Type and Starting Time, I also compare performance for each Problem Type for each of the three starting times for the sample (corresponding to data points in the columns of Figure 1). There were statistically different performances on "half an hour" and "thirty minute" tasks for each of the three starting times: x:30 ($p = .020$), x:10 ($p < .0001$), and x:40 ($p < .0001$).

Results thus far confirm that students interpret questions phrased as "half an hour" versus "thirty minutes" differently, with students performing better on tasks phrased as "thirty minutes." These results lead to the following question: How do students' responses vary on tasks based on the units of elapsed time? In order to consider these questions, I turn to an analysis of students' provided responses for across the six tasks.

Examining Patterns in Student Responses

I now consider patterns in incorrect responses for the six tasks presented in Figure 1 identify incorrect responses according to how the final response reflects a particular displacement from the starting time. Five types of displacements emerged across problems (see Figure 3): (1) Displacement by an hour (e.g., 4:40 with 30 minutes elapsed is 5:40); (2) Displacement by 1.5 hours; (3) Displacement to a x:00 or x:30 point (e.g., 4:40 with 30 minutes elapsed is 5:30); (4) Displacement to a multiple of 5 between 5 and 55 minutes after the starting time (e.g., 4:40 with 30 minutes elapsed is 4:50); (5) Accurate displacement with inaccurate notation (e.g., 4:40 with 30 minutes elapsed is 4:70). In addition, Figure 3 reflects students that provided as a response: (6) the starting time or one minute after the starting time (e.g., 4:40 with 30 minutes elapsed is 4:40 or 4:41); (7) Idiosyncratic responses with frequencies of 1 or 2 in the sample; and (8) "I don't know" or no response. I draw attention to parts of the graphs in Figure 3 in which particular incorrect response codes were provided for one Problem Type and not the other. Note that since Figure 3 presents the proportion based only on incorrect responses, the *n*s for each problem do not include correct responses and, therefore vary across problems. When possible for each code, I reflect on how responses relate to children's theory of measure and/or the role of tools and notation.

1. Displacement by 1 hour. The first incorrect response code reflected displacement by one hour. Students provided this response across problems regardless whether phrased as "half an hour" or "thirty minutes." The proportion was greatest for the starting time of x:30; this may be related to children treating the "o'clock" time (e.g., 4:00) as a zero-point rather than the provided starting time. In this case, solving problem A1 (3:30 plus 30 minutes) may treat the next hour mark as the zero point, thereby adding 30 minutes to 4:00 to solve the problem. While this code was applied most for when the starting time was x:30 and when the problem was phrased as "half an hour," this does not well explain how this particular response occurs for starting times of x:10 or x:40, leading to further questions regarding children's problem solving.

2. Displacement by 1.5 hours. The second strategy code reflected a 1.5-hour displacement from the starting time. Students drew upon the use of mathematical words in "half an hour" to quantify both *half* and *hour*, adding them together for a total displacement of 1.5 hours. Across the sample, only once did this response occur when the problem was phrased as "thirty minutes." In this particular case, the two units—hours and minutes—differently mediate children's problem solving.

3. Displacement to x:00 or x:30. The third code reflected a translation of either "thirty minutes" or "half an hour" into a final time with notation ending with x:00 or x:30. This category arose for

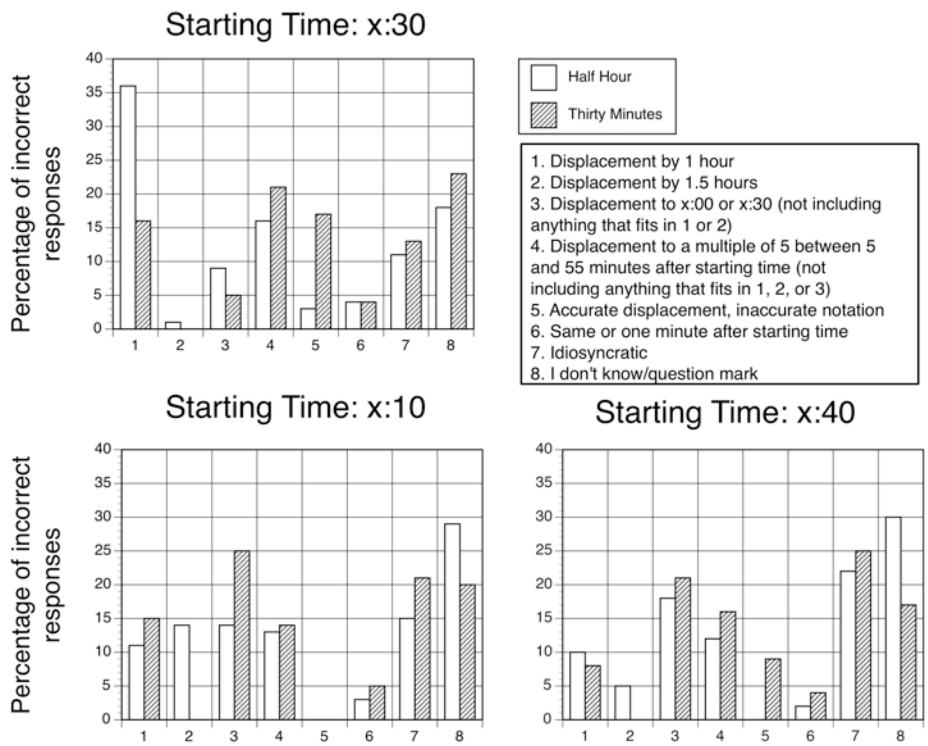

Figure 3: Proportions of incorrect response codes comparing tasks for "half an hour" or "thirty minutes" by starting time.

problem phrased both as "half an hour" and "thirty minutes." Students provided a variety of responses that resulted in this code. For example, for problem B3, students responding with "2:00" or "11:00" each received this code, though the different response choices do not reveal children's microgenetic constructions leading to these solutions. Assessment data alone do not well reveal the logic underlying the variety of responses receiving this code, though a conjecture underlying the frequency of x:00 or x:30 responses pertains both to the elapsed duration of 30 minutes/half an hour as well as the social relevance of times to the "o'clock" or "thirty."

4. Displacement to a multiple of 5. A common incorrect response for problems phrased both as "half an hour" and "thirty minutes" involved providing a new time at a multiple of 5. This particular category includes many different responses that do not fit into categories provided above. At the same time, the pattern of responses at multiples of 5 was striking. A conjecture is that the analog clock as a tool highlights time at five- minute intervals. Despite the fact that focal problems did not feature analog clocks, students' prior experiences with time, particularly the instructional and cultural emphases on identifying discrete positions in time to the five minutes (NGA Center & CCSSO, 2010), likely informed their problem solving leading to responses at five minute intervals.

5. Accurate displacement, inaccurate notation. The fifth strategy code reflected an accurate displacement with inaccurate notation, specifically adding minutes without regrouping such that the final minutes were greater than 60. This code was employed only for those tasks that reached or crossed the hour, and almost exclusively for those problems presented as "thirty minutes." This may be related to students' developing understanding of addition across elementary grades in which minutes to the right of the colon in digital notation are treated in terms of base-10 when asked to add thirty minutes. Conversely, "half an hour" tasks did not lead to this particular code. This may be because the unit "hour" cues for students spatial representations of time related to the analog clock, such that adding half an hour (as opposed to 30 minutes) to 12:40 involves does not cue the same base-10 reasoning. This suggests that issues with composition and decomposition in groups of 60 may result from problems phrased as “thirty minutes” in more prevalent ways than when phrased as “half an hour.”

Concluding Remarks

Results indicate that students treat problems phrased as “half an hour” and “thirty minutes” differently. While such displacements are mathematically equivalent, they pose differential challenges to children solving such problems. Based on assessment data, this paper makes the following claims and conjectures, though all must be further substantiated through interviews of children’s in-the-moment problem solving:

- Students treat “half an hour” and “thirty minute” displacements differently in their problem solving, despite the fact that such durations are equivalent;
- Students in each grade may translate an elapsed time of either “half an hour” or “30 minutes” into an hour of displacement;
- Students translate “half an hour” into 1.5 hours, but do not do this with “thirty minutes;”
- Students may translate displacements of “half an hour” and “thirty minutes” in coordination with treating the “o’clock” or “-thirty” time as a zero-point, leading to solutions ending in x:00 or x:30;
- Students draw upon the cultural emphasis of time to the five minutes when solving problems involving time;
- Accurate duration with inaccurate notation is observed when the resulting time either reaches or crosses the next hour, and when tasks are presented in minute- units.

While these findings provide information about children’s problem solving with time notation, the study leads to new questions that are yet unanswered in the current analysis. This paper reports on trends in data from an assessment given to elementary students, yet does not answer how or why students respond in such ways. For example, might students’ success on problems with a starting time of x:10 be related to whole number understandings, such that 30 is added to 10 without regard to time notation? Does “half an hour” cue for students part-whole relations and their understandings of fractions in a way that “30 minutes” does not?

Given the limited research in this area, the present paper begins to address questions about how elementary children respond to questions about elapsed time. Rather than answer such questions completely, results of this study lead to further questions. While clocks are certainly pervasive in culture, results of this study underscore that children’s conceptions of time in standard notation are quite varied. A concern of this project is in supporting all students in coordinating their developing notions of duration with standard time units, as such understandings are generative both in children’s developing theory of measure as well as engaging in any scientific investigation requires relying on time.

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IDENTIFYING SIMILAR POLYGONS: COMPARING PROSPECTIVE TEACHERS' ROUTINES WITH A MATHEMATICIAN'S

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This paper reports two prospective teachers' and a mathematician's ways of identifying similar triangles and hexagons through the analysis of routines, a characteristic of geometric discourse. The findings show that visual recognition was a common approach for the mathematician as well as the two prospective teachers. However, when asked for justification, their routines of identifying similar polygons diverged. The paper also discusses the implication of classroom discourse practices to enhance prospective teachers' communication and reasoning skills while learning geometric concepts such as similarity.

Keywords: Geometry; Classroom Discourse; Teacher Knowledge

In geometry, similarity is an important concept connecting many other mathematical domains such as spatial reasoning, ratio, proportion, and transformation functions. *Similarity* is defined as a relation between figures. For example, similar figures in a plane can be seen as pre-images and images of dilations (similar transformation) that preserve their shape but not necessarily the size. Similarity is not only a cross-cutting concept in geometry for middle school students (CCSSM, 2014), but also a concept that has many applications in science and engineering (NGSS, 2011). Yet, for some teachers, similarity is still a difficult topic to teach because their prior geometry coursework made little mention of it. Therefore, it is important for prospective teachers (PSTs) to know with the topic and connect it with other concepts such as ratio and proportion. However, there is little known about PSTs' geometric thinking in the context of similarity in recent research. The purpose of the study is to address this gap.

Theoretical Framework

Sfard's (2008) framework is used to analyze participants' interviews. She has proposed that mathematical discourses differ one from another in at least four features: 1) *Word use* (mathematical vocabularies and their use), mathematical words that signify mathematical objects or process; 2) *Routines*, these are well-defined repetitive patterns characteristic of the given mathematical discourse; 3) *Visual mediator*s, these are symbolic artifacts related especially for particular communication; 4) *Endorsed narratives,* any text, spoken or written, which is framed as a description of objects, of relations between processes with or by objects, and which are subject to endorsement or rejection, that is, to be labeled as true or false. These features are interwoven with one another in a variety of ways. For example, endorsed narratives contain mathematical vocabularies and provide the context in which those words are used; mathematical routines are apparent in the use of visual mediators and produce narratives. Sfard's framework provides an analytical tool to investigate how thinking is communicated through interactions. To have a better understanding of PSTs and mathematician's geometric thinking and their ways of identifying similar polygons, this paper reports findings that address these two research questions: 1) what are PSTs' routines for identifying similar triangle and hexagons? 2) What are the mathematician's routines for identifying similar triangles and hexagons?

Method

Six PSTs who participated in the study were enrolled in a Midwestern university teacher education program in the US. The mathematician was a visiting professor at the university at the time of the study. The interviews were designed to investigate participants' geometric thinking in a 30-minute one-on-one interaction. In one task during the interviews, participants were asked to identify similar figures among fifty pre-selected geometric shapes. These shapes included triangles, quadrilaterals, hexagons, and circles; all mixed together on an 11" x 17" piece ofpaper (see Figure 1). As shown in Figure 1, all shapes on the paper are labeled with numbers from 1 to 50. Among them, some are similar figures, and some are congruent figures. All interview tasks are designed to elicit participants' thinking about similar polygons. For instance, in a task of identifying similar polygons, the interview questions such as "How do you know they are similar?" "What do you mean when you say 'they are the similar'?" were asked to capture participants' geometrical thinking in identifying similar figures. All participants were given the same tasks and were asked the same initial interview questions by the same researcher following the interview protocol. However, each interview was guided by individual participant's responses to the tasks and questions. All interviews were video recorded and transcribed. All transcripts document what participants said and did during the interviews.

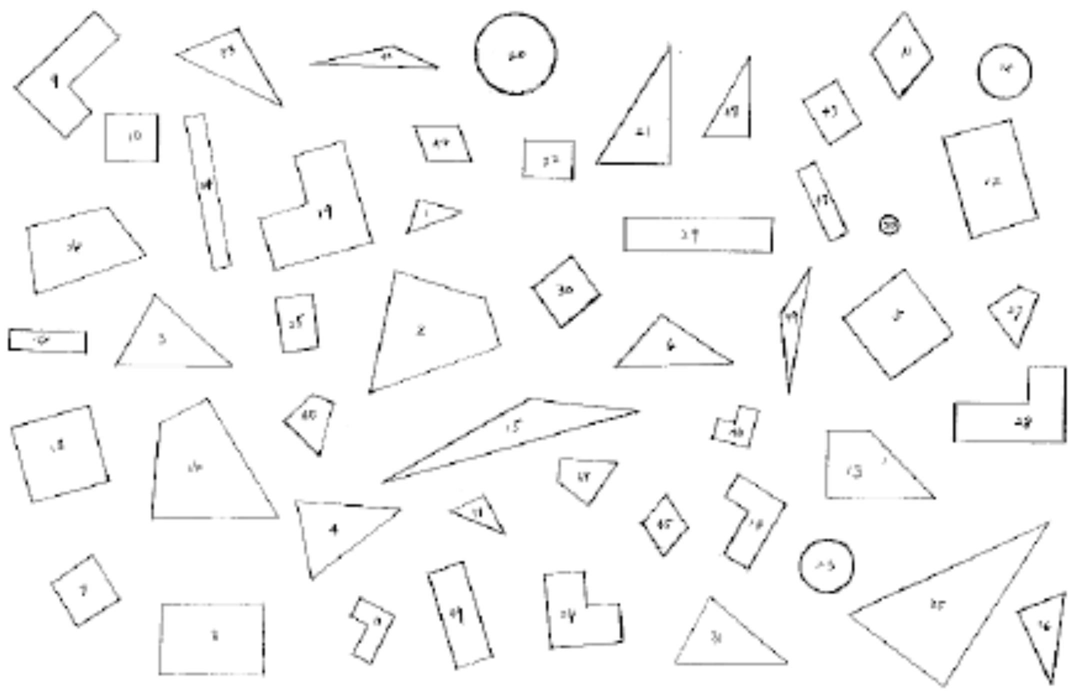


***Figure 1.* Identifying similar polygons**

Results

The results show the differences in PSTs' and mathematician's routines of identifying similar triangles and hexagons. In this paper, two PSTs' (PST1 and PST2) routines are used to illustrate these differences, and the three main differences are as follows:

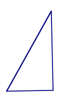
- The mathematician was aware of the abstraction of similar figures at the abstract level whereas PSTs only focused on using measurements to verify similar figures at the object level.

- The mathematician looked for two possibilities of ratios "internal versus external," whereas the PSTs did not, but used one of them instead. That is, the mathematician explicitly explained the two possibilities of ratios, internal ratios and external rations and how they could fit together to determine similar figures.
- The mathematician considered different approaches when the first attempt did not work, whereas the PSTs refuted their conjectures and did not consider other possibilities that could show the polygons were similar.

Matching Similar Triangles

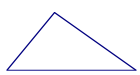
For this task, the participants were asked to identify all figures that are similar to a right triangle (#21), from the fifty figures given on the grid paper. The mathematician identified a scalene triangle (#33) as a similar triangle to this right triangle by direct recognition. When prompted by the researcher, "how do you know they are similar not based on visual?" the mathematician measured the sides of the trianglesand attempted to check for the ratio between them. She then measured the included angles of the two triangles to check for congruency. The conversation between the mathematician (M) and the researcher (I) is as follows:

1 I: Which following figures are similar to this [#21]?



#21

2M: Just looking visually, this one maybe [marked a circle on a triangle].

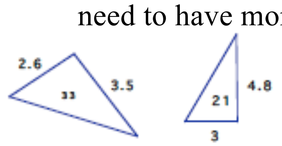


#33

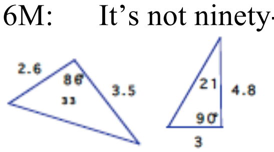
3I: If I would ask not just based on the visuals, what would you do?

4M: Can I do that [find measurements]? No, I can't necessarily; the best I can do is that to show it definitely is not similar.

5 M: 2.6 centimetres. Is that 3.5 centimetres [#33]? I would say 3 and 4.8 on this one [#21]. I need to have more information about the angle.



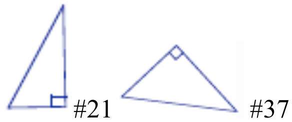
6M: It's not ninety-degrees, maybe eighty-six. That one is ninety degrees



7M: They definitely are not similar.

The mathematician identified a triangle (#33) as a candidate, which was similar to a given right triangle (#21). After measuring the sides and angles of the two triangles, she refuted her claim and concluded, "They definitely are not similar" based on different angle measures. The mathematician chose the two sides and their included angle to check if the triangles were similar. The mathematician did not calculate the ratio of the sides but checked measurement of the included angle first. When she found the angle measure did not match, she made her claim of "they definitely are not similar." In contrast to mathematician's responses to the task, PST1's routines procedures were different. She assumed the triangle (#21) was a right triangle by direct recognition. She identified another triangle (#37) that looked like a right triangle and concluded they were similar triangles:

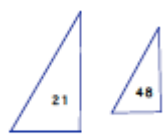
8I: Which following figures are similar to this figure [#21]?
9PST1: I am going to go with this triangle [#37] and marked right angle signs on both triangles.



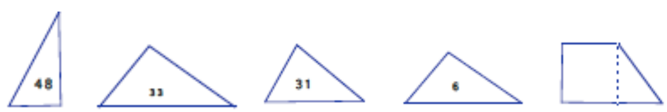
10I: How do you know this triangle[#37] is similar to this one [#21]?
11PST1: They both have a right angle.
12PST1: I am sure if you set up the proportions, they have the same angles.

PST1's routines procedures were direct recognition and it was self-evident. When prompted by the researcher, PST1 realized that two other corresponding angles of the two triangles were not congruent to each other and thenshe changed her claim about the two triangles were similar. Although PST1 did not say it explicitly, she used Angle-Angle-Angle (AAA) similarity criterion to check if the two triangles were similar. PST2's first reaction to same task was, "they should have same shape and same number of sides". That is she would consider all triangles were similar to each other:

13I: Which following figures are similar to this triangle [# 21]?
14 PST2: That is similar to it? It would be…number forty-eight.



15I: Why do you think they are similar?
16PST2: Because they have the same shape and same number of sides.
17I: Can you identify all the figures that are similar to this one [#21]?
18PST2: Circle them all?



#13

19PST2: Would that be considered a triangle if I divide a shape [#13]?
20I: Maybe not. We only focus on same number of sides like you said earlier.

PST2 identified triangle #48 as a similar triangle to #21, which was correct. However, her reasoning of why the two triangles are similar was incomplete. The following excerpt [18] showed that PST2 did not understand what similar figures meant mathematically. She focused on the visual appearance of the polygons (e.g., they have same shape) by counting the number of sides (e.g., all triangles are similar if they have same numbers of sides). To explore further, the interviewer prompted for a different verification:

21I: Why do you think these two triangles are similar to this triangle [#21]?

22PST2: By measuring their sides
23I: Show me
24PST2: 1-2-3-4-5.. I don't know.. I forgot how to write

25I: Are they similar to each other?

26PST2: They are not [similar]. They do have same number of sides, but their length measures are not the same. They are similar, but they are not the same. I am confused. It's Side-Side-Side (SSS).

Triangles #6 and #31 are not similar triangles. PST2 made a claim that the two triangles were similar because they had the same shape and the same number of sides. However, it appeared that she was confused about what "similar" meant because the two triangles had the same shape with same number of sides but different length measurements. PST2 remembered term "Side-Side-Side", but did not know how to use it in her reasoning. In addition to triangles, participants were asked to identify similar polygons to L-shaped hexagons.

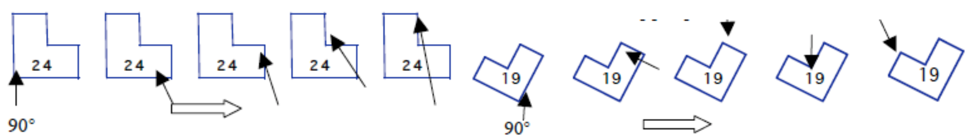
Matching Similar Hexagons

The mathematician identified an "L" shaped hexagon (#19) as a similar hexagon to the given hexagon (#24) by just looking visually again. When asked for substantiation, she measured all the angles in hexagon #24 and then in hexagon #19 accordingly, and checked if corresponding angles were congruent. After confirming the angles in the two hexagons were congruent, the mathematician proceeded to measure the sides in the hexagons, and to check the ratios between the two sides in each hexagon:

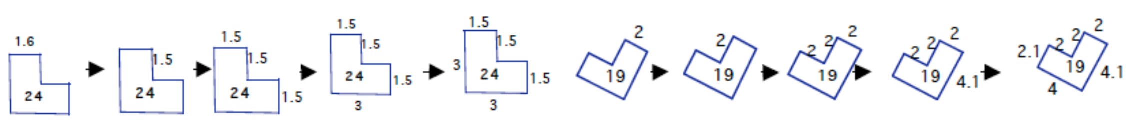
27 I: Which following is similar to this one [#24]?

28M: This one [a hexagon, #19] could be similar to [#24], just looking visually.

29M: [Used a protractor to measure the angles of the two figures].

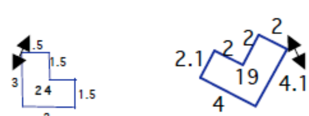


30M: Angle wise, they do seem to be all matched up. That would be not enough. I could do some measuring.



31M: [Used a ruler to measure the lengths of the two figures].

32M: This length is twice as much as that, and this is more than twice, so they can't be similar.

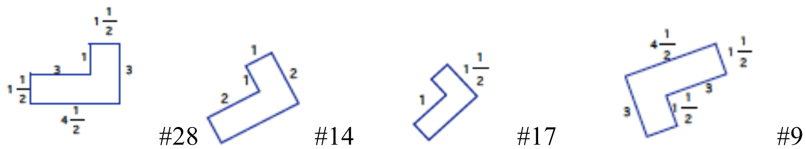


This excerpt highlights a set of actions describing how the mathematician substantiates the claim of "this one [#19] could be similar to [#24], just looking visually". She used direct recognition to identify a candidate that was similar to hexagon [#24]. After measuring the sides of the two hexagons, the mathematician found that the two hexagons did not share same ratios. She refuted her initial claim and concluded, "So they can't be similar." During the interview, the mathematician was concerned about the accuracy of the measurements. Her reaction to the use of measurements to determine similar figures was to "disconfirm" or to show "which is not."

In contrast, PST1 was asked to identify similar polygons to a hexagon [#28]. Her first reaction was to measure all the sides in hexagon #28, and then looked for L-shaped figures that had same shapes with hexagon #28:

33I: Which following figures are similar to this one [#28]?

34PST1: [Used a ruler to measure the sides of the given hexagon.

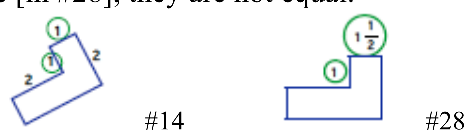


35PST1: This one [#9] is congruent to #28, so it is similar to #28 [a given hexagon].

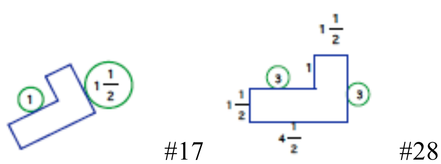
PST1 concluded that the two hexagons (#9 and #28) were similar because they were congruent. This is a correct response as congruent figures are special cases in similar figures because the ratios between all corresponding sides in congruent figures are 1:1. However, the task was not complete as the PST1 was expected to identify all similar figures. Therefore, the researcher prompted for more information:

36I: What about #14 (a hexagon) and #17 (a hexagon) are they similar to this one [#28]?

37PST1: I measured this side and this side [in #14], and they (ratios) are equal. When you compare it to this one [in #28], they are not equal.



38PST1: And this one [#17], I have these two sides, and they (ratios) are not equal to these two sides [in #28].



39:PST1: So they're not proportional to each other, and they are not similar to #28 [the given hexagon].

When verifying similar hexagons, PST1 focused on checking the ratios between the sides in one figure (#28) to the ratio of corresponding sides in other figures (17&14). She did not check any angle measures. PST2's response to the same question was different. PST2 first identified all L-shaped figures that would be similar to the hexagon (# 28) by direct recognition. When she was asked to verify, "How do you know this one (#24) is similar to this one (#28)?"(See Figure 2).

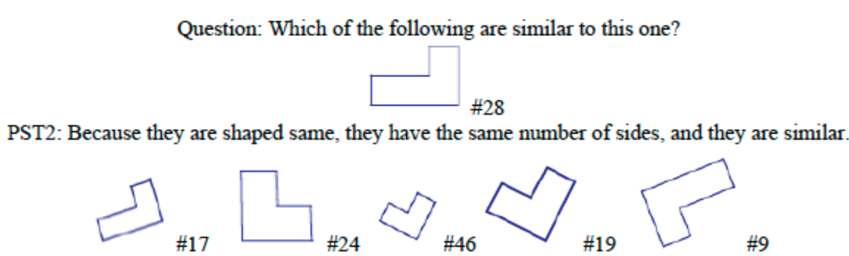


Figure 2.

PST2's responses of hexagons are similar to the given hexagon (#28). PST2 did not measure any sides and angles in the two figures, and replied by saying, "they're the same shape, but just different scale sizes," The researcher then asked, "Is this one [#9] also similar to this one [#28]?" PST2 replied, "It looks like it's the same size. Yes, I would have to say so. Yes." Figure 2 presents a collection of hexagons that PST2 considered similar to the given hexagon (#28). PST2's responses to this task showed a repeated pattern of focusing on the shape of the figures rather than checking the ratios of sides and measure of angles in the figures when identify similar hexagons.

Discussion

Findings suggest that when identifying similar figures, direct recognition is a common first approach to the PSTs and the mathematician based on the shapes of the figures and their orientations. This could be argued that geometry is one of the most intuitive areas of mathematics. However, procedures that are more rigorous are required to identify similar figures by checking the ratios of corresponding sides and the congruence of corresponding angles. Mathematically, congruence is a special case of similarity; however, findings show that one prospective teacher (PST2) did not think this is the case. When identifying similar triangles, the PSTs and the mathematician responded differently and their routines of identifying similar triangles showed the differences (see Figures 3, 4 & 5).

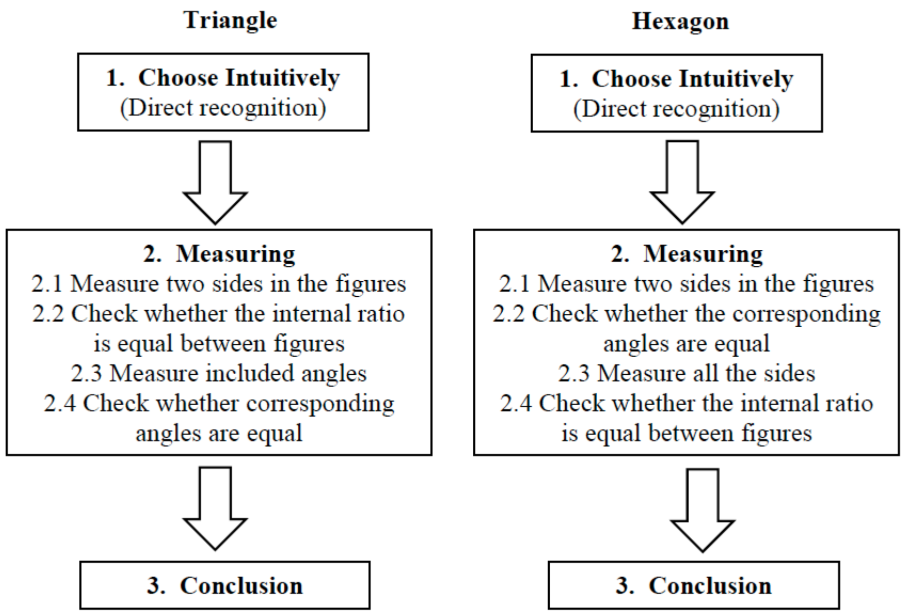


***Figure 3.* The mathematician's routines of identifying similar figures**

Between the two PSTs, PST1 demonstrated a better understanding in identifying similar triangles than she did in identifying similar hexagons, whereas PST2 showed difficulties and confusion in identifying similar triangles and hexagons. The findings also suggest the vague mathematical understanding the two PSTs had through their use of the words "similar" and "same shape." Note that both of them were aware of the informal definition of similarity, but there were confusions between the colloquial and mathematical meaning of the terms *same shape* and *similar* in the context of similarity. For example, PST2 did not understand what "same shape" meant when discussing similarity, and she interpreted the term as the figures shared the same number of sides, or that figures are shaped the same, which was not correct in the classification of similar figures. To the same word, similar, the two PSTs responded very differently: PST1 focused on the conditions of similarity (e.g., corresponding angles are equal, corresponding sides are proportional), which is more of the mathematical use of the term similar, whereas PST2 focused more on the "sameness" (e.g., same number of sides, shaped the same) between shapes than the mathematical reasoning (see Figures 4 & 5). The vague use of mathematical words such as *similar* and *same shape* provide information on how the concept of similarity was learned and understood by PSTs to make references to the wide range of relationships between shapes.

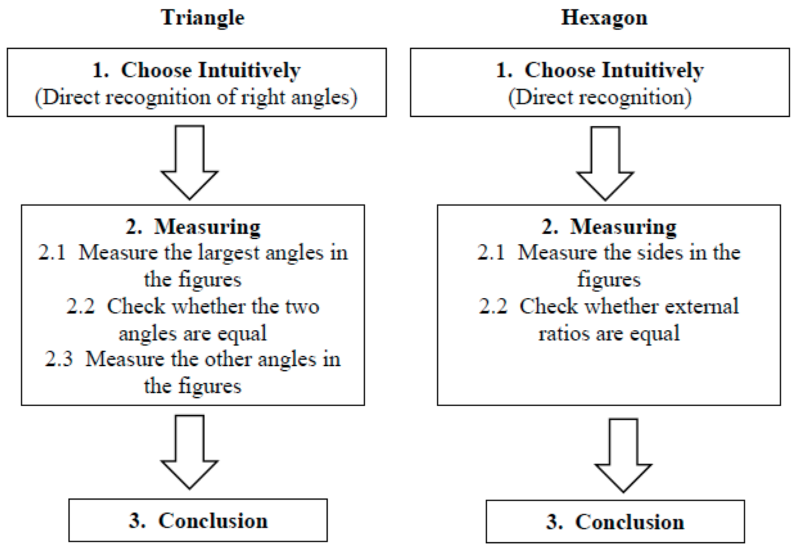


***Figure 4.* The PST1s' routines of identifying similar figures**

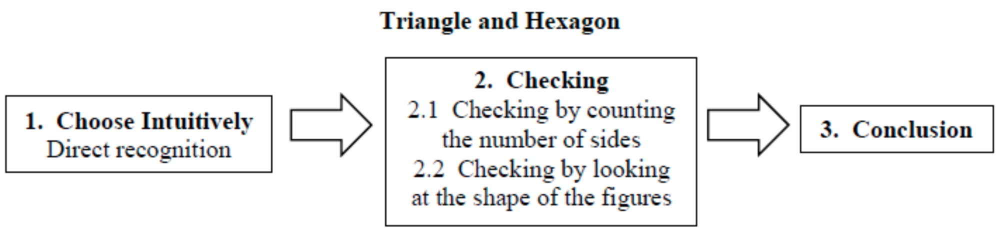


***Figure 5.* PST2's routines of identifying similar polygons**

Findings of the study suggest that many misunderstandings of a concept such as similarity will likely be missed if we only focus on the product or single answer of the task. When the PSTs' were asked to explain and to justify claims they made, and to clarify what they meant when mathematical terms (e.g., similar, same, proportion, etc.) were used, we started to see the vague understanding of similarity. It was through those interactions that the misunderstandings of the concept in similarity were detected by a series of questions that were designed to elicit their thinking. Therefore, one recommendation the study could make is to infuse discourse practices in our mathematics classrooms for PSTs to enhance their explaining, clarifying, and defining skills and to shed a light on their use of mathematical terms to ensure that the mathematical concepts are developed correctly. We need to ask more questions about "why" and "how" during the classroom discussions in order to help PSTs articulate their mathematical thinking and reasoning.

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CONNECTIONS BETWEEN PROVING PRACTICES AND CONCEPTIONS OF FIGURES OF SECONDARY PRESERVICE TEACHERS

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Reasoning and proof are an important part of the most recent curricular reform effort (NGA/CCSSO, 2010), but previous research has shown that many pre- and in-service teachers have insufficient conceptions of proof (Knuth, 1999; Martin & Harel, 1998). A limitation of studies of teachers knowledge of proof is that researchers have focused on whether or not the teachers work was or was not a proof without considering how those teachers understood the concepts with which they were being asked to prove. The intent of this study is to understand both the concepts that pre-service teachers (PST) have of triangles and quadrilaterals and how they applied those conceptions to proving conjectures about triangles and quadrilaterals. An interesting finding is that PSTs whose proofs displayed a restriction on the generality of their proof (Harel & Sowder, 1998) was a result of their conceptions of triangles or quadrilaterals.

Keywords: Mathematical Knowledge for Teaching; Reasoning and Proof

The Common Core State Standards for Mathematics place emphasis on reasoning and proof for instruction by including reasoning as one of their *Standards for Mathematical Practice* and putting proof in clusters in half of the domains of geometry (NGA/CCSSO, 2010). However research has shown that teacher's, whether pre- or in-service, elementary or secondary, have insufficient conceptions of proof (Knuth, 1999; Martin & Harel, 1998). Studies on teachers' knowledge of proof have focused on whether they consider an argument a proof or not or to what extent the arguments they create are proofs. Although it is relevant to understand how teachers reason about topics, what studies on teachers' conceptions of proof lack are an understanding of how their knowledge of the content affect their proving.

The purpose of this paper is to report findings from such a study. Specifically, this study found that pre-service secondary teachers (PSST) whose proof productions had a restriction on the generality of their argument was a result of how they thought about the figures on which they were proving. The organization of this paper includes a description of literature on conceptions of figures and proof, which is followed by a description of the study. Results from a PSST who was represented of others PSSTs will be presented along with supporting evidence from other PSSTs, but due to space restrictions a detailed summary of only one PSST will be given. Finally, implications for teacher education will be discussed.

Conceptions of Figures and Proof

There are two types of geometric definitions, partitional and hierarchical (de Villiers, 1994). Partitional definitions put figures into disjoint sets, for example a square is not a special case of rectangle. Hierarchical definitions classify figures so that more particular figures form subsets of more general figures. Hierarchical definitions are useful for proving (de Villiers, 1994) because if a conjecture is proved for a figure then all the figures in the subset inherit that conjecture.

A proof in this paper is meant to be an argument that convinces or persuades an individual or community (Harel & Sowder, 1998). Harel and Sowder (1998) created a taxonomy of proofs that they found individuals use. Two of their categories are relevant to this study so they will be the only categories described. The empirical proof scheme is marked by arguments that validate based on the conjecture being true for one or more examples. The transformational proof scheme contains three criteria (a) generality–understanding to justify for all cases, (b) operational thought–evidence of goals and sub-goals, and (c) logical inference. Individuals who reason in a transformational way can face restrictions on the generality of their proof. Meaning that proofs are proved in a specific case, like rectangle, but intended to prove for a more general case, like quadrilateral. The transformational proof scheme is considered the first deductive proof scheme.

Harel & Sowder (1998) suggest that it is necessary to have a dynamic conception of figure in order to prove, meaning that as one reasons about a quadrilateral, for example, it is important to understand the variety of images a quadrilateral can take on instead of focusing on one static or prototypical image. Prototypical image is meant to be a specialized case where even things like orientation are relevant. An example would be that a triangle has to have three congruent sides and a horizontal base. Having a diverse set of examples from which to reason on is considered an important skill for proving Sandefur, Mason, Stylianides, Watson (2013).

Method

Thirteen PSST agreed to participate in the study. All were enrolled in the second of two required methods courses and were in the last semester of their course work. One PSST was majoring in special education and getting an endorsement in mathematics, but all others were majoring in mathematics education. This population was targeted because they would be near the end of their mathematical course work and thus have finished their formal training in proof.

There were two phases of data collection. The goal of the first phase was to gain an initial understanding of the PSSTs' conceptions of triangles and quadrilaterals and what types of arguments they find convincing. The PSSTs were given a packet to take home and return in one week that contained thirty-four items. Items asked PSSTs whether definitions were valid or not, whether figures had or did not have certain properties, whether conjectures were valid or not, to interpret statements of inference, and whether certain arguments were or were not proofs. The items were in a multiple-choice format.

The second phase of data collection was an hour-long task-based interview. During the first ten to fifteen minutes of the interview, the PSSTs were asked to elaborate on some of their work from the take home packet. The rest of the time was spent asking the PSSTs to explore geometric conjectures that are typical of a high school geometry course. They were given a list of thirteen conjectures and told that some of them are true, some of them are false and all of them are based on Euclidean geometry. The majority of the conjectures (9) were false, but they were not entirely false meaning that they were often stated for the general case and were not valid for the general case but were true for a specific case. The purpose of this was to put the PSSTs in a position where they were first exploring whether the conjecture was or was not true in an effort to evoke transformational reasoning.

The PSSTs were asked to decide if the statement was or was not true. If they felt the statement was true, they were asked to provide a justification that they felt would prove the conjecture. If they felt the statement was false they were asked to give a counterexample and if possible amend the conjecture so that it would be true and prove that conjecture.

During the interview the PSSTs were given a ruler, straight edge, compass, protractor, a list of terms and their definitions taken from a high school geometry textbook, and geometer's sketchpad that had built in tools that would construct special cases of triangles and quadrilaterals. The PSSTs were told to choose the conjectures with which they felt most comfortable because there would not be enough time to finish all the conjectures.

The take home packets were scanned and marked for accuracy. Interesting patterns in the PSSTs answers were identified to be explored in the interview during the second phase. The interviews were video recorded and transcripts were made along with scans of written work. The PSSTs proofs were coded using Harel and Sowder's (1998) taxonomy of proof schemes. The constant comparison method of grounded theory (Corbin & Strauss, 2008) was used to analyze how the PSSTs conceptions of figures affected their proof productions. Initial passes created codes while continued passes through the data resulted in codes being removed, added, and refined until themes emerged. One particularly relevant theme that emerged is that the proof productions of the PSSTs that had a restriction on the generality of the proof were a result of the PSSTs holding a partitional, static conception of figures.

Results

During the first phase a PSST with the pseudonym Lara accepted all partitional definitions for rectangle and also accepted a definition that was not valid for rectangle. On another item Lara was asked if different classes of quadrilaterals had the property that the diagonals bisected each other. She drew prototypical images of the classes and based off of those images she accepted properties that were not true about quadrilaterals. This suggested that Lara did not have a strong concept of quadrilateral. Moreover, for Lara it seemed that any property of a figure was a defining property.

During the second phase, Lara worked on seven conjectures. She ran out of time on her last proof and we chose to stop before she finished. She proved one conjecture, had one purely empirical proof and had five proofs with a restriction on the generality of the argument. What was clear is that it was not Lara's inability to think deductively that lead to her struggles with proving the conjectures it was her limited conceptions of figures.

During the second phase, it became clear that Lara saw the general case, quadrilateral, as constituting only the special cases (e.g. rectangle, rhombus, parallelogram, and square). For example, she proved the conjecture "the diagonals of a quadrilateral cut the area in half" by proving that the triangles created by the diagonal of a quadrilateral would be congruent. When asked about her proof being for a rectangle she said, "A rectangle is a quadrilateral."

For that same conjecture, Ana was confused at first and said she would have to prove for each case. She then proved that the triangles created by a diagonal of a rhombus would be congruent using the Side-Side-Side Postulate (SSS). She stated that opposite sides being congruent was a property and that the third pair of congruent sides was the shared diagonal. She then drew a rectangle, parallelogram, and square and created the same proof that rested on SSS for all three cases. When asked if she needed each part for her proof to be formal she said "if you wanted to prove for all quadrilaterals you would need to show for each case." Another PSST did not think she could prove a conjecture for all quadrilaterals unless she proved each case. In total, five of the thirteen PSST reasoned as if quadrilateral only meant the special cases.

To compound this Lara's figures appeared to be static for her since she drew only one figure for each conjecture except one and on that conjecture her new drawing appeared congruent to her previous drawing she simply made her lines straighter. For example she proved the conjecture "isosceles triangles are congruent when their vertex angles are congruent" by creating two similar sized isosceles triangles and using SSS. She stated from the definition that two sides would be congruent and she falsely assumed that the two pairs must then be congruent to each other as well. She then reasoned that since the vertex angles are congruent the opposite sides would have to be the same length. After creating her argument she gave no indication of considering how her argument would stand up for other figures despite being prompted to do so. When asked if she felt there were other ways to draw her figure, she said "I don't think there's another way."

Two other PSSTs gave a similarly false deductive proof for the problem because they drew two isosceles triangles that appeared to be congruent. Two other PSSTs gave a proof that had a restriction on the generality of the argument. An example would be that one PSST took the measure of the vertex angle to be 40 degrees and then using that specific case showed that the other angle measures would have to be 70 degrees. Eventually she showed that all three pairs of sides and angles were congruent.

Finally, Lara's figures were in my opinion protypical. Her triangles were nearly perfect equilateral triangles. Her drawings for quadrilateral were either rectangles or squares. That is not to suggest that Lara would not have recognized a definition of triangle as a polygon with three sides. It is to note that because Lara often reasoned off of one figure and that figures often perceptually had properties that were not true for all figures in the class she often became confused as to whether or not the conjecture was true. For the previous proof she went back and forth three times before deciding it was true. Because she did not spend time exploring diverse examples of triangles she was likely convinced perceptually by her drawing.

Interestingly, Lara's only formal proof was for the conjecture "the diagonals of a rhombus are angle bisectors." She drew one diagonal and mentioned drawing either one would not matter. She then stated that because it was a rhombus the two triangles were isosceles since the sides were all equal. Because the triangles were isosceles she knew that the base angles were equal. She then proved the isosceles triangles congruent and showed that all four angles created by the diagonal were congruent. Lara, like the other PSSTs whose proofs had a restriction on the generality of the argument was able to create a formal proof, but the conjecture did not test her restriction as it was true for all rhombi.

Implications

It is often stressed that teachers need richer and/or more experiences with proof however, the struggle the PSSTs faced in this study was more a result of how they conceived of the figures. Therefore it may be critical to first improve teachers' conceptions of figures prior to improving their proving ability. Or PSSTs may improve both their understanding of figures and proving by exploring conjectures that are not necessarily true so that they need to be truly explored first.

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CHILDREN'S THREE-DIMENSIONAL MENTAL STRUCTURING USING A DYNAMIC COMPUTER ENVIRONMENT

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We investigated elementary children's three-dimensional mental structuring to measure volume using a dynamic computer environment constrained by a fixed sequence of length, width, and height. The results of three case studies indicate that children's attention to multiplicative reasoning over additive reasoning, recognition of the importance of three-dimensions, and interpretation of two-dimensional representations as three-dimensional were important in their development of reasoning about volume. The constrained order of dimensions proved critical.

Keywords: Geometry; Learning Trajectories; Measurement

Measurement of three-dimensional space is an integral topic in children's mathematical and scientific development. However, children's development of volume concepts is a long and complicated journey. Two important conceptual advances in children's understanding of volume involve unitizing and spatial structuring. Young children begin to quantify volume by unitizing space and iterating that unit throughout space (Van Dine, et. al., in press). Development of exhaustive and efficient counting schemes for their units involves the grouping of units, and the further grouping of units of units (Battista & Clements, 1996). This spatial structuring is foundational for computing volume.

The purpose of this investigation was to further understand how students develop spatial structuring schemes by grouping units. We instituted a dynamic computer environment that we believed would help students establish and apply structuring schemes to three-dimensional tasks presented in a two-dimensional format to answer the following question: In what ways does a dynamic computer environment constrained by a fixed order of operating in each of three dimensions affect students' three-dimensional mental structuring? Based on the research of Clements (1999), we expected that a computer-based visual environment might function as a concrete manipulative for students in much the same way as physical objects. In addition, the environment provided a restricted structuring scheme that would profer the recognition and incorporation of groups of units rather than single units in a broad collection. The restrictive aspects of our microworld afforded us the opportunity to investigate children's change in their application of spatial structuring schemes to operate with groups of units, much as dragging in dynamic geometry software was analyzed by Arzarello, Olivero, Paola, and Robutti (2002) to infer types of thinking about geometric ideas.

Theoretical Framework

We used a learning trajectory (LT) about volume measurement (Van Dine et. al., in press) to analyze student progression because we were investigating children's development of spatial structuring schemes. This LT consists of a developmental progression that includes specific instructional goals and tasks that support students' growth. These goals and tasks derive from Hierarchic Interactionalism's stance that children progress through domain-specific, hierarchic levels that can be characterized by specific mental objects and actions.

The following volume LT levels specify reasoning children exhibit. At *Volume Quantifier* (VQ) children distinguish volume from other spatial dimensions. At *Volume Unit Relater and Repeater* (VURR) children relate size and number of units explicitly and understand that fewer larger than smaller units will be needed to fill or packspace. At *Initial Composite 3-D Structurer* (VICS) children understand cubes as filling a space andexplicitly relate size and number of units to volume. Children also use additive reasoning, such as skip-counting, converting units requiring ratios even more complex than 1:2. At *3-D Row and Column Structurer* (VRCS) children begin using multiplicative comparisons to carry out comparative analyses of volume among several shapes or objects.

Methodology

As part of a larger study, we used a case study approach to investigate the thinking of three students at a private school in the Midwest. Each student participated in two interviews with one or two researchers. During each interview we asked the student to find the volume of rectangular prisms, first without, and then with the support of a dynamic computer environment developed using GeoGebra (International GeoGebra Institute, 2015).

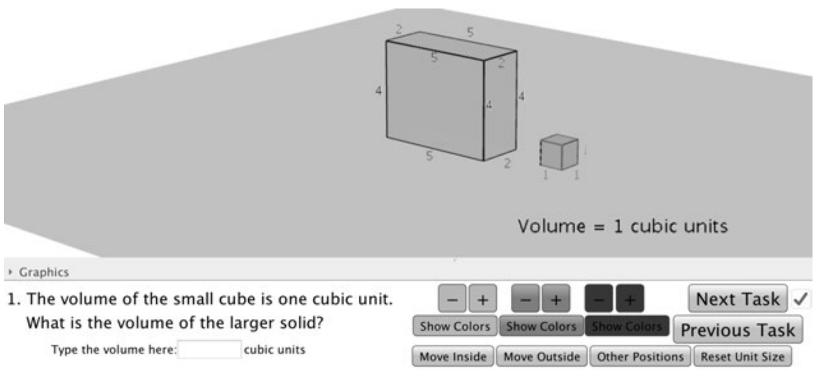

Figure 1: Dynamic Computer Environment

At the beginning of the first interview we introduced the student to the computer environment, (see Figure 1) asking him to push each color's '+' and '–' buttons and predict what would occur for each push. On each volume task, we first presented the student with a paper version identical to the task pictured in Figure 1. The interviewer said, "The volume of the small cube is one cubic unit. What is the volume of the larger solid?"

After the student solved the task on paper, the interviewer asked him to solve the same task using the computer environment. This allowed the computer environment to act as an intermediary between the two-dimensional representation and three-dimensional reality of the tasks. The computer environment allowed the student to move the unit cube inside a wire frame of the initial rectangular prism and add cubes to fill the wire frame controlled by the three sets of '+' and '–' buttons. Each set would add or subtract cubes in one of three directions.

We asked the student to use each set of buttons in succession, left to right, without returning to previous sets. When the student used the first set of buttons one cube at a time would be added or subtracted from the set of cubes, creating an n x 1 x 1 row of cubes. When the student used the second set of buttons new nx 1 x 1 rows would be added (or subtracted), one row at a time, resulting in an nx m x 1 layer. When the student used the third set of buttons new nx m x 1 layers would be added (or subtracted), one layer at a time. Any of the '+' and '–' buttons could be used at any given time, but we allowed few exceptions to the left to right succession. These restrictions created an environment conducive to students' spatial structuring development.

The interviewer asked questions of the student throughout the process of filling the wire frame on the computer. In a typical interview, just before using a set of '+' and '–' buttons, the student was asked to predict how many cubes there would be when he finished using that set of buttons. In some interviews, after filling the wire frame with cubes we asked the student to push one of the show colors buttons, which would alter the colors of the cubic units to show horizontal layers, vertical layers left to right, or vertical layers front to back in alternating colors.

Results, Discussion, and Conclusions

Peter, age 9, Tim, age 9, and Mike, age 10, were chosen as case studies because each demonstrated growth over the course of two interviews. Peter drew two-dimensional representations to solve three-dimensional problems and changed from a two-dimensional to three-dimensional counting scheme. Tim progressed from additive to multiplicative thinking and identified the importance of three dimensions in calculating volume. Mike changed his interpretation of the paper task and aligned his counting strategy with the computer.

Growth Along the Volume Learning Trajectory

At the beginning of the first interview Peter was using strategies typical of VQ when working on paper. For example, he was not accounting for hidden cubes, double counted cubes on the edge, and did not connect the displayed measures with the number of cubes that would fit along an edge. However, he also demonstrated strategies typical of VRCS when working with the physical 3-D objects and when using the computer environment. As Peter progressed through the two interviews he displayed fewer misconceptionstypical of VQ, such as double counting edge cubes, while working each task on paper. In their place, he began to use strategies typical of VRCS. This suggests that the analogous tasks in the dynamic computer environment helped Peter progress along the LT in his treatment of the paper tasks.

At the beginning of the first interview Tim demonstrated strategies and misconceptions ranging from VQ (failure to connect number with space) to VRCS (operating on a layer). Although Tim was able to calculate accurately on the final task, he did not demonstrate that he was operating at VRCS when solving the problem on paper. Using the computer environment Tim demonstrated, through consistent operation on composite units, that he was operating at VRCS. This suggests that the mediating power of the computer environment characteristic of Peter's experience might not have reached its full potential for Tim.

At the beginning of the first interview Mike was not operating at VRCS on the paper task, but was able to operate on a layer (typical of VRCS) using the computer environment. On the final task of the second interview Mike used VRCS strategies when solving the task on paper. This suggests that, in a manner similar to Peter, Mike progressed along the LT in his treatment of the paper tasks with the mediating influence of the computer environment.

Influence of the Dynamic Computer Environment

Students' three-dimensional spatial structuring schemes were influenced in several ways by the use of a dynamic computer environment that constrained students to a sequential use of length and width of base and then finally the height of a prism. First, the computer environment enhanced the opportunity for students to recognize the efficiency of multiplication for volume calculation. For example, Tim transitioned from additive to multiplicative strategies. Peter transitioned from a unit counting strategy to a combination of multiplication and addition.

Second, the computer environment highlighted the importance of three distinct measurements in the calculation of volume. Although all three students progressed from the use of many measurements to the use of only three, this was most apparent in the interviews with Tim. During the second interview Tim began writing the words tall, long, and wide with measurements ascribed to each.

Third, the computer environment acted as a mediator between two- and three-dimensional representations. Although it was difficult for all three students to interpret the two-dimensional representation of a volume task as a three-dimensional task, the computer environment facilitated a reinterpretation of the task that allowed the students to approach the task with a three-dimensional spatial structuring scheme.

Implications

We argue that there were three critical features of the study to support growth and we recommend that volume instruction include activities that incorporate these features. First, the computer environment implemented an implicit spatial structuring scheme by first adding single cubes to make a row, then adding additional rows to make a layer, and finally adding additional layers to create the full prism. This feature was important because it highlighted a reasonable spatial structuring scheme and the multiplicative nature of volume calculation.

Second, this iteration of rows and layers was accentuated by having students use each set of buttons in succession without repeated use of prior sets of buttons. This helped students visualize a beneficial structuring of space early in the process. It also accentuated the three-dimensional nature of volume and the use of those dimensions in sequence.

Third, the students were engaged in both paper and computer tasks. This was important because the computer provided a medium that allowed students to connect the two-dimensional paper representation with actual three-dimensional space. Although this took time, working the same task on paper and on the computer allowed the students to make connections between two-dimensional representations and three-dimensional space. Further research should investigate the efficacy of this intervention for instruction on volume measurement with elementary students.

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PRELIMINARY FINDINGS ON TWO NINTH GRADERS' CONCEPTIONS OF ANGLE AND ANGLE MEASURE AS GROSS QUANTITY

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Keywords: Geometry; High School Education; Curriculum

Students' difficulties learning angle concepts constitute an enduring problem for the field of mathematics education that has been well documented at the elementary, middle, and undergraduate levels (Lehrer, Jenkins, & Osana, 1998; Mitchelmore & White, 2000; Moore, 2013). Recent research with undergraduate participants (Moore, 2013) has illustrated that instructional sequences with a focus on quantitative reasoning (Thompson, 2011) are productive for developing an understanding of angle measure as ratio. However, the outcomes of such an approach to teaching angle measure have not been documented in the existing literature with participants in other stages of education. While a ratio understanding of angle measure is productive for undergraduate students enrolled in a precalculus course, ratio may be a difficult starting point for some pre-college students learning about angle measure. Thus, research is needed to examine the effects of an instructional sequence with an emphasis on quantitative reasoning designed for students at other educational levels.

In this poster session, I present tasks, data, and findings from individual initial interviews conducted with two ninth-grade students who participated in an eight-month teaching experiment (Steffe & Thompson, 2005) focusing on angle and angle measure. The tasks in the initial interview protocol were organized around three themes that align with key principles of quantitative reasoning. First, prompts were developed to explore students' conceptions of angles as objects. Second, students were asked to describe the quality of an angle that is described by an angle's measure. Finally, students were provided with a series of tasks designed to explore how students order angles based on varying extents of this quality. This final theme was designed to engage students who have yet to develop an understanding of ratio. Findings regarding substantial differences in students' conceptualizations of angles will be presented. Additionally, three distinct strategies for making gross quantitative comparisons exhibited by one student will be presented. Implications for teaching about angle concepts will be considered.

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ELEMENTARY STUDENTS' SPATIAL REASONING IN A MINECRAFT ENVIRONMENT

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Keywords: Cognition; Geometry; Elementary School Education; Technology

Spatial-visualization using three-dimensional objects from different viewpoints is an important aspect of geometric reasoning and has been shown to be positively correlated to mathematics achievement (Cheng & Mix, 2014). Emergent work within a dynamic geometry spaces (i.e., Battista, 2007) calls for further investigations in these environments. We report on an exploratory study that aims to examine student spatial reasoning with tasks involving cube buildings in a popular computer game. Our study extends how students reason about spatial tasks when using the computer game Minecraft, an open-world building game, as a dynamic geometry environment.

We examined 16 4th and 5th grade students working collaboratively on spatial tasks in a Minecraft environment during an after-school mathematics enrichment camp. We gave participants a variety of spatial tasks to determine how the game Minecraft could be utilized as a dynamic geometry environment. The tasks were oriented around providing participants with three orthogonal views (front, top as seen from the front, and right side) of a cube building (Battista & Clements, 1998). The participants were asked to construct these buildings from the given orthogonal views, as well as create a written procedure to communicate their building strategy.

Our results indicate that the participants had trouble initially coordinating the multiple views and integrating them into one structure. Consistent with existing work (Battista & Clements, 1996), most students were eventually able to coordinate and integrate the views by exploring the placement of cubes. Our participants found the top view to be the most difficult to integrate, as they were initially not coordinating the other two viewpoints to account for depth. They interpreted the top view as the top layer of the structure. However, our participants used in-game features to create a labeling scheme by placing sign-posts to help in orienting their perspectives so that each participant could keep track of the same orthogonal view. By using the dynamic features of Minecraft to rotate around the cube building and observe a variety of viewpoints, the participants were able to explore the placement of the cubes, and thus coordinate and integrate the three viewpoints. Our poster will encapsulate our data, research and analytic processes, significant findings, as well as provide a guide for using Minecraft as a teaching and research tool.

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INSTRUCTORS' PERCEPTION OF SPATIAL REASONING IN CALCULUS

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Keywords: Geometry; Post-Secondary Education; High School Education; Advanced Mathematical Thinking

Calculus may be regarded as a gatekeeper for students pursuing STEM careers, as a required course for many students in engineering and science. The calculus reform debates of the 1990s illustrated this ongoing struggle to improve the teaching and learning of calculus. There is a need for understanding what skills may be helpful for students to be successful in learning calculus.

One such skill is spatial reasoning, the ability to imagine manipulation of objects. Ferrini-Mundy (1987) investigated whether spatial reasoning training would improve students' achievement in calculus. She found that spatial reasoning relates to certain areas of calculus. Studying the relationship between spatial reasoning and calculus specifically has some grounding, because spatial reasoning is considered integral to studying engineering (Ferrini-Mundy, 1987). This study aims to answer the following research questions: What do calculus instructors perceive as topics from calculus that use spatial reasoning? What kinds of spatial reasoning are involved in learning calculus? The aim is to see (a) if and where spatial reasoning appears in calculus content and (b) identifying if there are certain types of spatial reasoning over others that are helpful to learning calculus.

Uttal et al. (2012) constructed a 2x2 framework for classifying types of spatial tasks. A task is (a) intrinsic or extrinsic and (b) dynamic or static. A spatial task is *intrinsic* if the focus of the task is about the object itself, whereas a task is *extrinsic* if the focus is on the relationship between an object and its environment. As a second dimension, a task is *dynamic* if the subject must mentally imagine movement, whereas a task is *static* if it does not involve movement.

Semi-structured interviews were conducted with four (N=4) calculus instructors at a large university. Participants discussed the role of spatial reasoning in single variable and/or multivariable undergraduate calculus. Then, they gave specific examples or topics they believed used spatial reasoning or where spatial reasoning would be helpful. These examples were analyzed using Uttal et al.'s (2012) framework to identify the types of spatial reasoning.

Some topics that the participants gave included the Fundamental Theorem of Calculus, epsilon-delta limits, integrals, level fields and planes, three dimensions in general, etc. This research is currently in the analysis phase, but preliminary results suggest that many of these calculus examples are dynamic and extrinsic in nature. This suggests that students who struggle with imagining movement, especially objects in an environment, may also struggle with understanding certain topics in calculus; this research hopes to illuminate how spatial reasoning is embedded in calculus.

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HOW DO GEOMETRY TEACHERS CONSTRUCT MEANINGS IN RELATION TO STUDENTS' PRIOR KNOWLEDGE

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Researchers have stated the importance of broadening students' opportunities to establish connections with their funds of knowledge when doing mathematical work (Drake et al., 2015). By funds of knowledge we refer to the knowledge that students accumulate from their life experiences (González, Andrade, Civil, & Moll, 2001). We identify teachers' actions for activating students' prior knowledge in typical geometry lessons and provide a description of these actions. Drawing from a social semiotics approach to examine meaning making processes in mathematics classrooms (Morgan, 2006), we ask: *What meanings do teachers construct when establishing connections between students' prior knowledge and mathematical content?* We provide examples of actions that teachers perform with detailed descriptions about the relationships that teachers establish to support students' mathematical understanding.

We analyzed eight lessons from five different teachers in four different high-needs schools. We focused on how the teachers established connections with three sources of prior knowledge: mathematical content, the context of the problem, and out-of-school practices. We identified sixteen instances where teachers activated students' prior knowledge with 13 actions. Seven actions pertained to mathematical content, six actions pertained to out of school mathematics, and no actions were related to contextual problems. This finding may respond to the fact that none of the lessons were problem-based, so the teachers did not need have to discuss relevant contexts as in other investigations (e.g., Jackson et al., 2013). We then conducted a thematic analysis (Herbel-Eisenmann & Otten, 2011) to examine how teachers constructed mathematical meanings. For mathematical prior knowledge, teachers established relationships between students' knowledge of mathematical definitions, concepts, theorems, and algebraic operations. For out-of-school practices, teachers established relationships between spatial or environmental themes, cultural references, and embodying mathematics with properties of angle pairs. Actions connecting with out-of-school practices created multiple entry points to develop mathematical understandings. Our goal is to show how these actions promote students' understanding of geometric concepts in relation to their funds of knowledge. Furthermore, cataloging teaching actions have implications for teacher educators and researchers in documenting examples of activating students' prior knowledge in secondary mathematics.

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STUDENTS' REASONING ABOUT CUBE-PACKAGE ENUMERATION PROBLEMS

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According to PSSM, "Spatial visualization—building and manipulating mental representations of two-and three-dimensional objects and perceiving an object from different perspectives—is an important aspect of geometric thinking" (NCTM, 2000, p. 41). In fact, almost all geometric reasoning and problem solving are intimately connected to spatial reasoning. More generally, the National Research Council claims that, "Underpinning success in mathematics and science is the capacity to think spatially" (NRC, 2006, p6), a statement backed by much research (Newcombe, 2010; Wai et al, 2009).

In this paper, we investigate the nexus of geometric and spatial reasoning by examining students' reasoning about 3D packing problems (Battista & Berle-Carman, 1996). For example: How many packages consisting of a line of 5 cubes fit into a 4-by-6-by-3 box? Such tasks add complexity to students' spatial structuring and enumeration because the packages may not completely fill the box, and students must properly visualize package orientations. Packing problems are not only important in mathematics, but provide an opportunity for students to deepen, clarify, and extend their reasoning about volume and geometric measurement.

In our analysis we use theoretical constructs from research on spatial visualization and research in mathematics education to investigate the reasoning, cognitive processes, and mental models students use to solve cube-package problems. We also analyze the cognitive obstacles students face. Our data sample consists of videotapes of four 5th grade boys who were working in pairs on cube-package problems in an inquiry-based classroom as part of a four-week unit on volume. In our data analysis, we discovered three cognitive processes above and beyond those used in standard cube-packing, volume problems—package-orienting, orthogonal projecting, and package-locating. We also found that students were using three types of mental models: layer composite-unit based, non-layer composite-unit based, and non-composite-unit based. Our research geometrically elaborates two general spatial reasoning strategies described in spatial visualization research—visualizing object transformations and decomposing objects into parts and reasoning about the parts (Hegarty, 2010).

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Chapter 6

Mathematical Processes

Research Reports

Brief Research Reports

Poster Presentations

A REFINEMENT OF MICHENER'S EXAMPLE CLASSIFICATION

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In this paper we propose a refinement of Michener's (1978) well-known example classification based on data from university mathematicians. The refinement takes into account the mathematician's perspective on the role of examples in doing mathematics. More specifically, our work provides insight into the ways in which mathematicians talk about using examples in their scholarly work and their work with students. The proposed classification has the potential to inform our work as teachers as we strive to create opportunities to engage students in authentic mathematical work.

Keyword: Advanced Mathematical Thinking

Examples play a significant role in teaching and learning mathematics. Often it is with a carefully formulated example that subtleties in a definition or an algorithmic process can be detected. Examples make it possible to consider generalities, but can also limit one's mathematical perspective. Ideally, mathematical learning experiences provide opportunities to develop a rich array of examples that contribute to students' problem-solving skills and understanding.

Interestingly, what is known about the role of examples in doing mathematics is not adequately informed by the mathematician's perspective. Few studies on examples or related topics have mathematician participants (c.f. Lockwood, Ellis, & Knuth, 2013). On the other hand, there is evidence that mathematicians frequently generate examples in the process of validating a proof given by a peer (Weber, 2008) and that mathematicians use different types of examples in conjecture-related work (Lockwood, Ellis, & Knuth, 2013). In addition, given the number of proofs without words publications, it is reasonable to assume that mathematicians value opportunities to "see the general in the specific" (Mason & Pimm, 1984). That is, mathematicians sometimes use an example to make a proof. However, we have little evidence that speaks to the mathematicians' perspective on the use of examples in teaching or how examples support their own mathematical work.

Michener's seminal work (1978) presented an epistemology of mathematics from her perspective as a mathematician that classified example types for the development of mathematical thinking: start-up examples, reference examples, model examples, and counterexamples. Classifying an example is necessarily tied to the purpose the example serves in supporting mathematical learning. Studies on the use of examples with students continue to reference Michener's classification (e.g., Alcock & Inglis, 2008; Watson & Mason, 2002; Zaslavsky & Shir, 2005). Because of this, we wondered whether Michener's classification is useful when examining the ways mathematicians use examples in their work. In particular, we asked: do mathematicians' articulations of the uses of examples align with these classes? Do the classes capture the range of the mathematicians' articulations related to the uses of examples?

We describe our initial refinement of Michener's epistemology based on data from an earlier study where we examined how mathematicians make sense of definitions (Kinzel, Cavey, Walen, & Rohrig, 2011). We then use preliminary analysis of data from a current study focused on mathematicians' use of examples more generally to propose further refinements.

Background

Michener's (1978) epistemology of mathematical knowledge presents three interrelated categories of items: results (theorems), concepts (definitions, heuristics, advice), and examples (illustrative material). The interrelatedness of these categories is described in terms of a predecessor/successor relationship: Examples may lead to the construction of definitions and/or theorems or alternatively may serve to illustrate a definition or theorem. These relationships are connected through the notion of "dual relations" and the identification of "dual items." The dual items of a particular example are the ingredient concepts and results needed to construct the example and the results motivated by the example. Similar duals can be defined for results and concepts. Thus, examples are integrally entwined with the other categories of items.

Michener further states that because not all examples are "created equal," those more noteworthy deserve further attention and can be grouped into epistemological classes. These classes are not intended to be disjoint: A single example can play more than one role, perhaps even within a particular context or situation. It is these classes of examples in which we are interested. Michener's four classes of examples are paraphrased below (1978, pp. 366–368).

- *Start-up examples* are used to introduce a new subject by motivating basic concepts and useful intuitions, such as using a specific picture or visual representation to highlight key features of graphs of monotonically increasing sequences.
- *Reference examples* are widely applicable and provide a common point of contact, and so tend to be referred to repeatedly. This class can also include standard cases used to verify one's understanding of a particular concept or result as when a particular function is used to illustrate what it means for a graph to be concave down.
- *Model examples* suggest and summarize the expectations and default assumptions about results and concepts. The absolute value function could be a model example for the idea that a continuous function need not be differentiable on its domain.
- *Counterexamples* sharpen the distinctions between concepts and are used to show a statement is not true. The function $f(x) = x^3, x \in \Re$ is a counterexample to the statement that all cubic polynomial functions have two local extreme values.

Goldenberg and Mason (2008) elaborated on Michener's classes, pointing specifically to the role of "pertinent nonexamples" (p. 184) in clarifying the sorting of items into things *that are* versus things *that are not*. Nonexamples are not equivalent to counterexamples in that there may not be a statement whose truth is in question; however, the identification of nonexamples can also serve to sharpen distinctions or interpretations of results and concepts. Nonexamples may also be used to develop one's intuition (start-up example) or to verify one's established understanding of a concept (reference). The primary use of nonexamples is to highlight contrast, which can overlap with the purposes of other classes. In this paper, we include nonexamples as a fifth class in our framework, paying particular attention to instances when nonexamples can be classified otherwise. As we describe in the next section, data from our earlier research supported the inclusion of this fifth class.

Methodology

Interview data from our earlier research, which focused on how mathematicians make sense of definitions, revealed themes related to the role of examples within that work (Kinzel, Cavey, Walen, & Rohrig, 2011). Nine mathematicians participated in interviews in which they were asked to first describe how they make sense of a new definition for themselves and then how they support students in making sense of new definitions. Examples, and the purposes for using examples, were prominent in the mathematicians' descriptions. Making sense of definitions is a key component of mathematical work but does not capture the full range of that work. We used data from the earlier study to begin a refinement of Michener's framework, then returned to the same group of mathematicians to explore the role of examples beyond the context of definitions. In the follow-up study we asked the mathematicians to review the classes of examples presented in the framework and consider (1) how well the framework reflects the ways in which they use examples, (2) if there are uses of examples not represented in the framework, and (3) if the framework is a useful means for thinking about mathematical work, either for themselves or for students. As we are still collecting data from the follow-up study, what follows are the methods used to establish the initial refinement.

Interview data were transcribed and pseudonyms were assigned to the participants. Individual research team members reviewed transcripts to identify instances where the mathematicians made a reference to the use of an example (*example-instances*); coded transcripts were then shared and example-instances verified. To be identified as an example-instance, an articulation by the mathematician either explicitly included the word "example" or referred to illustrating a concept or result in some way; for instance, a description of how the mathematician uses examples to clarify which items fit a definition and which do not counts as an example-instance. The instances we identified did not always include the articulation of a specific concept or mathematical object. Where possible, the specific concept or result within the instance was identified (e.g., articulating the usefulness of providing visual examples to illustrate the concept of collinear points).

The second stage of analysis focused on coding the example-instances based on Michener's (1978) classes of examples. We focused the data analysis through two questions: (a) What types of examples did the mathematicians describe? and (b) How did they describe the purpose of each example? We began this part of the analysis with a discussion of Michener's classes and a group analysis of one transcript to clarify the shared understanding of the classes. This involved using transcript data to clarify distinctions between the classes. These distinctions often focused on the perspective of the learner as well as the intended use of the example by the mathematician. After these criteria were established, the remaining transcripts were analyzed by individual team members; two other team members then verified the coding. Within this analysis, we encountered several articulations related to the construction or analysis of *things that are not*; a function that would *not* meet the criteria of a particular definition, for instance. We introduced *nonexamples* as a fifth class in the framework to capture these articulations.

To illustrate the coding process, we present several example-instances and the resulting classification. Consider the articulation from Adam in response to what helps him to make sense of a new definition: "I start off with things that are familiar to me . . . I would start going through the list of standard examples that I have in my head for these." This instance was coded as *reference* because we inferred from this statement that Adam's "standard examples" were widely applicable. On the other hand, Sam responded to the same question as follows: "the simplest thing to try first is just to look at the specific concrete examples … you kind of get a feeling of how this specific definition is working." This instance was coded as a *start-up,* since we inferred from "get a feeling of how this specific definition is working" that Sam's purpose was to develop his intuition. About three-fourths of the way into the interview with Greg he began explaining the importance of using examples with students where the "main feature" of the concept is "worked out that we actually want to transport by those examples." By this, we inferred that he was articulating the importance of using an example that illustrates critical features, and thus coded it as *model*.

After we coded all the example-instances, we examined the instances within each class to identify themes in how mathematicians talked about using examples to support either their own or their students' learning. Broad epistemological themes emerged within each class of examples. Recall that mathematicians were asked to reflect on their own processes/experiences as well as on those they intend for students. Because of this, it was necessary to identify the intended learner within example-instances. Each instance could potentially refer to the mathematician, to their students, or to a hypothetical learner. When we refer to the intended use of an example, we always mean in reference to the learner, whether it be the mathematician or a potentially hypothetical student. However, the epistemological themes that emerged address both the instructor's and the learner's perspective. It was the identification of these themes that led to the refinement of Michener's classification of examples.

Initial Refinement of Michener's Example Classification

In this section we provide a description of our initial refinement of Michener's example classification. Data analysis led to clarification on the classes already noted in the literature (Michener, 1978; Watson & Mason, 2005). Informal conversations with mathematicians about our proposed refinement indicate that these categories are useful but may not be exhaustive, especially in relation to work with results or theorems. For this reason, we anticipated data from the follow-up study to lead to further expansion and clarification of our initial refinement.

See Table 1 for a list of the classes in our initial refinement and a summary of the purposes associated with each class.

Table 1: Initial Refinement Of Michener's Example Classification

Class	Purpose
Start-up	develop intuitive notions consider what is and what is not to isolate concept check initial understanding through generating examples
Reference	widely applicable or standard case isolate new subclasses of mathematical objects
Model	demonstrate salient features consider interplay between features of definition and example
Counterexample	refute the truth of a statement in question
Nonexample	demonstrate what *is not* show "control" of a definition consider particular features of a definition

Start-Up Examples

In presenting Michener's classification, Watson and Mason (2005) describe start-up examples as those from which "basic problems, definitions, and results can be conjectured at the beginning of learning some theory and can be 'lifted' to the general case" (p. 64). The articulations from the mathematicians provide further detail into particular ways in which this conjecturing might be supported. For example, analyzing a collection of examples can help to isolate essential features of a new concept. The activity of generating examples of a new idea contributes to clarifying one's emerging understanding of the idea. Creating variations of known examples can serve to further demonstrate or clarify one's understanding. From the learner's perspective, examples in this class should be familiar objects that also illustrate key features of the new concept.

Mathematicians described using examples to develop intuition about the features of the concept that distinguish it from other related ideas. Greg articulated that students may not know to what to pay attention at first: "And then I might try to work out with them a little bit, you know, what could be mathematically interesting there? What are the features there? . . . Then state a formal definition and then go and do plenty of examples to kind of work on that." Ned articulated this use of examples in his own work: "But if it was foreign or I read it and realized that I didn't truly understand it, then I probably would try to come up with an example. Which is a hard thing because you need your example easy enough to understand, yet hard enough so that it eliminates what needs to be eliminated." Wes gave a similar response: "And how is it new? And can I think of something that does this as well as something that doesn't do this?" Across these instances, the focus was on using examples of *things that are* as well as examples of *things that are not* to develop intuition for the essential features of a newly encountered definition.

Reference Examples

Reference examples are intended to be widely applicable and available for consideration as a standard case. A learner may return to one or more reference examples while in the process of developing understanding of a new idea or subclass of mathematical objects; perhaps using a familiar example of a group to explore a new property, for instance. A learner may have a known and familiar set of standard examples that are consciously used to verify or extend understanding. As with model examples, the level of awareness of the learner with respect to the essential features of these examples is critical in their appropriate use. Overuse of a specific case could lead to confusion between aspects of the particular with aspects of the more general concept.

Wes and Adam specifically discussed using familiar objects to *understand an unfamiliar subclass of objects*. Wes shared an experience of reading a student thesis in which a particular property was introduced. In considering whether he understood the property as it related to the thesis, he asked himself: "It says it does this, or these things do that, so why is this (stably free module) different from this thing that we have (free module)." To make sense of the notion, he noted that he began thinking of examples of free modules with which he was familiar and tried to identify which would have the new property and which would not. Adam expressed this same idea in his own work; a general practice for him when encountering a new idea is to check his "list of standard examples" to see which of those objects illustrate the new idea. Adam also explained how he uses this process with students. The instance he shared related to introducing the concept of algebraic groups; he drew students' attention to familiar sets with operations (such as integers under addition) and then engaged them in determining which of these familiar things met the criteria of a group. In this way, Adam seemed to intend that these familiar instances could become reference examples for the students for the concept of group.

Model Examples

Model examples are intended as paradigmatic and generic, and can be used to convey salient features of an idea. As noted by Sam, a model example may emerge from the analysis of a collection of examples; one exemplar from the collection may serve as an illustration of the desired concept. The presentation of a model example can serve to highlight the interplay between the use of the example and one's understanding of the idea; that is, the activity of determining *why* the presented exemplar qualifies as a model example potentially interacts with one's understanding. Poor choice of a model example can cloud one's understanding of an idea, in that this awareness of essential features may be (perhaps implicitly) compromised.

In choosing examples to present to students as potential model examples, Greg emphasized the need for the example to illustrate the key features, and that it be "not too trivial," yet also "not too complicated" as that could interfere with "seeing" the features of the concept. Marc also spoke of choosing pictures to convey relevant features to students. Ned stated that he uses the interplay between example and concept as a means for checking students' understanding. After presenting a (model) example, he makes small variations, such as changing a positive slope to negative. In his experience, students who were able to see the key features were less likely to be distracted by the variation. That is, we would argue that Ned was determining whether students saw the initial presentation as a model example for the presented idea; if so, students were more likely to be able to identify key components and perhaps not be distracted by variations that did not alter the underlying concept.

Nonexamples

Nonexamples clarify distinctions between *what is* and *what is not*, and thus are used to demonstrate the importance of key features of a concept. The purpose of a nonexample can overlap the purpose of start-up, model, or reference examples. A collection of examples can be used to draw attention to common features; contrasting such a collection with nonexamples serves to sharpen distinctions. The successful generation of relevant nonexamples can be taken as an indication of the "control" one has with regard to a concept or result. A learner's explicit attention to aspects of nonexamples is an indication of depth of understanding.

Mathematicians discussed using nonexamples to refine one's understanding of a definition or concept. In particular, the generation of nonexamples can focus attention on salient features and the purpose of those features. Often, this involved dropping or violating one or more criteria within the definition and asking, in Ned's words, "how does that change the outcome of what is permissible?" Sadie stated that nonexamples give a "different perspective" through analyzing what "it can't be." Sam acknowledged that "find[ing] an example in which this fails" can be challenging for students, but can be a critical step in developing understanding of the definition by forcing one to "look deeper at what things are."

Counterexamples

Counterexamples are used to refute a statement. Some become well-known and used often. While not an explicit focus of our analysis, we see some common aspects. In particular, the learner's awareness of why an instance qualifies as a counterexample for a given statement is key, and could serve as an indication of the learner's understanding of the underlying ideas.

Current Work

We are currently in the midst of a follow-up study focused on mathematicians' use of examples more broadly. We asked the mathematicians who participated in the definitions research study to read the results of that work, including our initial refinement of Michener's example classification prior to a face-to-face interview. We also asked each mathematician to respond to the following questions in writing prior to the interview:

- How well does the proposed example-use framework reflect the way you think about the purposes of different types of examples in your mathematical work? Please address this question as it relates to your scholarly work (writing, research, etc.) and teaching.
- Is there is a type of example that you use that is not articulated in the framework? Is there an example type that you rarely use? Please explain.
- Is the framework useful as a means of thinking about the different purposes of examples in doing mathematics? If so, in what ways?
- Is the framework useful as a means of thinking about teaching mathematics? If so, in what ways?

The mathematicians we have interviewed articulated benefits associated with thinking carefully about the purposes of different types of examples in their teaching but not in relation to their own mathematical work. They describe using examples for the purposes described in the initial refinement along with other purposes more closely related to their scholarly work. Thus far, it appears that there may be one or two other example classes that warrant defining. In particular, two mathematicians noted how examples can be preceded by definitions and results, providing the impetus for *existence examples*. The purpose of this class is to demonstrate the existence of a mathematical object and is thus distinct from the other classes. Greg noted, "You may actually indirectly prove that a thing exists without actually constructing it" while Evan noted odd perfect numbers as an instance where the definition has preceded examples. Another possible category is that of *boundary examples*—those that support understanding of "where the boundaries are" (Evan) with a concept or result. It not yet clear from the data whether this category stands on its own. The data suggest that there is overlap with the purposes of reference and nonexamples in that boundary examples can be used to identify subclasses of objects or to demonstrate control of a definition. Further analysis is needed to clarify these distinctions. Moreover, careful consideration of the epistemological value of potential new classes is needed. As noted by Michener (1978), the classes are not meant to be exhaustive but rather particularly informative in relation to thinking about how to fully support students in learning mathematics.

Discussion

Research indicates that students benefit from generating their own examples rather than passively accepting examples given by the teacher (Dahlberg & Housman, 1997; Watson & Mason, 2002; Sowder, 1980; Weber, Porter, & Housman, 2008). In our work, we observed mathematicians attending to perceived benefits of example generation both for themselves and for their students. Generating examples was seen to help build intuition about a new concept (start-up), to sharpen distinctions (non-example), and to reveal or verify understanding (start-up and/or model). In general, the ability to construct an appropriate example was taken as evidence of some level of understanding. Being able to then modify or create variations of the example could be further evidence of understanding. Constructing nonexamples was seen as more challenging, but was also seen as evidence of working knowledge of a concept.

Watson and Mason (2005) describe *example spaces* as a metaphor for the psychological structure of the ideas and examples associated with a particular concept. An interesting feature within the mathematicians' articulations was the interplay between examples and related ideas. For instance, Adam talked of drawing on his set of "standard cases" of objects to make sense of a new concept. He may think of standard examples of groups to make sense of a newly encountered type of group. Using Watson and Mason's metaphor, he pulls reference examples from one example space to be used as start-up examples in a related space. In discussing their work with students, the mathematicians talked of beginning with objects that should be familiar or known to students, using these as start-up examples for a new concept, or establishing them as reference examples for a concept. These descriptions align with Watson and Mason's characterization of mathematical activity as the reorganization of example spaces.

Our work aligns with work by Lockwood and colleagues (2012, 2013) in which they analyzed mathematicians' articulations of the role of examples within the context of exploring conjectures. We particularly agree with Lockwood, et al. (2012) as to the importance of "intentional example exploration" (p. 157). Our application of Michener's classification relies on identifying the intended role of the example; for instance, a model example is most powerful when the learner recognizes the features that exemplify the concept.

Implications

In general, refinement of the example classification provides the mathematics education community with a common language about the role of examples in teaching and learning mathematics. Having a common language is important for future advances in the area. Of course, this work also raises important questions regarding the role of exemplification in the teaching and learning of mathematics. In particular, how well does the classification capture the role of examples within a broader context of mathematical activity? How might the classification be used to guide instructional design? Could deliberate attention to the purposes support the selection of, presentation of, and plans for student engagement with examples?

A common language for example types can support more intentional selection of examples within instruction. From an instructional design perspective, one might attend to the intended purpose of an example to determine its place within the unit of study. Following Adam's suggestion, for example, a textbook author might include one or more start-up examples of a concept prior to introducing a definition. Likewise, reference and model examples may be better placed after a definition has been presented. Further, a classroom teacher might be more explicit about the role of examples, providing support to students' interpretations and use of examples.

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STUDENTS' CONCEPTIONS OF DERIVATIVE GIVEN DIFFERENT REPRESENTATIONS

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In Calculus, students are often both presented problems with and taught to use three interconnected types of representations: symbolic, graphic, and numeric. However, students often fail to notice the relationship between mathematical objects (and even the same object) that are presented using different types of representations. Using the APOS framework and student interviews, this study explores ways in which students conceive of tasks involving derivatives that are posed using the different representational types. Patterns are drawn among: problem conception and representation type; problem conception and task type; schema level and representation type; and schema level and problem conception. Though there were patterns between representation and student conception, stronger patterns emerged between task type and student conception.

Keywords: High School Education; Cognition

Introduction

In calculus, the concept of derivative is often taught using three interconnected types of representations: symbolic, graphic, and numeric (Schwarz & Hershkowitz, 2001). However, students often do not recognize and use the relationships between them well (Asiala et al., 1997; Aspinwall, Shaw, & Presmeg, 1997; Schwarz & Hershkowitz, 2001). This may be for a variety of reasons, such as that algebraic representations are typically used more on tests (Aspinwall et. al, 1997), that instruction focuses on specific translation skills instead of a discussion of relationships between representations (Schwarz & Hershkowitz, 2001), and that students are often asked to translate in one direction, but not in reverse (Asiala et al., 1997).

Anna Sfard (1991) discussed the interaction between visual (graphs and pictures) and analytic (equations and numbers) representations and grouped students' conceptions into two complementary categories: operational and structural. She claimed that analytic notation encourages an operational conception, for which the focus is the procedure, whereas visual notation encourages a structural conception of the mathematical entity as an object. Therefore, students might tend to think about derivative differently depending on the representation presented in a problem. For example, if a student is presented with a graph of a function, he may tend to use a structural conception, but if presented with an equation he may hold an operational conception. This does not give information about the student who is presented with a tabular representation, or that needs to translate between representations in order to solve a problem.

Similar to Sfard's operational and structural conceptions, Dubinsky posits the APOS theory, with which student understanding of a mathematical entity falls into one of four conceptions: action, process, object, and schema (e.g. Dubinsky & McDonald, 2001; Maharaj, 2013). The action and process conceptions align with Sfard's operational conception, while the APOS object conception aligns with Sfard's structural conception (Maharajh, Brajlall, & Govender, 2010; Tall, 2013). However, while a student may possess several different conceptions of a mathematical entity, the student may not access all of these equally in all situations (Baker, Cooley, & Trigueros, 2000), particularly when comparing visual and analytic representations, both during instruction and problem solving (Maharajh et. al, 2010).

Dubinsky & McDonald (2001) and Baker et. al (2000) also discuss the notion of the *triad*, a fixed order of three stages of schema development (intra, inter, and trans) in which students progressively

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see more connections between events or objects. Students begin at the *intra* stage, where they see mathematical entities as disconnected objects, and as they develop connections between objects, they progressively advance through the *inter* stage to the *trans* stage, in which they possess a coherent schema with which they can reason about one or more related objects in a meaningful way. One might expect this triad to exhibit a correspondence of stages to the APOS conceptions of cognition, with the action conception corresponding to the intra stage, the process conception corresponding to the inter stage, and the object conception corresponding to the trans stage. This observation brings into focus the following research question: how are students' conceptions of derivative related to the ways in which they make connections within and among different representations? For example, is a student who is operating at the intra stage unable to utilize an object conception for a mathematical entity because that would require connections that are not yet made? Or, if a student displays an object conception of derivative when presented a graph, but an action conception when presented a table, would the student be able to draw the necessary connections between the information given by the graph and the table?

Framework

APOS theory has been used by a variety of researchers to analyze students' procedural and conceptual knowledge (e.g. Asiala et. al, 1997; Baker et. al, 2000; Hähkiöniemi, 2006; Maharaj, 2013). APOS stands for *action*, *process*, *object*, and *schema*, which are categories of conceptions that an individual might hold of a mathematical entity (Maharaj, 2013). Understanding APOS conceptions allows us to form a *genetic decomposition*, or model of the specific conceptions that a learner might make (Asiala et. al, 1997) to better describe students' understanding of mathematical concepts (Dubinsky & McDonald, 2001; Maharajh et. al, 2010).

An individual who has an action conception is able to carry out a step by step set of instructions or a written algorithm, but is able to do little more (Asiala et. al, 1997; Dubinsky & McDonald, 2001) or to control the action (Hähkiöniemi, 2006). While an action is always in response to a stimulus that is perceived as external, a process performs the same operation as an action, but may be contained completely in the mind of the individual (Maharaj, 2013). This internalization allows the individual to be able to reverse or compose operations (Dubinsky & McDonald, 2001). An object is formed as a complete entity in the individual's mind when a process can be transformed and acted on as a totality (Dubinsky & McDonald, 2001; Hähkiöniemi, 2006; Maharaj, 2013). A schema is a collection of actions, processes, and objects coordinated by the individual (Baker et. al, 2000; Dubinsky & McDonald, 2001).

Because an individual's schema is an evolving composition of actions, processes, and objects, it is more of a meta-cognitive development for problem solving than a conception in itself (Baker et. al, 2000; Dubinsky & McDonald, 2001). This means that at any given moment, when an individual uses either an action, process, or object conception to solve a particular problem, the individual is concurrently operating within their schema as well. For this reason, this study describes only the APOS conceptions of action, process, and object, and assumes that the individual is always operating within some personal schema.

Table 1 gives examples of what might be observed of a student using each conception if a problem involving derivative is given using each of the three main representations: table, equation, and graph. It is important to note that even though these conceptions seem to evolve from one to the next, the development of action, process, and object is not necessarily in that order (Dubinsky & McDonald, 2001). Though this study explores the student's conceptions as a snapshot, and not being developed over a substantial period of time, this is still a valuable reminder for the current study because, for example, it is possible for an individual to use an object conception without having ever thought through the action or process involved first.

It is a purpose of this study to determine if a student will tend to use different APOS conceptions given different problem representations. For example, if given a table, a student can perform steps to approximate a derivative at a point only when prompted for the steps, the student would be at the action conception. However, if the student is given the same question with a graphical representation and has no difficulty recalling that the derivative at a point being the slope of the line tangent to the graph, drawing the line, and approximating the slope, the student would be at categorized as having a process conception. This genetic decomposition is used in the analysis and discussion of students' responses to interview questions in this study.

Table 1: APOS Conceptions of Derivative Given Different Representations.

Conception	Table	Equation	Graph
Action	The student follows steps when prompted to find slope and write equation of tangent line.	The student can use steps to find equation for derivative and tangent line.	The student can use steps when prompted to find slope from graph and draw the tangent line.
Process	The student can easily apply procedures using tabular data, and can catch errors and explain reasoning.	The student can easily manipulate equations dealing with derivatives and provide some rationale.	The student can notice and use parts of the graph to answer questions about tangent line and derivative.
Object	The student uses the table to reason about derivative and solve problems.	The student uses the equation to reason about derivative and solve problems.	The student uses the graph to reason about derivative and solve problems.

APOS theory has been further refined through the introduction of the *triad* to recognize the evolving nature of schemas (Baker et. al, 2000). The triad consists of three stages, *intra*, *inter*, and *trans*, which occur in that order (Baker, Cooley, & Trigueros, 2000). As is suggested by their names, at the intra stage, the individual is concerned with and isolates the single action, process, or object of focus, while at the inter stage, relationships are formed between cognitive entities (Dubinsky & McDonald, 2001). For example, a student given a graph at the intra stage may be able to draw a tangent line but not look for numerical information to find its slope. The student at the inter stage would use the graph to find numerical values or an equation that could then be used to solve the problem, but these connections would still be isolated and superficial and while the student may reason about why a particular representation is chosen, the student does not reason about more than one representation at a time. An individual operating at the trans stage exhibits a schema with the most coherence, for example, recognizing all situations requiring the computation of a derivative at a point as interconnected (Baker et. al, 2000). Table 2 describes what might be observed of a student at each stage given different representations.

Methods

Four students were interviewed using a series of calculus questions to determine their understanding of derivatives. The researcher recruited volunteers from an AP Calculus class at a mid-sized, southern high school. The volunteers from this class are considered to be typical students who take AP Calculus at this school. They have all taken two courses in algebra, a course in geometry, and a preparatory class for calculus (Pre-calculus). Two are female and two are male, with one of the females in twelfth grade and the other three students in eleventh grade. All four of the students interviewed perform well in their class, with an average grade of either an A or high B. At

Table 2: Triad Stages for Derivative Given Different Representations.

Triad stage	Table	Equation	Graph
Intra	The student makes decisions using only a tabular representation.	The student makes decisions using only equations.	The student makes decisions using only a graphic representation.
Inter	The student uses the values in the table to construct a graph or equation.	The student uses the equation to find numerical values or construct a graph.	The student uses the graph to find numerical values or an equation.
Trans	The student reasons about connections between the table and an equation and/or graph to solve the problem.	The student reasons about connections between the equation and table and/or graph to solve the problem.	The student reasons about connections between the graph and the equation and/or numerical values to solve the problem.

the time of the interviews, students had been in the calculus class for about eight months, and were studying integrals. They began their study of derivatives several months before the interviews took place, completed a unit focused on derivatives, and have continually used and applied them to other calculus problems since.

Each interview was conducted outside of school hours at a local public library, and lasted between 30 and 45 minutes. Interviews were recorded using a Livescribe pen and paper, which records the interview and connects the audio recording to the written work.

Each of the three interview questions presented the student with information about a problem using a different representation (table, equation, or graph). Question 1 (tabular representation) is shown in Figure 1 as a representative of each of the questions asked. Though the representation is different across the three questions, the tasks in each part are the same across questions. The reason for the repetition of the four tasks across interview questions is to determine if students tend to use a particular conception over another for a task when given different representations. In addition, these tasks were designed to assess the student's ability to translate between representations, thus assessing the triad stage that the student uses.

#1: The following table gives values of a differentiable function f(x) for different values of x:

X	-1	0	1	2	3	4	5	6
f(x)	-2	2	5	8	12	8	4	7

a. Approximate f'(1).

b. Approximate $\lim_{h \to 0}\left(\frac{f(4+h)-8}{h}\right)$

c. If f'(-1) = 5, write the equation of the line tangent to f(x) at x = -1.

d. Is f'(x) probably increasing or decreasing at x = 3?

Figure 1: Interview Question #1

The data from each interview was then analyzed to determine each student's APOS conception and triad stage for each interview question. The results were then reviewed to determine if there is a correlation between the representation given to the student, the APOS conception that the student used, and the triad stage at which the student operated.

Results

While there was at least one instance for each APOS conception given each representation, some strong patterns emerged. Table 3 compares the given representation to the APOS conception that the student used. Since each problem contained four tasks, and each was asked to four students, the total for each row is sixteen. The *no determination* column was added because there were some problem parts where the student either said that they did not know how to do it or else created an incorrect "rule" with no justification.

Table 3: Students APOS Conceptions Across Given Representations.

Representation	No determination	Action	Process	Object
1. Table (numeric)	7	2	3	4
2. Equation (analytic)	1	3	9	3
3. Graph (visual)	5	1	5	5

Students felt the most confident when given the equation, with which they almost all immediately started writing out answers, compared to where they generally stopped to think about a course of action given the other representations. They exhibited a strong process conception in that there was a routine and they were following it, but they also knew why they followed each step and could give appropriate justifications for each part of their answers.

A stronger pattern that emerged was that between the type of information asked for in each task and the students' APOS conception used in approaching the task. Table 4 compares the task type to the APOS conception used.

Table 4: Students APOS Conceptions Across Task Type.

Task Type	No determination	Action	Process	Object
a. Find derivative (at a point)	3	1	5	3
b. Find derivative by limit.	9	3	-	-
c. Write equation of tangent line.	-	1	10	1
d. Is the derivative increasing or decreasing?	2	-	2	8

Students had the most difficult time with the limit part of each question. All four students verbally said that they had forgotten how to find limits, but this again was dependent on the given representation. In the questions that give the table and graph, students could describe what a limit is, but when actually approaching the problem, mostly said that they do not "remember how to do it." Given the equation representation, three students of the four made much more of an attempt to try different algebraic rules until they came to an answer for the limit. The student in the following excerpt demonstrates an action conception in trying to find the limit of a difference quotient given an equation (A denotes the student, R the researcher):

A: [draws graph] I feel like this is not the way to do it, but if you plug in -1… you get 33, and at 1… it's 51. [sighs] I know there is another way to do this, it's been so long.

[The student skipped to parts c and d and then came back to b.]

R: So when you were doing limits, did you just plug numbers in, or did you cancel first and then plug numbers in? Did you do anything with simplifying?

A: Oh, wait, wait, ok. Ok, um, you factor, not factor, but, um… 3 times 2, 6, plug 3 times h equals 3h [continues to simplify part b at the right of Figure 1]… Yes! Ok, sorry, yeah, so then you factor h out, you have 9h plus 42 over h, those cancel, and nine times 0 plus 42, so the limit at that point is 42!

In contrast, the students held a process conception in ten of the twelve instances of writing the equation of a tangent line, and did this part of each question with the least effort, regardless of representation, and also regardless of whether they could find the derivative. The student who struggled with the limit went through the solution to using the equation of the function to find the equation of the tangent line at x = 0 almost without pause:

A: Ok, so all you gotta do is plug in 0 to the derivative so that's 6, so that's the slope of your tangent line. And to find y, you plug it into your original equation, that's 1, so I need slope-intercept form, b = 1, the equation of the tangent line is 6x + 1 at x = 0.

The students used an object conception in eight of the twelve instances when thinking about how the derivative changes, regardless of given representation. In this excerpt, the student is reasoning about if the derivative is increasing or decreasing given the graph of the function:

B: All right, k' is the rate of change of the graph itself, and it's decreasing the entire time.
R: What's decreasing?
B: Well, the original function is decreasing all throughout this set, but it's getting less and less steep of a decrease, which means that the, uh, the derivative or k' is increasing on this period because it is uh, it's sort of like, it's approaching 0 which it does, it hits at 0 or somewhere near there it looks like, and it's negative before that because it's going downwards, and it's positive afterwards because it's going up afterwards. So in between here it's negative, or increasing, because it's approaching zero from a negative number.

However, the way that students thought about finding the derivative to begin with depended more on the representation. All of the students used the chain rule quickly and easily and could give appropriate justifications when given the equation. In contrast, there is only one instance of a student that was given a graph or table who held a process conception of finding a derivative. For these representations, the students were much more likely to reason about the function as an object, or to decide that they did not know how to get the answer. Students also looked at the graph and table in much the same way. One student actually remarked when they got to #3 (the tasks using the graphical representations) that they looked a lot like #1 (the tasks using the tabular representation), even though the questions were identical for all three representations:

C: Ok, so this is pretty similar to the first one.
R: Ok, in what way?
C: Well, I mean it gives you data, except it's in a graph, not a table. And it uh, in general it's pretty much the same thing. When it says approximate k'…

The given representation did not seem to make much of a difference in the schema level that the students employed (see Table 5). The intra schema, for which students stay within the given representation, seems to be more common with equations; however, that is affected by the students being willing to try the limit question when given the equation, effectively moving three scores from

Table 5: Schema Stage Across Representation

	No determination	Intra	Inter	Trans
1. Table (numeric)	4	6	3	3
2. Equation (analytic)	1	8	4	3
3. Graph (visual)	4	4	6	2

no schema to intra. The inter and trans schemas show similar amounts of instances across representations.

There was a very strong relationship between APOS conception and the schema employed. Table 6 shows the amount of instances of reasoning within each conception at each schema level.

Table 6: Comparison of APOS Conception and Triad Stage

	No determination	Intra	Inter	Trans
No determination	9	3	1	-
Action conception	-	3	3	-
Process conception	-	10	7	-
Object conception	-	2	3	7

Not surprisingly, most instances in which the conception is not determined also had no determined schema with which to deal with the problem. This occurred most often with the limit task, but also in two instances of finding the derivative at a point and in determining whether the derivative was increasing or decreasing. At the intra and inter level, students with undetermined conception substituted in numbers or came up with some false "rule," which they applied to the problem.

All of the students interviewed revealed an action or process conception at least part of the time, and in all instances that the action or process conception was held, the student operated at the intra or inter schema level. This means that the student either stayed entirely within the given representation, or that they translated part of the information given to a different representation without reasoning about the relationship between representations. For example, when asked to find a derivative given an equation, all four students operated under a process conception and simply applied the chain rule. They stayed within the given representation, and when prompted, they explained how the chain rule works, but did not include other representations in their description. When asked to find the equation of the tangent line at a point given an equation, all four again used a process conception, and one student stayed in the intra schema level, but the other three moved into an inter level, translating between representations to either complete the task or describe it, but not reasoning about the connections between representations.

Finally, students who used an object conception did use all three schema levels, but tended to be at the trans level more often, reasoning about connections between representations in order to complete the task. For example, in reasoning about whether the derivative is increasing or decreasing a student set the derivative equal to zero and solved. She then explained that this would help her find whether there is a local maximum or minimum. She sketched graphs of a minimum and maximum and gestured the slope along each graph to reason about which derivative would be increasing or decreasing. The student then went on to sketch a number line and populate the number line with the solution she found by setting the derivative to zero. She related the number line to her graphs by cross-comparing the points she had made before determining that the derivative is going from negative to positive so it is increasing. Though it seems like she answered the question several times in different ways, in her mind she only answered it once because her answer consisted of connecting all of the pieces of her schema in forming her object conception of the derivative to complete the task.

Discussion

This study uses APOS theory to guide the theoretical framework for analyzing students' conceptions of derivatives in calculus. Four students participated in task-based interviews in which tasks were presented using three different representations. While findings agree with the existing literature that students tend to use a process conception more when given an equation and less when given a graph, there is also evidence to suggest that a relationship may exist between task type and student conception. In particular, students tended to use an action or process conception when determining the derivative of a function or the equation of the tangent line, and an object conception when reasoning about the change in derivative. The students in this study also used an intra or inter schema in conjunction with their action or process conception, and only operated on the trans schema level when using an object conception. However, even with the object conception, the trans schema level was only employed in seven out of 12 instances.

This study contributes to our understanding of students' conceptions of derivative, which can be useful for instructors in planning course instruction and assessment. For example, teachers may find it helpful to be cognizant of the conceptions that students tend to have when given different tasks such as writing the equation of a tangent line or reasoning about the change in derivative for a function. It may also be useful to consider students' schematic levels for thinking of derivative when considering problems to pose that may lead to action, process, or object conceptions by students.

Students complete many other tasks in calculus that were not analyzed in this study, and further research is needed to determine if the relationship between task type and APOS conception relies on cognitive demand, type of representation, class routine, or other contributing factors. Further research is also needed to determine the strength of the relationships between representation and conception, between task type and conception, and between conception and schema level for students in a variety of calculus classes.

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A MATHEMATICAL MODELING LENS ON A CONVENTIONAL WORD PROBLEM

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Given the Common Core's dual emphases on mathematical modeling, there is a need to understand modeling as a practice and content standard to develop students' mathematical modeling skills. This study of 12 students from differing levels of mathematics instruction and English Language proficiency includes analysis of their modeling with mathematics and a focus on their transitions through a mathematical modeling cycle. Findings suggest that students were engaging in critical processes that support mathematical modeling. We posit that conventional word problems can augment the benefits of using mathematical modeling tasks and can help educators explore a process-oriented approach to mathematical modeling.

Keywords: Modeling; Standards; Cognition

Globally, research in the teaching and learning of mathematical modeling spans the past 40 years. In the US, the models and modeling perspective (Lesh & Doerr, 2003) has given rise to model-eliciting activities (MEAs) (Lesh, Hoover, Hole, Kelly, & Post, 2000), model-development sequences (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003), and a design approach to developing these sequences. These research programs have focused on developing novel classroom instructional tools in order to teach significant mathematical concepts and on providing transformative teacher professional development.

Elsewhere, other mathematical modeling research theorizes the modeling *process* carried out by the modeler. The modeling process transforms a real world problem into a well-posed mathematical problem that can be analyzed mathematically. The results are then interpreted in terms of real world constraints and the model is validated. The mathematical model, or its representation in conventional mathematical terms (e.g., equations, graphs, etc.) is then iteratively refined. One such widely adopted mathematical modeling cycle (MMC) was posed by Blum & Leiß (2007) and has been used as a framework for examining and developing students' modeling skills It also serves as the basis of the CCSSM's description of mathematical modeling (CCSSM, 2010). Two complementary perspectives on mathematical models in the secondary curriculum are offered in the Common Core: one with the Standard of Mathematical Practice #4, Modeling with Mathematics and one with mathematical modeling as a Standard for Mathematical Content. Missing in the current literature is research on how to link the research findings (models of students' mathematical modeling) to the daily practice of solving conventional word problems in the secondary classroom.

The goal of this paper is to offer insights on students' mathematical thinking generated from a *process*-oriented view of their work on mathematical modeling tasks. It is a timely contribution as the mathematics education research community embarks on emphasizing mathematical modeling in the K – 12 U. S. classrooms. We choose to begin from the stance that the status of the task and the modeler's work is determined by the research lens rather than intrinsic properties of the task and we pose the question: What does a mathematical modeling perspective on a conventional word problem afford us?

Mathematical Modeling Cycle

The MMC is a description of the modeling process in terms of stages of model construction and modeling activities that are transitions between the stages. (See Figure 1.) The MMC was adopted as a research framework and focus was on the observable mathematical activities underlying each of the transitions in order to understand what a process view reveals about students' mathematical modeling. The MMC was operationalized via an observational rubric which conceptualizes each of the six transitions as a suite of mathematical activities (Czocher, 2013). The rubric was developed, refined, and validated via the method of constant comparison. Table 1 summarizes the transitions, the process they capture, and a sample indicator from the rubric. We offer an analysis of student work in the results section below. It is important to note that students may not exhibit all stages and transitions in order, or at all. Focusing on the transitions is intended to draw attention to mathematical thinking and activities being carried out, not to serve as a checklist of requirements.

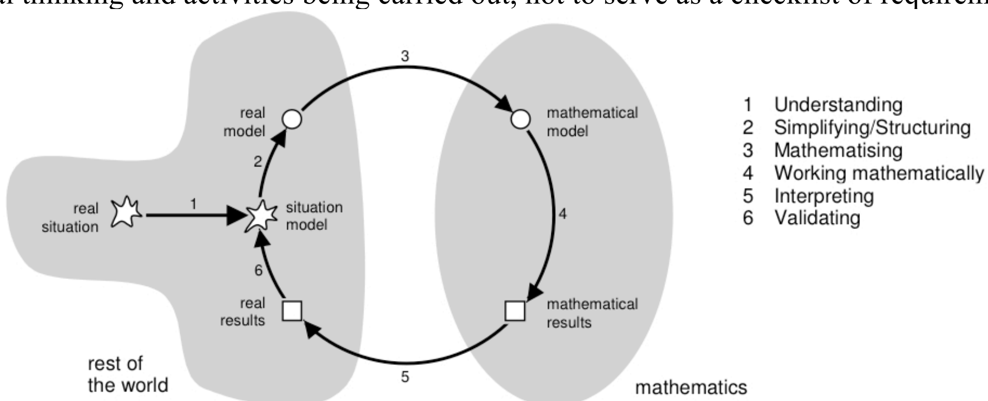

Figure 1 Schematic for a mathematical modeling cycle (Blum & Leiß, 2007)

Table 1 Modeling cycle transitions and sample indicators from the observational rubric

Activity	Trying to Capture	Sample Indicator
Understanding	Forming an initial idea about what the problem is asking for	Reading the task
Simplifying/structuring	Identify critical components of the mathematical model; i.e., create an idealized view of the problem	Listing assumptions or specifying conditions
Mathematizing	Represent the idealized model mathematically	Writing mathematical representations of ideas (e.g., symbols, equations, graphs, tables, etc.)
Working mathematically	Mathematical analysis	Explicit algebraic or arithmetic manipulations
Interpreting	Recontextualizing the mathematical result	Speaking about the result in context of the problem or referring to units
Validating	Verifying results against constraints	Implicit or explicit statements about the reasonableness of the answer/representation

Methodology

This study draws on a larger qualitative research design project aimed at understanding students' mathematical modeling processes. Participants were 12 middle and high school students who were selected so that four each were pre-algebra (6th grade), algebra (9th grade), and post-algebra (two in 10th grade; two in 12th). From each mathematical level, two had at some point in the U.S. schooling been identified as an English Learner (EL). All performed at the satisfactory or advanced level on the Texas mathematics standardized exam (STAAR) in the most recent year taken. 81% of the students at the middle school and 27% of the students at the high school that participated in this study are eligible to participate in the free or reduced price lunch program. The sampling plan was developed to be inclusive of mathematical approaches and representative of the diverse student population in

Texas. Tasks were provided in Spanish for EL students if they preferred. All of the EL students elected to conduct their interviews in English and explained their thinking in English. Due to the research design, this study reports qualities of students' modeling processes rather than an exploration of student characteristics.

The students participated in a series of three semi-structured, task-based interviews. For this study we consider their work on a conventional word problem, attempted by all 12 students, and similar to problems that appear in the algebra curriculum and on standardized tests. The students were asked to solve the following Turkeys & Goats problem (T&G): *A nearby farm raises turkeys and goats. In the morning, the farmer counts 48 heads and 134 legs among the animals on the farm. How many goats and how many turkeys does he have?* The task has value as a cornerstone of the institution of classroom algebra, but we do not claim that T&G is itself a modeling task. It is best classified as a concept-then-word problem (English, 2010) because it is designed and assigned to encourage students to practice already-learned procedures. T&G has the potential to reveal student thinking about making and justifying assumptions (e.g., one-to-one correspondence between heads and animals), working with real-world-imposed constraints (e.g., only whole-number animals should be considered), creating representations (e.g., equations or algorithms), and validating the resulting mathematical model. The ubiquity of similar tasks in mathematics classrooms suggests that analysis of student work via a mathematical modeling lens may be valuable in helping educators identify mathematical modeling processes carried out by students and therefore using such tasks to help students develop modeling skills.

The objective of each interview was to elicit the students' mathematical thinking and reasoning about the task, not to teach mathematics nor to teach mathematical modeling. The students were reassured that we were interested in their responses and explanations and not in whether the obtained the correct answer. Interviewer interjections were kept to a minimum. The interview sessions used design research principles of *cross-fertilization* and *thought experiments* (Brown, 1992). Cross-fertilization is when information and experiences from one interview session inform interviewer sensitivity and follow-up questions in another session. The interviewer posed clarifying and follow up questions ("Can you help me understand what you did here?") or asking the student to think aloud ("What are you thinking about?") or to provide additional reasoning ("Can you say more?"). Another set of interventions could be classified as scaffolding questions. For example, the interviewer adjusted the numbers in the problem if they proved too large for the student to reason about or calculate with. Thought experiments pose *what-if* questions that tweak the task as follow-ups to student explanations. For example, the interviewer could change the kinds of animals present on the farm so that the number of legs could not be realistically distributed among the animals.

Interviews were audio and video recorded and transcribed. The transcripts were summarized holistically according to the following dimensions: what approach did the student use, how it was executed, and what result was obtained. We used the observational rubric calibrated to the transitions in the MMC to look for evidence that the students were engaging in mathematical thinking that supported mathematical modeling (which is described below). When a mathematical activity was observed in the transcript or in the student's writing, it was tagged with a descriptive indicator and then coded with the associated transition from the modeling process. For example, if the student was observed to be carrying out a mathematical activity that could be described as an *explicit algebraic or arithmetic manipulation* that segment of transcript was coded *working mathematically*. In this way, the transcripts and students' written work were microanalyzed according to the MMC. Coding was carried out individually and then in pairs. All discrepancies were resolved. We then conducted a cross-case analysis looking for patterns in the students' modeling and to generate a list of questions and insights that arose from the two-layer analysis. This list guides our discussion of the implications of using a mathematical modeling lens on tasks currently used in mathematics K-12 classrooms.

Results

A solution to the Turkeys & Goats problem satisfies two conditions simultaneously (a fixed number of heads and a fixed number of legs). An algebraic solution strategy was defined as an attempt to write symbolic, algebraic expressions describing the relationships among the total number of heads and the total number of legs. A partitioning strategy was defined as some intention to separate the total number of heads into goat heads and turkey heads (or separate the total number of legs into those belonging to goats and those belonging to turkeys) and work out how many legs (heads) must belong to each group. The 6 students who attempted a partitioning strategy began with a halving strategy that there were 24 goat heads and 24 turkey heads. Based on typical student explanations, this is because there were two kinds of animal and $48/2 = 24$.

The partitioning strategy led to a guess-and-check approach where the number of heads (legs) was adjusted based on the outcome from the previous assumption. This approach handles the two conditions sequentially and iteratively. The strength of the algebraic approach is that it generalizes the guess-and-check strategy by simultaneously evaluating the cases that arise and so it is more mathematically efficient. However, using an analytic, algebraic approach requires that both conditions are made explicit via the implicit assumption that each animal has one head and the constraints that goats have one head and four legs and turkeys have one head and two legs. The modeling lens recognizes that this transformation is nontrivial and it allows us to decompose students' difficulties in formulating mathematical constraints. Regardless of student level, the task has the potential to afford insights into their mathematical thinking. Table 2 summarizes the students' work on the task according to mathematical level and EL status.

Table 2 Summary of student strategies and answers

Level	EL	Algebraic Strategy	Partitioning Strategy	Both Constraints	Any Answer	Correct Answer
Pre	EL	0	2	1	2	0
	Non EL	0	1	1	2	1
Algebra	EL	0	1	1	1	0
	Non EL	2	0	1	2	1
Post	EL	1	1	1	1	1
	Non EL	1*	1*	1	2	1

* The same student used a partitioning strategy and then transitioned to an algebraic strategy.

At each level there was at least one student who executed neither solution strategy. Of the 9 students who attempted either strategy, only 6 recognized the need to satisfy both constraints at once. This discrepancy may point to a reason why the other 3 students were unable to use a solution strategy and they may have been unable to see a way to handle both constraints simultaneously. Ten of the 12 students obtained an answer but only 4 obtained a correct answer. For the pre-algebraic students, this may not be surprising since solving the task without algebraic techniques is cognitively demanding and computationally inefficient. However, of the 8 students who were in algebra or post-algebra classes, this is quite surprising given that such systems-of-equations tasks are a focus of instruction and high stakes tests.

Due to space constraints, we present a synopsis of one student's work on the Turkeys & Goats problem. Bree was a pre-algebra, non-EL student. Her work was selected because she developed a non-algebraic of the situation and because she focuses autonomously on satisfying both constraints. Thus her work illustrates what the students, regardless of mathematics level, are capable of doing. Transitions from the MMC are marked in the first part of the synopsis to demonstrate how the MMC is present in the student's work. Though we highlight one student's work, all students exhibited transitions from the MMC, demonstrating that they were engaging in mathematical modeling when working on the task even if no answer or conclusion was ultimately reached.

Bree began by reading the problem aloud (*understanding*) and then questioned the number of legs on a turkey (*simplifying/structuring*). She initially tried 48×4 (*mathematizing, working mathematically*) to obtain the total number of legs (*interpreting*) because "it'd be easy." She abandoned this approach because "some of them have two legs" (*validating*). She continued, "since there's turkeys I have to figure that out. So it's either that most of them could be like, I'm going to try half" (*simplifying/structuring*) and decided to find the number of legs among 24 turkeys and 24 goats by combining 24×2 with 24×4 (*mathematizing*) and using the standard algorithm (*working mathematically*) to get 144 legs (*interpreting*). Bree noted that she needed 134 legs. She wondered what to do about the "10 extra" legs. The interviewer asked her what she was thinking. She responded "What I was thinking first is if I, maybe if I div…cut the 48 in half, 24 turkeys and 40, 24 goats, then I figure I would, so 48 legs for the turkeys and 96 legs for like goats…I got 144. But I think we needed 134 legs, so I'm trying to figure out right now how I could put that down." Bree switched her focus from the heads constraint to the legs constraint. As a follow up, the interviewer asked how she would reduce the number of legs. She concluded that there would have to be more goats. Next, Bree tried 22 turkeys and 26 goats and realized that she added legs. She reversed her guess and tried 22 goats and 26 turkeys to obtain 140 legs. She shook her head because she recalled that "We're trying to find 134."

Bree changed her approach yet again: "So I guess I could try to find how many times 4 goes into 134 or how many times 2 goes into 134." She proceeded to carry out long division via the standard algorithm for $134/4$ and $134/2$. She obtains 32 goats and 6 turkeys, but did not realize that she was missing 13 heads. She became confused on how to figure out the number of legs for a given number of heads and backpedals to her computation $134/2$. She states that this would yield 67 turkeys and no goats which is impossible because there were only 48 animals. At this point, Bree announced "There has to be an answer" and began working quietly doing various multiplicative computations.

The interviewer asked, "Could you summarize for me what you know about what the possible combinations [of turkeys and goats] might be?" This prompts Bree to organize her work into a list of goat/turkey combinations she had tried and begin systematically adjusting those values. She stated her strategy, "I've tried 22 goats and 26 turkeys and I got 50, 52, and 88 goats, but that equaled 140. So that one didn't work. And 9, 8, I guess I could try other ones just like that. I just have to keep on going down 'til I got to 134." She used her algorithm developed from trying 24 goats with 24 turkeys and 22 goats with 26 turkeys to "keep on going down" until she found the combination 19 goats with 29 turkeys yielded 134 legs. When asked to explain how she knew to increase the number of turkeys and decrease the number of goats, she responded that "4 doesn't go into many numbers as much as 2 does 'cause it's bigger," indicating that she had some sense of needing the net number of legs to decrease.

Implications and Conclusions

The Turkeys & Goats problem is a conventional word problem similar to those found in algebra textbooks and on standardized tests. We presented two layers of analysis on students' work: one focused on solution obtained by the student and how the student carried it out and one that examined students' mathematical activity in support of mathematical modeling processes. The first is a product-oriented perspective that focuses on representation of the solution and its correctness. According to this perspective, many of our students were unable to solve a conventional word problem. However, this kind of surface level analysis is limited in that it does not reveal the deeper complexities of student's mathematical thinking and understanding which are useful for informing classroom instruction. A closer examination of students' mathematical thinking showed that students took an algebraic approach or used a partitioning strategy. Analysis of student strategies reveals how they were thinking about the task.

The process-oriented perspective focused on mathematical modeling as a process. Operationalizing the MMC in terms of mathematical activity and then applying the rubric to students' work showed that all students were engaging in mathematical modeling processes. Analysis revealed how students progressed through the task even if they did not ultimately obtain a right (or any) answer. Bree's work demonstrated that students are capable of modeling with mathematics, a Standard of Mathematical Practice (CCSSM, 2010) without necessarily using an explicitly algebraic approach. She carried out a guess-and-check strategy, but her guesses were not haphazard. She developed an algorithm based on constraints implied by the problem statement in order to relate the number of heads of each kind of animal to the total number of legs among the animals. To do so, she introduced two parameters: the number of turkey heads and the number of goat heads. After each trial, she adjusted her estimates in ways that anticipated their effects on the outcome (number of legs). That she was capable of organizing her work into a systematic algorithm after some prompting and encouragement suggests that she was at a participatory stage of forming a concept about satisfying multiple constraints simultaneously but that she had developed an activity-effect relationship between adjusting the head parameters and verifying the number of legs (Tzur & Simon, 2004). In essence, she was able to apply her mathematical model as an algorithm dependent on input parameters.

Thus a product-oriented perspective on a conventional word problem can impede educators from fully understanding students' mathematical activity because it emphasizes what students could accomplish. Therefore, it reveals student struggles in achieving objectives in mathematics instruction. In contrast, a process-oriented perspective on mathematical modeling offers more information that educators can base decisions on because it shifts attention to the students' mathematical activity and allows for articulation of what students are capable of doing. The shift has implications for helping classroom mathematics teachers identify and harness student success in mathematical modeling in a way that is grounded in students' current mathematical activity rather than solely on obtaining a correct answer.

Our analysis suggests that independent of where the student is in their formal mathematical instruction, they are capable of and spontaneously do exhibit the kinds of thinking that support mathematical modeling. These results parallel the findings of Cognitively Guided Instruction (CGI) research: children do not need explicit direct instruction in modeling (Carpenter, Fennema, Franke, Levi & Empson, 2015). Children come to school with ways of thinking about mathematics that do not need to be formally taught. Instead, the goal is to understand their thinking and original strategies and guide them toward more efficient and proficient ways of solving problems. Likewise, the goal in teaching modeling should be providing opportunities for the students to develop their judgment to become comfortable making assumptions that will satisfy a complicated situation. As demonstrated by Bree, the students may already have a sense of needing to validate their models and revise their assumptions. Reinforcement of the idea that assumptions and representations may require revision could be provided by working on techniques for validation.

Thus, classroom mathematics teachers may not need to consider the modeling standard and practice articulated by the CCSSM as something "new" to add to the curriculum. Our analysis shows that even on conventional word problems, students are already thinking in ways that support mathematical modeling. The challenge becomes helping teachers identify what the modeling process looks like so that they can recognize when a student is developing a model (not just that the strategy is inefficient) and ask specific questions of students at specific times. A comparison between process- and product-oriented views of students' mathematical activity can help educators focus on mathematical structure instead of just correcting mistakes in carrying out procedures. Such emphasis may also push students toward process-oriented views of their own mathematical activity instead of focusing on obtaining the correct answer. .

Using the MMC to see beyond solution strategies on a seemingly straightforward task led to conversations about the role of the interviewer (and by extension, the role of the teacher in a classroom setting). Though all students were capable of making progress in the modeling process, the realization that the models needed revision and some encouragement as to how to do so was influenced by interviewer questions. In Bree's case, to fully develop her algorithm she was prompted to summarize what she had done so far. Perhaps realizing that their models may need revision later contributed to some students' unwillingness to commit to a solution strategy. Future research must critically examine the role of interviewer prompts in scaffolding mathematical modeling. Such analysis would have implications for when and how teachers may most productively intervene in students' modeling processes while respecting the students' ideas and model development. In addition, due to the ELs opting to conduct their interviews in English, we can continue to question and think about the implications of past schooling on their mathematical modeling. One of the challenges in implementing the CCSSM has been that teachers may not share the same vision of how to operationalize the standards in the classroom as the standards writers intended. There is a need for research-based understandings of what teachers may be currently doing in classrooms which may need to be reconsidered or adjusted in order to fully realize a mathematical modeling perspective. We suggest that mathematical modeling need not be a wholly new undertaking. The participants in this study demonstrate that K-12 students are engaging in mathematical modeling processes whether it is taught explicitly or not. The challenge becomes helping teachers identify it – it's not foreign -- and how their students are doing it.

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STUDENTS' STRATEGIES FOR ASSESSING MATHEMATICAL DISJUNCTIONS

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This paper presents results from three teaching experiments intended to guide students to reinvent truth-functional interpretations for mathematical disjunctions. The initial teaching experiments revealed that students' emergent strategies for assessing disjunctions did not entail or facilitate the development of a relevant partitioning of example space (comparable to Venn diagrams). Students were unable to form generalizable strategies for finding relevant exemplars to evaluate quantified disjunctions. The latter teaching experiment, in contrast, successfully prompted students' to attend to reference and partitioning of the referent space through an alternative instructional sequence. I set forth the methodology and findings of this study to demonstrate how conventions of mathematical logic can emerge within students' mathematical activity toward the end of their apprenticeship into proof-oriented mathematics.

Keywords: Cognition; Advanced Mathematical Thinking; Post-Secondary Education

In proof-oriented mathematics, mathematicians embed mathematical meaning in mathematical language (definitions, theorems, proofs, etc.). This requires a high level of clarity and precision in mathematical language, which is why mathematicians were the first to invent formal languages (Azzouni, 2009). Formalizing language requires attending to the relationship between linguistic form and meaning. For mathematicians, this involved 1) creating equivalences – *but* and *and* are mathematically equivalent connectives, 2) disambiguation – distinguishing *or* and *either…or* as capturing the inclusive and exclusive everyday meanings of *or*, and 3) creating truth-functions relating truth-values of component and compound predicates. These aspects of formal language and its acquisition stand in contrast to natural language, which is learned through many more implicit or preconscious processes and which entails looser relations between form and meaning (see Stenning, 2002).

How then can mathematics students in proof-oriented mathematics courses learn formal, mathematical language, specifically as it pertains to the relation between form and meaning? Using the guided reinvention heuristic of Realistic Mathematics Education (Gravemeijer, 1994), I sought to engage students in conscious and effortful systematization of their use of mathematical language. I guided students to reinvent truth-functional definitions for mathematical disjunctions and conditionals in a series of short teaching experiments (Steffe & Thompson, 2000). In this paper I report on the major patterns of student interpretation of mathematical disjunctions, how failure to partition example spaces inhibited their ability to reinvent normative interpretations of quantified disjunctions, and an alternative instructional sequence that supported the emergence of normative interpretations of quantification.

Studies of students' interpretations of linguistic form

Because there is ample evidence that students' untrained interpretations of mathematical language differ significantly from that of mathematicians (e.g. Durand-Guerrier, 2003; Epp, 2003), many Introduction to Proof courses include a unit on logic. However, my method departs from many of the common approaches to teaching these topics in mathematical logic because 1) I want students to impose logical form on meaningful mathematical statements rather than abstract or nonsensical ones and 2) I problematize students' *reasoning toward an interpretation* (Stenning, 2002; Stenning & van Lambalgen, 2004) of language. The majority of psychological and mathematics education studies of student's interpretations of linguistic form tend to elicit students' preconscious interpretive processes (Evans, 2005; Inglis & Simpson, 2008), but assess those interpretations against a single, formalized linguistic meaning. This assumes some logical structure is embedded in language or semantic content and that people are irrational for reasoning alternatively (e.g., Stanovich, 1999). As Stenning (2002) eloquently argues, everyday linguistic interpretation is far too complex and varied for this approach. One may distinguish three views of *logic*'s relation to language use that clarify my stance. Logic can be thought of as a description of language use (common in later 20th century logic, Stenning, 2002), a prescription for proper language use (as many psychologists deem it), or the constructed product of a learning process of systematizing language (my proposal). I call this third view students *reasoning about logic* to denote its conscious and reflective nature. I adopt this lens because many studies suggest formal logic is a poor model of most students' reasoning, but proof-oriented mathematics requires that students conform their reasoning to mathematical norms. Stenning's (ibid.) findings support this study's use of meaningful mathematical statements, as he states, "formal teaching can be effective as long as it concentrates on the *relation* between formalisms and what it formalizes" (p. 187) and "logic teaching has to be aimed at teaching how to find form in content" (p. 190). So, I operationalize logic not in terms of students learning formalisms (e.g. truth-tables, Venn diagrams), but as their progressive systematization of their interpretations of mathematical statements till they impose a consistent, generalizable, and normative form.

Methods

I recruited three pairs of Calculus 3 students from a medium-sized Midwestern university to take part in short teaching experiments. I chose this course to find students who were mathematically proficient, could benefit from learning formal mathematical language, and who had not taken proof-oriented mathematics courses. I identified their background in learning logic via an online survey. Students met in pairs with the author for six one-hour sessions, and were compensated monetarily for their participation. The first three sessions focused on mathematical disjunctions and the latter three sessions on mathematical conditionals.

The guided reinvention approach helped to identify the interpretations students imposed upon the statements, how those interpretations shifted upon reflection, and which tasks elicited reasoning that approximated normative interpretations. I asked students to determine whether provided mathematical statements were true or false, then to systematize and describe their method, before asking them to negate the statements. It was initially anticipated that disjunctions would be easier for students to formalize and provide a foundation for interpreting conditionals (known to be a problematic linguistic form, Evans, 2005). Instead, disjunctions were quite challenging for mathematically important reasons. Thus, I only report on data from the first three days of each teaching experiment. The first two pairs met simultaneously and employed the same instructional activities. The third met several months later using modified activities. As such, data is presented as two experiments distinguished by their anticipated learning trajectories and instructional tasks. Table 1 presents a selection of the statements used in the study, with "D1:3" denoting the third statement on Day 1. An apostrophe denotes an item from Experiment 2.

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), during Experiment 1 the author served as teacher/researcher and another researcher served as outside observer. The observer kept field notes during each session and the researchers debriefed after each teaching session. Video recordings were also reviewed each day to form and test hypotheses about student learning to inform the teaching activities for the following session. Full retrospective analysis commenced after the experiment ended. The author coded data in the open and axial method of grounded theory (Strauss & Corbin, 1998). Codes related to 1) truth-value assessment strategies (e.g., one condition false makes the disjunction false), 2) paraphrases of provided statements (e.g., introducing "either…or" language), 3) modes of reasoning about logic (e.g., attending to the meaning of the *or* connective), 4) clarification of semantic information (e.g., identifying relevant warrants such as “all squares are rectangles”), and 5) negating actions (e.g. negating [A or B] with [not A or not B]). I report on common trends in students’ interpretive behavior and emergent links among their strategies, interpretations, and particular disjunctions.

Table 1: Sample disjunctions provided to study participants

D1:1 “Given an integer number x, x is even or x is odd” D1:2:“The integer 15 is even or 15 is odd. D1:7 “The real number 0 has a reciprocal $\frac{1}{0}$ such that $0 \cdot \frac{1}{0}$ 1 or 0=0.” D1:9 “Given any even number z, z is divisible by 2 or z is divisible by 3” D2:6 “Given any triangle, it is equilateral or it is not acute.” D2:7 “Given any triangle, it is acute, or it is not equilateral.”	D3:3 “10 is an even number or 20 is an even number” D3:4 “13 is an even number or 6 is an even number.” D2’:6 “For which integer numbers *z* is it true that ‘*z* is divisible by 4 or *z* is divisible by 3’” D2’:10 “For which real numbers y is it true that “y<3 or y>5” D2’:12 “For which triangles is it true that ‘it is equilateral or it is not acute.’”

Experiment 1

Of the four participants in this experiment, one pair had no training in logic and the other pair had completed a philosophy course in logic. Their patterns of reasoning were nearly identical. When initially assessing the truth-values of non-quantified disjunctions, both groups declared a disjunction with a false component false (e.g. D1:2 and D1:7). Only on D1:9 did either group begin explicitly attending to the connective *or* and its role in the statements’ meaning. Both groups at this point also began distinguishing the truth-values of the two components from that of the disjunction. For instance, Ron said, “This just got interesting cause when you say “or” only one of ‘em has to work, not both of ‘em.” When asked to revisit their initial decisions, both groups reinterpreted D1:2 as true because of the *or* connective. Students were more reluctant to affirm D1:7 as it seemed more mathematically absurd, but later decided that the 0=0 condition also made it true. By the end of the first session, both groups were consistently interpreting non-quantified disjunctions in a manner consistent with the normative truth-functional definition. As Ovid said, “Cause “or” for me means either it could be one or the other or both.”

Students generally had more trouble with quantified disjunctions where the truth-function was not sufficient to assess the truth-value of the statement. Some statements afforded semantic affirmation without testing particular cases as with D1:1 where the categories are exhaustive. In other cases, though, students used a *sentential testing strategy* in which they picked examples and reread the statement to evaluate whether it “covered” the given case (often reading left to right). This strategy reduced quantified disjunctions to a sequence of non-quantified disjunctions, but it also necessitated a strategy for picking cases and organizing the example space.

Partitioning the example space

Experiment 1 participants did not spontaneously develop an intentional way to partition the spaces of examples because each space was pre-organized according to familiar mathematical categories (e.g. even, acute, rectangle). The normative logical partitioning of examples (as portrayed in Venn diagrams) distinguishes cases that satisfy each component condition of the disjunction such that the examples fall into four categories (TT, TF, FT, and FF). Because students’ reasoning stayed focused on the statements themselves, they failed to attend to how the statement provided a novel partitioning of example spaces. This was not problematic for cases that could be easily and exhaustively seriated such as the integers or even integers. Students tested cases sequentially (2, 4, 6, 8…), usually assigning a truth-value after 3-5 examples. However, both pairs of students struggled to assess geometric statements such as D2:6 and D2:7. This is because they reasoned about triangles in terms of familiar semantic categories – equilateral, acute, right, obtuse – rather than treating right and obtuse as equivalent relative to the given statements – non-equilateral, non-acute. Without a simple way of exhausting the example space, they were never sure if a statement was true of all triangles.

Students developed some other normative and non-normative strategies that helped them resolve quantification issues. First, recognizing that all equilateral triangles are acute, some students incorrectly concluded that not equilateral meant not acute, implying that statements like D2:6 were true. Students struggled in other ways with how to interpret negative predicates, often substituting a positive predicate that was non-complementary, such as replacing "<" with ">" or "not acute" with "is obtuse." In general, students did not seem aware that a negative predicate ("not acute") could be thought of as denoting the complement of the set of cases satisfying the positive predicate ("is acute"). As a result, students were disinclined to reason about negative predicates without paraphrasing ("can't be acute") or substituting positive conditions.

However, students' sentential testing strategy led to some other strategies that more closely approximated normative interpretations and led students to make appropriate determinations of truth-values. Two participants began anticipating that anything satisfying the first condition made the statement true. So, they began ignoring such cases, as when Ron interpreted D2:6 as, "if it's not equilateral, it must be obtuse." I call this strategy an *"if not…then" paraphrase*. While not identical to the Venn diagram partitioning of examples, this strategy allowed students to reduce the set of cases they had to attend to by excluding cases satisfying the first predicate. Their reasoning also implicitly approximated the negation of the disjunction – negating both predicates – because it led students to question whether anything failing the first condition must necessarily satisfy the second. For instance, Ron rejected that any non-equilateral triangle must be obtuse, which led him to find the non-equilateral, acute counterexample[1]. Though they used it repeatedly, students in Experiment 1 did not consciously identify or abstract their "if not…then…" strategy. Furthermore, without specific guidance students did not reinvent the Venn diagram partitioning of examples and consistently used sentential testing of a few cases.

Negating disjunctions

One of the challenges in reinventing logic is to find experientially real activities (Gravemeijer, 1994) that foster language systematization as entailed in *reasoning about logic*. Assessing truth-values successfully prompted students' reinvention of truth functions. Negating statements appeared a natural next activity, but it was unclear how to describe logical negation to participants unfamiliar with the notion. In Experiment 1, I asked students to find a systematic way to produce an opposite statement that always had the opposite truth-value. This description of the negation of a statement proved to be underspecified for reinvention.

Experiment 1 participants commonly engaged in syntactic manipulations of the statements to produce a negation such as 1) negating both conditions with the same connective ($\neg(A \vee B) \leftrightarrow \neg A \vee \neg B$) or 2) negating with a non-complementary property ($\neg(x < y) \leftrightarrow x > y$). Even when provided with various statements intended to dissuade such strategies, students were unperturbed to negate different statements in different ways. Ron and Drew negated "10 is even" with "10 is odd," but more appropriately negated "π is even" with "π is not even." Anticipating that the *and* connective in the negation would not be obvious, I asked participants to negate D3:3 and D3:4. Students in both pairs negated D3:3 as "10 is an odd number or 20 is an odd number," which yielded the opposite truth-value as desired. They then recognized a problem when the same transformation of D3:4 yielded another true statement. Drew responded by negating D3:4 with "13 is an even number or 6 is an odd number." This statement is false, as required, but Drew did not show evidence of anticipating whether this method would work for any other statements. Ron and Drew proposed and tested various transformations of the statement before introducing an *and* connective, seemingly by trial and error. Upon testing, they recognized that this method properly negated the given statements, but could not semantically justify why. Contrary to the researcher's intentions, study participants did not perceive any connection between why a counterexample falsified a disjunction (it satisfied neither condition) and a systematic method of negating a disjunction (not A and not B). There was also no evidence that they understood why the negation of a universally quantified disjunction should be an existentially quantified conjunction, though the counterexample heuristic might suggest it. It appears from these two teaching experiments that the negation criterion of merely having the opposite truth-value was too underspecified to foster students' recognition of a generalizable method. They relied on trial and error syntactic transformations of the statement in lieu of any intentional, semantic strategy.

An alternative criterion for mathematical negation

One episode from the third interview with Ovid (his partner Eric was absent) suggested an alternative approach to reinventing negation. Ovid was already comfortable with abstracting from each non-quantified disjunction the two component truth-values and applying their truth function. Ovid recognized that negating a disjunction with a disjunction would not work because each of the two components would reverse truth-values. He said, "If this is false-true, then the opposite would be a true-false statement." Like Drew, Ovid initially wanted to change the way he negated the components rather than changing the connective. I invited him to explore all such component patterns and the desired outcomes for the negation, which produced the table in Figure 1. Analyzing this representation, Ovid said, "So we would have to do, probably would be an *and* statement. Because then it would have to fit both criteria rather than either, or, or both." He went on to check that "13 is not an even number and 6 is not an even number" properly negated D3:4 (i.e. produced the opposite truth-value as desired).

Parts	St	Parts	Neg
TT	T	FF	F
FT	T	TF	F
TF	T	FT	F
FF	F	TT	T

Figure 1: Reproduction of Ovid's truth table for negating disjunctions.

Ovid's discovery that the negation of a disjunction must be a conjunction was significant for two reasons. First, this was one of the clearest cases where the formalization and abstraction of the truth-value structure of the mathematical statements led a student to reinvent a normative logical theorem. As Stenning (2002) discussed, Ovid learned from the relation between the formalization and what it formalized. Ovid translated the semantic statements into a logical representation system (a truth table), deduced the appropriate pattern from that representation, and then translated it back to the semantic system of mathematical statements.

The second reason I highlight Ovid's discovery is that it suggested an alternative way of characterizing negations. Each of the statements Ovid reasoned about in this episode could be viewed as a case of the condition "x is even or y is even." As in Ovid's truth table, the negation of the condition must yield the opposite truth-value for each pair of numbers. Thus the negation of a quantified disjunction must be a case-wise negation (yielding opposite truth values for each example) in addition to a global negation (having the opposite truth-value overall). This insight, combined with the need to guide students to attend to partitioning the example space suggested the revised teaching activities employed in Experiment 2.

Experiment 2

The teaching activities on the first day of this teaching experiment were nearly identical to those in the former, except that some uninformative items were removed and the geometry items from the second day were added. The participants in Experiment 2, Cid and Macy, attended to the *or* connective much earlier (on D1:2), but still rejected D1:7 as false due to its apparent absurdity. Unlike the participants in Experiment 1, Macy had been taught logic in a mathematical context. Despite this, she consistently imposed a non-normative "exclusive or" interpretation, though it took her some time to recognize when it applied to quantified disjunctions. Like the first group, Macy and Cid distinguished and coordinated the three truth-values in a non-quantified disjunction according to truth-functions, though they disagreed about the output when both predicates were true. Like the previous pairs, neither Cid nor Macy developed a generalizable strategy for finding example cases, especially for the geometric items. Cid himself explained that he was merely "stabbing at examples in [his] head." He implicitly used a *sentential testing strategy* and *"if not…then"* paraphrases, but did not recognize or abstract these approaches. Macy attempted semantic substitution, inappropriately paraphrasing D2:6 as "is acute or is not acute" because "equilateral triangles are acute."

Alternative activities intended to emphasize quantification of predicates

On day 2, many of the same conditions were presented to the students, but the activity was reframed from assigning truth-values to quantified disjunctions to finding the set of cases that satisfied a disjunctive predicate (see the D2' items in Table 1). This sequence of tasks was intended to guide students to associate classes of examples with each condition rather than single examples, leading to the normative interpretation that mathematical predicates entail sets (rather than simply describing cases). The close association between "divisibility by 2" and $\{x \in \mathbb{Z} : 2|x\}$ (the set of even integers) is commonplace in proof-oriented mathematics, but analysis of Experiment 1 suggested that it was not a natural association for study participants.

I anticipated that visually representing some of the sets described by these disjunctive predicates might lead students to an interpretation approximating the Venn diagram for *or*, the principle of which is that a disjunctive predicate entails the union of the sets of cases entailed by the two component predicates. For this reason, I included items such as D2':10 that would easily lend themselves to visual representation.

The third reason to shift from declaring quantified disjunctions true or false to identifying the set of cases that satisfied a disjunctive predicate was to provide a natural segue to case-wise negation. Rather than negating statements by other statements that have opposite truth-values, the negation of a disjunctive predicate "X or Y" is the predicate that entails the complement of the cases. Thus I anticipated modifying the task to, "For which integers is the condition false?"

Results of the alternative instructional activities

Cid and Macy approached the second day's activities initially using verbal strategies as they had done the day before. In some cases, they could describe the set easily as "all real numbers" or "all even integers." They ran into difficulty when they tried to use "and" to denote the union of two sets, which confused the intended meaning of the connectives. Beginning with the geometric items, they began instead describing the cases that do not satisfy the condition. Regarding the correlate task to D2:6, Cid said, "An acute triangle doesn't satisfy it. I think." Macy clarified, "An acute triangle that's not equilateral." The interviewer invited them to extend this strategy and specify the set of cases making each condition false. By the next item, Macy generalized the strategy, saying "I am trying to think if there's any counterexamples where you can make both of those statements false. Cause then you can exclude some of the triangles." In this way, Macy recognized that the statement was false for cases that failed both component predicates. Both Cid and Macy later noted that it was much easier to describe the set of counterexamples to disjunctive conditions than describing the cases that satisfied them. They could not articulate why this was easier, though. In addition, they implicitly recognized the complement relation between the satisfying and falsifying cases.

I intended for a visual representation to suggest the relationship that the cases satisfying a disjunctive condition consist of the union of the cases satisfying each condition (as the Venn diagram suggests). Cid and Macy drew number lines for D2':10, but they described the resulting set in spatial terms ("It's false when y is between 3 and 5.") rather than in inequality language. This dissociated the set from the negations of the two component predicates and led to no generalizable strategy. The interviewer then revisited D2':6 and invited the students to create two number lines that demonstrated which numbers satisfied and falsified the given condition. The students did so (Figure 2) including the truth-values of the two components for each number. This led them to rediscover Ovid's observation that the component truth pattern of the negation will invert that of the original and that the connective *and* will ensure the proper pattern of truth-values for the disjunction and negations overall. While this alternative visual representation did not emphasize the union property, it clearly fostered truth functional analysis leading Cid and Macy to reinvent the case-wise negation of a disjunction.

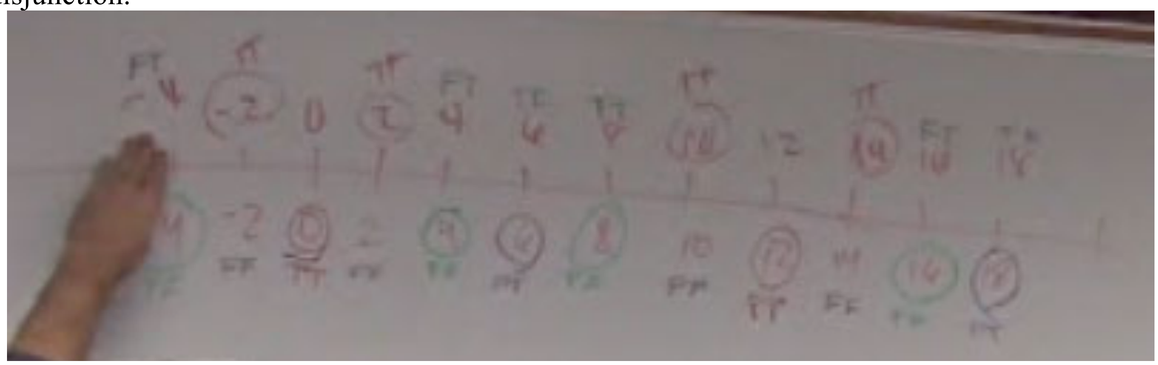

Figure 2: Representing the sets entailed by a disjunctive predicate and its negation.

Summary and discussion

From the diversity of strategies employed, and the frequency with which study participants paraphrased the given statements in various ways, it was clear that study participants had not systematized the meaning of *or* in mathematical sentences prior to the teaching experiments. Rather, students *reasoned toward an interpretation* (Stenning & van Lambalgen, 2004) of each sentence trying to find some way to have the language or content suggest a means of assessing each statement. Several key strategies emerged repeatedly and independently such as sentential testing and "if not…then," but students did not apply such strategies consistently. Due to the methodological choice to work in meaningful mathematical contexts, participants had to impose logical form in their mathematical interpretations. Participants only slowly developed meta-language for describing and abstracting patterns and strategies across various contexts.

In each study, students reinvented the standard truth-function for inclusive or exclusive *or* in one session. The activity of assessing truth-values did not lead study participants to develop strategies for quantified disjunctions that approximated the normative Venn diagram partition of examples. Students selected examples according to the semantic structure of the content of each sentence (numbers, triangles, etc.) rather than according to the predicates in the disjunction. This suggests that instruction in proof-oriented classes must help students begin associating any property or predicate with the set of examples satisfying that predicate and its complement. Properties describe single cases, but they also organize or partition sets of examples. None of the study participants approached the given tasks in this quantified way without being guided to do so. Motivated by the need to attend to quantification of predicates and to develop a case-wise meaning of negation, I developed the instructional sequence used in Experiment 2. This approach successfully led Cid and Macy to formulate the normative negation of a disjunctive condition and to identify that a condition and its negation entailed complementary sets.

A theoretical goal of this project is to recast mathematical logic within students' activity. These results provide several instances of students *reasoning about logic* such as problematizing linguistic interpretation, reinventing the standard definition of *or*, comparing interpretations across statements, developing meta-language to abstract patterns, and truth-table analysis leading to new discoveries. These data support the hypothesis that students can reinvent the structures of logic when engaged in the activity of logic: systematizing language. However, further studies are needed to better understand reinventing logic's instructional affordances and implications.

Endnote

[1]While Ron's paraphrase also falsely suggests right triangles are counterexamples, his line of reasoning led him to the correct counterexample.

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LINGUISTIC NORMS OF UNDERGRADUATE MATHEMATICS PROOF WRITING AS DISCUSSED BY MATHEMATICIANS AND UNDERSTOOD BY STUDENTS

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We studied the linguistic norms of mathematical proof writing at the undergraduate level by asking two mathematicians and five mathematics undergraduate students to read seven partial proofs based on student-generated work and to identify and discuss uses of mathematical language that were out of the ordinary with respect to standard mathematical proof writing. By asking participants to discuss the seriousness of each breach, we not only identify and discuss some of these linguistic norms, but also describe important differences between the ways in which mathematicians and students understand them.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking

In professional mathematical practice, proofs are an essential type of communication. In an influential paper on the role of proof in mathematics, Rav (1999) wrote that proofs "are the heart of mathematics" and that they play an "intricate role […] in generating mathematical knowledge and understanding" (p.6). As a result, fostering undergraduate mathematics students' abilities to understand and construct valid proofs is one of the primary goals of mathematics instruction at the advanced undergraduate level. However, evidence of undergraduate students' difficulties when reading and constructing proofs is pervasive in the mathematics education literature (Weber, 2003). One potential difficulty students have when constructing proofs concerns students' inability to understand and use mathematical language and notation (Moore, 1994).

Mathematical language has been studied and interpreted in a variety of different ways: as a foreign language composed almost entirely of technical symbolic representations (Ervynck, 1992), as a combination of natural language and a system of mathematical symbols (Kane, 1968), and as a set of meaning that is created and expanded with the creation of terminology and the designation of technical definitions to natural English words (Pimm, 1987). In particular, researchers have focused on the differences between mathematical language and neutral or common language. For example, Veel (1999) discussed the precision necessary when implementing certain verb phrases in mathematics and Halliday (1978) noted the nominalization of mathematical language, in which a mathematical action or phenomenon becomes an object (e.g., differentiation). As these aspects of precision and rigor in mathematical writing may cause difficulties for students, a number of mathematics educators have suggested ways to improve students' use of mathematical language (e.g. Veel, 1999; Moschkovich, 1999; Lemke, 2003). However, these suggestions have focused on school level mathematics. Research on how mathematicians and undergraduate students understand the language of mathematics is lacking: to our knowledge, there are only two studies (Konior, 1993; Burton & Morgan, 2000) to date that have explicitly and empirically investigated the language of mathematical proof writing, and neither study investigates how undergraduate students understand such language.

Objectives of the Study

This qualitative study is a first attempt to address this gap in the literature by examining how mathematicians and undergraduate students understand the linguistic norms of mathematical proof writing. By interviewing both mathematicians and undergraduate students, this study not only identified various linguistic norms of undergraduate mathematics proof writing, but also illustrated how students understand these norms. In particular, this study addressed the following research questions:

1. How do mathematicians view and describe the linguistic norms of mathematical proof writing at the undergraduate level?
2. How do undergraduate mathematics students taking an introduction to proof course understand these norms?

Theoretical Framework

This study is informed by Herbst and Chazan's (2003) body of work on practical rationality. In particular, Herbst (2010) described norms as statements that articulate practice, as made by an observer of the practice. Since participants may not be fully aware of the norms they follow, Herbst and Chazan (2003) adapted the ethnomethodological concept of breaching experiments (Mehan & Wood, 1975) to study these norms. Herbst and Chazan hypothesized that when a participant of a practice is engaged in a situation where a norm has been breached, the participant will attempt to repair the breached norm highlighting not only what the norm is, but also expounding on the role that the norm has in the practice (Herbst, 2010). Adapting this methodology, this study used the concept of breached norms to investigate the linguistic norms of mathematical proof writing at the undergraduate level.

This study also employed the use of Scarcella's (2003) conceptual framework for academic English. Scarcella defined academic English as a "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (Scarcella, 2003, p. 9). This framework has previously been applied to mathematics education with regards to English learners studying mathematics (e.g. Silva et al., 2008; Heller, in press). However, we propose to apply the framework as a tool to begin to investigate important aspects of the mathematics sub-register of academic English. The framework of academic English specifies that there are three dimensions of language; the linguistic dimension, the cognitive dimension, and the sociocultural-psychological dimension. For the sake of brevity, we focus on the linguistic dimension and its components.

The linguistic dimension of academic English involves phonological, lexical, grammatical, sociolinguistic, and discourse components. The phonological component "includ[es] stress, intonation, and sound patterns" (p. 11) and "knowledge of graphemes (symbols) and arbitrary sound-symbol correspondences" (p.13).

The lexical component requires knowledge of the words used in a field. In particular, Scarcella (2003) distinguished between general words used in everyday language, academic words used across academic fields, and technical words that are field-dependent. She also included knowledge of fixed expressions, which are "expressions that tend to stick together and cannot be changed in any way" (p. 14), as part of the lexical component.

The grammatical component of academic English entails knowledge of "the grammatical co-occurrence relations that govern the use of nouns" (Scarcella, 2003, p.15). For instance, Scarcella (2003) noted students need to learn the associated grammatical features for these technical words, that "certain nouns […] are generally followed by prepositional phrases" and that some "verb + preposition combinations […] cannot be changed" (p. 16).

The sociolinguistic component involves developing competence in a variety of functions of language, including an understanding of the appropriateness of a given sentence in a particular context. Scarcella (2003) noted that "signaling cause and effect, hypothesizing, generalizing, comparing, contrasting, explaining, describing, defining, justifying, giving examples, sequencing, and evaluating" (p. 18) are examples of different academic language functions.

The discursive component entails understanding and using linguistic forms necessary to communicate successfully and coherently. For instance, in every day language, greetings and parting phrases indicate to speakers the beginning and end of conversations. Scarcella (2003) noted academic English "includes specific introductory features and other organizational signals" and that "writers' presentation of ideas must be orderly and convey a sense of direction" (p. 19).

This conceptual framework of academic English highlights important aspects of learning an academic language, and the components of the linguistic dimension provide a lens for analyzing mathematicians' and undergraduate mathematics students' understandings of the linguistic norms of proof writing in undergraduate mathematics.

Methods

Based on the methodology used by Herbst (2010) and Herbst and Chazan (2003), this study investigates the linguistic norms of undergraduate mathematics proof writing by showing participants student-generated proofs and asking them to identify and describe uses of mathematical language that are out of the ordinary with respect to standard mathematical proof writing at the undergraduate level. By identifying these non-standard uses of mathematical language, mathematicians and undergraduate students discuss their understanding of the linguistic norms of proof writing at this level.

Two mathematicians and five undergraduate students were interviewed for this study. Both mathematicians had experience teaching an introduction to proof course at the large research university in the United States. The undergraduate students were all enrolled in an introduction to proof course at the same university at the time of the study. Interviews with individual participants were conducted by the first author, videotaped, and lasted one to two hours.

Materials

The materials for this study include seven partial proofs that are based on student-generated work and truncated to help participants focus on the use of mathematical language and not the attempted proof's logical validity. One example of a partial proof used in the study is presented in Figure 1.

Let R and S be relations on a set A. Prove: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Suppose $(S\circ R)^{-1}$ such that $(x,z) \in (S\circ R)^{-1}$, $x,z \in \mathbb{Z}$.

Since $(x,z) \in (S\circ R)^{-1}$, then $(z,x) \in S\circ R$.

Since $(z,x) \in S\circ R$, then $(y,x) \in S$ and $(z,y) \in R$.

Figure 1: Example of the partial proofs presented to participants

These partial proofs were chosen from student exams given in an introduction to proof course at the same university of the study. For each one of these proofs we created a copy with markings of what we considered to be breaches of linguistic norms of mathematical proof writing at the undergraduate level. An eighth partial proof was constructed to illustrate the interview procedure prior to the beginning of the interview.

Procedures

Mathematicians were presented with the student-constructed partial proofs one at a time and were asked to mark the partial proofs for anything that was out of the ordinary with respect to the use of mathematical language in the writing of mathematicians.

The interviews made two passes through the materials. In the first pass, mathematicians were

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asked to explain why they made each mark indicating an unconventional use of mathematical language. For each of those unconventional uses, the mathematician was asked if the issue at hand was a logical issue, if it would affect the validity of the proof, if it was an issue of mathematical writing, if it was definitely unconventional or a matter of personal preference, if it significantly lowered the quality of the proof, and if the mathematician would have deducted points based on this issue when grading the student generated proof in an introductory proof course. These prompts were designed to elicit the mathematician's views on mathematical language with respect to proof writing. In particular, the prompts addressed the severity of each breach and enabled a differentiation between issues of logic and issues of mathematical writing in the analysis of the data.

In the second pass through the data, mathematicians were presented with marked copies of partial proofs presented in the first pass. That is, they were presented with a copy of a partial proof marked with one instance of an unconventional use of mathematical language as identified by the authors. These marked partial proofs were presented for each of the predicted instances of unconventional use of mathematical language that had not been previously identified by the participant in the first pass of the data. In particular, mathematicians were asked whether or not they would agree with a colleague of theirs who had suggested that these were indeed unconventional uses of mathematical language. If the participant agreed, the interviewer would then prompt the mathematician to discuss the breach as in the first pass of the data.

The structure of the interviews with the undergraduate students mirrored the structure of the interviews with mathematicians. However, the instructions and prompts used in the student interviews specified that the participants were to indicate and describe what they believed a mathematician would find out of the ordinary with respect to the use of mathematical language in the writing of mathematicians in formal settings. The only two other differences with the previously described protocol was that in the first pass students were not asked to make an assessment regarding the deduction of points in the grading of the partial proofs, and that in the second pass they were asked whether or not they would agree with a *classmate* of theirs who had suggested each unconventional use of mathematical language not identified in pass 1.

Analysis

Interview videos were transcribed and materials generated in the interviews were scanned for analysis. Data was analyzed using memoing and grounded theory in the style of Strauss and Corbin (1990). Memos were used to conceptualize the breached linguistic norms and to categorize the broken norms.

Results

The analysis of the mathematicians' interviews identified ten breaches of linguistic norms in the partial proofs presented to them. The number of mathematicians and the number of students who identified each one of these breached norms in the first part of the interview (pass 1) are listed in Table 1. Analysis of the interviews provided us with participants' descriptions of each breached norm, their perceived seriousness of each breach (whether they considered the breach to be definitely unconventional, or a matter of personal interest or context), and in the case of mathematicians, whether or not they would deduct points when grading a proof containing such a breach. A summary of some of this information is included in Table 1. However, for the sake of brevity, we only describe how four of these ten norms emerged from the data and how the student participants understood these four norms.

Include Necessary Antecedents

In the interviews, both mathematicians identified an unclear referent in a partial proof. Specifically, the proof included the word ‘it’ referring to a function that had not been explicitly defined in the partial proof. One of the mathematicians said:

Table 1: Identification of Norms and Their Severity by Mathematicians and Students in the First Pass

Linguistic Norms of Mathematical Proof Writing	Mathematicians*	Students**
Specify each variable	2 – 2 – 2	2 – 1
Include necessary antecedents	2 – 2 – 2	0 – 0
Use proper imperative sentence structure	2 – 2 – 2	4 – 3
Use standard mathematical vernacular	2 – 2 – 2	3 – 2
Write in full sentences	2 – 2 – 2	0 – 0
Make relations between statements clear	2 – 2 – 1	3 – 3
Do not use abbreviations in formal settings	2 – 2 – 0	4 – 1
Do not use formal propositional logic	1 – 0 – 0	0 – 0
Explicitly indicate the structure of a proof	1 – 1 – 1	1 – 1
Use notation appropriately in texts	1 – 1 – 0	3 – 1

*Number of mathematicians (out of 2) who identified the norm breach in the first pass (1st number), deemed the breach definitely unconventional (2nd number), and would deduct points for this type of breach (3rd number).
**Number of students (out of 5) who identified the breach of the norm in the first pass (1st number) and deemed the breach definitely unconventional (2nd number).

> Yes, I mean it’s always, in writing in general, you need to have… the pronouns have to have an antecedent. And so, not just mathematical writing, but in particular you have to be careful about mathematical writing.

In this quote the mathematician indicated the necessity of including antecedents to avoid unclear referents and suggested that it is not only the case that the rules of English grammar apply to mathematical language, but also that the adherence to this rule in mathematics is particularly important. Using Scarcella’s (2003) framework, we identify the inclusion of necessary antecedents as a linguistic norm regarding the grammatical component of the language of mathematical proof writing. Both mathematicians judged the use of pronouns lacking clear referents as definitely unconventional in mathematical proof writing and indicated that they would deduct points from an exam in an introductory proof course based on that breach.

However, none of the students identified the unclear referent in the first pass through the data. After presented with the marked partial proof indicating the unconventional use of mathematical language, four students agreed that the word ‘it’ introduced ambiguity. But these students did not see this issue as severely as the mathematicians. In particular, the students indicated that they believed a mathematician would view this issue as a matter of personal preference. Failing to view the grammatical necessity of including antecedents could be interpreted as evidence that these students are not proficient in this aspect of the grammatical component of mathematical proof writing.

Use Proper Imperative Sentence Structure

The interviewed mathematicians indicated that the following phrases from the partial proofs were ungrammatical and meaningless: 1) “Suppose $(R \circ S)^{-1}$ s.t. $(x, z) \in (R \circ S)^{-1}$” and 2) “Let $\forall n \in \mathbb{Z}$”. These are both imperative phrases beginning with transitive verbs. As such, English grammar dictates that the phrases need both a direct object and an object complement to be a complete sentence. That is to say, that the sentence must suppose the direct object in relation to another object or a property about the direct object.

Both mathematicians judged these two statements to be definitely unconventional and worthy of deducting points on an exam in an introduction to proof course. In particular, one mathematician said that the former "just doesn't make any sense at all, I don't know what they were… what they mean to be saying." He continued to say "Suppose what about $(R \circ S)^{-1}$? Following it with the 'such that' is not a statement about $(R \circ S)^{-1}$, so you can't say suppose", indicating that the sentence is incomplete. Concerning the phrase "Let $\forall n \in \mathbb{Z}$", the second mathematician said:

> It's not a statement. I mean, let something implies that the something is a statement that's being assumed and this is not a statement. […] Linguists probably have a word for this; it's just not a sentence. […] It's not a correctly constructed meaningful thing.

Here the mathematician attempted to describe that using the word 'let' needs to be followed by a meaningful statement. While the mathematicians had difficulty explaining the breached linguistic norm exactly, it is evident that both believed the phrases included unconventional use of mathematical language.

Four of the student participants agreed in the first pass that these sentences are unconventional uses of mathematical language. Moreover, one of the students indicated that the statements were incomplete sentences and were grammatically incorrect. However, they did not agree with the mathematicians' severe opinions of this type of breach. While three students indicated that this was indeed an unconventional use of mathematical language (and not simply a matter of writing style), each of students also believed such a use of language was harmless to the quality of the proof.

Make Relations Between Statements Clear

Both mathematicians discussed the importance of using verbal connectives to show the relations between different statements within the partial proofs. Moreover, they both agreed that lacking these verbal connectives was definitely unconventional and would merit point deductions on an exam in an introductory proof course. In particular, one mathematician said:

> No, it's not harmless. […] I mean if I'm worried about a student actually getting broken of a habit of making these kinds of, here's a statement, here's a statement, here's a statement, without drawing the connectives in between. Then that's not a logical issue, but it's a serious presentation issue.

In this quote, the mathematician indicated that he did not want students to write proofs as a series of unrelated statements, and deemed this breach as a serious presentation issue. Such relations between statements show the flow of the argument to readers, which is an aspect of the discursive component of mathematical language.

However, there was less agreement among the students. The three students who indicated in the first pass that one must make relations between statements clear each agreed that this use of mathematical language was definitely unconventional. One student said:

> I don't know if he's just stating it or if he means it to be part of that [assumption], but I think if you're writing it out formally, you should be more clear about it, like put an assume.

This student indicated that lacking verbal connectives could lead to ambiguity, which puts unnecessary stress on the reader to decipher the flow of the argument. On the contrary, two of the five students indicated in the second pass that there was no need for verbal connectives and a proof lacking words was conventional. One of these students said, "in some homework assignments, I have done this before and it's not wrong. You could write it in words or you could write it like this." This student did not seem to believe in the necessity of connecting the various statements in a proof.

Use Notation Appropriately within Text

One partial proof included the sentence "So there are 19 possible differences in $\mathbb{N}\times\mathbb{N}$ that are ≥ -9 and ≤ 9". One of the mathematicians indicated in the first pass that this sentence used the mathematical symbols $\geq$ and $\leq$ inappropriately. In particular, the mathematician said:

I would say that those should have been said in words rather than using the symbols. Or complete expression – that is, there are 19 possible differences, *d*, such that $d \geq -9$ and $d \leq 9$. [...] Do it in a symbolic phrase or do it in words, but don't kind of mix it in.

In this quote, the mathematician indicated that the binary operators $\geq$ and $\leq$ require notation on both the left and right sides of the symbols. Relating the to lexical component of academic English, this quote suggests that using the binary operator without notation on both sides of the operator is an unconventional use of mathematical language. This mathematician indicated that inappropriately mixing the symbolic notation with text would significantly lower the quality of the presentation and that he would deduct points from an exam based on this use of language. While the other mathematician agreed in the second pass that such a use would be unconventional, he believed that mixing the symbolic notation with text is a personal preference and would not deduct points off of an exam. This mathematician however discussed that one should not technically use this type of "short hand" in formal settings, but that it is commonly used despite it being unconventional. This indicates the sociological component of mathematical proof writing includes understanding the appropriateness mixing mathematical notation and prose.

Three of the five students' responses in the first pass also indicated that mixing the notation and text in this way was an unconventional use of mathematical language. For example, one student said:

> I think it's just more the notation. It looks odd to just have the greater than sign without having something like directly in front of it. So I think it should be written out in words.

In this quote, the student described that the binary operators $\geq$ and $\leq$ require notation on both the left and right sides of the symbols, suggesting that the students agreed with the first mathematician on what was the appropriate way to use symbols in prose. On the other hand, many of the students also believed that this mix of notation and prose was harmless. In particular, one student said that "it's faster than writing greater than or equal to or less than or equal to [… and] I've seen lots of teachers use that when like they do lectures." So this student emphasizes that since teachers mixed the notation and prose when writing on the board, there is no reason why he should not do so as well.

Discussion

As this qualitative study considers only a small sample of mathematicians and undergraduate students, the findings are simply suggestive of how mathematicians and undergraduates view the language of undergraduate mathematics proof writing. In particular, based on mathematicians' interviews we have identified ten linguistic norms of mathematical proof writing. Our analysis suggests that while interviewed students showed some competence in certain aspects of the lexical and grammatical components of mathematical proof writing, there were significant differences between these students' understandings of some aspects of the sociological and discursive components of mathematical proof writing and the corresponding understandings of the mathematicians interviewed in this study.

While the nature of this study precludes any claim of sample-to-population generalization of these findings, we believe the linguistic norms of mathematical proof writing identified in this study, as well as the method suggested for studying such norms, opens interesting avenues for future research in an area that is both under-researched in mathematics education and important in our attempt to understand students' difficulties reading and writing mathematics. In particular, these findings suggest the following research questions: To what extent does the larger community of undergraduate mathematics professors agree on the linguistic norms of undergraduate mathematics proof writing identified in this study? To what extent do students in transition to proof courses (and beyond) agree on the described views of these norms, and to what extent do students' perceived views of these norms align with those of the professional mathematical community? How do students' understandings of these norms develop and change throughout a semester of an introduction to proof course? We are currently designing studies that address some of these questions.

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STUDENT UNDERSTANDING OF DIRECTIONAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

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Action-Process-Object-Schema (APOS) Theory is applied to study student understanding of directional derivatives of functions of two variables. A conjecture of the main mental constructions that students may do in order to come to understand directional derivatives is proposed and is tested by conducting semi-structured interviews with 26 students who had just taken multivariable calculus. The interviews explored the specific constructions of the genetic decomposition that student are able to do and also the ones they have difficulty doing. The conjecture, called a genetic decomposition, is largely based on the elementary notion of slope and on a development of the concept of tangent plane. The results of the empirical study suggest the importance of constructing coordinations of plane, tangent plane, and vertical change processes in order for students to conceptually understand directional derivatives.

Keywords: Advanced Mathematical Thinking

Introduction and purpose of the study

The calculus of functions of several variables is of fundamental importance in the study of mathematics, science, and engineering. Some work has been published regarding functions of two variables (see for example, Trigueros and Martínez-Planell, 2010; Martínez-Planell and Trigueros, 2012). However, there are very few publications on the differential calculus of such functions. The only publication we'll refer to, Weber (2012), includes a discussion of the rate of change of functions of two variables focusing on the use of covariational thinking to help students build a notion of rate of change in space which centers on students' development of a symbolic representation of the directional derivative. In this paper we focus on students' geometrical understanding of directional derivatives and its relationship to other important ideas in the schema of the differential calculus of functions of two variables. Our research questions are: What are students' conceptions of directional derivatives after taking a Multivariate Calculus course? What are the main mental constructions involved in learning this concept?

Theoretical framework

APOS Theory is used as a theoretical framework to study the cognitive development of students who completed a course using a traditional lecture/recitation model, as discussed in Arnon et al. (2013, p. 106). As APOS is a well-known theory it is briefly discussed (see Figure 1). In APOS, an Action is a transformation of a mathematical object that is perceived by the individual as external. It could be the step by step implementation of an algorithm according to explicit instructions or the application of a fact or result that has only been memorized. Activities that lead students to repeat and reflect on an action can help them to interiorize the Action into a Process. A Process is characterized by the individual's ability to imagine doing the main Actions and to anticipate their result without having to explicitly perform them; In a Process the Actions are perceived as internal. A Process may be coordinated with other Processes, and may also be reversed. As an individual needs to apply Actions on a Process, he/she may become aware of the process as a totality. In this case, when the individual applies or can imagine applying Actions to the Process, then it is said that the Process has been encapsulated into an Object. Actions, Processes, and Objects may be organized into Schemas. A Schema for a given mathematical notion is a coherent collection of Actions, Processes, Objects, and other Schemas that are related in the individual's mind to the notion that is being considered. Actions on a Schema may result in its being thematized into an Object. Schemas develop as relations between new and previous Actions, Processes, Objects and other Schemas are constructed and reconstructed. Their development may be described by the Intra-, Inter-, Trans- "triad": At the Intra- stage relations among the Schema components are being constructed but they remain for the most part isolated from one another. At the Inter-stage, transformations between some of the Schema components are recognized. The Trans- stage is defined in terms of the construction of a synthesis between them, so that the Schema is coherent and the individual can decide when its use in problem solving is needed.

Also, although it might be thought that in APOS theory there is a linear progression from Action to Process to Object and then to having different Actions, Processes, and Objects organized in Schemas, this often appears more like a dialectical progression where there can be partial developments, passages and returns from one conception to another. What the theory states is that a student's tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept is different depending on whether the student understands the concept as an Action, a Process, or an Object or has constructed a coherent Schema.

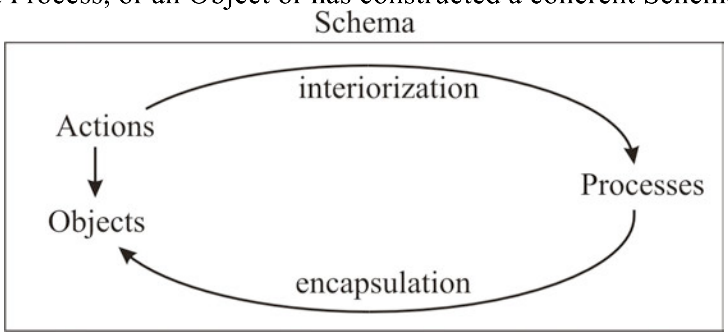


Figure 1: Mental structures and mechanisms

The application of APOS theory to describe particular constructions by students requires that researchers develop a genetic decomposition - a model that describes the specific mental constructions a student may make in understanding mathematical concepts and their relationships. As a model, a genetic decomposition predicts the constructions needed to learn the concepts of interest. It is proposed by researchers and needs to be tested experimentally. The genetic decomposition that follows was developed from reflection on the mathematics itself, considering what it takes to make the idea of a directional derivative understandable to students, and the classroom experience of the researchers implementing initial versions of the idea for several consecutive academic years.

Our genetic decomposition of the directional derivative is essentially based on the notions of directed slope in R^3 and vertical change on a plane. Moreover, the genetic decomposition of vertical change on a plane can be used as a starting point to describe the mental construction of several important concepts of the differential calculus of functions of two variables including tangent plane, differentials, and directional derivatives. Hence, while based on the elementary idea of slope, it can potentially provide a unifying framework for the description of the main ideas of the differential calculus of two-variable functions, thus contributing to help students construct, at least, an Inter-Schema stage of development for the differential calculus of functions of two variables.

The genetic decomposition is as follows: Given a non-vertical plane, the Processes of slope of a line and fundamental plane (planes of the form $x = c$, $y = c$, $z = c$) are coordinated into new processes of vertical change in the x and y directions, where it is recognized that vertical change in the x direction can be described as a function of the horizontal change in the x direction ($\Delta z_x = m_x \Delta x$), and similarly for vertical change in the y direction ($\Delta z_y = m_y \Delta y$). These processes are coordinated into a Process of total vertical change on a plane, so that total vertical change in any plane is given in terms of the sum of vertical changes in the directions of the coordinate axes: $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$ (see Figure 2). Actions and Processes which are treatments and conversions in and between representations (Duval, 2006) are performed on the Process of total vertical change to encapsulate it into the Object conception of plane in three dimensions. In particular, the point-slopes formula for a plane, $z - z_0 = m_x\left(x - x_0\right) + m_y\left(y - y_0\right)$ may be seen as the vertical change from an initial point $\left(x_0, y_0, z_0\right)$ to a final generic point $\left(x, y, z\right)$ on the plane.

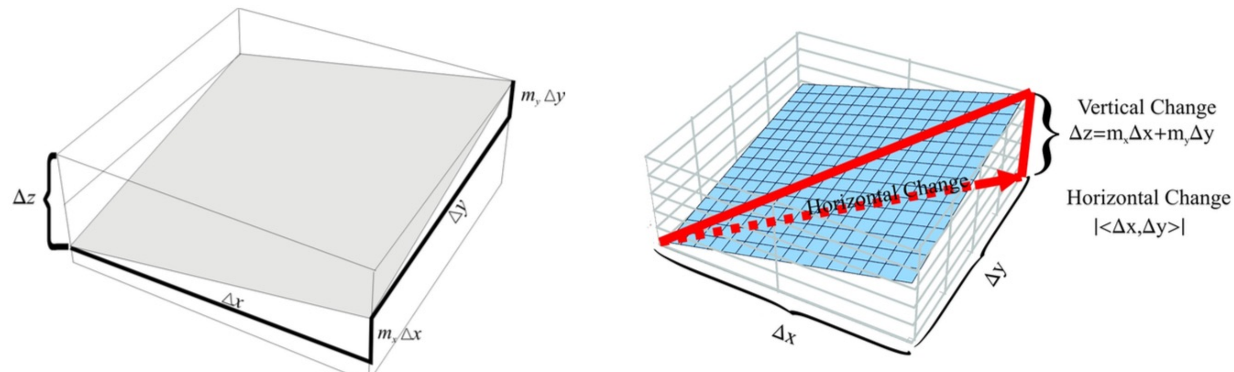


Figure 2: Vertical change on a plane and directional derivative

The Process of partial derivative is coordinated with that of plane in three-dimensional space into a new Process where tangent planes to any surface at different points can be considered and calculated. When there is a need to consider particular tangent planes and perform actions on them to describe the surface in terms of the behavior of partial derivatives, this Process is interiorized into an Object conception of tangent plane.

To do the mental construction of $D_{\vec{v}} f(a,b)$, the directional derivative of f in the direction $\vec{v} = \langle \Delta x, \Delta y \rangle$ (not necessarily unitary) at the point (a,b), the student may coordinate the Process of three-dimensional space with the Process of function of two variables in order to locate and represent in space or imagine the point $\left(a,b,f\left(a,b\right)\right)$. Further coordination with the Process of vectors allows use of the point $\left(a,b,0\right)$, or more generally any point of the form $\left(a,b,z\right)$ as a starting point from which to represent the direction vector $\vec{v} = \langle \Delta x, \Delta y \rangle$ in space as $\langle \Delta x, \Delta y, 0 \rangle$. Then, the Processes of vector, slope, and derivative of function of one variable are coordinated to represent physically or geometrically, and recognize, the directional derivative as the slope of the line tangent to the graph of the function at the point $\left(a,b,f\left(a,b\right)\right)$ in the given vector direction (a directed slope). To obtain the value of $D_{\vec{v}} f(a,b)$ (see Figure 2), the student would then coordinate the Process of slope of a line with the Process of tangent plane to obtain the vertical change as $f_x(a,b)\Delta x + f_y(a,b)\Delta y$, and with a Process of vector magnitude to obtain the horizontal change as the magnitude of the direction vector $|\vec{v}| = \left|\langle \Delta x, \Delta y \rangle\right| = \sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}$, and thus obtain $D_{\vec{v}} f(a,b)$, as $\dfrac{f_x(a,b)\Delta x + f_y(a,b)\Delta y}{\left|\langle \Delta x, \Delta y \rangle\right|}$. These

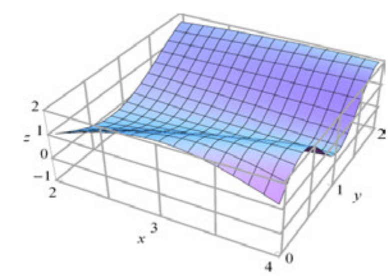
Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Processes and coordinations must be constructed in different representations. The computation of $D_{\vec{v}} f(a,b)$ at a fixed point (a,b) for different direction vectors $\vec{v}$ and at different points (a,b) for a fixed direction vector $\vec{v}$ allows the encapsulation of the directional derivative Process as a function that depends on the direction vector $\vec{v}$ while also recognizing the functional dependence of the directional derivative on the starting point (a,b).

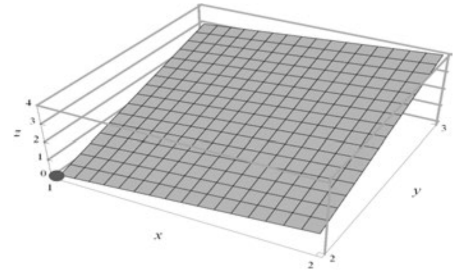
Method

An instrument consisting in 6 multi-task questions was designed to conduct semi-structured interviews with 26 students to test their understanding of the different components of the proposed genetic decomposition. Of these 6 questions we will only report on the 2 which directly considered directional derivatives. All participants were students of science and engineering that had just finished a course on multivariable calculus. Nine (9) of them came from a group where a traditional teaching approach was followed, and the other 17 students came from two groups where activities designed in terms of the genetic decomposition were used. All students used the same textbook (Stewart, 2006) and covered the same material in the course. The three instructors of the groups were asked to choose 9 students from each of them, considered as above average, average, or below average based on their performance in the class, providing as balanced a distribution as possible, to participate in the interviews. One of the students did not show up to the interview and hence we were left with a total of 26 students. All participating professors were experienced (with at least 20 years of experience and having repeatedly taught the course during those years), popular with students (judging on how fast their sections fill up and on student evaluations), and had, throughout the years, shown concern about student's learning. Each interview lasted on the average from 40 minutes to 1 hour. The interviews were recorded and transcribed. Data was independently analyzed by the researchers and conclusions were negotiated among them. We now discuss the questions that dealt directly with the directional derivative and that were analyzed in terms of the structures described in the genetic decomposition:

1. Suppose the graph of $z = f(x,y)$ is as follows. State the sign (positive, negative, zero) of $D_{<-2,1>} f(4,0)$.
   (This is the second part of a question in the original interview instrument. In the first part, students used the same graph to determine the sign of $f_y(4,0.7)$.)



2. The following plane is tangent to the graph of $z = f(x,y)$ at the point (1,2,0). Use the given figure to find $D_{\langle 1,1\rangle} f(1,2)$.



It is important to note that in the first question, the function initially is increasing in the given direction and thus the directional derivative is positive, $D_{<-2,1>} f(4,0) > 0$, while in the second one, thinking of the directional derivative as a slope, the vertical change may be obtained looking at the given figure as $\Delta z = 4 - 0$, and the horizontal change as $|\langle 1,1\rangle| = \sqrt{2}$. Hence the value of the directional derivative is $4/\sqrt{2}$. Using the Process of vertical change on a plane this may also be obtained as $D_{\langle 1,1\rangle} f(1,2) = \dfrac{m_x \Delta x + m_y \Delta y}{|\langle 1,1\rangle|} = \dfrac{1(1)+3(1)}{\sqrt{2}} = \dfrac{4}{\sqrt{2}}$.

Results

It seems to be commonly assumed in instruction that students can readily represent the vector direction in a directional derivative. However, we found that frequently this is not the case. Tania, one of the best performing students, could quickly and without hesitation identify the sign of $f_y(4,0.7)$ in problem 1, but seemed unable to represent the vector direction and thus was not able to give the sign of $D_{\langle -2,1\rangle} f(4,0)$.

Interviewer: Will that directional derivative be the slope of a tangent line or not?
Tania: Yes, it is the slope of a tangent line.
Interviewer: Of what line?
Tania: That's the tricky part. That's the line I'm looking for.

In the case of David it was observed that in problem 1, he made the necessary coordinations to identify the base point and correctly represented the direction vector. However, he did not coordinate the vector and derivative of a function of one variable Processes. Furthermore, he remembered a formula that would give him a correct answer but did not seem to have constructed a Process of vertical change on a plane that would enable him to give geometric meaning to it:

David: The directional derivative at the point $(4,0)$ … x is -2, y is 1, it goes this way…

[Correctly representing the vector direction with a dashed line starting at the base point; see Figure 3.] Then the directional derivative at this point will be equal to … $\dfrac{f_x(4,0)(-2)+f_y(4,0)(1)}{\sqrt{5}}$ … [He went on to interpret this as the slope of the secant line he drew in Figure 3.]

After a while, the interviewer suggested to David that he think of a tangent line. He then managed to correctly draw a tangent line (see Figure 3), however, the slope he came up with was not the **directed** slope:

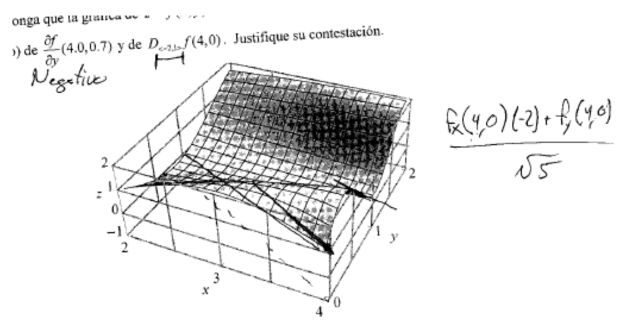


Figure 3: David's drawing on problem 1

Interviewer: And if I were to tell you that the directional derivative is the slope of a tangent line, could you draw the tangent line to that graph at that point in that direction?
David: At this point in that direction? [He draws a tangent line that seems to be correct.]
Interviewer: Will that slope be positive or negative?
David: it will be a negative slope… it is negative because… while the value of z decreases, the value of x increases, so it would, it would be negative, it would be coming down.

Even though David manages, without hesitation, to obtain the correct answer to problem 2, he gives no evidence of having coordinated the Processes of vertical change and plane or of tangent plane to give geometric meaning to his conception of directional derivative:

David: Use the given figure to find … it would be the partial with respect to x at the point (1,2) times 1, the partial with respect to y at the point (1,2) times 1, over the square root of 2… 4 over the square root of 2 (see Figure 4).

Overall we can only say that David seems to show a conception of directional derivative that is in transition from Action to Process because, even though he showed some evidence of having constructed a Process conception of directional derivative, as evidenced by his ability to locate base points, represent vectors' directions, and do some computations, he was still mostly dependent on a memorized formula.

$$\frac{\overbrace{f_x(1,2)}^{1}(1)+\overbrace{f_y(1,2)}^{3}(1)}{\sqrt{2}}$$

$$\frac{4}{\sqrt{2}}=\frac{4\sqrt{2}}{2} \qquad 2\sqrt{2}$$

Figure 4: David's written answer to problem 2

Some students like Luis, were able to do these coordinations to explain correctly the expected answer. His behavior on problem 1 is consistent with a Process conception of directional derivative. Of course, his behavior on just one problem is not sufficient to guarantee he has constructed this type of conception of directional derivative, since a student's conception can only be ascertained by considering the student's **tendency** to deal with different problem situations involving directional derivatives.

Luis: [On problem 1] The directional derivative, I'm given (4, 0). Here I have located the point. I have to look for the direction $\langle -2,1\rangle$ which would be 2 units to the left on the graph, that vector, and 1 unit to the right in y. It should be positive there since the slope of z [he probably means the value of z] in that direction is increasing.

Continuing with the genetic decomposition, after the student was able to locate the base point and the vector direction, and after coordinating the Processes of vector and that of derivative of function of one variable to think of the problem in terms of tangent lines and directed slopes (in problem 1), the student was expected (in problem 2) to coordinate the Process of slope with the Process of tangent plane in order to compute the vertical change along the plane and to coordinate this last Process with that of vectors in order to compute the horizontal change as the magnitude of the direction vector. However, in problem 2, Luis was not able to coordinate the Processes of vertical change, horizontal change, and slope of a line to deal with the directional derivative, but rather seemed to be depending on a memorized and unconnected formula (used in the class textbook; Stewart, 2006) which is valid only for unit direction vectors. After writing:

$$D_{\langle 1,1\rangle} f(1,2) = m_y\Delta y + m_x\Delta x = 3(1) + 1(1) = 4$$

Luis: The directional derivative should have the value of 4. I'm not completely sure, but I am quite sure this should be the value of the directional derivative.

Interviewer: Does the fact that the vector is not unitary play any role?

Luis: … Well, in this case, since it is a plane… I believe that in this case since it is a plane it won't make much of a difference… the directional derivative will have the same value as long as it is in the direction $\langle 1,1\rangle$… maybe if it were a more complex graph… in this case it shouldn't be. Maybe there's a problem but I'm not sure.

Although Luis showed that he was certainly on his way to constructing a Process understanding of directional derivative, he did not show he was able to coordinate the Process of vertical change on a plane needed to interpret $D_{\langle \Delta x,\Delta y\rangle} f(a,b)$ as a slope with vertical change given by $f_x(a,b)\Delta x + f_y(a,b)\Delta y$. The reason for this is, probably, that in his classroom and in the textbook, $D_{\vec{u}} f(a,b)$ was defined only for a unit direction vector $\vec{u}$ and the geometric interpretation of the directional derivative as a slope remained hidden both in a graph and in the formal development of the mathematics (see Stewart, 2006).

Daylin was able to compute the directional derivative in problem 2 while justifying geometrically her computations. Hence she seems to have constructed the conjectured coordinations needed to be able to think of the directional derivative as a slope and obtain the necessary vertical change and horizontal change (see Figure 5). Further, since she also gave evidence of doing the conjectured mental constructions required to solve and explain problem 1, one may reasonably state that she seems to have constructed a Process conception of directional derivative.

Daylin: Then the height at the new point will be 4, if I put this triangle [see Figure 5] my height here is 4… this is the direction $\langle 1,1\rangle$ and I want the slope… here z is zero, I have the rise, I need the run, the run will be … the square root… so the slope will be 4 over $\sqrt{2}$.

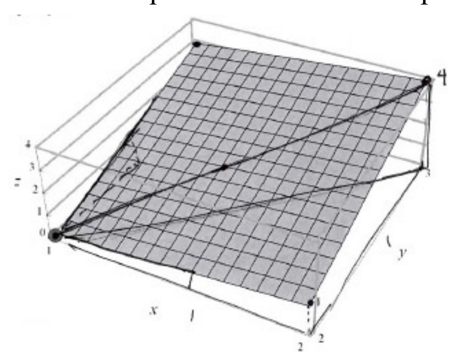


Figure 5: Daylin's drawing on problem 2

Summary and Discussion

Only 4 of 26 students gave evidence of having made all or most of the mental constructions required in the genetic decomposition. This tells us that the idea of a directional derivative is difficult for most students and they need more help understanding even the most elementary notions associated with this idea. It also suggests that much work still remains to be done designing and improving activities to help students do the conjectured constructions.

We just saw that although teachers probably frequently assume that students will be able to imagine the base point and the role of the vector direction to understand directional derivatives, this seems not to be the case for many students, and that explicit attention to the necessary coordinations can help students make the desired mental constructions. Also, many students did not show a Process conception of derivative of function of one variable to be used in the proposed mental constructions of directional derivative. This suggests that rather than assume they have constructed this Process, instruction can start by, once again, explicitly considering ways to help them construct it, but now in the context of functions of two variables. The construction of a Process of vertical change on a plane

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

was shown in this study to be important to facilitate the mental construction of the processes of tangent plane and directional derivative that help students give geometrical meaning to directional derivative, and that students may not be able to construct it without explicit attention in their classroom instruction.

Overall, results show that most of the students who had successfully finished a course on multivariable calculus did not have a deep understanding of the concept of directional derivative. Many students relied on memorized facts and formulas and thus showed difficulties when responding to questions that needed a deeper conceptual understanding. Most students lacked geometrical understanding of the basic components involved in the definition of a directional derivative. The lack of understanding of those concepts impaired them from realizing that the symbol $D_{\vec{v}} f(a,b)$ denotes a special type of derivative and that, as such, it can be represented as a slope of a tangent line in the given direction, as well as understanding that, in three-dimensional space, the notion of slope would be ambiguous unless it is a directed slope, and recognizing that for a vector direction $\langle \Delta x, \Delta y \rangle$, which is not necessarily unitary, the directional derivative represents the directed slope of a line on the tangent plane with vertical change given by $f_x(a,b)\Delta x + f_y(a,b)\Delta y$ and horizontal change given by the magnitude of the direction vector $\langle \Delta x, \Delta y \rangle$. This is necessary to make sense of the formula traditionally used in textbooks, where the direction vector $\langle u_1, u_2 \rangle$ is unitary.

The assumption that students will, look at textbook figures and on their own, come to understand the geometric ideas involved in directional derivatives, apparently held by many teachers who just present students with the formula $D_{\langle u_1,u_2 \rangle} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$, seems not to be valid since the simple geometrical explanation of the notion of directional derivative as a slope remains hidden. Of course, any instructional approach will need to eventually consider only unitary direction vectors since this formula can be expressed as the dot product of the gradient vector and the unitary direction vector, an observation which is crucial in exploring the properties of the gradient vector. Results obtained suggest that instruction might start by exploring the basic property of slope where the vertical change is seen as the slope times the horizontal change, $\Delta V = m\Delta H$, and then interpreting this in the case of a plane to obtain the basic idea of vertical change on a plane, symbolically represented by $\Delta z = m_x \Delta x + m_y \Delta y$. This is essentially the idea inspiring the genetic decomposition presented. A possible virtue of this approach is that the notions of plane, tangent plane, the differential, and the vertical change in a directional derivative are all explained and inter-related by this simple geometric idea, thus potentially helping students build a coherent schema for the differential calculus of functions of two variables. But, this remains to be investigated.

Acknowledgements

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CHANGING CONES: THEMES IN STUDENTS' REPRESENTATIONS OF A DYNAMIC SITUATION

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Researchers have identified challenges students face when modeling dynamic situations. This report discusses the results of semi-structured clinical interviews with ten prospective secondary mathematics teachers who were provided with a dynamic image of a growing and shrinking cone. We asked the students to graph the relationship between the surface area and the height of the cone. We identify four themes in the students' solution approaches and discuss the implications of these approaches. Specifically, we discuss the themes with respect to relationships between the students' solutions and their images of the growing and shrinking cone, including the extent that they leveraged this image to determine their solutions.

Keywords: Modeling; Problem Solving; Cognition

The authors of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices, 2010) argued students should have the opportunity to construct and compare a multitude of relationships including those that have constant rates of change and those that have varying rates of change. They also identified modeling and reasoning quantitatively as two practices that should permeate students' mathematical experiences. These policy calls are in line with researchers who have identified students' *quantitative* and *covariational reasoning*–students conceiving situations as composed of measurable attributes that vary in tandem–as critical to numerous K-16 concepts (Ellis, 2007; Johnson, 2015a; Moore & Carlson, 2012; Thompson, 2011). These same researchers have argued that much is to be learned about how students approach dynamic situations, including the extent that these approaches are rooted in constructing structures of quantities and relationships between these quantities. Further, researchers have called for increased attention to exploring students' activities as they make sense of situations where they conceive of multiple quantities covarying (Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 2011).

We discuss ten prospective secondary mathematics teachers' (heretofore referred to as students) solutions to a task in which they graphed a relationship between the surface area and height of a cone as the cone changed in size (but maintained a constant slant-angle). Using our analyses of the students' activities in semi-structured clinical interviews (Clement, 2000), we illustrate four themes in their solution approaches. With respect to these themes, we discuss how they used their images of the situation to determine a relationship between the surface area and height of a cone. Collectively, these themes provide glimpses into students' thinking as they construct relationships between quantities in what we intended to be a three-dimensional context.

Literature Review and Motivation

Saldanha and Thompson (1998) described covariation to include, "imagistic foundations for someone's ability to 'see' covariation" (p. 298). By 'see' covariation, we infer Saldanha and Thompson did not mean that covariation and quantities are independent of the mind or merely perceptual objects. Instead, we interpret them to refer to someone constructing and re-constructing a dynamic situation to the point that they envision it as entailing measurable attributes and an understanding or anticipation of how these attributes change in tandem. Additionally, we characterize a sophisticated image of covariation to include the capacity to 'replay' one's image of the dynamic situation while holding in mind how these attributes are changing in tandem. In this regard, how students operate and reason in what they come to understand a covariational situation is directly tied to their images of that situation.

Students' images are important for their construction of mathematical objects (Izsák, 2004; Thompson, 1994). Focusing on students' images in applied problems, Moore and Carlson (2012) examined students' activities for the purpose of determining distinguishing features of students' images with respect to the formulas and graphs that the students produced when modeling and representing situations with covarying quantities. Most relevant to the present work, the authors noted that despite the students' creations of mathematical products that an observer might deem incorrect with respect to the intended situation, these mathematical products were consistent with the *students'* images of the situations. For instance, on a task that included a box with varying base dimensions, some students envisioned a box with fixed base dimensions. As a result, these students determined a volume formula that captured a base with fixed dimensions. Based on their findings, the authors (Moore & Carlson, 2012) argued that researchers and educators should give more attention to students' images of problem contexts including determining how these images play a role in the mathematical products students construct.

Theoretical Perspective

We leverage tenets of radical constructivism by approaching knowledge as actively built up by the individual in ways that are idiosyncratic to that individual and fundamentally unknowable to another individual (von Glasersfeld, 1995). Hence, we approach quantities as personally constructed measurable attributes (Steffe, 1991). Likewise, relationships between quantities are constructed by the individual, with these relationships being influenced by the individual's understanding of each quantity and their image of the relevant situation (along with other potential influences). It follows that we do not assume that students see situations that we provide them with in the same way that we do or intend for them to do (Thompson, 2011). For instance, a student may imagine a cone growing smoothly as a video suggests, growing in discrete snapshots corresponding to adding to the cone in sections, or physically changing in some other fashion (e.g., stretching the cone as if it is made of malleable rubber).

Although we take the stance that students' knowing and thinking is fundamentally unknowable to us as researchers, we can make inferences about students' thinking based on our interpretations of their words and actions. Steffe and Thompson (2000) referred to such inferences or models as *the mathematics of students*. Our goal was to characterize students' images of a situation and the mathematical products they created based on our inferences of their activities when given a dynamic situation as described in the following sections.

Methodology

We conducted a series of three semi-structured task-based clinical interviews (Clement, 2000) with ten students (eight female, two male). The students were enrolled in a secondary mathematics teacher education program at a large university in the southeast United States. At the time of the third interview–the interview we focus on here–these students had completed their first content course in the secondary mathematics education program, as well as at least a full calculus sequence and two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with a grade of C or better. Some students had completed several additional education and mathematics courses.

The interviews consisted of a series of tasks and problems with many tasks asking the students to construct and represent a relationship between quantities in a dynamic situation. Each interview was videotaped and these videos were digitized for analysis. Two members of the research team were present at each interview, and for each interview, the interviewers took field notes and discussed observations and insights afterwards. Upon completion of the interviews, members of the research team viewed the videos and selected instances of student activity that revealed insights into the students' thinking. The research team then met to discuss their observations and used an open (generative) and axial (convergent) approach (Strauss & Corbin, 1998) to construct tentative themes we observed across students. Upon further analysis, themes were refined by comparing and contrasting different students' activities. Through this process of constructing, refining, and re-refining, the research group reached a consensus on themes that characterized the students' activities on the cone problem.

Task Design – The Cone Problem

At the start of the interview, we presented students with a video of a growing and shrinking cone with a fixed slant angle. The height of the cone increased and decreased at a constant rate with respect to the video playback (Figure 1). We then gave the students the following prompt, "Watch the video, which illustrates a cone with a varying height. Sketch a graph of the relationship between the height of the cone and the outer surface area of the cone."

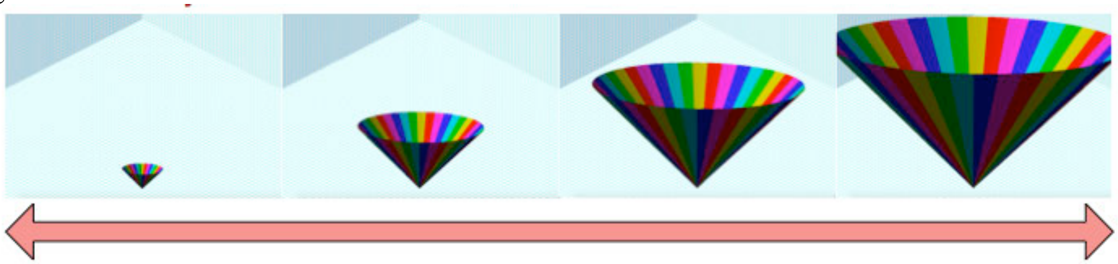


Figure 1: Dynamic Image of Cone

Saldanha and Thompson (1998) described a student's activity as he considered the relationship between two similar quantities (e.g., length)–a car's distances from two fixed points as the car traveled along a straight line. We extended this type of situation to involve the covariation of two quantities of different attributes: length and surface area. We also designed the task so that the situation lent itself to reasoning about amounts of change between the two quantities (e.g., for successive equal changes in height, the outer surface area of the cone increases and the change in the outer surface area also increases). We consider Saldanha and Thompson's (1998) task to be much more complex (imagistically) in this regard. We also note that we designed the task so that the students would not have a memorized formula readily at hand, although we hypothesized that students may attempt to construct or recall such a formula.

Results

We organize the students' solution approaches to the cone problem into four themes (Table 1). In what follows, we describe these themes including the relationships between students' images of the situation and their solution activity.

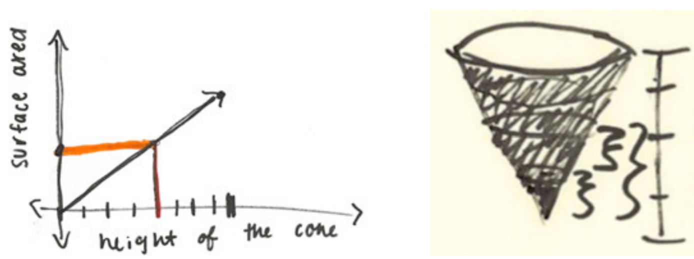
Predetermined Relationship

Students classified in the "Predetermined Relationship" theme used their initial image of the situation to dictate all further actions and conclusions. These students quickly came to a conclusion about the relationship between height and surface area based on some imagistic or physical aspect of the cone (e.g., the cone growing and shrinking in a 'smooth' or 'constant' manner). Instead of further analyzing the situation in an attempt to justify their claimed relationship, the students assimilated all

Table 1: Student Solution Approach Themes

Theme Name	Theme Description	Students
Predetermined Relationship	The student uses some imagistic aspect of the situation to reason about the relationship of the quantities. No fundamental changes of image occur from his/her initial observation.	Polly, Alice, Roz
Formula Values Determine Relationship	The student derives a formula to obtain specific numbers to determine and investigate the relationship between the quantities.	David, Kate, Terrence
Formula Structure Determines Relationship	The student derives a formula and uses properties of the formula to determine the relationship.	Angela, Audrey
Images of Covariation Determine Relationship	The student uses their image of the situation to determine the directional change and amounts of change of the two quantities under consideration.	Caroline, Trish

subsequent actions and products in terms of their initial claim. For instance, if the students determined a formula, they described the formula in terms of their initially stated relationship between the quantities; the students did not intend the formula to be for testing or verifying their relationship. Most notably, the students maintained their initial conclusions even when we directed them toward particular aspects of the situation or their activity that we thought would contradict their initial conclusions.



(a) (b)

Figure 2: Roz's Solution to the Cone Problem

For example, after concluding the relationship was linear and drawing a graph to reflect that relationship (Figure 2a), Roz drew the 3-D cone in a way that identified (from our perspective) amounts of change of surface area for successive equal changes of cone height (Figure 2b). When asked to describe how the changes in surface area for each successive height were changing, she responded, "going off this idea that it's a constant change in surface area," and then described that for equal changes in height, there was an equal change in surface area. Roz maintained that the surface area changed by constant amounts, even after the interviewer repeatedly prompted her to identify and shade different sections that represented the change in surface area. This response illustrates that Roz already had a pre-determined linear relationship in mind when reasoning with her picture of the situation (Figure 2b) to identify and explain how the changes in surface area varied.

Formula Use

We identified two themes in which students constructed formulas to model surface area (but not always in relation to only the height of the cone) and used their formulas to reason about the relationship between height and surface area. We make two distinctions between these students' solutions based on how they used their formula to make conclusions about the relationship.

Formula Values Determine Relationship. Some students used formulas to compute numerical values. Students in this theme constructed an initial image of how the quantities were related (e.g., both the surface area and height *increase* as the cone grows), but were then perturbed by whether the quantities covary at a constant or changing rate of change. Each student then moved to determine a formula by which they could calculate paired (height, surface area) values. As no actual values were given to the students to describe the cone, the students created hypothetical values. After determining a formula, each student calculated the area for several specified values of height, with these height values increasing in equal increments. Each student then determined the relationship by comparing her or his calculated values (e.g., determining the difference between successive surface areas). We note that despite each student creating a different, technically incorrect formula, all students in this theme concluded (accurately) that the changes of surface area increased for equal changes of height.

David is one of the students who engaged in this type of reasoning. After watching the video, David conjectured that the surface area is increasing at an increasing rate with respect to height. Having difficulty using the situation or a diagram to justify his conjecture, he created a formula to compute surface area. David used the height, h, and average radius, $r_{ave,}$ (which he described as half the radius) of the cone in combination with his prior knowledge of surface area (SA) of a cylinder to derive $SA = 2\pi(r_{ave})h$. He assumed the height and radius were equal to get the final formula, $SA = 2\pi(h/2)h$. After first trying to use this formula to determine changes of surface area for arbitrary equal changes of height, he moved to using specific (numeric) height values to compute surface area values (Figure 3). He then computed differences in surface area values to conclude that the changes of surface area increase for equal changes of height.

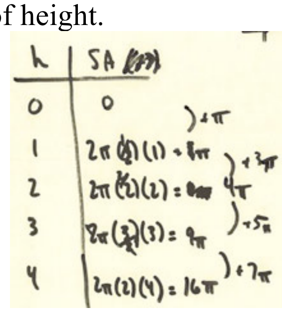
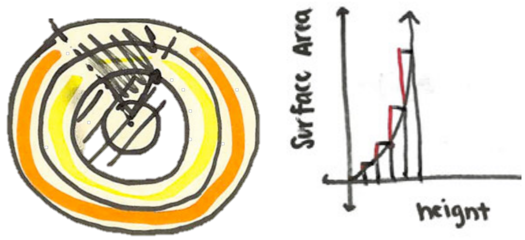


Figure 3: David's Solution to the Cone Problem

Formula Structure Determines Relationship. Whereas students grouped in the previous theme used their formulas to calculate and compare numerical values, students classified in this theme inferred how the quantities covaried based on the structure of their formulas. Specifically, the students determined a surface area formula with a multiplicative relationship between the quantities (i.e., height times height or height and radius of the base multiplied together with an assumption that the height and radius were proportionally related) and thus concluded that as height increased, surface area increased at an increasing rate.

To illustrate, after watching the video, Angela concluded that the surface area of the cone increased as the height increased. She then created a 2-D image to represent surface area by drawing a circle with a wedge removed (Figure 4a). Angela wrote the formula $A = \pi r^2$, where r represents the radius of her circle (or, equivalently, the slant height represented by x). Although she understood that the formula would not produce accurate surface area values without further information or modification, she reasoned that the formula was correct in its general structure and drew a graph (Figure 4b) that she associated with that formula. She claimed, "I know that since this is r-squared that it's going to be, the surface area is going to be a quadratic…path of a parabola" After using a linear function to relate the radius of the circle (or, equivalently, the slant height) to the cone's height, she further justified her graph by noting that the quadratic nature of her modified formula (*A= 'some quadratic in h'*). She added, "I know what a parabola looks like and I know it's increasing at an increasing rate." Although Angela drew several diagrams to reason about an increasing surface area (see Figure 4a), she did not use these diagrams to describe *changes* in surface area for equal changes in height. Rather, Angela relied on the quadratic nature of the formula for surface area to make conclusions about the graph and relationship; she reasoned with a formula structure-covariational relationship association.



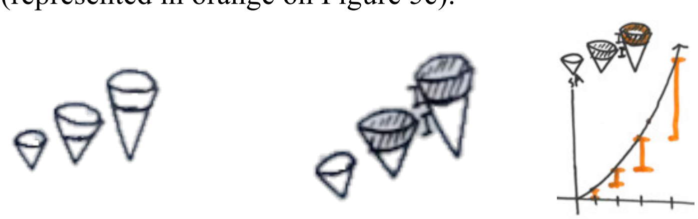
(a) (b)

Figure 4: Angela's Solution to the Cone Problem

Images of Covariation Determine Relationship

The students classified in this theme relied solely on their image of the situation to determine how the surface area and height of the cone covaried. The students maintained a 3-D image of the situation, and they were the only students to exclusively leverage a 3-D image of the situation to describe the rates of change of surface area and height of the cone. Specifically, Caroline and Trish imagined the surface area and height increasing continuously. Each student also imagined changes of surface area as successive strips for equal changes of height. The students used this image of the changes of surface area to compare successive changes in surface area.

As an example, after reading the prompt, Caroline drew three cones corresponding to equal changes in cone height (Figure 5a) and used these diagrams to determine how the quantities covaried. Caroline stated, "When you add some height, you add an extra strap around it [*re-draws 2nd cone bigger than 1st, draws strap with bottom of the strap at the height of 1st cone, seen in Figure 5b*]. Then you add some height, then you add a strap above that [*re-draws 3rd cone, bigger than 2nd with strap starting at height of top of 2nd cone seen in Figure 5b*]." She continued, "And if I were drawing these, to scale, this [*shades in strap in 3rd cone*] would have more surface area then this [*shades in strap in 2nd cone*]. So that means for equal changes in height [*marking changes of height in her diagram see in Figure 5b*], the change in surface area increases." Caroline used this reasoning to conclude that the surface area increases at an increasing rate with respect to height, and she produced a graph to reflect this relationship. Further, she identified how changes in surface area were represented on her graph (represented in orange on Figure 5c).



(a) (b) (c)

Figure 5: Caroline's Activity and Solution to the Cone Problem

Discussion

We identified themes in the students' activities that provide insight into ways these students modeled a dynamic situation. These themes are not intended to be evaluative or exhaustive; we do not claim preference of one theme over another, and there are many other possible solution approaches to this task. By comparing the students' activities across these themes, we note differences in how students' solution approaches are related to and influenced by their images of the situation. Three of the ten students (classified in the first theme) focused on a particular physical phenomenon of the situation (the constant or smooth growth of the cone), and they generalized properties of this phenomenon to the relationship between height and surface area. Five of the ten students relied on values produced from (second theme) or the attributes of (third theme) a formula to describe the relationship between the quantities. Although these five students leveraged an image of the situation that was attentive to the quantities of height and surface area, their use of this image was primarily static (e.g., using a fixed state to determine a rule). Only two of the ten students (fourth theme) used their images of the situation exclusively and continually reconstructed these images to reason covariationally.

We were surprised that only two of the ten students relied exclusively on leveraging images of the situation to represent amounts of change. Many students (e.g., Roz, Angela) engaged in activities that we believed had the potential to support them in reasoning emergently about the relationship (e.g., drawing diagrams and shading), but such activity was often assimilated in terms of previously made conclusions. This result was especially unexpected considering that in previous interview tasks several of the students classified in the first to third themes modeled relationships in dynamic situations by strictly leveraging images of the situation. One possible explanation for this result is that these students had learned the formula for the surface area of a cone in their prior school experiences. Hence, the students aimed to remember or to derive this formula rather than attempt to use images of the situation to construct the relationship.

Future Research

Several students' initial images of the situation relied on the video showing a constant change of height with respect to (implicit) time (first theme). Thus, we ponder how these students would engage in a task in which the cone's height grows at a non-constant rate or the students are able to control how the height varies. Additionally, several students persistently attempted to derive a formula for the relationship with their subsequent actions relying on their formula. Future researchers might be interested in comparing how students engage in situations that lend themselves to formulas and situations that do not. This would give insight into how students reason and rely on formulas and the consequences of such reliance, especially with respect to nuances in students' covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2015b).

Although some students maintained viable images of the quantities in the situation (from our perspective), they had difficulty reasoning about and comparing corresponding changes in the quantities. Moreover, even when we directed the students to consider how changes in the quantities might be identified in the context of the situation, some students identified what we perceived to be increasing amounts of change, yet argued that these amounts of change were equal. We envision that further investigation into students' images of change (Castillo-Garsow et al., 2013) would help explain these seemingly contradicting activities.

Acknowledgements

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PRODUCTIVE USE OF EXAMPLES FOR PROVING: WHAT MIGHT THIS LOOK LIKE?

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The study reported in this paper is part of a larger study on the roles of examples in learning to prove. We focus here on manifestations of students' productive use of examples for proving in the course of exploring conjectures and proving or disproving them. In this context, we define productive use of examples for proving as students' utterances that indicate that working with examples led them to realize and gain insights into some aspects of the key ideas for proving (or disproving) the conjecture. There were a total of 39 participants (12 middle school, 17 high school, and 10 undergraduate students). Each took part in an individual one-hour task-based interview. We identified 77 cases of productive use of examples, 41 based on an interviewer's provision of example(s) and 36 based on students' spontaneous generation of examples. These cases serve to characterize students' strengths that are not directly fostered in school.

Keywords: Reasoning and Proof

Proof and Proving in Mathematics Education

It is commonly agreed among mathematicians and mathematics educators that mathematical proof and proving are at the heart of mathematics, and that the activity of mathematically proving is dauntingly difficult even for most good undergraduate students. A continuing concern in mathematics education is that students do not sufficiently understand the nature of evidence and proof in mathematics and that they struggle with providing logically sound justifications and arguments to support the validity of mathematical conjectures or claims (e.g., Healy & Hoyles, 2000; Kloosterman & Lester, 2004; Knuth, Choppin, & Bieda, 2009). This concern has been guiding numerous studies, as it reflects a deficiency in one of the key elements of mathematics and mathematical practice (e.g., Harel & Sowder, 2007; Knuth, 2002; Sowder & Harel, 1998). Consequently, there have been calls for proof to play a more central role in mathematics education, by researchers (e.g., Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002), as well as reform initiatives (the *Common Core State Standards for Mathematics* and the NCTM *Principles and Standards for School Mathematics*). However, despite these calls, research continues to indicate that students' understanding of proof is far from being satisfactory (Harel & Sowder, 2007; Healy & Hoyles, 2000).

A major source underlying students' difficulties in understanding proof and proving is related to their treatment of examples (e.g., Healy & Hoyles, 2000; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012). There is evidence that an inhibiting factor in students' proving (at all levels) is an over reliance on examples. They often infer that a general claim is true for all cases on the basis of checking just a number of examples that satisfy this claim. This tendency has been recognized as a stumbling block in the transition from inductive to deductive arguments, and the progression from empirical justifications to proof (e.g., Fischbein, 1987). This tension between empirical and formal aspects of proving suggests that understanding the logical relations between examples and statements is a non-trivial task that is critical for proving. However, this kind of understanding is not usually explicitly addressed in the course of learning mathematics, in general, and learning to prove, in particular.

There have been attempts to help students learn the limitations of examples for proving in order to reduce their tendency to infer from examples more than is logically valid (e.g., Sowder & Harel, 1998; Stylianides & Stylianides, 2009; Zaslavsky et al, 2012). While these attempts address the limitations of examples for proving, they overlook the potential of example-based reasoning strategies in enhancing conjecturing and proving. In fact, less attention has been given to facilitating students' ability and inclination to build on the potential strengths of using examples for proving. More specifically, there is scarce research on using examples generically, i.e., in a way that allows to see the general through the particular, make sense of a mathematical statement, and gain insight into all or some of the main ideas of its proof (e.g., Knuth, Kalish, Ellis, Williams, & Felton, 2011; Leron & Zaslavsky, 2013; Mason & Pimm, 1984; Rowland, 2001). Mason and Pimm's (1984) terms of *generic example* and *generic proof* capture the essence of what we mean by using examples generically. Accordingly, "A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general." (ibid p. 287); and a "generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number." (ibid p. 284). Example-based reasoning strategies encompass this way of thinking with and through examples.

We believe that students' failure to engage productively in example-based reasoning strategies, to think about examples generically, and to analyze examples when engaging in activities related to proving, accounts for many of the difficulties they encounter in learning to prove. Our study stems from the stand that students should learn to use and analyze examples analytically and generically, not only in order to gain a better understanding of the conjectures (or statements) that they explore but also in order to learn to develop proofs (or dis-proofs) of these conjectures.

Very little research has focused on the nature of middle school, high school, or undergraduate mathematics students' thinking about and use of examples in generating, making sense of, and proving mathematical conjectures. Alcock and Inglis (2008) argue that such studies are needed in order to effectively develop instructional practices that foster the development of students' learning to prove. We aim at better understanding the nature of example use across grade levels, and in particular, how example use may support students' reasoning and proof development.

Zaslavsky (2014) distinguished between three settings of example use: spontaneous example use, evoked example production, and provisioning of examples. The spontaneous setting highlights what may come naturally to learners and experts, and how productive their choices and what they make of them are. The evoked example production allows us to study what choices learners make when pushed to use examples and also how productive they are. This setting has a strong diagnostic power, as it may evoke students' strengths as well as their weaknesses with respect to exemplification and proving. The provisioning of examples by a researcher allows us to examine what learners see in these examples, and in what ways they are able to build on the given examples to gain insights about how to justify or prove a claim. This setting also may shed light on possible mis-matches between intentions (of a teacher/researcher and a learner). In our study, we distinguish between example uses that involve student generated examples and those that involve researcher provided examples.

The Study

The study reported in this paper is part of a larger study of the roles of examples in learning to prove. Its purpose is to better understand the roles examples play in the development, exploration, and justification of mathematical conjectures, with the overarching goal being to help students appreciate the need to prove and to learn to prove. In this portion we focus on ways in which students use their own examples or examples provided by the researchers to support their claims about the validity of mathematical statements and conjectures. More specifically, we examine cases in which the use of examples can be considered productive for proving. For productive use of examples for proving we consider indications of gaining insights from an example or set of examples about the main idea(s) of a proof. It can be manifested, for instance, by a shift from no clue why a conjecture works (or doesn't work) to an articulation of an idea why. Some of the manifestations of this type of example-use may be seen as generic proving (Leron & Zaslavsky, 2013).

Data Collection

This study was based on individual task-based interviews with 12 middle school (MS) students, 17 high school students (HS), and 10 undergraduate (UG) mathematics majors. The interviews lasted approximately 1 hour and were comprised of a series of tasks in which participants were given the opportunity to conjecture and prove.

Task 2: The Sum of Consecutive Integers

Part 1:
This question involves consecutive numbers. For example, 2, 3, and 4 are consecutive numbers, but 2, 3, and 8 are not consecutive numbers.
Tyson came up with a conjecture about consecutive whole numbers that states: If you add any number of consecutive whole numbers together, the sum will be a multiple of however many numbers you added up. *At this point the interviewer suggests that the participant give an example of how the conjecture works for 5 consecutive whole numbers.*

Tyson thinks that this conjecture will always be true no matter how many consecutive numbers you use or which consecutive numbers you choose. So he thinks that if you add any 3 consecutive numbers, the answer will be a multiple of 3, or if you add any 6 consecutive numbers, the answer will be a multiple of 6, and so on.
Do you think the conjecture is true for any set of consecutive numbers, not just when you pick five consecutive numbers?

Part 2:
Let's come back to the Question 2 [i.e., Part 1 above] conjecture that the sum of five consecutive numbers is a multiple of 5.
At this point the interviewer says while writing the example: Another student had an idea of how to explain it. For the five consecutive numbers 5, 6, 7, 8, and 9, she decided to write the sum as (7-2)+(7-1)+7+(7+1)+(7+2), and writing it that way helped her to explain why the sum must be a multiple of 5. How do you think that helped her see why the conjecture is true for any five consecutive numbers?

The interview protocol for middle and high school participants included 8 tasks total each, and the one for undergraduate participants included 7 tasks total. Three tasks were shared across all participant populations. In this paper we focus mainly on one of these three shared tasks (Task 2 above): The Sum of Consecutive Integers Task. Similar versions of this task served researchers in other studies (e.g., Tabach et al., 2011). For several reasons (mainly due to time constraints, and protocols' modifications done after a number of interviews had been conducted), not all students got to engage in all the tasks that were included in the final interview protocols.

Data Analysis

While we started out looking for example-uses and focusing on whether each example-use was productive for proving, other categories emerged as we were analyzing the data, thus, in part, we used a grounded theory approach to analyze participant responses. The units of analysis were the tasks (except for Task 2, for which we coded each part separately). For each participant, we coded his or her performance on each task according to several categories (productivity, example-source, proof-exhibition). We began by identifying cases in which we observed productive and non-productive example-use for proving. For example, in task 2 part 1, if a participant was able to use many numerical examples to determine that Tyson's conjecture was true only for odd numbers of integers but was not able to produce a legitimate argument as for why this was the case, we considered this "non-productive" for proving (although this activity was clearly productive for conjecturing). In order to be categorized as "productive" for proving, a participant had to use examples to make an argument that showed not only *that* Tyson's conjecture was true only for odd numbers of integers, but *why* this conjecture holds or does not hold, based on the parity of the number of integers involved. Instances of productive example use included participants' use of examples to generate an argument, and were related to a shift in their ability to provide a valid justification that would hold for any other such case.

Additionally, we looked at the source of the example, and distinguished between cases in which an example was provided by the student (spontaneously) or by the interviewer (non-spontaneously). This distinction is important as cases in which productive example-use was based on provided examples, may have pedagogical implications in the classroom.

For each participant we coded his or her performance on each task as productive (P), non-productive (NP), or indecisive. There were two main reasons for considering a case indecisive: (i) if a student came up with a proof but it was unclear whether the examples were helpful in reaching the proof; or (ii) if it was not clear whether an argument qualified as proof. Altogether, 222 cases were analyzed, of which 24 were indecisive.

We also coded cases according to whether or not a proof was exhibited (even a partial or informal one), and whether or not examples were used. When examples were used, we distinguished between cases that included just examples generated by the students (Exp. by St.) and cases where examples were provided also or solely by the interviewer (Exp. by Int.). Note that for those who completed Task 2 Part 1 with a full proof that the conjecture holds for all odd numbers and does not hold for even numbers, and used the same reasoning as in the prompt for Part 2, did not receive the second part (to eliminate redundancy).

Findings

Scope of Productive Example-Use for Proving

The findings in Table 1 include all cases that were coded either as productive or as non-productive (excluding the 24 indecisive cases). There were a total of 198 cases, 62 MS, 89 HS, and 47 UG. Of the 198 cases, 77 (39%) included productive use of examples for proving. Of these, more than half (41) the cases were based on examples provided by the interviewer.

Table 1: Distribution of Cases by Example-Use and Productivity for Proving in all Tasks

Grade Level	**Came up with a Proof** (or partial proof)			**No Proof**			**Total**
	Productive (P) Use of Examples for Proving		No Exp. (NE)	**Non-Productive (NP) Use of Examples for Proving**		No Exp. (NE)	
	Exp. by St.	Exp. by Int.		Exp. by St.	Exp. by Int.		
MS	10	13	2	27	10	0	**62**

HS	11	18	2	37	17	4	**89**
UG	15	10	3	12	4	3	**47**
Total	**36**	**41**	**7**	**76**	**31**	**7**	**198**

In terms of productivity, middle school and high school students performed similarly, as 37% of MS cases and 33% of the HS cases exhibited productive use of examples for proving, while the undergraduate students exhibited considerably more productive use of examples (53%).

Table 2: Productive and Non-Productive Example-Use in Task 2 (the Sum of Consecutive Integers)

Grade Level	Task	Productive (P) Use of Examples for Proving		Non-Productive (NP) Use of Examples for Proving		Total
		Exp. by St.	Exp. by Int.	Exp. by St.	Exp. by Int.	
MS	**Part 1**	4	0	6	0	**10**
	Part 2	0	4	0	2	**6**
HS	**Part 1**	4	0	13	0	**17**
	Part 2	0	7	0	5	**12**
UG	**Part 1**	5	0	3	0	**8**
	Part 2	0	5	0	1	**6**
Total		**13**	**16**	**22**	**8**	**59**

In table 2 we present the findings related to Part 1 and Part 2 of Task 2. Looking at the task as a whole, for this task there are larger differences between the extent of productive use of examples for proving between the three groups: MS – 50% (8 of the 16 cases), HS – 38% (11 of the 29 cases) and UG – 71% (10 of the 14 cases).

Figure 1 examines the trajectory of students who completed both parts of Task 2, and for which none of their performances was indecisive (this reduced the total number of cases in Table 2 by 15).

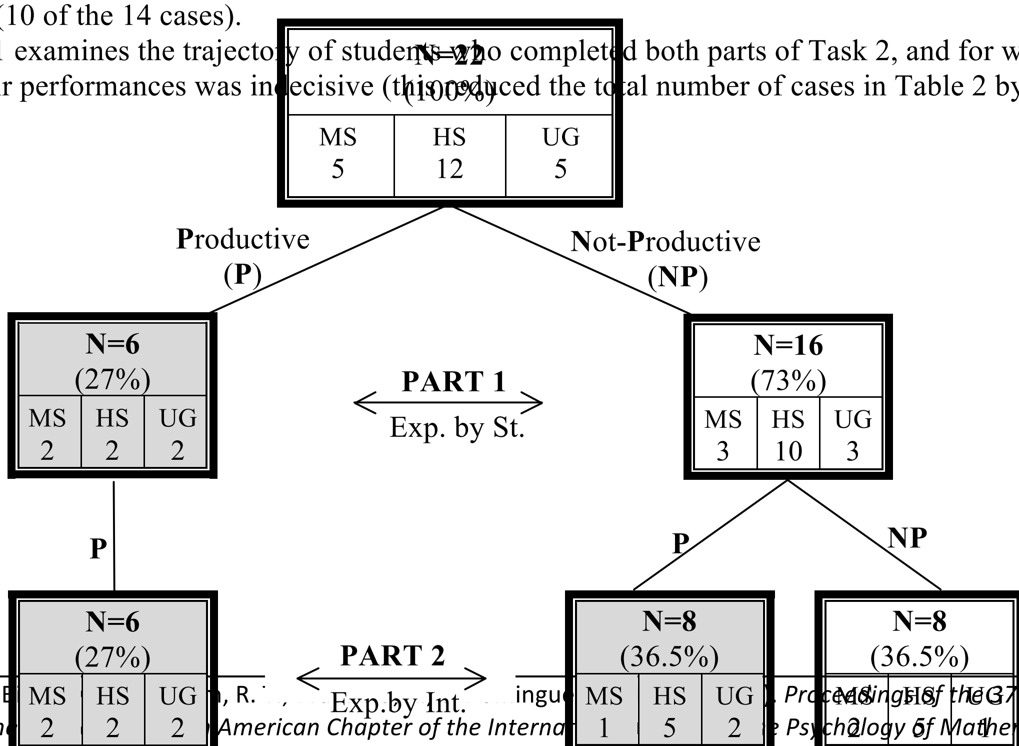

Figure 1: Paths of productive and non-productive uses of examples in Task 2
(Excluding cases that were indecisive or incomplete; N – no. of students)

All of the cases that dealt with Task 2 were cases in which examples were explicitly used. In Part 1, the examples were generated by the students, while in Part 2 – a generic example was provided by the interviewer. Not surprisingly, students who used examples productively in Part 1, on their own, were also able to use the generic example productively. However, interestingly, half of the students who were not able to use examples productively in Part 1, were able to reason productively with the generic example provided by the interviewer in Part 2.

Characteristics of Productive Use of Examples for Proving

We turn to two cases that convey what productive use of examples for proving may look like.

Case #1: A HS student's use of his own examples productively. In Part 1 of Task 2, Sam tries three sets of examples: 1 + 2 + 3 + 4 +5 ; 2 + 3 + 4 + 5 +6 ; 3 + 4 + 5 + 6 + 7. He calculates the last two sums, and immediately is able to make an argument in support of the truth of the conjecture for any five numbers with the following observation:

"I tried a few examples, and then I realized that, well, if you add 1 to every number, then you're ultimately adding 5, because there's 5 numbers. And if the first- and the first- and if the first example 1, 2, 3, 4, 5 is- equals a multiple of 5, then by adding 5 to- to every case, it'll stay a multiple of 5." [10:43 - Time Stamp]

In other words, Sam is able to present a pseudo-inductive argument that emerges from the observation that the conjecture is true for a base case (1+2 +3+4+5), and the mechanism by which this sum changes from this case to the "next" case (where "next" is defined as increasing each term in the sum by 1) does not change the divisibility property of the sum with respect to 5. By looking at the sequence of his three examples Sam is able to both see and utilize modular reasoning when considering the divisibility properties of this sum, as increasing a number by multiples of 5 does not change its remainder (in this case, zero) upon division by 5. As Sam puts it: "if you subtract 5 from a multiple of 5, it'll still stay a multiple of 5."[13:00]

Sam is able to take advantage of the generality of this observation by answering a question that he himself had posed earlier in the interview, namely, whether negative integers were allowed in Tyson's conjecture. He is able to leverage his reasoning about the modular distribution of integers that are divisible by 5 into a correct claim that Tyson's conjecture works just as well for negative integers as it does for positive. In other words, Sam was able to extend the domain of the conjecture by creating an argument that relied solely on three numerical examples. In this case we do not consider each one of his three examples in isolation as "generic," as that clearly does not reflect his thinking. However, we consider all three seen in conjunction with each other as one generic example, as the insight that Sam gained from these examples was located in the relationship between them.

Sam is able to use this insight to create a legitimate argument for why Tyson's overall conjecture is incorrect for four consecutive integers. He reasons: "I thought of a number like 4, which is 1, 2, 3, 4, which adds up to 10. And then it's not a multiple of 4, so, and… even if you add or subtract from that, it'll always be uh, it won't be a multiple of 4."[16:40] In effect, he argues that his base case is a counter-example, and this counter-example does not just hold for the particular example of 1 + 2 + 3+4, but in fact holds for *any* four consecutive integers. Although he does not explicitly discuss remainders, we can interpret his argument as noticing that the remainder upon division by 4 is invariant under increasing a number by multiples of 4.

Case #2: A HS student's use of an interviewer's example productively. In Part 1 Isaac is clearly operating empirically, as he chooses a wide range of examples and uses them for verifying *that* the conjecture is true (e.g., for 5 numbers), without being able to offer a logically valid explanation of *why* it is. He explains that it is because "I did a bunch of trials that go really far into the depths of numbers, including negatives which kind of sealed the deal for me, because negatives are really different from positives."[24:26]

During the second part of this task, the interviewer presented Isaac with a generic example, rewriting 5 + 6 + 7 + 8 + 9 as (7-2) + (7-1) + 7 + (7+1) + (7+2). Immediately, Isaac is able to see the generality within this particular example and apply it to a generic argument. Isaac uses this argument to explain why it must work for any odd number of consecutive integers, and also why it must not work for an even number of consecutive integers. He also produces a parallel algebraic representation, which is the first time that he has done so within this task. Perhaps this is due to the visual salience of the invariance of the 7's in the representation provided by the interviewer. Isaac's immediate response to the interviewer's prompt is reproduced below:

"Okay, so I can see now- this is pretty good proof for why it's, uh, for why it has to be a multiple of 5 or just a multiple of an odd number in general. Or, uh, so um, um the thing with this is, what happens-- ... Um, these numbers cancel each other out. The 2 cancels the 2, the 1 cancels the 1. And you just end up getting 7+7+7+7+7 and, um, and the reason this wouldn't work with an odd- with an odd pair, like if you added- if you added a, uh, 10 [The interviewee wrote +10 at the end of 5+6+7+8+9] to this and then you- and then you added plus 7 plus 3 [The interviewee wrote +(7+3) at the end of the (7-2)+(7-1)+7+(7+1)+(7+2)], then these would cancel out. Then you would be left with (7×5) + (7+3) plus... Yeah, plus (7+3), which would just give you, um, 3- 3 numbers off of what you want. So, yeah. And I guess you could do it with an equation, using *x*... (x-2) + (x-1) + (x) + (x+1) + (x+2) + (x+3). These cancel each other out, these do it as well, and this is just left there as a kind of like, almost like it ruins the party or something. So, yeah."

He later uses the word "symmetry" to explain this argument. In other words, Isaac was initially "stuck" and the generic example provided by the interviewer helped Isaac create a deductive argument as for why Tyson's conjecture was only true for odd numbers of integers.

Concluding Remarks

While the vast majority of studies on students learning to prove focus on their difficulties and suggest ways to address these difficulties, our study identifies numerous cases of students treating examples generically on their own. Moreover, these cases capture shifts from not being able to explain why a mathematical statement is true (or false) to being able to see clearly why it must work in general. Thus, the findings add to our understanding of processes by which students may learn to prove, and at the same time suggest that for some students under certain conditions this may come (almost) naturally with minimal interference. In other words, these cases could be inspiring for teachers who want to build on students' strengths.

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STUDENT CONCEPTIONS OF WHAT IT MEANS TO BASE A PROOF ON AN INFORMAL ARGUMENT

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In this paper we explore how students construe what it means for an informal argument to be the basis of a formal proof and what students pay attention to when assessing whether a proof is based on an informal argument. The data point to some undergraduate mathematics students having underdeveloped conceptions of what it means for a proof to be based on an argument. These underdeveloped conceptions limit what students pay attention to during informal-to-formal comparison tasks and may have adverse effects on students' ability to use their own informal arguments to construct proofs.

Keywords: Post-Secondary Education; Reasoning and Proof

Introduction

The constructs we use throughout this paper revolve around the observation that proofs are expected to be written in a verbal-symbolic representation system but may not be generated wholly within that system. Following Weber and Alcock (2009) we refer to reasoning that stays solely within this system as syntactic reasoning, and reasoning that falls outside of it as semantic reasoning. Similarly, we conceptualize a formal proof as a deductive argument that establishes the result to be proven and conforms to the norms of the representation system of proof. This is a characterization of the end product, not the reasoning that led to it. Additionally, we conceptualize an informal argument as a deductive argument that establishes the result to be proven, but does not conform to the norms of the representation system of proof. We refer to the use of informal arguments to inform the construction of formal proofs, or more generally the process of using semantic reasoning to inform syntactic reasoning, as formalization.

Research relevant to formalization can be partitioned into two non-disjoint categories. The first category focuses on the role of semantic reasoning in informing proof productions. The second category examines the semantic-to-syntactic formalization process needed to use semantic reasoning to generate a formal proof.

Research falling into the first category has illustrated the important role that semantic reasoning can play in proof generation. Various types of semantic reasoning have been shown to inform proof generation (Gibson, 1998, Sandefur, Mason, Stylianides, & Watson, 2013, Zazkis, Weber, & Mejia-Ramos, in press). This first body of work has perpetuated the recommendation that students should use semantic reasoning during proof construction. This recommendation has gained considerable traction in mathematics education (e.g., Garuti, Boero, & Lamut, 1998; Raman, 2003).

A second set of studies has focused on the formalization process itself. This research has provided evidence that mathematics majors struggle to use semantic reasoning to inform proof generation (Selden & Selden, 1995; Alcock & Weber, 2010; Zazkis et. al., in press).

These two sets of studies point to a discrepancy between researcher recommendations (that students should generate proofs using informal arguments) and students' behavior and abilities. In order to better understand this discrepancy we examine what mathematics majors pay attention to when attempting to determine if a formal proof is based on an informal argument and how these determinations compare to normatively correct interpretations.

In order to operationalize these notions we consider a formal proof to be *based on* an informal argument if there is a mapping between these two chains of inferences that has two properties: (1) it is meaning preserving to the extent allowed for by the rules of the verbal-symbolic system, and (2) corresponding inferences (or chains of inferences) appear in the same order. We use the acronym FBI-judgment, to refer to student judgments of whether Formal proofs are Based on Informal arguments.

The Study

We are interested in what it means for a proof to be based on an informal argument from a mathematics major's perspective. Thus we create a model of what mathematics majors pay attention to when making FBI-judgments and use this model to create a plausible explanation for why these students may have difficulty with formalizing informal arguments. In particular, we want to illustrate how prioritizing a particular subset of attributes when making informal to formal comparisons influences how students view what it means for a proof to be based on an argument and by extension, affects their ability to formalize.

Participants

Participants were pursuing undergraduate degrees in mathematics at a large state university in the Northeastern United States and had completed a proof based second course in linear algebra, an introduction to proof course, and an introductory analysis course. Thus, the eight participants were familiar with reading and writing proofs in a variety of mathematical contexts. Participants were selected to have roughly equal amounts of A, B, and C grades represented.

Procedure

The second author conducted one-on-one clinical interviews with each participant that lasted between 90 and120 minutes. This involved presenting participants with both informal arguments and proofs and engaged them in a series of comparison tasks. More specifically, he presented participants with triples that consisted of one informal argument and two formal correct proofs, only one of which, from our perspective, was based on the argument.

The informal argument in each triple was presented in the form of a video. Each video lasted approximately 30 seconds and involved the first author justifying the result with a combination of verbal argumentation, graph generation and gestures. The two correct formal proofs in each triple were presented in written form.

At the beginning of each one-on-one interview participants were told that they would be shown triples. They were told that they were to understand and compare the three parts of the triples, but that they did not need to validate correctness. In the first round of the interview participants were shown each of the three triples one at a time. They watched the video and read the two proofs out loud. After participants verified that they felt they understood the result, they were asked to make side-by-side comparisons of what they noticed in terms of similarities and differences between the three parts of the triple. This was done with each of the three triples. Note that during the first stage only the participants' impressions and what they noticed was elicited.

The second stage involved revisiting each triple and judging whether each of the proofs in the triple was based on its informal argument, how confident they were in their assessment and on what information they based their conclusion. Students were not informed that, from the researchers point of view only one of the proofs in each triple was a formalization of the informal argument. This made it possible for participants to conclude that neither or both of the proofs in each triple were based on the informal argument.

Analysis

A set of minimum criteria, which was agreed upon prior to the interviews, was used to determine whether student FBI-judgments were consistent with mathematical norms. We were also interested in what students paid attention to when making FBI-judgments. Each interview was transcribed and grounded theory (Strauss & Corbin, 1990) was used to categorize interviewee responses in terms of what they paid attention to during FBI-judgments.

Materials

We briefly mention that Task 1 involved proofs that $\int_{-a}^{a} \sin^3(x)\,dx = 0$, for all real numbers a, and Task 3 involved proofs that the derivative of a differentiable even function is odd. For space reasons we discuss only Task 2 in detail. To clarify our discussion each step in the informal arguments is labeled with an "I2," each step in the formalization of this argument is labeled with an "F2," and each step in the distractor proof is labeled with a "D2."

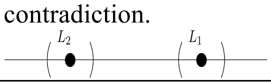
<table>
<tr><th colspan="2">If a sequence (a_n) has a limit, it is unique.</th></tr>
<tr><td>Informal argument</td><td>Proof F_2 (Formalization of informal argument)</td></tr>
<tr><td rowspan="3">(I_2-1) If the limit wasn't unique then we could have two limits, say L_1 and L_2.
(I_2-2) But we control the size of the ε-neighborhood around these.
(I_2-3) and if we make it small enough these two ε-neighborhoods are going to not overlap.
(I_2-4) So when we get far enough down the sequence we're going to be both in here [pointing an neighbor hood around L_1] and in here [pointing an neighbor hood around L_2].
(I_3-5) But since we can't be in two places at the once we get a contradiction.
</td><td></td></tr>
<tr><td>Proof D_2 (Distractor)
(F_2-1) Toward a contradiction assume the sequence (a_n) does not have a unique limit.
(F_2-2) Then there exists L_1 and L_2 such that $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} a_n = L_2$ and $L_1 > L_2$. $\Big\}$ (I_2-1)
(F_2-3) Let $\epsilon = \frac{L_1-L_2}{2} > 0$. By definition there exists an N such that if $n > N$ then $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$.
(F_2-4) $|a_n - L_1| < \epsilon \Rightarrow a_n > L_1 - \epsilon = \frac{L_1+L_2}{2}$
$|a_n - L_2| < \epsilon \Rightarrow a_n < L_2 + \epsilon = \frac{L_1+L_2}{2}$ $\Bigg\}$ $(I_2-2/3)$
(F_2-5) But then for $n > N$ we have that $a_n > \frac{L_1+L_2}{2}$ and that $a_n < \frac{L_1+L_2}{2}$ which is impossible. $\}$ $(I_2-4/5)$</td></tr>
<tr><td>(D_2-1) Toward a contradiction assume the sequence (a_n) does not have a unique limit.
(D_2-2) Then there exists L_1 and L_2 such that $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} a_n = L_2$ and $L_1 \neq L_2$.
(D_2-3) By definition for every $\epsilon > 0$ there exists a N such that if $n > N$ then $|L_1 - a_n| < \epsilon/2$ and $|L_2 - a_n| < \epsilon/2$.
(D_2-4) Consider $n > N$.
$|L_1 - L_2| = |L_1 - a_n + a_n - L_2|$
(D_2-5) $\leq |L_1 - a_n| + |a_n - L_2|$ via the triangle inequality
$= |L_1 - a_n| + |L_2 - a_n|$
(D_2-6) $< \epsilon/2 + \epsilon/2$
$= \epsilon$
(D_2-7) So $|L_1 - L_2| < \epsilon$. Since ϵ can be made arbitrarily small $L_1 = L_2$, contradicting the assumption that the limit was not unique.</td></tr>
</table>

Figure 1: If a sequence (a_n) has a limit, it is unique.

Unlike F_2, D_2 is not intrinsically an argument by contradiction. The contradiction was artificially added to proof D_2 to make D_2 and F_2 superficially similar. F_2 starts off assuming, toward a contradiction, that the limit is not unique (I_2-1 and F_2-1) and because of this we may choose two limits, L_1 and L_2 (I_2-1 and F_2-2). Although the assumption that $L_1 > L_2$ (F_2-2) does not explicitly appear in the informal argument, it is implied by the accompanying diagram. Next I_2-2 and I_2-3 argue that the ε-neighborhood around L_1 and L_2 can be made small enough to not overlap. In proof F_2, this "not overlapping," is achieved by choosing ε to be exactly half the distance between L_1 and L_2 (F_2-3), and then showing this choice of ε places a_n both above and below the midpoint (F_2-4) for all n sufficiently large. Finally, both Proof F_2 and the informal argument end by arguing that being in two places at once leads to a contradiction (F_2-5 and I_2-4/5). In the proof this is done formally by arguing that a term in the sequence, a_n, cannot be both above and below the mid point. Proof D_2 demonstrates that any two limits of a sequence can be made arbitrarily close to each other, and thus must be equal, but does not rely on the "two places at once" idea.

Results

A top-level view of the connections the participants made relative to our pre-agreed standards and hence, from our perspective, made normatively correct FBI-judgments for normatively correct reasons can be found in Table 1. As can be seen from the table, the first task was relatively unproblematic for the students in this study. The other two tasks were more difficult. Only 2 of the eight students met our minimum standard on both parts of task 2 and none of our students met our minimum standard on both parts of task 3.

These data point to mathematics majors' difficulties with FBI-comparison tasks. The ability to recognize the final product of formalization is crucial. Without this a student cannot recognize when the formalization process is complete and thus cannot effectively formalize.

Table 1: Student FBI-judgments relative to the minimum standard

	Task 1	Task 2	Task 3
Met criteria for D-I judgment	7/8	5/8	2/8
Met criteria for F-I judgment	8/8	3/8	1/8
Met both F-I and D-I criteria	7/8	2/8	0/8

A model of what students pay attention to when making FBI-judgments

In our analysis we identified four different aspects of arguments/proofs that students focused on when making FBI-comparisons. Two of the foci outlined below are adaptations of Pedemonte's (2007) structural distance and content distance constructs. The reframing of these constructs was necessary because the focus of Pedemonte's research is different from our own. The four foci of comparison are described below:

(1) Structural foci involve noticing global similarities and differences in which inferences follow from one another (i.e., structure). We conceptualize this as an adaptation of Pedemonte's (2007) notion of "structural distance" to a FBI-comparison context. In a broad sense structural foci can be seen as an attempt to evaluate what kind of relationship exists between the structure of an informal argument and the structure of a formal proof.

(2) Content foci involve noticing which specific elements (i.e., inferences, assumptions, data and claims) are present or not present within both an argument and proof. This can be seen as an attempt to evaluate the relationship between the content of an informal argument and the content of a formal proof. This focus can be seen as an adaptation of Pedemonte's (2007) notion of "content distance" to an FBI-comparison context.

(3) Methodological foci involve noticing the proof method used (e.g., contradiction, contrapositive, induction, construction, etc.) as well as the role this method plays in the proof.

(4) Holistic foci involve noticing similarities and differences in terms of goals, style purpose or overarching idea. These comparisons focus on proofs and arguments as a whole and overlook specific structural, content and methodological details.

Here we show that prioritizing one of the four foci of comparison in lieu of others has detrimental effects on students' ability to make informal to formal comparisons. It is important to note that these foci are not necessarily static. Some students may shift foci when moving to a different task.

Content foci. Next we discuss content foci. This involves paying attention to the assumptions and inferences within a proof/argument, but largely overlooking the roles and structure of these elements within a proof. Here the focus is localized to specific steps in the proof. In other words, the details of a proof are examined, but the bigger picture is ignored. We illustrate this focus with an excerpt of S8's work on Task 2. In the excerpt below S8 is asked to compare D_2 and F_2:

Int: Any… any other differences you can see? Looking at the proofs again?
S8: Umm… L_1 is defined to just not be equal to L_2 in proof D_3 and in proof F_3, they say L_1 is greater than L_2.
Int: I, I guess, … do you consider those significant differences? The ones that you mentioned?
S8: Yea, yea definitely. Those are significant differences.

The assumption that $L_1>L_2$ is not explicitly made in the I_2, however, it is implied by the accompanying diagram. Following the above excerpt S8 proceeded to use the fact that the $L_1>L_2$ assumption is part of proof D_2 and not F_2 as a justification for why the distractor proof is based on the informal argument while the formalization proof is not. In focusing in on a particular piece of content which is present in only one of the proofs he overlooks the bigger picture and consequently, makes an FBI-judgment that is in conflict with the normatively correct interpretation. Hence, we contend that prioritizing content in leu of structure is also insufficient for making normatively correct FBI-judgments.

S8's assessment is consistent with his content foci. He is looking for specific inferences that are present in the proofs in order to compare them to the informal argument. Thus, his expressed de facto conception of what it means for a formal proof to be based on an informal argument involves the formal proof using similar assumptions and similar inferences to the informal argument. Within this conception, a difference in assumptions used is sufficient evidence that a proof is not based on an informal argument.

Methodological foci. It is useful to note that our anticipation of methodological foci influenced our task design. Methodological foci were the motivation for making proof D_2 artificially a proof by contradiction. If we had not artificially made D_2 a proof by contradiction students may have concluded that D_2 was not based on I_2 solely based on the fact that I2 is a proof by contradiction without working to make other connections. Thus our task design intentionally discouraged surface level methodological foci. However, one participant did notice this feature of D2:

S7: Wait, what? … There is no point in this [D_2] being a proof by contradiction. That is completely redundant I could have just crossed this out here, "Assume it does not have a unique limit." You can cross that out.

Our task design intentionally prevented superficial methodology based assessments but left the tasks open to deeper assessments like the one in the excerpt above. Since only one of the participants noticed this feature of D_2 , we argue that artificially adding or removing particular methodologies from proofs has the potential to lead students to make incorrect FBI-judgments. That is, if we used a direct proof as a second distractor in place of F_2, we anticipate that the majority of students in our study would have incorrectly used "only one of these is a proof by contradiction" as a justification for why D_2 was based on I_2. This highlights the limitations of a strictly methodological foci.

Holistic foci. The final foci we discuss involves examination of holistic traits. The word trait here is construed broadly and may include attribute such as elegance, efficiency, style, pedagogical purpose or overarching idea. In short, this is intended to capture any treatment of a proof as more than the sum of its parts. Proofs have purposes and can be qualitatively compared to both each other and to the general genre of proof writing.

First we begin by discussing the work of S1 on task 1. The excerpt below begins after S1 reads proof D_1 (He has already read proof F_1).

S1: I feel like D_1 was kind of lamer than the other one
Int: Lamer?
S1: This one [F_1] was a little prettier, it was… I mean over here, we had uh, we were using that it was… this [D_1] felt very… brute force

S1 treats the two proofs in task 1 as aesthetic entities and does not solely focus on the internal (line-by-line) workings of the proofs. He expresses the belief that elegant proofs are more desirable than brute force proofs and judges D_1 as less desirable. Later in his interview, when he was asked to compare F_1 and D_1, he discussed the two proofs relative to the genre of proof as a whole.

S1: Okay, what do they have in common? Clearly they have the goal in common, but the guy on the left, proof D_1, proof D_1 felt more like uh… I don't really know if there's an actual distinction in the math world between a "prove something" and a "show something," but if there was, this [D_1] definitely feels like, you know, just show that it's 0. But this [F_1] was like a really… this felt like it had more behind it here… whereas this [D_1] was like, let's just evaluate it and see where that takes us. Okay? Which is fair, you know? It just doesn't give you any insight into why that's the case.

S1 expresses the belief that proof D_1, does not provide any mathematical insight regarding why the result holds. It is simply an exercise in implementing well-established calculation techniques. Implicitly, he expresses that he often looks for what insights he can gained from presented proofs, in this case he did not find any.

However, one cannot effectively make FBI-judgments by focusing on holistic attributes of a proof in lieu of other attributes. For example, there may be multiple elegant arguments that justify a result. Thus, elegance alone is insufficient for making comparisons.

Multi-focus comparisons. In the previous subsections we argued that prioritizing only one of the foci of comparison in lieu of others was insufficient for making normatively correct based on judgments. In this section we illustrate what comparisons that utilize all four foci look like and how they may yield normatively correct FBI-judgments. To clarify we are arguing that balancing ones attention to these foci greatly increases the likelihood that a student consistently generates correct FBI-judgments but does not guarantee normatively correct judgments.

Below we examine S7's immediate reaction after reading proof F_2 for the first time:

Int: General impressions.
S7: F2 is just literally the proof version of I2…. Uh, so the idea behind this is that, okay if we are trying to show that this sequence has a unique limit, which we want to show that it can't have two limits. So we suppose there is two limits, basic proof by contradiction. So both are proofs by contradiction. And the contradiction occurs when epsilon is small enough. Here they show it intuitively but it's pretty clear from the picture that what they use was a number that's less then half way in between. Here [F_2] is that function, the average. So once we have the. It's not the average it's close enough so that it doesn't even reach the average. And that way the two have no overlap. And then by definition of sequence it should eventually get far enough that it's in this region always and once you get far enough it's in this region always but then it's therefore always in both these regions once it passes that specific end that we defined. And that's the contradiction. Which is what they said here. When we get small enough down we can't be in both but it has to be in both.

It is important to emphasize that S7 realizes that F_2 is based on I_2 before he is asked to make any kind of comparison. The above is simply his initial response. The part of the interview where he will be specifically asked FBI-questions occurred 30 minutes later. Also, he immediately jumps into the comparison when he states that both the proof and the informal argument have the same idea behind them (holistic foci). He then shifts to discussing how this idea manifests itself in terms of structure of both I_2 and F_2 (structural foci). He notes that both the proof and informal argument are necessarily arguments by contradiction with the contradiction in both cases being that you cannot be in two places at once (methodological foci). This is then related to the specifics of the proof, with being both above and below the midpoint of L_1 and L_2 corresponding to being in two places at once in the informal argument (content foci). S7 makes all four types of comparisons, does this without any specific prompting to make a comparison and relates the four types of comparison foci in his discussion.

We believe that the fact that S7 saw the relationship between I_2 and F_2 before he was asked to compare them to be particularly important. Mathematics is often discussed metaphorically as a language. Here S7 recognized that the informal argument and proof were metaphorically telling the same story. This is akin to being shown two paragraphs that tell the same story in two different languages, both of which one is fluent in. The fact that the same story has been presented twice, as well as the multitude of parallels between its two presentations is salient even without being asked to compare the two paragraphs.

On the other hand, if one is learning the second language and is asked the same question the comparison is very different. The comparison becomes an exercise in finding parallels between the words and phrases used, as well as, the order in which these appear. In this case one is likely to grasp onto only a fraction of the similarities and differences between the two paragraphs and make a determination based on only this subset. This is analogous to what we observed students doing when they prioritized one of the foci over others.

Discussion

This paper contributes to the literature on proof and proving both methodologically and theoretically. First, from the perspective of theory, this paper introduced a four-part model of the aspects of arguments/proofs students focus on when attempting to determine whether a particular argument is based on a particular proof. The components of this model are content foci, structural foci, methodological foci and holistic foci. We illustrated that comparisons where students prioritized one of these categories of comparison in lieu of others were prone to incorrect or incomplete conclusions regarding whether a proof was based on an informal argument. Furthermore failure to see the rich connections between informal arguments and proofs point to students having under developed conceptions of what if means for a proof to be based on an informal argument. These underdeveloped conceptions account for difficulties students have with generating proofs based on informal arguments (e.g., Zazkis et. al., in press) and in understanding the connections between informal arguments and proofs presented in lecture (e.g., Lew et al., 2014). These underdeveloped conceptions also account for some of students' resistance to generating informal arguments during proof production.

Methodologically, the triples method introduced in the study is a valuable research tool for those interested in research on the connections between informal arguments and formal proofs. Examining how students compare and contrast ready-made informal arguments and formal proofs provides valuable insights regarding what they notice when making FBI-judgments. In turn, what students' notice during FBI-judgments provides a valuable lens into how they conceptualize formalization and how they might view formalizing their own informal arguments. This method was able to reveal that students' conceptions of what it means for a proof to be the basis of an informal argument are not as rich as an expert conception—often only encompassing a fraction of the connections that exist between informal arguments and formal proofs.

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THEORIZATION OF INSTRUMENT DESIGN IN RESEARCH ON MATHEMATICAL PROBLEM SOLVING

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In this paper we report the interview questions used in a study of middle school students' mathematical problem solving behaviors which were chosen based on Vygotsky's concept formation theory and Berger's appropriation theory. We discuss the task design/selection process along with the findings associated with the use of these tasks so to provide new direction for gauging research on mathematical problem solving.

Keywords: Research Methods; Problem Solving; Geometry

Introduction

Improving the teaching and learning of mathematical problem solving relies heavily on development of a theory of mathematical problem solving, which is currently missing from the field (Schoenfeld, 2007). As Schoenfeld articulated, the focus of mathematics education researchers interested in mathematical cognition may need to shift on building theoretical capacity that may account for human decision making in the course of problem solving (2013). We posit that progress towards building such a theory might demand greater attention to theorizing instrument design and task development in conducting research on mathematical problem solving, an area currently absent from the field. Discussions surrounding tasks have frequently focused on defining problems: a question where an individual does not have a ready- to-use approach to find the answer (Wilson, Fernandez, & Hadaway, 1993). Agreement exists that whether a question is a problem depends on the individual working on the task (Schoenfeld, 1985). This description has imposed constraints on researchers' ability to theorize specific principles of instrument design/selection when undertaking research on mathematical problem solving. Existing studies have generally selected questions based on the targeted subject area and whether the question is appropriate for the participants in the study, referring to the participants' educational background as a standard to determine whether a question is beyond their capability or not (e.g. Elia et al., 2009; Kuzle, 2011). These efforts, although useful in providing a profile of expert problem solvers, do not provide a coherent perspective on instrumentation as a methodology.

In a larger study of middle school students' mathematical problem solving behaviors, we used research-based assessment to capture students' ways of knowing by unpacking the relationship among mathematical concepts, cognitive behaviors, and metacognitive behaviors as evidenced during clinical interviews. We reported the findings from the pretest instrument in the study, which suggested that Vygotsky's concept formation theory could serve as an effective framework for designing novel assessments to provide researchers with more precise tools to articulate intricacies of students' understanding of mathematical concepts (Zhang, Manouchehri, & Tague, 2013). In this paper we will report on theoretically based criteria for design of tasks to be used in our research on mathematical problem solving. We will report findings related middle school students' performance on a selected sample of these tasks so to provide direction for future research on mathematical problem solving.

Theoretical framework

As a starting point in our task design, two issues were of particular concern: (1) establishing theoretical capacity that would allow us to document and analyze both cognitive and metacognitive behaviors of the participants in the course of their problem solving; (2) selecting a specific mathematical concept so to develop appropriate tasks surrounding it. To address the first issue we incorporated two theoretical perspectives to design/select interview questions used in this study: Vygotsky's (1962) concept formation theory and Berger's (2004) appropriation theory. Vygotsky's theory proposes a framework for an individual's concept development within a social environment, while Berger's theory proposes an interpretation of Vygotsky's theory in the domain of mathematics by adjusting certain stages. Both theories break down any concept development into three phases: heap, complex, and concept. In the heap phase, the learner associates a sign with another because of physical context or circumstance instead of any inherent or mathematical property of the signs. In the complex phase, objects are united in an individual's mind not only by his or her impressions, but also by concrete and factual bonds between them. In the concept phase, the bonds between objects are abstract and logical. For the purpose of the study we also decided to focus on the concept of Area due to its critical role in school mathematics. Hence, research-based formation stages for the concept of area were assembled as illustrated in Figure 1.

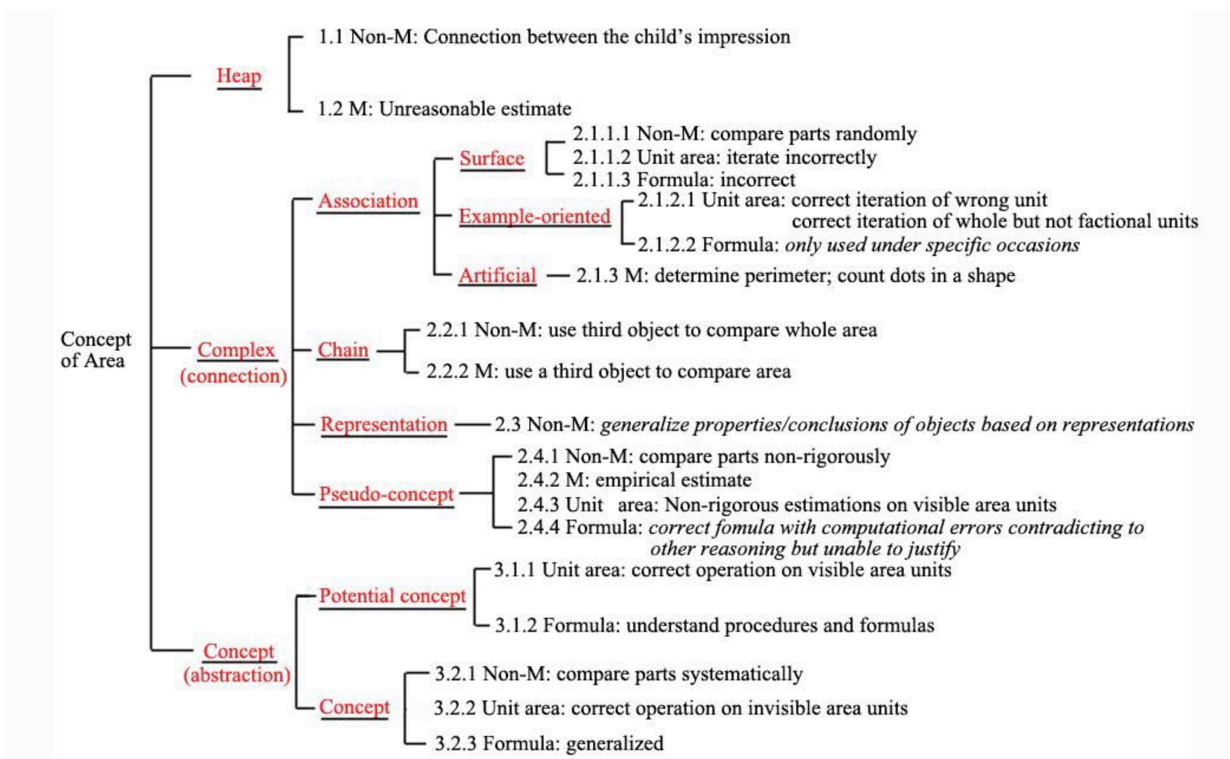


Figure 1. Developmental Stages of the Concept of Area

This structural model served as our primary analytical tool for qualifying the students' work as well as the tasks selected to be used in study. This model was further refined upon a short pilot study in which responses to tasks were obtained from 44 middle school students. The students were asked to responds to five items. Upon analysis of their responses we considered revisions of some of the tasks so to assure ambiguities that could lead to irrelevant answers were removed. Analysis of the relationship among mathematical concepts, cognitive and metacognitive behaviors emerged from the interview results based on our final selection of items which were used in in-depth interviews with five middle school students. The current report is based on our findings of these interviews.

Methods

Participants

Five individuals from a population of 44 sixth grade students were selected to participate in interviews. The original group of 44 were enrolled in three distinct class periods of an algebra course taught by the same teacher. All three classes had been observed by the lead author for 6 months prior to data collection. A pretest was administered to the 44 students. The process of determining and classifying the pretest responses is described as the following. 1) Two authors independently reviewed all responses to identify and document enacted approaches and coded developmental stages associated with each approach based on an earlier version of the developmental stages framework. Notes were compared for consistency in coding. 2) Students' approaches that were ambiguous or non-anticipated were discussed. The framework was adjusted based on the analysis of these responses; five more stages were identified and added to the original framework. 3) Based on the analysis we built a reference list of detected stages for each student to inform participant selection. Each individual's developmental status revealed in the responses was categorized as "overall low" (all responses were rated as Heap and non-Pseudo- concept Complex stages), "varied" (responses were rated across Heap to Concept stages), and "overall high" (Reponses were rated as Pseudo-concept Complex and Concept stages).

Participant selection was deliberate. The following criteria guided our choices: 1) willingness to be involved in the study and had signed the consent forms; 2) the participants would need to represent a range of different developmental attributes pertaining to the target concept. This data was collected through the students' pretest responses; 3) the participants would need to be comfortable with thinking aloud. Five individuals met these criteria: Shana exhibited a *low* status, Andy exhibited a *high* status, Sandy, Allen, and Ivan exhibited *varied* status.

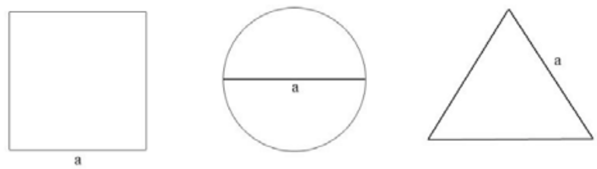
Data collection

The five participants were interviewed individually. Each interview consisted of two parts: During the *background interview* part the participants' mathematics background information, their beliefs about mathematics, and their views on the value of mathematics for their lives were elicited. The second part, *problem solving session* the participants worked on specific mathematical tasks. During the problem solving sessions, interviewer interventions were limited to eliciting clarifications, explanations, or justifications, when needed.

Instrument design

F ive problems were used during the interviews. All problems were related to the concept of area and allowed the participants to tackle the tasks from different stages of concept development. The problems could potentially cover a wide range of concept stages. Peripheral concepts were integrated in several problems in order to enable some degree of interactions between different concepts (e.g. transformational reasoning and variable), since we believe authentic problem solving should involve more than one concept yet the number of concepts needs to be controlled for a deeper analysis on the behaviors acting upon them. When selecting tasks, concept stages along with the corresponding exemplar approaches that could be elicited by each problem were predicted by the authors. An example of this conceptual mapping is shown in Table 1.

Table 1. Predicted stages and exemplar approaches

<table>
<tr><td colspan="2">Compare areas problem
Which of the regions shown below has the largest area? How would you order them?
</td></tr>
<tr><td>Predicted stages</td><td>Exemplar approach</td></tr>
<tr><td>Surface Association Complex – Formula</td><td>Using an incorrect area formula</td></tr>
<tr><td>Chain Complex</td><td>Fitting the circle and the triangle into the square</td></tr>
<tr><td>Potential Concept - Formula</td><td>Using a correct area formula</td></tr>
</table>

Data analysis

Data collection consisted of four phases. First, each participant's key cognitive behaviors during each problem solving episode were mapped and documents. Second, a summary of observed concept stages and metacognitive behaviors during the episode were catalogued and noted. Detailed analysis of the individual's problem solving behaviors according to the relationship among concepts, cognitive behaviors, and metacognitive behaviors was constructed. Finally, cross analysis of the observed concept stages, metacognitive behaviors, and the relationship between them concluded the analysis phase. This process was followed for each of the five tasks used.

Results

The larger study analyzed the participants' performances on each interview question from four perspectives: 1) each participant's point of entry, including identified task elements/objectives and his/her initial approach, 2) types of approaches the study participants used, 3) concept stages revealed when working on the problem, and 4) metacognitive behaviors revealed from the participants during the problem solving episode.

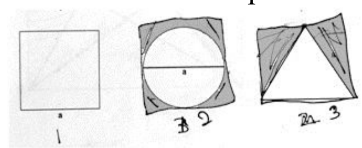
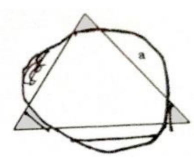
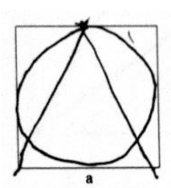
To focus on the research question in this paper, we only report on: 1) each participant's point of entry, including identified task elements/objectives and his/her initial approach, 2) types of approaches the participants used, and 3) concept stages revealed. The problem in Table 1, Compare Areas problem serves as an illustrative example of research-based tasks used in the larger study reported elsewhere (Zhang, 2014).

Point of entry

In the Compare Areas problem, one task element and one task objective are essential to solving this problem: the variable a that represents the equal length of sides and diameter, and the comparison of the areas, respectively.

Table 2 summarizes the task elements/objectives (i.e. key conditions and goals to solve the question) identified by each participant and by the interviewer before each of the participants started solving the problem. Initial approaches adopted by each participant are also outlined.

Table 2. Summarization of initial response/approach adopted by each participant

Participant	Task elements or objectives identified by the participant	Task elements or objectives highlighted by the interviewer	Initial approach
Shana	Equal measure of sides and diameter	NA	The circle and the triangle could fit into the square. 
Sandy	Just area	The *a* meant equal measure of sides and diameter	Computed the areas by formulas.
Ivan	NA	NA	The circle could fit into the square (verbal conclusion); the triangle had the same area as the circle since the three vertices of the triangle “pokes out.” 
Andy	Around is the circumference.	The *a* meant equal measure of sides and diameter	Computed the areas of the square and the triangle by formulas.
Allen	The area inside	NA	The circle and the triangle could fit into the square. 

Among the five participants, Sandy’s understanding of variable greatly influenced her initial behaviors. She focused on the actual amount of area instead of the condition that the lengths of the sides and the diameter had the same measures. This prevented her from adopting either a visual or a numerical approach to start tackling the problem. Allen, whose understanding of variable was restricted, chose to ignore it so to avoid confusion. Ivan overlooked the variable; his later behaviors

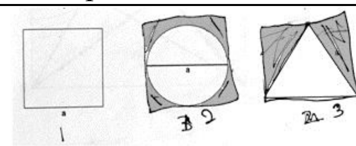
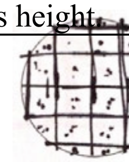
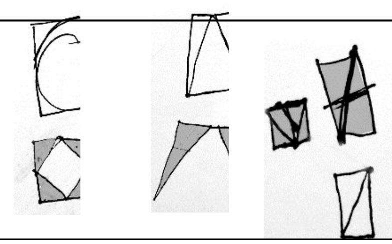
Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

were influenced by this element rather than at the beginning. Interestingly, Shana (who was assessed as low concept development) was the only participant who explicitly identified the task element of equal measure, whereas the others were either confused by the information, overlooking it, or needed to be reminded by the interviewer.

Documented approaches

Table 3 illustrates the approaches that the participants used when solving the Compare Areas problem.

Table 3. Approaches adopted by the participants solving the Compare Areas problem

	Approach	Description	Example from interview
1	Fit into the square	Showed the circle and the triangle could fit into the square.	
2	Formula	Used area formulas to compute the areas.	Assigned a number to *a* and used the measure of the side of the triangle as its height.
3	Unit squares	Drew and count unit squares to approximate the area.	
4	Compare leftover	Transformed the leftovers of the circle and the triangle into manageable shapes for comparison.	

The third approach was adopted when a specific area formula (i.e. the area formula of circles) was not available to Andy who used the second approach as his initial heuristic. The fourth approach was usually adopted when the participants, who used the fit-into-the-square approach as their initial approach, were prompted to further compare the areas of circle and triangle.

Revealed concept stages

When designing the interview instrument, the researchers had predicted six concept stages that could be revealed in this problem: 1) *Surface Association Complex – Formula* wherein a student uses an incorrect formula, 2) *Artificial Association Complex* where perimeter is compared instead of area, 3) *Chain Complex - Non-measurement* wherein the circle and the triangle can fit into the square, 4) *Pseudo-concept Complex - Formula* where visual reasoning contradicts to computational answers, 5) *Potential Concept - Formula* where a number is plugged into a formula, and 6) *Concept – Formula* where variable *a* is plugged into the formula to reach a generalized answer. Among these stages, 1, 3, and 5 were revealed during the interviews, while 2 was observed during the identification of task objectives prior to one participant's initial approach.

Table 4 summarizes the concept stages of Area concept revealed from the five participants when solving this problem. The approaches associated with each stage as used by them are noted as well. The stages listed in italic are the ones predicted to be revealed by the researcher.

Table 4. Concept stages of area and associated approach revealed by participants

	Revealed concept stage	Associated approach
	Surface Association Complex – Non-measurement	Compare leftover
	Surface Association Complex – Formula	Formula
	Chain Complex	Fit into the square
	Pseudo-concept Complex - Non-measurement	Compare leftover
	Potential Concept – Unit area	Unit squares
	Potential Concept - Formula	Formula
	Concept - Non-measurement	Compare leftover

All other stages were observed when the participants sought an alternative way to either refine or complement their initial approaches. The participants' problem solving behaviors were much more novel during this process (thus less predictable) than their behaviors during the initial attempt (i.e. using fit-into-the-square or formula approach).

Discussion

The study results suggested that metacognitive behaviors of the participants are closely linked to their perceived level of complexity of the task and whether they found it appropriately challenging. That was the case regardless of the participants' level of development pertaining to the concept under study (in our case, area, for instance) or their personal preferences for particular strategies. That is, when a task was perceived as too challenging, it was treated as enigma. This limited our capacity as researchers to access cognitive or metacognitive behaviors of the participants. The same occurred when the problem was perceived as trivial. Because of this, we argue that description of what constitutes a problem (in the context or problem solving) needs to be more precisely defined. The common description "whether a task is a problem depends on the individual" (Schoenfeld, 1985) too broad to inform research instrumentation when studying mathematical problem solving. We posit that benchmarks for selection/design/ development of appropriate problems, according to the concept formation stages specific to those embedded in the task, can enable researchers to more adequately elicit student problem solving. This issue is particularly critical if inferences are to drawn regarding the individuals' understanding of a concept in presence of their mathematical problem solving performance.

Since existing literature does not provide a guide that can inform task design/selection for research purposes, focused scholarly efforts towards construction of such a theoretical platform are needed. Towards that goal several key issues merit elaboration, among many include: 1) task elements and objectives that are essential to understanding the problem in light of learning progression, 2) specific concept stages as entry points to the tasks, 3) desired concept stages embedded in solutions, and 4) potential shifts/paths between concept and its associated concept stages.

Discussion of task elements need to identify conditions under which a problem may suit different populations according to their experiences with the concepts involved. Naturally, if the problem solver has had limited or no experience with a particular concept their interpretation of the task or what is expected as an appropriate answer may not match those of the researchers. Discussion of specific concept stages as entry points granted by the tasks allows the researcher to use the same task with different populations who may not share the same concept developmental status. Such an elaboration allows us to determine whether the task might be too challenging/impossible to solve given a specific concept developmental stage. A description of potential shifts and paths can serve as an aid for the interviewers/researchers to gauge their interventions during the interviews (potential prompts/probing questions when observing problem solving process). These descriptions mainly determine the capacity of a task. If a task allows only one entry point and one desired stage for the solution, it mostly assesses whether an individual knows the procedure or not, which is commonly used in proficiency tests. The development a detailed theoretical model of task design is a necessary tool towards development of a theory of mathematical problem solving.

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DEVELOPMENT OF VERTICALLY EQUATED PROBLEM-SOLVING MEASURES

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This manuscript describes initial research on the process of vertically equating two problem-solving instruments: one for sixth-grade students and another for seventh-grade students. We connect this study to research aimed at developing measures to assess students' problem-solving performance as framed by the Common Core State Standards for Mathematics. Rasch modeling results indicated that three of the four anchor items worked as planned. Thus, scores on the seventh-grade measure indicate students' performance on seventh-grade content but also their performance on sixth-grade content-focused items.

Keywords: Problem solving; Assessment and Evaluation; Middle School Education

The Common Core State Standards for Mathematics (CCSSM; National Governors Association, Council of Chief State School Officers, 2010) highlight problem solving in some fashion across grade-level Standards for Mathematics Content (SMCs). Furthermore, problem solving is found in several titles and descriptions of the Standards for Mathematical Practice (SMPs). Assessments should address the depth and focus of instructional standards (Wiliam, 2011). Thus, new assessments are needed that reflect the call for problem solving expressed in the CCSSM. Our first objective for this study is to briefly describe psychometric results of a new assessment for seventh-grade students. The second objective is to explore the vertical equating process with respect to this assessment and a previous one for sixth-grade students.

Related Literature

Problem Solving and Problems

We draw upon Lesh and Zawojewski's (2007) notion of problem solving as a process including "several iterative cycles of expressing, testing and revising mathematical interpretations – and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics" (p. 782). Problem solving occurs only when learners work on a problem. Schoenfeld (2011) characterized a problem as a task such that (a) it is unknown whether a solution exists, (b) the solution pathway is not readily determined, and (c) more than one solution pathway is possible. Problems are unique from exercises. Exercises are tasks intended to promote efficiency with a known procedure (Kilpatrick, Swafford, & Findell, 2001). The CCSSM emphasize problem solving in the SMCs and SMPs, hence teachers are expected to engage students in problem solving during everyday instruction.

Researchers have suggested that students should experience non-routine word problems as part of their regular mathematics instruction (Boaler & Staples, 2008; Matney, Jackson, & Bostic, 2013; Palm, 2006; Verschaffel et al., 1999). These word problems ought to be open, realistic, and complex (Matney et al., 2013; Verschaffel et al., 1999). An *open* problem can be solved using multiple strategies and offer problem solvers numerous entry points to the task. A *realistic* problem allows an individual to draw upon his/her experiential knowledge and also fosters engagement through a connection to the real world. Finally, a *complex* task supports perseverance and sustained reasoning, which are highlighted in SMPs #1 and 3. Our research draws upon established ideas for problems and problem solving, as well as prior published work in developing CCSSM-focused mathematics assessments (Bostic & Sondergeld, in press, 2015), to highlight the pilot testing process as well as preliminary findings of a second measure, the Problem Solving Measure for seventh grade (PSM7). We used Rasch modeling (also known as one-parameter item-response theory) and vertical equating while developing the Problem Solving Measure for sixth grade (PSM6) and PSM7 in order to create a sound system for measuring students' problem-solving of items addressing the CCSSM across two grade levels.

Assessment: IRT and Vertical Equating

Oftentimes assessments are delivered as incidents independent of future or previous testing. This is acceptable if the goal is to assess student skills at a single point in time. However, if the purpose of testing is to measure student growth across grade levels then multiple assessments must be given, each related to their appropriate grade level content standards. How then might teachers and researchers determine student growth in learning if the content is different at each grade level? One solution is vertical equating (or scaling) (Wright & Stone, 1979). Vertical equating can only be done when investigating a single unidimensional construct such as problem solving (Lissitz & Huynh, 2003). An attempt to measure student growth from one year to the next using two linked assessments requires common items (called anchors) on both measures (Wright & Stone, 1979). These anchor items are typically moderate in difficulty from the lower level assessment, and placed on the higher-level assessment to be evaluated for change in item difficulty (or displacement). If the items remain at approximately the same difficulty level (within ±.40 logits – or log odds units) on the higher-level test then they are acceptable anchors that will work well in linking different grade level tests (Kenyon, MacGregor, Ryu, Cho, & Louguit, 2006). There is no standard for the number of required anchor items when vertical equating. It is more important to ensure anchor items are of high quality (displacement within range of ±.40 logits) than to have a greater quantity of poorly functioning anchors (Pibal & Cesnik, 2011). In fact, Pibal and Cesnik showed that as few as three anchor items can work well to vertically link tests together provided they are of high quality.

Method

Building upon prior research related to developing the PSM6, we created a problem-solving measure to address the seventh-grade SMCs. The aim of this paper is to provide evidence for vertical equating of scores between the PSM6 and PSM7. Our research questions are (a) Is the PSM7 psychometrically sound? (b) What psychometric evidence suggests that scores on the PSM6 and PSM7 might be vertically equated?

Instrumentation

PSM6. Previously peer-reviewed work by Bostic and Sondergeld (in press, 2015) discusses the validity and reliability of the PSM6. It has 15 items, three from each domain of the CCSSM (i.e., Expressions and Equations (EE), Geometry (G), Number Sense (NS), Ratio and Proportions (RP), and Statistics and Probability (SP)). Two sixth-grade classroom teachers, one mathematician holding a Ph.D., and two mathematics educators with terminal degrees reviewed the items for connections with the SMCs and SMPs, developmental appropriateness, and use of complex, realistic, and open problems. This review panel was also tasked with considerations about bias on the measure. Results of the review indicated that the items addressed the SMCs and numerous SMPs, were developmentally appropriate, all tasks could be solved with at least two strategies, drew on real-life contexts, and were sufficiently complex to be considered problems. A group of sixth-grade students representing different ethnicities, socioeconomic backgrounds, and cognitive abilities (as measured by classroom grades) volunteered to serve on a second review panel to examine the PSM6 for potential bias. The review panel consisting of adults and children found no bias that might impact students' performance.

Data for the PSM6 were collected from 137 sixth-grade students located in a Midwest state. None of the respondents were English Language Learners. Test administration took approximately 75 minutes. Students' responses were scored as correct and incorrect. These scores were used to examine the psychometric properties of the PSM6 using Rasch modeling for dichotomous responses (Rasch, 1980), which is also called one-parameter Item Response Theory (see Bond & Fox, 2007).

PSM7. Development of the PSM7 followed a similar process as the PSM6. The aim of this manuscript is to share the process of vertically equating scores across grade levels on the PSM6 and PSM7. To meet that aim, we selected tasks from the PSM6 that represented core ideas of sixth-grade mathematics and were of moderate difficulty for sixth-grade students. For our study, four total items from the PSM6 were selected as anchors on the PSM7: two items addressing the EE domain and two items addressing the RP domain. These items were within one standard deviation of the mean item difficulty for the PSM6 as well as the one standard deviation of respondents' mean (making them average difficulty items). EE and RP draw upon the number fluency ideas that are central to support the progression pathway to algebraic reasoning and thinking that is at the core of CCSSM content development across grade levels (Smith, 2014). Therefore, items addressing this content were deemed acceptable content for both grade levels.

Data Collection and Analysis

Four-hundred and eighty-seven students located in a Midwest state completed the PSM7. Seventh-grade students took approximately 80 minutes to complete 19 items, four of which were the items for vertical equating purposes. Responses were scored as correct and incorrect. These data fueled the quantitative analysis using Rasch modeling for dichotomous responses. Rasch methods are often viewed by many social science researchers as the best method for instrument development and refinement because they convert ordinal data into conjoint, hierarchical, equal-interval measures that place both person abilities and item difficulties on the same scale allowing them to be directly compared to each other (see Bond & Fox, 2007). Winsteps Version 3.74.0 (Linacre, 2012) was used for all Rasch analyses. Psychometric evaluation of the PSM7 in terms of its reliability and validity evidence were conducted similar to that which was previously done for the PSM6. Assessment of anchor item functioning and displacement was also performed.

Results

Psychometric Findings

Unidimensionality is a fundamental quality of measurement. Items with negative point biserial correlations, infit mean-square (MNSQ) statistics falling outside 0.5 – 1.5 logits, or outfit MNSQ statistics greater than 2.0 logits are not meaningful for measurement (Linacre, 2002), and should be removed from a test as they do not contribute to a unidimensional latent trait. No items on the PSM7 had negative point biserial correlations (.06 – .65). All items fell within Rasch MNSQ fit parameters for infit statistics (.76 – 1.25). Only one item had an outfit statistic greater than 2.0 (.10 – 3.13). Item reliability was high at .98 suggesting strong internal consistency for items. Further, item separation was high (6.90) and item measures ranged from 3.03 to 4.69 indicating a meaningful variable (i.e., problem-solving ability) was created. Collectively, these statistics suggest all items worked together to form a unidimensional measure capable of assessing a wide range of problem solving abilities among seventh-grade students.

Vertical Equating Findings

Of the four items used as anchors from the PSM6 to the PSM7, three fell within the appropriate range for displacement (-.06 – .18). One RP item had a displacement of -.43 and was thus deemed unacceptable for anchoring purposes. Therefore, three of the PSM6 items will remain on the PSM7 assessment during further testing administrations for linking purposes.

Conclusions

Overall, the PSM7 showed acceptable psychometric properties during its first administration. We were able to develop a continuous construct of problem solving assessed across two grade levels by using Rasch modeling methods and vertical equating. This construct aligns problem-solving items on two tests with appropriate grade level SMCs. It allows evaluation of a student's sixth-grade problem-solving skills while providing an opportunity to assess a student's growth in competency when moving into grade seven. Such linked measures like the PSM6 and PSM7 respond to calls for assessments to reflect the depth and focus of standards, especially new ones adopted across multiple regions of the USA.

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STUDENT RECORD KEEPING FOR COGNITION AND COMMUNICATION

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This study explores how the records that students make during problem solving assist their cognition and communication. Grounded in the problem-solving literature and cognitive load theory, we examine the records that 14 middle grades students make as they solve geometry problems in one-on-one task based interviews. We identify features of record keeping that assist the students with cognition and communication and discuss the implications of this work.

Keywords: Cognition; Geometry; Middle School Education; Metacognition

This paper investigates how records students generate during problem solving assist in their cognition and communication. Solving mathematical problems and communicating thinking are fundamental parts of learning and doing mathematics (NCTM, 2000; NGA & CCSSO, 2010). Helping students learn and use strategies for solving mathematical problems and communicating their thinking are essential elements of successful mathematics instruction.

In this paper we describe a study of *record keeping* (RK) that middle school students used to help them think about a problem and then communicate their thinking. We were guided by the following research question: How do middle-grades students use record keeping during problem solving for cognition and communication?

Record Keeping

In reviewing the literature we found diverse terminology that intersected with our idea of record keeping. For our purposes, *record keeping* (RK) is the act of capturing pieces of information developed during the process of solving a mathematical problem in a manner that allows the problem solver to retrieve this information later. Such pieces of information include: important aspects of the problem statement, additional information possibly needed for solving the problem, characteristics of possible solutions, ideas about solution strategies, and/or partial solutions. The record could be physically inscribed or electronically captured, and it might take various forms including words, symbols, equations, drawings, or diagrams. Its defining characteristic is that RK secures information external to the mind of the problem solver; it does not take the form of a mental note.

Theoretical Perspectives

Our study was guided by foundations in problem solving (Polya, 1957; Schoenfeld, 1992) and Cognitive Load Theory (CLT) (Sweller, 2003). CLT is a learning and instructional theory based on the temporary and limited nature of working memory and the permanent and essentially unlimited capacity of long-term memory (Sweller, 2003). Working memory draws on long-term memory, but can only store about seven chunks of information and process only two or three chunks of information at a time. If these limits are exceeded, working memory gets overloaded.

From a CLT perspective, problem solving can generate a high cognitive load for students due to the information that needs to be stored and processed simultaneously. Literature on problem solving suggests that a means of reducing this load is to offer heuristics that students can use, including writing an equation to describe the problem situation or drawing a diagram.

Using equations, diagrams, and other external representations as records during problem solving can serve two purposes (1) bolster the capacity of working memory by offloading part of students' thinking onto the environment (Tabachneck-Schijf, Leonardo, & Simon, 1997), and (2) focus attention on key quantities and relationships needed to solve the problem (Zhang, 1997). Successful problem solvers are adept at working with a representation that captures the essential relationships needed to solve the problem (Diezmann & English, 2001; Nunokawa 1994). Expert problem solvers also construct many more visual representations than novices do during problem solving (Stylianou & Silver, 2004). Once created, diagrams can be useful for facilitating communication for students. Multistep explanations are generally challenging to describe, and also hard for listeners to follow (Hufferd-Ackles, Fuson & Sherin, 2004).

Modes of Inquiry

We conducted 14 one-on-one problem-solving and interview sessions with middle school students to understand how RK helped them in cognition and communication. The sessions lasted about 45 minutes each while the student worked on three challenging geometry tasks. The sessions were videotaped and the students' written work was also collected. After the students worked on the problems, the researcher asked follow-up questions.

Analysis of student work and interviews involved detailed examination of each student-generated record, specifically describing the record created, how it was used in problem solving, and the student's reasoning related to the record. A combination of emergent codes and codes based on the literature were used to analyze data from the sessions. The research team together created and shared analytic memos describing how each student's RK aided cognition and communication. Two main themes supported by multiple examples arose from this analysis – *RK fosters exploration and supports persistence* and *RK for communication aids recounting and referencing*. We discuss these two themes drawing on examples from one student's work on the Shelf task (Fig 1). In this task the students were given a cross-section of Joseph's bedroom with some pertinent measurements and were asked to find the length of the shelf.

Results

RK for cognition fosters exploration and supports persistence

RK featured prominently during the initial stages of work on problems when the students did not seem to have a clear approach to a problem and were exploring various avenues. In such cases they drew on prior experiences like using the numbers provided to do some initial computation, and labeling or marking figures. Based on these initial results, the students may have continued with their approach or may have abandoned it and tried something new, in either case gaining some traction on the problem. Below we illustrate how Carol, an 8th grader, initially engaged in exploratory RK which eventually led to a successful solution of the Shelf task.

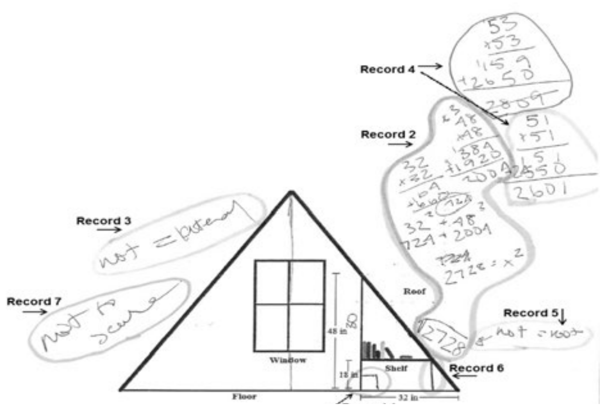

Figure 1: Record Keeping for Shelf

Carol began the task by marking a right angle (Record 1, Fig. 1) which prompted her to explore the use of Pythagoras' theorem. She made several records to determine (incorrectly, due to computational errors) the length of the hypotenuse of the key triangle is the square root of 2728 (Record 2, Fig 1). After observing that evaluating this square root would involve further calculations, she explored other properties of the triangle. Using her pen as a measuring device she observed that the triangle in the figure was not equilateral and made a note – "not = lateral" (Record 3, Fig. 1). Returning to the Pythagorean approach, she tried to find a whole number square root of 2728 (Record 4, Fig 1). While making these calculations, she realized that no whole number, when squared, could have 8 in the units place. Carol wrote “not = root” (Record 5, Fig 1) to remind herself of this fact. At this point she asked if the figure is drawn to scale and was told that it was not necessarily. After using her pen to make measurements near the 18 inches mark and a segment she had drawn before (Record 6, Fig 1), Carol wrote “not to scale” (Record 7, Fig 1).

Carol next turned the paper over and proceeded to create a scaled drawing of the figure. She noted 4 as a common factor of 32 and 48 and made equally spaced dots representing 4 inch intervals along both the vertical and horizontal parts of the diagram (Record 8, Fig 2). She used this approach, making several additional records (Records 9-12, Fig 2), to determine the length of the shelf as 20 inches.

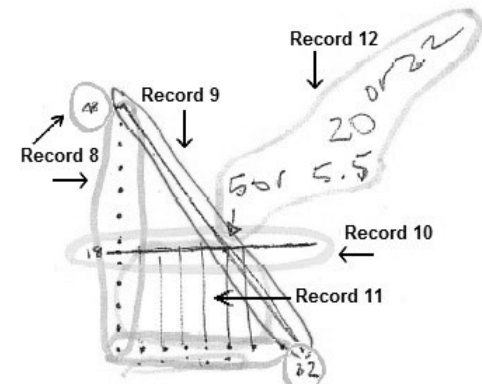

Figure 2: Record Keeping for Shelf

In our analysis of the interview with Carol, we note that her RK supported two approaches – one numerical and one diagrammatic. In the numerical approach she used Pythagoras’ theorem and attempted to work out the length of the roof. Through her calculations she noted that this would not be a whole number and thus decided to try a diagrammatic scaling approach. Carol began by trying to determine if the original figure was drawn to scale. When she discovered that the figure was not necessarily drawn to scale, she was motivated to create a scaled figure, leading to an approach that yields the solution. At each step, her RK such as “not to scale”, “not = lateral”, and “not = root” acted as signposts that guided her towards another approach that eventually led to a successful solution. By offloading her work and keeping track of the dead ends, Carol was able to use her RK to attempt possible approaches to the task, but set them aside when she found them unproductive and considered other approaches.

RK for communication aids recounting and referencing

The records that the students generated usually mediated their interactions with the researcher during the session interviews. The records served to remind students of the work they had done and also grounded many comments and gestures that students used to communicate their thinking to the researcher.

In Carol’s explanation of her work on the Shelf task, she used the records to walk the researcher through a chronology of her thinking, including all of the ultimately not successful work to find the length of the roofline of the shelf structure, and to find the length of the shelf via comparative measurement with known values. She gestured to each record to draw the researcher’s attention to the calculations she completed and parts of the diagram she had marked as she related each episode of her work. In this way, the records provided Carol a set of prompts for the chronology and points of reference for the researcher as the audience.

Carol’s culminating diagram (Fig 2) served to ground her numerous gestures as she explained how she determined the number of dots that would represent the length of the shelf. Carol gestured repeatedly to Records 8-11 when describing the diagrammatic scaling method she used to solve the problem, using her pen as a pointer to indicate the dots and their meaning as measures used to count by 4 inch increments both vertically to locate the height at which the shelf would be set, and horizontally to determine its length. Pointing to the records also emphasized the estimating decision that Carol made about whether the end of the shelf that intersected the roof corresponded to 5 or 5.5 dots (20 or 22 inches) horizontally.

With the diagram as the background, Carol was able to point to various parts and combine these gestures with her speech to develop a coherent explanation that the researcher was able to follow. The diagram as a record was vital to this interaction; without access and reference to it the researcher would not have understood the thinking behind Carol's approach.

Discussion

In our data we found that RK effectively supported both the students' cognition and communication. Based on the analysis of their records the students experimented with various approaches to the problem. In some cases, the records brought to the fore an approach that they used to successfully solve the problem. We conjecture that RK allowed the students to first offload their thinking onto the paper to free up more of their working memory. Further, by having the records in front of them, the students were able to draw on pertinent information from their long-term memory to solve the problem.

Besides aiding the students' cognition, RK proved useful in communicating their thinking. The students used their existing records or sometimes generated records to ground their interaction with the researcher. We conjecture that by providing a constant outlet for the students' ideas, the RK prompted students to explore further, consequently exhibiting more persistence in their problem solving.

Given the benefits we have outlined for RK, the larger goal of our study is to understand what features of mathematics task presentation promote RK that aids students' cognition and communication. Further, we are interested in deriving principles for designing tasks to promote useful RK, and general development of students' RK strategies for problem solving.

Acknowledgments

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OPTIMIZATION PROBLEMS IN FIRST SEMESTER CALCULUS

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Optimization problems require students to use a combination of calculus and pre-calculus skills to solve word problems, which are historically challenging for students. Although they are often able to learn strategies for arriving at the correct answer, the goal is for students to understand how and why these strategies work. Using Tall and Vinner's (1981) concept image as our theoretical perspective, we examine barriers to students' understanding of the role of the optimizing function as they are in the orienting and planning phases of Carlson and Bloom's (2005) Problem Solving Framework.

Keywords: Problem Solving; Modeling

Most standard optimization problems in first semester calculus require students to read and interpret a problem, set up an appropriate function to model the situation, differentiate the function to find critical values, prove the critical values give a maximum or minimum, and answer the specific question from the problem. White and Mitchelmore (1996) observed that students were more successful at solving such problems when the function is already given than when they must construct the optimizing function first. Here we examine barriers to constructing an optimizing function and to understanding the significance and meaning of the answer.

Literature Review

As part of a larger study, White and Mitchelmore (1996) presented students with one of four versions of two standard optimization problems. There were significantly more correct answers to the versions given explicitly than there were to the versions given in the form of a word problem, requiring the students to define the variables themselves, supporting research stating that a weak concept image of variable often plays a role in students' difficulties with calculus (Clement, Lochhead, & Monk, 1981; Kaput, 1987; Malisani & Spagnolo, 2009; Orton, 1983). There is a large body of literature devoted to student understanding of functions (Dubinsky & Harel, 1992; Sfard, 1992; Vinner & Dreyfus, 1989). Eisenberg (1992) and Even (1998) found that students are often reluctant to consider the graphical representation of a function, and when they do, they frequently have trouble interpreting the information correctly.

Theoretical Perspective

Carlson and Bloom's (2005) problem solving framework allows us to describe students' activity as they solve optimization problems. The framework is divided into four phases: orienting, planning, executing, and checking. In the orienting phase, the student deciphers the problem and assembles the tools he or she thinks may be required. In the planning phase, the student uses conceptual knowledge to determine an appropriate course of action, which is then implemented during the executing phase. Finally, during the checking phase, the problem solver goes back to the original problem to see if the answer makes sense.

In this paper, we will focus on the concept images that are evoked while the students are in the orienting and planning phases of Carlson and Bloom's (2005) Problem Solving Framework. Tall and Vinner (1981) define concept image as "all the cognitive structure in the individual's mind that is associated with the given concept" (p. 1). This cognitive structure may be incomplete, incorrect, or logically inconsistent and may have very little to do with the formal mathematical definition of the concept. Tall and Vinner (1981) address the possibility of logically inconsistent components of the concept image by defining the “evoked concept image.” When the two conflicting parts of the concept image are simultaneously evoked, the individual experiences confusion known as cognitive conflict, often leading the student to resolve the conflict, developing a better understanding of the concept, and other times leaves the student feeling uneasy and frustrated but ultimately unable to address the conflict.

Methods

Data was collected through semi-structured interviews, which were video recorded and transcribed for analysis. We began with a small pilot study with three second semester calculus students during Spring 2014 (Arthur, Brent, Carl) and collected data two more times with first semester calculus students during the Summer 2014 (Franz, Sam, Tracy, Lars) and Fall 2014 (Ashod, Brandi, Cy) semesters. All three iterations of the interviews involved the students solving optimization problems and answering some questions about related prerequisite material. For the first interview, we gave three problems, but scaled back to two problems for the second and third interviews to allow time for more detailed, focused questions. In the second and third interviews, we explored students’ beliefs about the relationship between perimeter and area of rectangles (for example, does an increase in perimeter guarantee an increase in area?).

In the interest of space, here we only present the results from Question 1, which was common across all interviews. The students’ responses to subsequent questions reflected reasoning similar to their responses to Question 1. Question 1 was: *A rectangular garden of area 200 ft^2 is to be fenced off against rabbits. Find the dimensions that will require the least amount of fencing if a barn already protects one side of the garden.*

Results and Discussion

One of the three second semester calculus students and four of the seven first semester calculus students were able to solve Question 1 without intervention from the interviewer. Three of these five students solved the problem as though the barn did not exist, but when they became aware they had forgotten the barn, all three expressed surprise and disgust that they had neglected that detail, indicating that it was likely due to inattention.

The other five students were eventually able to solve the problem as well, but they all needed help, in the form of leading questions, hints, or explicit suggestions to try a different approach. The assistance was given with the intention of continuing through the interview to assess the students’ understanding at as many phases of the process as possible.

Why Do We Need This Function?

All of the students were aware that they needed to construct some function that could be differentiated, and they all knew they needed to do something to eliminate one of the variables. Beyond that, seven of the ten students demonstrated that they did not have a well-developed concept image for the optimizing function. We would like their concept images to include the following: 1) The optimizing function should be a single variable function representing the quantity we would like to maximize or minimize and 2) Our goal is to find the absolute maximum or minimum of this function.

Franz and Tracy had Property 1, but not Property 2; they knew they were supposed to differentiate a function representing perimeter, but neither knew why. Franz said, “It’s just a standard thing that we do,” and Tracy said, “I honestly don’t remember why that works, I just know that it does.” Arthur, Carl and Tracy all attempted to differentiate an expression for area at first, indicating that Property 1 was not part of their evoked concept image.

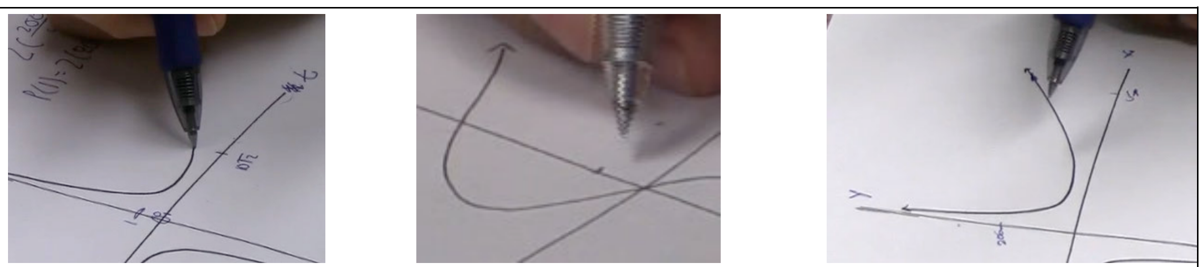
Ashod, Arthur and Cy became very focused on the key word, “dimensions” in the problem and discussed using the perimeter only as a means for finding dimensions, not the maximum or minimum. These three students did not display evidence that Property 2 was part of their evoked concept image; instead, they were simply using the perimeter equation because it provided a convenient way to find dimensions. Consider Ashod's response, "Basically what we would do is we solve for either variable to plug it in to find the perimeter of this. That's, to find the perimeter which would give us the dimensions of it." We use the terms function and equation intentionally here, because these students were focused on the perimeter equation as something they could use to solve for something, rather than as something they would need to differentiate. Their initial evoked concept image for area and perimeter limited them to equations or formulas, rather than expressions that could be thought of as functions.

Franz, Tracy and Lars demonstrated a lot of difficulty with thinking of these equations as functions, implying difficulty with Property 1. Franz wrote, "$2x + y =$ " with nothing written to the right of the equal sign. When asked about it, he said, "I don't know what it's equal to now. I think, uh..." and then gave up. After moving on from her initial attempt to use the area equation, Tracy wrote "$\text{minimize materials} = 2y + x$" which seemed like a good first step, but when it came time to express perimeter as a function that could be differentiated, she did not know what to do. Even after the interviewer suggested she use the letter *M* to represent "amount of material," she remained hesitant, saying, "Even then, I don't think that'd help me. That's just adding another variable." Lars had a similar problem.

How is This Function Related to This Graph?

Only one of the ten students mentioned the graph without being prompted, supporting Even's (1998) observation that students resist transitioning to different representations of functions. Cy said, "It's weird to, like, make the jump from numbers into a graph sometimes. In some situations, the numbers are really just a stand-in for the work I'm doing in my head with graphs, but in this situation, it's more, the numbers are all I really ever thought about this."

It is possible to solve optimization problems without considering the graph, so we do not fault the students for avoiding the graphical representation. However, we would like the students to be able to correctly interpret and identify key information on the graph of the optimizing function, even if they would not automatically use it. We noticed some of the pilot study students having trouble with this, so we asked pointed questions of the other seven students.



Brandi solved Question 1, found $y = 10\sqrt{2}$, and marked it here.	Franz solved a different question, found $w = 2\sqrt{3}$, and marked it here.	Ashod solved Question 1, found $x = \sqrt{200}$, and marked it here.

Figure 1: Students Labeling Graph

When they marked the point in the domain corresponding to the critical point, these students did not place it below what was clearly the absolute maximum or minimum of the graph. Even though they were able to find the correct answer, being unable to correctly place it on the graph suggests that the students either do not know what the answer signifies or that the significance of the answer does not play a dominant role in their minds as they are solving the problem. When these students were in the planning phase of the problem solving process, they did not plan ahead based on the goal of finding the absolute minimum of the optimizing function, rather, they were just looking for a function to differentiate without thinking about why.

Conclusion

All of the interviewees were eventually able to solve Question 1, generally considered to be an easy optimization problem, but their incomplete concept images for the optimizing function signal poor understanding of the optimization process. Variables, function notation, and the role of the optimizing function are important to the construction of an optimizing function. Several students did not display evidence of working towards the goal of constructing an optimizing function with Property 1, evidenced by their unwillingness or inability to use appropriate function notation. For many students, the evoked concept images of the role of the derivative when constructing an optimizing function appeared to be limited to the knowledge that the derivative was useful, and that setting the derivative equal to zero would somehow lead to the correct answer, demonstrating that Property 2 was not part of their evoked concept image for the optimizing function. We need to allow students to explore the role of the optimizing function while they are in the orienting and planning phases of the problem solving process to facilitate better understanding of the entire optimization process. We suggest an increased emphasis on constructing the graph of the optimizing function and activities exploring the optimizing function, such as sketching several rectangles with the appropriate area, but different perimeters.

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STUDENTS' REASONING OF LOGICAL EQUIVALENCE BETWEEN AN IMPLICATION AND ITS CONVERSE

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This study examines students' reasoning of logical equivalence between an implication and its converse. In particular, it focused on the cognitive schemes underlying students' logical comparison of both implications. Past research found that students experienced difficulties in these tasks. From 120 justifications written by 60 Singapore Secondary Three students, a hierarchical ordering of five types of logical comparison schemes was derived. The schemes ranged from disregarding the order of antecedent and consequent to comparing syntax, truth values, and counterexamples. Research for instructional support on logical equivalence is needed.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking; Problem Solving; Number Concepts and Operations

This study examines students' reasoning of logical equivalence between two mathematical propositions, specifically, an implication and its converse since implication is at the heart of mathematical reasoning (Durand-Guerrier, 2003). Students' competence in determining the logical equivalence between an implication and its variants (e.g., converse or contrapositive) is a key aspect of mathematical proving (Epp, 2003; Stylianides, Stylianides, & Philippou, 2004). It is the first step towards building their knowledge of proof strategies and ability to discern valid and invalid proof methods. However, little attention is given to this aspect in proof instructions.

Research had shown that students experienced difficulties in understanding the logical nonequivalence between an implication and its converse (Hoyles & Küchemann, 2002). They encountered the most struggle in interpreting an implication as a bi-conditional ("if and only if") and regarded the implication and its converse as equivalent. However, it remained unclear what cognitive schemes were underlying their reasoning about logical equivalence between implications. Understanding their schemes is a first step to inform the role of logical reasoning instruction in proof instructions (Epp, 2003).

Theoretical Framework

Reasoning of logical equivalence between propositions is grounded in logical reasoning of propositions. One aspect of logical reasoning in mathematics concerns the reasoning of mathematical objects in relation to the truth or falsity of mathematical propositions, and in particular, implications. A mathematical implication is an open sentence of the form "If P then Q" where the antecedent P and the consequent Q are propositions themselves. Since the counterexamples satisfy the antecedent but not the consequent, the order in which the mathematical propositions appear as the antecedent and the consequent in the syntax will determine what constitute as counterexamples. Variants of the implication can be generated by means of negating and/or changing the order of the mathematical propositions in the syntax,(e.g., "If *not* P then *not* Q," "If Q then P," and "If *not* Q then *not* P.")

Durand-Guerrier (2003) proposed that students' should adopt an understanding of the implication as generalized conditionals. Based on this epistemic stance, two propositions are logically equivalent when students determine mathematically that the set of example that satisfies and the set of counterexamples that falsifies each proposition are the same.

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Logical comparison schemes

Just as students hold schemes of mathematical proofs in proving conjectures (Harel & Sowder, 2007), they also hold logical comparison schemes when determining the logical equivalence between propositions. A student's logical comparison scheme is a cognitive scheme underlying one's process of constructing mathematical justifications to determine whether two or more mathematical propositions are logically equivalent. One type of logical comparison scheme may be characterized as logical reasoning errors due to students' interpretation of the "if-then" syntax based on its everyday meaning (Epp, 2003). However, other schemes that students may demonstrated remain unclear. The research question pursued by this study is: what types of logical comparison schemes do students use to determine the logical equivalence between an implication and its converse?

Method

The data of this study came from the written work of 60 Singapore Secondary Three (equivalent to U.S. 9th Grade, 14 to 15 years old) student, who participated in a larger study about logical reasoning and mathematical proving. The students in this study were taking double mathematics classes at that grade, one introduced basic contents in algebra, trigonometry, arithmetic, rate and proportion, and graphs while the other emphasized algebraic thinking and computations, quadratic and trigonometric functions and graphs, and basic calculus.

Each student attempted one logical equivalence tasks for two separate occasions. In the logical equivalence tasks, students were presented with both an implication (e.g., "Joe says: If the product of two whole numbers is even, then their sum is odd") and its converse (e.g., "Eva says: If the sum of two whole numbers is odd, then their product is even.") spoken by fictitious characters and asked to decide whether both propositions were expressing the same mathematical idea and were logically equivalent. Two different versions of the tasks using similar implications were given to each student. In each version, one of the pair of propositions is mathematically true (Eva's) while the other is falsifiable by counterexamples (Joes'). Every student attempted one version during the first occasion and the other during the second version. In between the occasions, students worked on a set of self-paced training materials that provided work-examples of implications of different logical combinations and mathematical proofs.

Coding and Analysis

A total of 120 students' written justifications were analyzed. A set of external codes was initially generated based on past research, for example, students' scheme of interpreting an implication as a bi-conditional statement (Hoyles & Küchemann, 2002), by analyzing their writing and mathematical reasoning. Generally, students' responses included use of examples and use of mathematical arguments based on properties and inferences followed by a conclusion about the logical equivalence. The type of justification used could be distinguished by its linguistic features. New codes were then generated and then refined based on emerging distinctions between students' justification approaches. Based on the extent to which students demonstrated sophisticated reasoning and their consideration of the examples and counterexamples associated to the implications, a hierarchical order of the coded categories was also developed as a result. 20% of (n = 12) students' written justifications were randomly selected for inter-rater reliability scoring. A second coder, who was a high school mathematics teacher, coded each justification. Seven justifications were coded as "others" since the students either misinterpreted the task or provided irrelevant responses (inter-rater reliability - 79.2%).

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Findings

Students' logical comparison schemes consisted of four types. Each scheme was demarcated by distinctive characteristics of logical reasoning. They were then ordered hierarchically based on the extent to which examples and counterexamples were considered in their logical reasoning.

Table 1: Students' logical comparison scheme for the implication and its converse

Scheme (Level)	Students' justifications about a pair of an implication and its converse
Order-irrelevance (0)	Conclude that the pair of implications is the same because: • The order of the antecedent and consequent in the syntax did not matter • Both propositions are referring to the same pair of symbolic equations though in different order • Examples are constructed to satisfy the pair exist though one of them may be falsified by counterexamples
Syntax Comparison (1)	Conclude that the pair is different because different mathematical objects were referred by different antecedents
Truth Comparison by examples (2)	Conclude that the pair is different because one is shown true using examples and the other is shown false using counterexamples
Truth Comparison by Proofs (3)	Conclude that the pair is different because one is proved true using deductive proofs and the other is shown false using counterexamples.
Counterexample Comparison (4)	Conclude that the pair is different because their general counterexamples to the implication and its converse are different

At level 0, students exhibited an "Order-irrelevance" scheme underlying their justifications in support of both the implication and its converse being logically equivalent. To that extent, an implication could be interpreted as either a bi-conditional statement (if and only if) or a conjunction proposition ("…and…"). They arrived at this logically invalid conclusion via three verification approaches: syntax, symbolic, or example. In the first approach, students compared the mathematical ideas expressed in the antecedents and consequents of both implications. In the second approach, students re-formulated the mathematical ideas expressed in the antecedent and consequent using systems of equations and variables for comparison. In the third approach, students justified that both implications were the same as if the task was demanding them to verify the propositions and constructed the same or different examples.

At level 1, students exhibited a "Syntax Comparison" scheme underlying their justifications for concluding that both the implication and its converse are logically nonequivalent. They arrived at this logically valid conclusion by comparing the mathematical ideas in the antecedents of the pair and noted that they were different. At level 2, students exhibited a "Truth Comparison by Example" scheme underlying their justifications for concluding the implication and its converse being logically nonequivalent. They arrived at this logically valid conclusion by comparing the truth values of both implications to determine one was true and the other was false. Typically, students constructed a counterexample and concluded that Dewey's (or Joe's) implication was false but used only examples to reason that Gabriel's (or Eva's) implication was "true," the latter of which exhibited an Empirical-based proof scheme (Harel & Sowder, 1998).

At level 3, students exhibited a "Truth Comparison by Proofs" scheme underlying their justifications to reach a conclusion of logical nonequivalence. Similar to the "Truth Comparison by Example" scheme, students compared the truth values of the implication and its converse. Instead of using examples to show one of the implications was true, they constructed a deductive proof. However, such a scheme might be invalid for tasks involving two true but logically nonequivalent implications, such as an implication and its inverse. At level 4, few students exhibited a "Counterexample Comparison" scheme underlying their justifications to arrive at a conclusion of logical nonequivalence. Contrary to the prior two "Truth Comparison" schemes, students did not compare the truth values of the implication and its converse. Instead, they analyzed and compared the kind of counterexamples needed if each implication were false. This usually led to a description of different general counterexamples for each implication. Students would then conclude that the implication and its converse were logically nonequivalent.

Discussions and Conclusions

How do students reason about the logical equivalence between a mathematical implication and its converse? This study revealed various logical comparison schemes underlying students' determination of their logical equivalence. These schemes involved students' proof schemes reported in past studies (Harel & Sowder, 1998), which implied that students' logical comparison schemes were closely knitted to their proof schemes. Also, the schemes assumed a hierarchical order according to the extent of validity in their logical reasoning in terms of the examples, counterexamples, and inferences made.

The above findings exposed the superficiality of students' logical reasoning of variants of implications: little emphasis is given to the counterexamples when analyzing the logical equivalence between implications and propositions. Students need instructional support on how to conceptualize logical equivalence by considering the different cases and the counterexamples related to the implication and its variants. It is important to teach logical reasoning from a semantic point of view, that is, to analyze the examples and counterexamples (Durand-Guerrier, 2003). This is the epistemic first-steps for understanding alternative but equally valid proof methods. Further research on students' logical comparison involving contrapositive and other logical propositions are thus required towards this purpose.

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MATHEMATICIANS' ASSESSMENT OF PROOFS WITH GAPS DEPENDING ON THE AUTHOR OF THE PROOF

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There is little research on mathematicians' assessment of mathematical proofs. The current paper contributes to this literature by reporting on professors' assessment of two mathematical proofs depending on whether a student or colleague authored the proof. By interviewing nine professors, we investigate mathematicians' perceptions of gaps in proofs depending on who authored the proofs. We also investigate whether mathematicians evaluate the gaps in the proofs differently depending on the author and their consistency in this judgment.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking

Proof plays a significant role in the advanced mathematics courses that mathematics majors complete. Proof is the dominant form of pedagogical explanation that professors use to convey mathematical insight to their students (Lai & Weber, 2014; Weber, 2004) and it is the primary way in which professors assess student performance (e.g., Weber, 2001). However, the important issues of how the professor chooses to present proofs in instruction and how they evaluate the proofs that their students present are under-researched areas in undergraduate mathematics education (Mejia-Ramos & Inglis, 2009 and Moore, 2014).

In this paper, we address these issues by exploring the permissibility of gaps in a proof. As proofs would be impossibly long if every logical detail is included (Davis & Hersh, 1981), it is common for mathematicians' proofs to contain gaps (Fallis, 2003) and it is understood that the reader of the proof might have to construct a sub-proof to bridge these gaps when checking the proof for correctness (Selden & Selden, 2003; Weber & Mejia-Ramos, 2011). Mathematics majors' proofs may also contain gaps (cf., Selden & Selden, 2003), perhaps because mathematics majors are uncertain of what requires justification. The goal of this paper is to address four related questions: What are mathematicians' perceptions of gaps in proofs given by professors during lecture? What are mathematicians' perceptions of gaps in the proofs that their students submit for credit? Do mathematicians evaluate gaps differently in these two contexts? How consistent are mathematicians in their judgment?

Related Literature

There is evidence that mathematicians are not uniform in their judgment on how they evaluate gaps in students' and mathematicians' proofs. With regard to student proofs, Inglis and Alcock (2012) and Weber (2008) presented mathematicians with a student-generated proof that contained what Selden and Selden (2003) called "the gap" (because the proof was valid except for a large gap) and asked the mathematicians to judge if the proof was valid. Across the two studies, 14 mathematicians evaluated the proof as invalid, five as valid, and one said he could not judge without further context. Some participants in Weber's (2008) study remarked that the proof would be invalid for students but valid if it were written by a mathematician, as less justification is required in a non-classroom context. These results are consistent with the qualitative study by Moore (2014), who also found mathematicians were not consistent with their evaluations of student-generated proofs and further, that whether a gap in a proof was significant depended on the understanding of who wrote the proof.

With regard to mathematicians' proofs, Inglis et al (2013) presented 110 mathematicians with a proof containing a gap and found substantial variation in their judgment of how the gap affected the validity of the proof. With proof presentation, mathematicians expressed a tension in deciding how much detail should be provided in a proof (Lai & Weber, 2014). Presenting too much detail can mask the more important ideas in a proof and deny students the learning opportunities that arise as they construct sub-proofs, but presenting too little detail can leave students' confused. In a survey asking if bridging a particular gap in a proof improved its pedagogical quality, about 40% of the mathematicians claimed it would while roughly the same number thought bridging the gap made it worse (Lai, Weber, & Mejia-Ramos, 2012).

Methodology

Participants

Nine mathematicians in either pure or applied mathematics who currently work as members of a Research I level University near the East Coast agreed to participate in this study.

Materials and Procedure

Participants met individually with the first two authors for their interview. The interview was broken up into two phases: 1) assessment of three proofs and 2) general open ended probing questions. For the first phase, the interview was further broken up into to two contexts. In **context A**, we told the professor that a colleague had written the proofs in an introduction to proof class on the board during lecture or as a solution handed out to students, and in **context B**, the students in your introduction to proof class hand the proofs in for homework.

The interview began with Context A. The participants were given the first proof and sufficient time to read the proof before discussing it with the interviewers. The interviewers asked the participants the following questions and to explain their reasoning.

1) Do you think that this proof is correct?
2) How would you rate the pedagogical quality or appropriateness of this proof?

This procedure was repeated for two more proofs (third proof won't be discussed for brevity). For Context B, the participants were given the proofs in same order. For proof 1, after sufficient time to re-read the proof, participants were asked two questions and to explain their reasoning.

1) Do you think that this proof is correct?
2) If grading this proof on a 10 point scale, how many points would you give it?

If the participant stated that the proof was correct, but did not give it a score of a 10, then the interviewers probed further for their reasons. At this point, the interviewers gave the participant another student proof for proof 1 without any obvious gaps, and asked the same two questions as before. The participants were given an opportunity to reassess the way they scored the first proof that had a gap and, if the first proof was reassessed, the participants were probed about the reasons for the changes. Next the participants were given both student versions of the second proof and asked the same questions for each proof.

The Proofs

Proofs 1 and 2 from Context A (proofs with gaps) are:

Proof 1. If n^2+1 is a prime number greater than 5, then the digit in the 1's place of n is 0,4, or 6.

Proof: Suppose that n^2+1 *is a prime number greater than 5. Thus* n^2+1 *is odd. Then* n *is of the form* $10k+0, 10k+2, 10k+4, 10k+6,$ *or* $10k+8$. *Then* n^2+1 *is of one of the following forms:*

$$(10k+0)^2+1=100k^2+1=10(10k^2)+1;$$
$$(10k+2)^2+1=100k^2+40k+4+1=10(10k^2+4k)+5;$$
$$(10k+4)^2+1=100k^2+80k+16+1=10(10k^2+8k+1)+7;$$

$$(10k+6)^2+1 = 100k^2+120k+36+1 = 10(10k^2+12k+3)+7;$$
$$(10k+8)^2+1 = 100k^2+160k+64+1 = 10(10k^2+16k+6)+5.$$

The only choices for n*, then, are those numbers with 1's digits of 0, 4, or 6.*

Proof 2. There exists a sequence of 100 consecutive integers, none of which is prime.

Proof: We will show that 100 consecutive integers $101!+2, 101!+3, 101!+4, \ldots, 101!+101$ *contain no primes. Since both terms of* $101!+2$ *contains a factor of 2, 2 is a common factor of* $101!+2$ *and thus* $101!+2$ *is not prime. We can do this for each number in the list. Thus, the sequence of 100 consecutive integers given above contains no primes.*

We refer the reader to the results to see the majority of the gaps in each proof.

Results

Context A

All but one of the participants evaluated Proof 1 as correct, but they had comments about the pedagogical quality or appropriateness of the proof. In referencing the pedagogical quality of the proof, many comments were directed that some of the gaps should be filled or time in class should be spent explaining how to bridge the gap. For example, if presenting this proof Professors A, C, F, and G said they would note the gap from n^2+1 being odd to n being even and Professors B, D, E, G, and H stated that there should be some explanation of why n can take on the stated forms. Professor C and G suggested that their colleague could have said something about why those cases exhaust all possibilities and Professors A, C, D and G pointed out that there should be some explanation of why we can dismiss the forms ending with a 2 and 8.

All of the participants evaluated Proof 2 as correct, and participants had a variety of comments about the pedagogical quality of the proof. Several had comments about the statement that the numbers in the sequence has a common factor. For example, Professors A, F and H stated (or implied) that there should be a general formula showing there is a common factor for each term of the sequence, Professor B said there should be motivation on why we choose this specific sequence, Professor C commented that there should be a few more examples of why it can be factored and should be motivated, and Professor G emphasized that this sequence is only one possible sequence. Professor B thought that the pedagogical quality of the proof would be improved if it had stated a more general sequence involving $k!$. We see that participants focused on different aspects of the proof and there was not a consensus on any one aspect. Hence, at least for these proofs, the mathematicians noted that although gaps in a proof were not sufficient to render them invalid, in these contexts, they found them pedagogically undesirable and were reasonably consistent with their judgment.

Context B

In this context, participants were to assess the proofs with respect to their students producing them for an assignment. During the interview, the participants were asked to grade the proofs on a 0 to 10 scale. All participants were given proof 1 (with gaps – proof 1(A)) and asked questions, followed by the same proof (without gaps – proof 1(B)). This was followed by giving both versions of proof 2 (with and without a gap) and asking the participants questions. All participants assessed the proofs on a 0 to 10 scale (Table 1).

Table 1: Mathematicians' Assessment of Proof with Gaps Authored by Students

	Prof A	Prof B	Prof C	Prof D	Prof E	Prof F	Prof G	Prof H	Prof I	Average
Proof 1(A)	6	9	8	9	9	7	10	6	8	8.00
Proof 1(B)	9	10	10	10	10	9	10	9	9	9.56
Proof 2(A)	10	10	8	10	10	8	10	8	10	9.33
Proof 2(B)	10	10	10	10	10	10	10	10	10	10

The lower average grade and overall scoring on proof 1(A) reflected that this proof had a number of gaps at various places in the proof, causing participants to assess it more critically than proof 2, which only had one obvious gap. When the majority of these gaps were filled in on proof 1(B), most participants were convinced that the proof was completely correct by giving it a 10. This data provides evidence that professors are more critical of students' proofs that have gaps in their reasoning, even if they are not large gaps. In contrast, the grading shows that mathematicians' were not too concerned (grading wise) about proof 2(A) not having a general formula that $\mathbf{101}! + \boldsymbol{j}$ has a common factor of $\boldsymbol{j}$ and were okay with the statement "We can do this for each number in the list." Even though mathematicians' did say that it would be better for the students to spell this out, they did not feel like they needed to take off points because it was evident from the written proof that the "student understood." We found that with Proof 1, most participants would deduct points from Proof 1(A) even though they judged it to be valid. There was substantial variation with how many points they would deduct for this proof as well as if they would deduct any points with Proof 1(B).

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PRESERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' CONCEPTIONS OF PROOF

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Developing students' understanding of proof has become an important task of mathematics educators (Hanna & de Villiers, 2008). Teachers' competence in creating opportunities for their students and enhancing their experiences with proof is considerably affected by their own conceptions of proof (Knuth, 1999). Therefore, this study investigated junior preservice middle school mathematics teachers' (PST) conceptions of proof through their responses on a written task where conceptions of proof referred to conceptions of what made an argument a mathematical proof (Knuth, 1999).

Data were collected from 32 PSTs enrolled in Elementary Mathematics Education program at a Turkish public university. PSTs took algebra and geometry courses, however, geometry courses did not include proof practices. Two mathematical statements (one algebra and one geometry) and three mathematical arguments which were trying to prove these statements were presented for each statement. PST's were asked to determine whether given arguments were valid proofs for the statements or not and explain their reasoning. Their responses were deductively coded according to Cobb's (1986) sources of conviction as *authoritarian* or *intuitive*, where the former addressed an outside authority (such as a book) as the source and the latter an individual's "uncritical belief" (Almeida, 2001, p.56) for "a proposition [which] makes intuitive sense, sounds right, rings true" (Cobb, 1986, p.3). Participants' emphasis on the distinction between empirical and general arguments was validated by the literature, as *generality* was considered as an important criterion of proof (Balacheff, 1988).

Findings showed that PSTs mainly relied on *intuitive* and *authoritarian* reasons and general terms in the argument while explaining their reasoning for accepting an algebraic argument as a proof. Some participants stressed that mathematical proofs should not be exemplifying specific cases. PSTs employed *intuitive* reasons and idea of *generality* in geometry task. However, they did not mention *authoritarian* reasons.

PSTs relied on similar reasons for evaluating both algebraic and geometric arguments. They were able to transfer their understanding of proof formed in algebra courses to the case of geometry. Not relying on *authoritarian* sources of conviction in geometry task might be due to the lack of experience of a geometry content course in which they would learn about authorities' practices and preferences. Mathematics content courses could be enhanced to have more influence on PSTs' conceptions of proof.

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ONE MATHEMATICS TEACHER'S ARGUMENTATIVE KNOWLEDGE CONSTRUCTION IN AN ASYNCHRONOUS GRADUATE COURSE

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Due to perceived "convenience," 100% online courses (synchronous and asynchronous) are progressively becoming more popular, particularly among professionals interested in advancing their education. In-service teachers are no exception. In asynchronous courses one of the most commonly used ways in which students interact with each other is via discussion board posts. Marttunen and Laurinen (2001) argued that since learners have more time and flexibility to formulate their arguments, asynchronous experiences such as discussion board posts might better facilitate knowledge construction.

In this poster presentation, a partial report of a larger study, we will share one in-service mathematics teacher's knowledge construction process based on the analysis of his posts on the discussion board in a graduate level online geometry course. In the larger study, we used Vinner's (1997) framework to identify pseudo-conceptual and pseudo-analytical thought processes of participants in this course. Kevin (pseudonym), the focus student of this poster presentation, showed predominantly pseudo-analytical thought processes in his coursework through the 8-weeks semester. To better understand his knowledge construction process and identify repeated practices, we used Weinberger and Fischer's (2006) framework to analyze his engagement on the discussion board. According to this framework there are four dimensions that may "extend and refine our understanding of what kind of student discourse contributes to individual knowledge acquisition" (p. 73): participation, epistemic, argumentative, and social mode of co-construction.

Our preliminary results show that in the participation dimension Kevin is one of the most active participants. However, in epistemic dimension his activities often fell into the category "construction of inadequate relations between conceptual and problem space" (Weinberger & Fischer, 2006, p. 74). He tends to argue "simple claims" and within social modes dimension he often "accepted the contributions of the learning partners in order to move on with the task" (Weinberger & Fisher, p. 77). In this presentation, we will share excerpts in each category and propose some strategies to engage learners in focused activities that will enable them to develop relations between prior knowledge and problem space with grounded claims.

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CONSTRUCTING PROBLEM-SOLVING STRATEGIES DURING COLLABORATION WITH PEERS OF EQUAL AND MORE ADVANCED KNOWLEDGE

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Past research has suggested that when two students who have partial but complementary knowledge of a concept work together, they can build a more advanced understanding of the concept (Ames & Murray, 1982). Schwarz, Neuman, and Biezuner (2000) suggest that this effect is most prevalent when the two students have different strategies and when they discuss their disagreement. The current study investigated how collaboration with more advanced peers and similarly skilled peers affects students' problem solving in the domain of mathematical equivalence, specifically with problems of the form 3 + 4 + 5 = 5 + __.

Eight pairs of second and third graders completed an individual pretest of four problems prior to working together to solve two additional problems. Finally, participants completed an individual, four-problem posttest. Of the eight pairs, four consisted of one partner who solved all pretest problems incorrectly (i.e., was nonequivalent) and one who solved most of the pretest problems correctly, and four consisted of two partners who were both nonequivalent at pretest.

Three (75%) of the four participants paired with a more skilled peer solved at least one problem correctly at posttest. In comparison, four (50%) of the eight participants paired with a peer of similar ability solved at least one problem correctly at posttest. When a participant adopted a correct strategy, the generation of that correct strategy occurred during collaboration. In all nonequivalent-nonequivalent pairs, if one student improved, so did his or her partner. This suggests that particular aspects of the interactions led to learning.

We investigated how the sole equivalent-nonequivalent interaction that did not lead to learning differed from the other equivalent-nonequivalent interactions. We found that this particular interaction was unusual in that the two children barely spoke.

Next we investigated how two interactions between two nonequivalent students led to learning for both partners. In three of four collaborations, one of the partners read all or part of the problem aloud, possibly highlighting the atypical location of the equal sign. This led to the generation of a correct strategy. In two of these partnerships, both students adopted the correct strategy, but in the third, the more dominant child dismissed the correct strategy in favor of an incorrect one. When neither student read the problem, the pair did not generate a correct strategy.

In sum, pairing a nonequivalent student with a more advanced peer was more beneficial than pairing two nonequivalent students together, but two nonequivalent students frequently did generate correct strategies during collaboration when they read the problems aloud. Ongoing work examines whether this pattern holds in a larger sample.

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IDENTIFYING AND PRIORITIZING UNKNOWNS IN MATHEMATICAL MODELING

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Keywords: High School Education; Modeling; Cognition

According to the Common Core State Standards for Mathematics (CCSSM), mathematical modeling serves as an essential link between classroom mathematics and decision making in everyday life (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Learning to do mathematical modeling presents challenges to students and educators because it requires coordination of mathematical and real world knowledge sources. Successful modeling requires students to identify and prioritize unknown information when constructing a specific model, such as those presented by open beginning and/or open middle problems. As such, it is important to understand student thinking during mathematical modeling tasks, especially when they are open to interpretation.

Open-beginning questions are defined as those which have no clear starting point and which do not cue a method of solution (Monaghan, 2009). Open-middle problems are those that can be solved in a variety of ways and which may have missing or ambiguous data (Sheffield, 2006). Where standard word problems tend to begin with an idealized, real-world situation that is couched in mathematical terms (Gould, Murray, & Sanfratello, 2012), open-beginning and open-middle questions challenge the student to assemble a strategy using his or her own mathematical toolkit, problem solving skills, and knowledge of the real world situation.

Using a process view of Mathematical Modeling (Blum & Leiβ, 2007), we explored how students identified and prioritized variables and conditions when solving open-beginning and open middle problems. We conducted task-based interviews with high school students. The students readily identified variables and constraints common to school mathematics (Watson, 2008) but had difficulty determining which or how they should be incorporated into a mathematical model and which should be ignored. Moreover, sometimes authentic real-world factors were excluded altogether. Analysis suggests that students may benefit from additional support in prioritizing variables, conditions, constraints, and assumptions so that they may be articulated in a mathematical representation.

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STUDENTS VACILLATE BETWEEN THE F-C-S LEVELS OF GENERALITY

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Keywords: Algebra and Algebraic Thinking; Cognition; Equity and Diversity; Learning Theories

Introduction & Background: Previous semiotic–cultural research on student mathematical reasoning (Gutiérrez, 2013; Radford, 2003) has identified three modes of action—Factual, Contextual, and Symbolic ("F-C-S"), representing different levels of generality—that students appropriate as means of dealing with pattern-generalization problems. The F-C-S framework, in its current formulation, might suggest that students monotonically progress from one mode to the next, yet recent findings indicate this is not necessarily the case. To help clarify the issue, this study asks: How do students *navigate* the F-C-S levels of generality?

Methods: The empirical context is a year-long ethnography of a high-school classroom community, conducted over the 2013-2014 school year. This poster presents the analysis of three students' situated reasoning during one particular pattern-finding problem involving geometric objects called Spiralaterals. Data consist of a 35 min. span of classroom video. I produced and analyzed a detailed transcription of students' verbal, gestural, and other semiotic actions as they referred to mathematical objects and attempted to construct generalizations. All student utterances were coded for their semiotic mode, and I analyzed whether each utterance reflected a generalization *to* a particular mode or merely a reiteration *in* the mode (Gutiérrez, 2013).

Results & Discussion: Students' reasoning *vacillated* between the F-C-S modes (Figure 1).

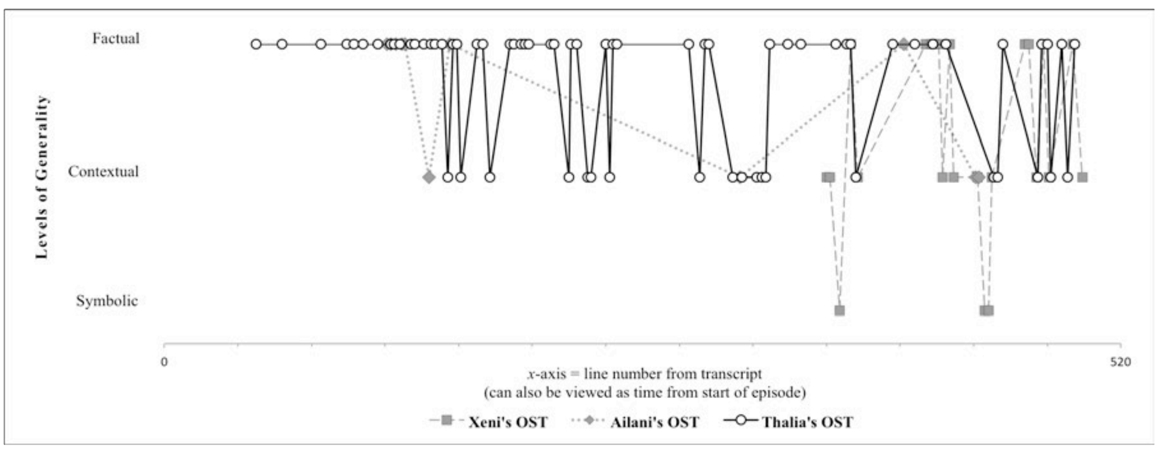

Figure 1. Juxtaposed Student Ostensive Semiotic Trajectories.

Chi-square analysis of total student utterances revealed a difference between the students with respect to their participation within each of the semiotic modes ($X^2(4, N = 103) = 14.78, p<0.01$). The Factual and Contextual modes made up the bulk of sense-making activity and were crucial for all three students' articulating generalizations. However, only Xeni operated in the Symbolic mode, which in situ resulted in a social-mathematical status hierarchy; further studies closely examine this semiotic-linked power dynamic and its impact on student agency and identity.

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HOW ARE COGNITIVE DOMAINS CORRELATED IN MATHEMATICS AND SCIENCE?

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STEM movement calls for a development of cross-disciplinary education integrating mathematics, science, technology, and engineering. However, a critical issue is that there is no understanding on how each discipline relates to another in K-12 education (Dugger, 2010). Dugger's discussion of STEM education necessitates further studies on interconnectedness between disciplines to build STEM curriculum. This research investigated relationships of cognitive skills used in learning of mathematics and science.

Table 1: Hypothesized relationships between cognitive domains

Linear Regression	Reasoning				Applying		
Model	R1	R2	R3	R4	A1	A2	A3
Dependent (Mathematics)	Analyze	Generalize/ Specialize	Integrate/ Synthesize	Integrate/ Synthesize	Select	Represent	Model
Independent (Science)	Analyze	Generalize	Integrate/ Synthesize	Draw Conclusion	Find solution	Interpret information	Use model

A data analysis began with establishment of theoretical relationships based on descriptions given by TIMSS 2011 assessment framework (see Table 1; Mullis et al. 2009). Assessment data of seventh and eighth graders from 2006 to 2010 in Iowa Tests for Basic Skills (ITBS) were collected. Generalized DINA (GDINA; de la Torre, 2011) was used to generate individual students' probabilities to master cognitive skills, which allows observing relationships of cognitive domains. Based on the results, correlations and linear regression equations were applied to describe relationships (see Table 2).

Table 2: Results of linear regression models

Grade		R1	R2	R3	R4	A1	A2	A3
7	N	11515	11515	5814	11515	11515	N/A	5701
	R	.539	.399	.029	.221	.494		.205
	R^2	.290	.160	.001*	.049	.244		.042
	Unstandardized slope	.504	.348	.025*	.177	.560		.278
	Constant	.360	.373	.514	.442	.245		.356
8	N	11596	11596	11596	11596	11596	5884	5712
	R	.474	.272	.239	.216	.414	.247	.379
	R^2	.225	.074	.057	.047	.172	.061	.144
	Unstandardized slope	.473	.211	.233	.204	.478	.331	.373
	Constant	.199	.404	.360	.328	.186	.263	.308

$^{*}p>.05$

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STUDENTS' REASONING AND SENSE MAKING ON CONCEPTS OF VARIABLES

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Algebra has an important role for students' mathematical learning. Expressions and equations are two algebraic contexts that students have to start learning from the 6^{th} grade (CCSS, 2010). Reasoning and sense making about mathematical contexts, including algebraic topics, are very important habits that secondary level students should develop in Algebra classes (Rasmussen et al., 2011). To understand students reasoning and sense making abilities on algebra problems, especially problems asking the behavior of variables in expressions and equations, the relationships between and among equations/expressions, and the effect of varying the value of the variable, I examined 9^{th} grade students responses on conceptual measures. Results revealed several patterns in solution methods that give a clue about the nature of students reasoning and sense making on algebra problems, hence their understanding on the basis of why a computation or algorithm works.

Participants were 300 9^{th}-grade students from different high schools in three states in the U.S. Participants included students with and without disabilities. Data collection consists of assessment forms (concept of variable forms) that Algebra students took at the end of school semester. Conceptual assessments has been linked to the Common Core State Standards, from grade 6 through high school grades. Concept of variable form includes 19 probes with 11 multiple-choice and 8 open-ended algebra problems. As an illustration, an example for multiple-choice item is "Carl simplified $6h - h$. He said an equivalent expression was $5h$. Do you agree with Carl?

A. Agree, because $6h - h$ can be factored as $h(6 - 1)$ to simplify it.
B. Disagree, because h is a common term in both so $h - h$ is 0, that leaves 6.
C. You cannot tell if he is correct because you do not know the value of h.
D. Carl is only correct if h is a positive number." and an example for open-ended item is "Bart said, '$w + 3$ is less than $5 + w$.' Circle one: Always true Sometimes true Never true
Explain your answer." Students responses were scored based on their correct choices on multiple-choice answers and their true, partially true or wrong explanations on the open-ended items. Data analysis focused on the open-ended items to investigate the nature of students' reasoning and sense making on algebraic concepts.

Findings show that students have a pattern of the misconceptions on open-ended algebra problems. For example, more than half of the students think that they cannot compare two expressions (in above question) because they do not know the value of the variable. This study identifies several misconceptions that students have on the concepts of variables, equations, and expressions.

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CONSISTENT OR INCONSISTENT STANDARDS? PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' EVALUATIONS OF MATHEMATICAL ARGUMENTS

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Current reforms have called for a stronger emphasis on proof across content domains in secondary school mathematics (Common Core State Standards for Mathematics, 2010). However, many pre-service secondary mathematics teachers (PSMTs) have considerable difficulty determining an argument's validity (e.g., Bleiler, Thompson, & Krajčevski, 2013) and constructing their own proofs (Ko & Knuth, 2009). To date, less attention has been given to PSMTs' criterion used in evaluating the validity of their own arguments and given mathematical arguments for the same problems in the domains of algebra, geometry, and number theory. To address the research gap, this study examines criteria that PSMTs perceive for determining whether or not their own argument and a given argument are convincing and are mathematical proofs in these content domains.

This study had 14 PSMTs from one Midwest University in the United States, all of whom had taken Calculus I, II, III, as well as an introduction-to-proof course. The instrument used in the study included three mathematical statements and six arguments adapted from the existing literature. The participants were asked to complete a two-part semi-structured interview. The first part of the interview focused on PSMTs' own proof productions along with their validations. The second part of the interview focused on PSMTs' validations of given arguments. Our data source included transcriptions of PSMTs' interviews and their written work. To analyze criteria that PSMTs perceived for discerning the validity of their own work and given arguments, we used Knuth's (2002) taxonomies of convincing arguments and mathematical proofs. The results of the study showed that some PSMTs tended to use different criteria to decide if their own arguments and given arguments were convincing and were mathematical proofs. Although all PSMTs had successfully passed an introduction-to-proof course, they still experienced difficulty identifying if a given argument was a mathematical proof across content domains.

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CONNECTING COLLEGIATE MATHEMATICS TO SCHOOL MATHEMATICS: PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CONSTRUCTION OF ABSTRACT MATHEMATICAL CONCEPTIONS

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Prospective Secondary Mathematics Teachers (PSMTs) have been typically required to take a significant number of collegiate mathematics courses to be qualified to teach school mathematics, but neither was there a strong positive correlation between student achievement and the number of collegiate mathematics courses taken by their teachers (e.g., Begle, 1979) nor did teachers seem to conceive the connections between collegiate mathematics and the mathematics they teach (e.g., Zazkis & Leikin, 2010). This contradictory evidence raises questions about the ways in which PSMTs develop their knowledge in collegiate mathematics.

This study posits that engaging in abstraction process within the context in which PSMTs can actively reflect on their school mathematics knowledge is necessary for PSMTs to connect collegiate mathematics and school mathematics. By adopting a theoretical framework 'Abstraction in Context' (Dreyfus, Hershkowitz & Schwarz, 2015), tasks for teaching interviews were designed so that PSMTs can engage in abstraction process through successions of activities and continuous transformation of their knowledge. The purpose of this study is to investigate PSMTs' abstraction process and their perceived connections in a series of teaching interviews. The data consist of teaching interviews with four PSMTs. Each PSMT participated in five 90-minute interviews in which they constructed and consolidated their knowledge of polynomial ring, irreducible polynomial and the prime factorization of polynomial building on what they had previously known about polynomials and factoring polynomials. The RBC+C model (Dreyfus et al., 2015) was used as an analytical lens to understand how PSMTs constructed abstract mathematical conceptions and connected collegiate mathematics and school mathematics.

PSMTs in this study showed their initial tendency to restrict their conceptions of factoring to the 'factoring by hand' rather than encompassing factoring over complex numbers as well. Through constructing their conceptions of an irreducible polynomial and polynomial ring, it seemed that not only were they able to recognize some applicability of these abstract mathematical notions to school mathematics context but also integrate and unify some school mathematics ideas in a vertically reorganized way.

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DEVELOPING MATHEMATICAL PROCESSES THROUGH COMMERCIAL GAMES

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Games have been recommended as a way for students to develop a meaningful understanding of mathematical ideas before they move toward abstractions (Diénès, 1971). Ernest (1986) identified three educational uses of games in mathematics class: gaining skill-based fluency, developing conceptual understanding, and refining problem solving approaches. Limited systematic research addresses the use of commercial board and card games that teachers and parents are already playing with their children, which contain opportunities to experience mathematical ideas. The poster addresses the research question: What mathematical learning do students demonstrate through their experiences with games and with each other?

Using Deweyan-inspired action research (Stark, 2014), experience is viewed as activity transformed through thoughtful reflection (Dewey, 1938/1997). The aim, methodologically, is transformative practice by cyclically clarifying meaning of (inter)actions. Theoretically, the dialectic of students' game play and encouragement to *notice* (Mason, 2002) their mathematical thinking in action contributed to understanding their growth in mathematical learning.

Located in two urban elementary schools, 65 students in grades 4 to 6 volunteered to participate. Students explored games such as *Gobblet Gobblers*, *Othello*, *Quartex*, and *Set*. Students' data included photographs, pictorial and written responses to reflective prompts, and individual interviews. Field notes and teacher interviews enriched the data analysis as multiple perspectives were included to assess fit between constructed meaning and experiences.

Results indicate an additional intention of incorporating commercial games in mathematics class: the crucial development of mathematical processes. *Communicating* mathematically was inherent in negotiating and defending specific plays, and was refined as students developed language for moves and saw their peers as an authentic audience. *Reasoning* through informal deductive statements emerged as students convinced themselves and others of best moves. *Visualizing* improved through first physically testing playing pieces on game boards toward an ability to predict moves in advance or find the best spot to place a game piece. Additionally, the students' development within the three processes was interconnected as their ability to see logical moves were verbalized to peers. I will use photographs and children's statements to illustrate these mathematical processes in the presentation. Opportunities for children to develop mathematical processes are imperative if these processes are to be used to understand specific mathematics content. Results of this project respond to two enduring challenges in mathematics education: 1) to provide opportunities for children to experience mathematical ideas so that they can build conceptual understanding through formal instruction; and, 2) to invite parents to re-engage in children's mathematical learning through informal game play at home.

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PRECALCULUS STUDENTS' PROBLEM SOLVING PROCESS ON A CALCULUS-CONCEPT TASK

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Introduction. In 2004, the MAA's Committee on the Undergraduate Program in Mathematics (CUPM) recommended that all mathematics courses should help students develop "analytical, critical reasoning, problem-solving, and communication skills". Calculus persists as a roadblock course and might be the first course in which some students are asked to conceptually reason. To investigate precalculus students' preparedness for the type of reasoning required of them in calculus, this study examined precalculus students' ability to reason given a mathematical modeling task based on calculus concepts which could be solved using previous knowledge and problem solving skills.

Methods. Five STEM major precalculus students participated in the study. During a task-based interview, the participants were asked to graphically represent the height of water in a bottle as a function of the amount of water in the bottle without numerical values, units, and rates given. The interviews were open-coded to explore common themes in the students' reasoning and their ability to work through the task, and coded according to Carlson and Bloom's (2005) problem solving framework to illuminate how students progressed through the problem solving cycle.

Results. Consistent with Carlson and Bloom's (2005) framework, the students repeated the planning, executing, and checking cycle. However, the checking phase was weak, and their problem solving was not as refined as the mathematicians' in Carlson and Bloom's study. All five students recognized the increasing relationship between height and amount of water, and that the shape of the bottle would result in a non-constant rate of change in height. However, students struggled to correctly model the non-constant rate of change graphically; only one produced a correct graph. Two of the students drew concave up or exponential graphs, and the other two drew piece-wise linear graphs. The results show that students had difficulty understanding exactly how the shape of the bottle would affect the rate of change. This could suggest a lack of conceptual understanding, which could hinder the students' success in a Calculus I course. Implications for teaching suggest a need for earlier activities exposing students to nonlinear rates of changes and covariation in order to prepare them for advanced mathematical reasoning.

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TEACHERS' AND THEIR STUDENTS' ENGAGEMENT IN MATHEMATICAL PRACTICES

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Mathematical habits of mind are both a means and an end of mathematics education (Lim & Selden, 2009) and learning can be viewed as the process of coming to participate in these mathematical ways. In the U.S., certain Standards for Mathematical Practice (SMPs) have been codified in the *Common Core State Standards* (2010). To better understand the challenges of enacting these SMPs, we examined the relationship between teachers' own engagement in SMPs and the ways in which they provide opportunities for their students to engage in SMPs. Results take the form of cases of two secondary mathematics teachers—Mr. Forrest and Mr. Mingley—who were similar in terms of exhibiting SMPs while working on mathematical tasks themselves but who were different in terms of the implementation of SMPs with their students.

Data for the teachers' engagement in SMPs consisted of written work and recordings of discourse from eight teachers (grades 5–12), including Mr. Forrest and Mr. Mingley, as they worked on 5 mathematical tasks. Classroom data involved three lesson observations each for Mr. Forrest and Mr. Mingley. Analysis involved identifying instances of the SMPs in both data sets, followed by qualitative coding of patterns within those instances. Within the classroom data, tools from discourse analysis (Truxaw & DeFranco, 2008) were also used to identify the nature of students' participation in the SMPs. The two cases were then constructed by looking across all instances of Mr. Forrest or Mr. Mingley engaging in SMPs during the teacher tasks and characterizing general themes evident in their classroom observation data.

Findings reveal that Mr. Forrest and Mr. Mingley conjectured, generated arguments, and criticized others' justifications. They also used some SMPs in conjunction, such as attending to precision as a way of refining conjectures. Although both teachers exhibited SMPs themselves, there were differences in their students' engagement in the SMPs. In Mr. Forrest's class, his mathematical tasks required students to explicitly engage in SMPs and, during the implementation, the students were held responsible for engaging in SMPs. In Mr. Mingley's class, however, SMPs occurred regularly but typically involved Mr. Mingley preparing opportunities for the SMPs, only to end up directly modelling the SMPs for the students.

This study contributes to the ongoing discussion about implementing the SMPs. Although teacher facility with the SMPs may be a necessary condition for successful implementation, the case of Mr. Mingley shows it is not sufficient. The study also prompts discussion about the potentially complementary opportunities of students (a) witnessing SMPs from a more knowledgeable other and (b) actually engaging in SMPs themselves.

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CHANGING CONES: STUDENTS' IMAGES OF A DYNAMIC SITUATION

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A growing number of researchers have indicated that students' quantitative reasoning is important for their engaging in modeling, problem solving, and generalizing (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis, 2007; Thompson, 2011). More recently, researchers have argued for increased attention to the images students construct when modeling and reasoning quantitatively. Moore and Carlson (2012) illustrated that students' mental images of a situation influenced how they related quantities of that situation. Students often held images of the situation inconsistent with those the researchers expected, but the students' solutions were compatible with the students' images of the situation. Other researchers (Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 2011) argued that students' images of situations and their envisioning how quantities change in tandem are intrinsically related. Addressing the call for more attention to students' images of dynamic situations, we used semi-structured task-based clinical interviews (Goldin, 2000) with 10 students to explore the images students constructed as they made sense of the relationship between the height and surface area of a cone as the cone's height increased and decreased in a proportional relationship with the cone's radius.

In this poster, we characterize the images students relied on when imagining the cone growing and shrinking. Two of ten students solely relied on a general image of the cone changing 'smoothly' with respect to time. The other students developed more sophisticated images with four students using 3D-images, three students using 2D-images, and one student using a combination of the two. We also discuss the implications of the students' images with respect to their determined solutions. Collectively, our results indicate that the students developed idiosyncratic images that greatly influenced how they solved the problem. An implication of our findings is that teachers need to be attentive to students' images of situations in order to support students in reasoning quantitatively as they model these situations.

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AN EXPLORATORY ANALYSIS OF PRE-SERVICE MIDDLE SCHOOL TEACHERS' MATHEMATICAL ARGUMENTS

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Calls for increased focus on mathematical argumentation in school mathematics are critical for teacher educators charged with preparing prospective K-8 teachers (PST) to foster mathematical argumentation in their future classrooms. This is because research suggests that middle school years are crucial for students to gain proficiency in creating and critiquing mathematical arguments. Supporting PSTs' strong understanding of what it means to *do* mathematics necessitates that teacher preparation programs place an emphasis on mathematical and pedagogical preparation that facilitates PSTs' strong understanding of mathematical argumentation and proving in school mathematics.

Based on problem-based interviews with 22 K-8 PSTs, we report on PSTs' ability to create mathematical arguments in the context of problems that involve reasoning and generalizing about patterns. The research question was: *How do pre-service middle school teachers reason about change in problem situations that require analyzing patterns and what is the nature of the arguments they create?* We selected this curricular topic because the reasoning about change and patterns is essential for learning concepts related to functions, algebraic thinking, and measurement. Records of the interviews were analyzed qualitatively (Strauss & Corbin, 1998) to first identify the specific problem solution trajectory. Then, using Toulmin's argumentation framework (2003) as a guide, we examined the structure of arguments pre-service teachers generated. Due to the space limitation, in this proposal, we only report the selected results from one task where participants were asked to generalize about the number of matchsticks to make a rectangle made up of R-rows and C-columns.

The vast majority of the PSTs reasoned about this task by first noticing the invariant characteristics of the given structure and focusing on horizontal and vertical change in the number of sticks. In the process of initially limiting the number of rows to only one and "unfolding" the consecutive numbers of columns they were able to develop the correct rule e.g., $4+3(c-1)$, or $4 + 3(r-1)$ if reasoning about change in rows, and later account for change in the second variable arguing that the number of sticks in *any* figure could be determined by a rule $4 + 3 (c-1) + [3 + 2(c-1)] (r-1)$. Overall however, they demonstrated limited ability to justify *why* their rule works. Most of those PSTs who used this strategy did not articulate clear links between the context of the problem and their rules. Instead, they supported the validity of their claims with specific examples (43% used one example, 21% two or more specific cases). Only a few were able to contextualize their rules using specific characteristics of the problem they solved. While in this proposal we only highlight selected findings from one task, this study documents reasoning paths PSTs at the beginning of their teacher education program generate and contributes to the knowledge of PST's ability to create mathematical arguments.

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Chapter 7

Statistics and Probability

Research Reports

Brief Research Reports

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Poster Presentations

A FRAMEWORK FOR ANALYZING INFORMAL INFERENTIAL REASONING TASKS IN MIDDLE SCHOOL TEXTBOOKS

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Building upon the work of Zieffler, Garfield, DelMas and Reading (2008) and others, we developed a framework for assessing informal inferential tasks in middle school mathematics textbooks. The framework both embodies the key recommendations for developing informal inferential reasoning and captures common trimming attributes, which lower the cognitive demand and opportunities to learn. Researchers believe that introducing inferential reasoning informally will assist students later in developing argumentation structures necessary for understanding formal methods (Wild & Pfannkuch, 1999). Inferential reasoning has long been a key learning goal of statistics education and provides access to viewing knowledge of core statistical concepts and reasoning about data distributions. Tools are needed to assess the fidelity of tasks in alignment with both national and research-based recommendations.

Keywords: Curriculum Analysis; Data Analysis and Statistics; Middle School Education

Background

Inferential reasoning has served as a unifying theme and goal of introductory statistics courses at the tertiary level for a number of years (Konold & Pollatsek, 2002). With the recent emphasis of statistics as a core component of the middle and secondary mathematics curriculum, the role of inference is gaining in prominence (NGA Center & CCSSO, 2010). Current recommendations for middle and secondary statistics education outlined in the *Guidelines for Assessment and Instruction in Statistics Education [GAISE]* report support the introduction of informal inferential reasoning at the middle school level and formalizing inferential reasoning during the secondary years (Franklin et al., 2007). These recommendations are evident in the articulation of the Common Core State Standards for Mathematics (CCSS-M) adopted throughout the United States (NGA & CCSS0, 2010), but not explained in an equally detailed manner. In response, middle school textbook publishers quickly produced curricular materials intended to align with the need for informal inferential reasoning in grade 7. Yet, many teachers, especially at the middle school level, do not have experience teaching informal inference. We argue that guidance is needed on how to assess the fidelity of inferential reasoning tasks contained within these curricular materials. While this may seem to be a narrow focus, inferential reasoning is a key learning goal of statistical education and incorporates knowledge of core statistical concepts and reasoning about data distributions. In this paper, we describe a framework we developed for characterizing informal inferential reasoning tasks based on recommendations of statistics education research, and then share how we analyzed tasks from three widely available seventh grade textbooks.

Informal Inferential Reasoning

In order to define and situate informal inferential reasoning for the purposes of this paper and framework, two broader concepts must be described: statistical inference and statistical reasoning. *Statistical inference* refers to moving beyond the data at hand to make decisions about some wider universe, taking into account that variation is everywhere and conclusions are therefore uncertain (Moore, 2004). *Statistical reasoning* is defined "as the way people reason with statistical ideas and make sense of statistical information" (Garfield & Ben-Zvi, 2004, p. 7). Hence, *inferential reasoning* is the way people make sense of statistical ideas and information with the goal of generating a conclusion that extends beyond the data at hand.

Generally, two types of problems fall under the broad definition of inferential reasoning: (a) generalizing from samples to populations, and (b) comparing samples to determine significant differences in populations (Garfield & Ben-Zvi, 2008). While students can address these problems with formal hypothesis tests, they can also formulate responses based on informal approaches that do not involve set procedures, but rather coordination of prior knowledge, statistical concepts, and the context of the problem. *Informal inferential reasoning* allows students in upper-elementary grades to engage and successfully draw inferences (Stohl & Tarr, 2002; Watson, 2002; Watson & Moritz, 1999).

Informal Inferential Reasoning Task Framework

Building upon the work of Zieffler, Garfield, DelMas and Reading (2008) and others, we developed a framework for assessing informal inferential tasks in middle school mathematics textbooks that both embodies the key recommendations for developing informal inferential reasoning and captures common trimming attributes, which lower the cognitive demand and opportunities to learn (See Table 1). While the recommendations from leaders in statistics education and other disciplines provide a comprehensive list of requirements for inferential reasoning tasks, our framework acknowledges a spectrum within each task dimension (i.e., inference, ill-structured, open-ended, context, and visual representation) that reveals nuances in tasks and ultimately pedagogical choices made by textbook authors and publishers that directly impact students' opportunities to learn

Table 1: Informal Inferential Reasoning Task Framework

Task Dimension	Low (Deterministic) - Limited/No reasoning required	Medium – Some inferential reasoning required	High – Inferential reasoning required
Inference	A population is utilized or the type of the data is unspecified. No requirement is needed to infer beyond data provided.	Sample data is utilized with the acknowledgement of variation.	Sample data is utilized with the acknowledgement of variation, and students are required to infer beyond the data at hand.
Ill-Structured	A prescribed procedure is desired with specified descriptive statistics computations.	A procedure exists that can be adapted in order to coordinate core statistical concepts with a choice of statistical measures.	Coordination of core statistical concepts is required to fully address the task without a prescribed solution path.
Open-Ended	Only one acceptable or "correct" solution exists.	Multiple numerical solutions with similar interpretations are possible or limited numerical solutions exist with a variety of possible interpretations.	Multiple numerical solutions are possible and a variety of conclusions.
Context	The task can be addressed fully by removing the context.	The context is helpful for generating an inference, but not required.	The problem context must be considered in order to generate a viable inference.
Visual Representation	Visual representations are neither provided nor encouraged.	Visual representation are provided or created, but mask the original data.	Raw data is provided and organized in graphical representations.

and reason about statistics. For each task dimension, we created a tiered set of categories based on the level of inferential reasoning required for the task: low (deterministic), medium, and high.

Inference

The first task dimension, inference, relates to how sample and population data are presented and utilized in tasks. Based upon a synthesis of research from educational psychology, science education, and mathematics education, statistics educators recommend informal inferential reasoning tasks require students to:

> 1) make judgments, claims, or predictions about a population based on samples, but not using formal statistical procedures or methods, 2) draw on, utilize, and integrate prior knowledge (formal or informal) to the extent that this knowledge is available, and 3) articulate evidence-based arguments for judgments, claims, or predictions based on samples. (Zieffler et al., 2008, p. 46-47).

A key facet of these recommendations relates to the need for students to experience and think about the differences between complete populations and samples. If a complete population is provided or the source of the data is unknown, then the task does not require inferential reasoning and is reduced to simply computing the differences in measures of center or another statistic of interest to draw a concrete and certain conclusion. Only through sample data is uncertainty introduced, which is the nature of statistics versus a deterministic mathematical problem.

Ill-Structured

Ill-structured tasks require informal reasoning versus applying formal approaches. Reasoning effectively to generate informal inferences requires prior knowledge of core statistical ideas, such as measures of center, variation, skew, outliers, shape of data distribution, and sample size, and an understanding of the relationships between them (Garfield & Ben-Zvi, 2007). Many statistical questions require coordination of both a measure of central tendency, such as mean or median, with a measure of variation such as range, interquartile range, or mean absolute deviation (MAD). In addition, middle school textbooks include tasks that require coordinating and comparing two measures of center, two measures of variation, or other combinations.

The second criterion for this dimension relates to the extent that the task is either well- or ill-defined in nature. Informal approaches to reasoning are needed when problems either do not align with known solution methods or are presented before students possess the knowledge of such methods. One would expect that students possess varying repositories of knowledge, which would result in a diversity of solution strategies when administered similar inferential reasoning tasks. This knowledge might consist of prior statistical knowledge, life experiences related to the context, and informal reasoning skills. As Means and Voss (1996) state, “Informal reasoning assumes importance when information is less accessible, or when the problems are more open-ended, debatable, complex, or ill-structured, and especially when the issue requires that the individual build an argument to support a claim” (p. 140).

When students approach ill-structured problems, they generally progress through four phases: problem structuring, preliminary design, refinement, and detailing (Goel, 1992). As ideas are flushed out in more detail, students become more committed to their solution strategy. The omission of one correct answer or lack of problem constraints is the key factor for encouraging informal reasoning. Watson and Moritz (1999) describe an iterative process that students embarked upon when comparing two data distributions involving: comparing measures of center, then considering other characteristics of the data distribution such as skew or range, and finally coordinating all possible data comparisons together to produce a detailed and integrated response. These steps provide a view into students’ statistical reasoning beyond traditional tasks that are highly structured in nature and seek a predetermined solution. The ranking for this category requires that no prescribed solution path is provided in advance and that students must compare at least two core statistical concepts.

Open-Ended

Open-ended tasks directly connect to the goal of eliciting informal approaches to inferential tasks (Bakker, 2004; Cobb, McClain, & Gravenmeier, 2003; Garfield & Ben-Zvi, 2007; Watson, 2002; Watson & Moritz, 1999). According to Leathman, Lawrence, and Mewborn (2005), open-ended problems “elicit reasoning, problem solving, and communication” (p. 413). Characteristics of high quality, open-ended tasks include the involvement of significant mathematics, the potential to solicit basic to sophisticated responses, and a balance between too much and too little information. Clearly, the bounds of ill-structured tasks and open-ended tasks overlap to some degree as the descriptions of both include common characteristics.

Many teacher-researchers initially introduce open-ended tasks to hone students’ thinking and reasoning about a situation. Through whole class discussion, the open-ended tasks become closed as taken-as-shared meanings develop (e.g. Cobb, 1999). In one study, students were asked to determine which of two ambulance service providers was better and provide justification for their reasoning (Cobb, McClain, & Gravenmeier, 2003). During a lengthy whole class discussion, students determined a process for reasoning about the information provided and agreed upon a final conclusion. Hence, the initially open-ended task became closed through the instructional process of establishing norms for acceptable justification.

By understanding this natural instructional sequence of tasks initially being open-ended in nature and over time becoming close-ended through the course of learning and whole class discussions, we anticipate not all tasks in a textbook would meet this requirement within an instructional unit. As students see relationships between tasks and establish ways of reasoning, the variety of conclusions will decrease with experience. However, if prescribed answers are provided for all inferential tasks, then the textbook is not allowing adequate room for students to engage in informal reasoning. Therefore, open-ended tasks require students to decide what is relevant and what constitutes acceptable justification without prior instruction. For example, if a textbook supports a range of answers as acceptable or incudes a clause, such as “Answer will vary”, then the task is deemed to be open-ended in nature. In addition, high quality, open-ended tasks require some level of justification or explanation to accompany the conclusion based on the selected relevant information. Therefore, we attend to both the open-ended nature of the response and the need for justification.

The Role of Context

The authors of the *GAISE* recommendations (Franklin et al., 2007) state, “In mathematics, context obscures structure. In data analysis, context provides meaning” (p. 7). Hence, the use of context is the norm in statistics education and instructors commonly introduce data sets in relation to some real-world phenomena or situation. However, the way statistics educators use context in their tasks varies substantially. On one hand, several have created problem scenarios familiar to students in an effort to increase accessibility and leverage prior knowledge and experiences (Bakker, 2004; Garfield & Ben-Zvi, 2007; Watson & Moritz, 1999; Watson, 2002; Watson, 2008). For example, Watson created a sequence of tasks based on measures of actual students’ heart rates and arm-span lengths. Creating data sets close to the knowledge and experiences of students helps focus the tasks on the reasoning process.

On the other hand, some researchers advocate tasks based on real-world contexts. Cobb (1999) and Cobb, McClain and Gravemeier (2003) created a variety of real-world contexts such as ambulance response times, success of speed traps, effectiveness of AIDS treatments, battery life spans, SAT scores based on school expenditure, and response time versus alcohol intake. Cobb, McClain and Gravemeier (2003) state that students must find the context of the problem both plausible and important before they will engage in reasoning about the data. In our framework, we attend to the inclusion of context and the role it plays in terms of generating an inference. Because we cannot be certain of which contexts will be either familiar or engaging to students, we focus only on the role of the context in the problem. If the context can be stripped away and/or ignored, the task is coded as low on the framework. If the context facilitates reasoning about the task, but is not needed to generate a response, then it is coded medium. Tasks that require attending to the context and incorporating it are ranked high.

Visual Representations

Visual representations shift students' thinking away from local attributes or summary statistics towards global characteristics and relationships. Tasks involving small sets of data (n<50) encourage the use of dot plots and bar graphs to depict the data distributions (Bakker, 2004; Garfield & Ben-Zvi, 2007; Watson 2002; 2008; Watson & Moritz, 1999). In addition to shifting students' thinking toward the entire distribution versus individual data values, visual representations facilitate coordination of core statistical concepts in a way that is extremely difficult with only summary statistics and little prior experience with statistical reasoning. The most useful representations for novices are graphical displays that reveal the raw data, in addition to organizing it visually, such as dot plots (Franklin et al., 2007). Therefore, we privilege representations that reveal the raw data and do not restrict the students' reasoning.

In the cases where only raw data is provided without a graphical display or a prompt to create a graphical display, the task is coded low. If the task contains graphical displays that mask the original data values (e.g. box-plots), it is coded medium. We acknowledge that box-plots serve an important role in inferential reasoning, by providing a lens in which to view the data that is useful. However, reasoning is restricted to some degree, as characteristics of the original data distribution are hidden from view. Lastly, if the data values are provided or generated by the students and visual representations are either provided or encouraged, the task is coded high.

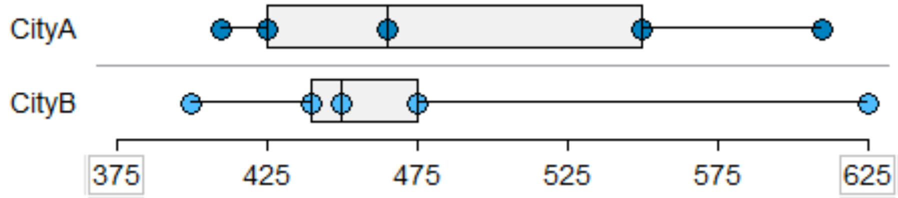
Application of the Framework

Analysis of Teacher Materials

We examined the teacher's editions of three commonly used 7th grade textbooks and identified the chapter(s) on statistics. In the chapter(s), the textbooks often reference examples for students' problems. Therefore, we analyzed the task based on the cited example. We acknowledge that hypothetically the task could be solved in a variety of ways; however, the example implies a set procedure path. In addition, if the answer key requires only a numerical answer, the task was classified as close-ended. Finally, if the task could be completed fully without considering the context, we coded the task low. The purpose of the following section is not to provide representative or typical tasks of the textbooks, but rather to demonstrate how the framework can be applied to a variety of informal inferential tasks found in CCSS-M aligned grade 7 textbooks.

Applying the framework, the task in Figure 1(problems 1, 2, and 3 inclusive), does not meet the requirement for inference since the source of the data is unspecified. One might assume this representation includes all the data of rental costs for each city, as there is no verbiage to the contrary. In regard to the task being ill-structured, prior examples in the textbook provide an approach to this problem of comparing the inner quartiles and the ranges of the box-plots. Since the inner quartile of CityB is smaller than CityA, yet the range of City B is larger than CityA, students will need to decide how to proceed. Therefore, this task is medium in terms of being ill-structured. A specified path exists but can be modified to accommodate coordination of core statistical concepts based on student's discretion. Next, the task is high in terms of being open- ended in nature, as the textbook notes that answers will vary. Depending on the decisions made when comparing CityA to

To answer the following problems, use the box-and-whisker plots of apartment rentals in two different cities.



Rental Cost ($)

1. *Which city has a greater median apartment rental cost?*
2. *Which city has a greater interquartile range of apartment rental costs?*
3. *Which city appears to have a more predictable apartment rental cost?*

Figure 1: Task adapted from Holt McDougal (2012)

CityB, students may arrive at different justifications. The context of the problem does not appear to be needed or facilitate reasoning, so it is rated low. Although a visual representation is provided, the original data is masked, leading to a medium ranking. Overall, we conclude that this task provides some opportunities for students to engage in aspects of informal inferential reasoning, but falls short of requiring all aspects.

The double dot plot below shows the quiz scores out of 20 points for two different class periods. Compare the centers and variations of the two populations. Round to the nearest tenth. Write an inference you can draw about the two populations.

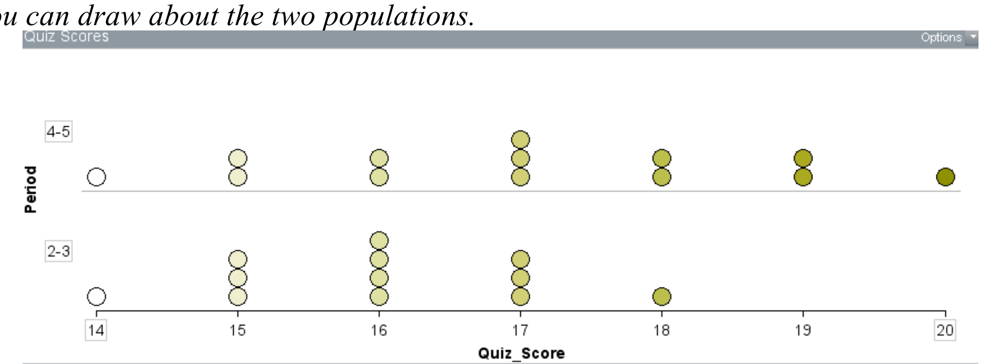


Figure 2: Task adapted from Glencoe (2013)

Applying the framework to this task, we conclude that it does not meet the requirement of an inferential task. The task implies that the dot plots represent the population for the two groups of class periods. In regard to the task being ill-structured, prior examples in the textbook provide a procedure of first comparing mean values and then comparing MADs. Students are steered to conclude that periods 4-5 have a higher mean and a larger MAD or more variation. Therefore, periods 4-5 scored higher on average, but the scores varied more and were spread out. In terms of being open-ended, the task is low because one correct answer is noted in the teacher's edition. In addition, the context is not needed for the problem and perhaps inhibits reasoning by grouping the data of two class periods. Lastly, in terms of visual representation, the task ranks high with the raw data visible and organized in a way that facilitates coordination of core concepts and informal reasoning. Overall, this task ranks low in terms of providing students opportunities to informally reason about inference.

Make a Conjecture *The box plots show the distributions of mean weights of 10 samples of 10 football players from each of two leagues, A and B. What can you say about any comparison of the weights of the two populations? Explain.*

Distribution of Means from 10 Random Samples of Size 10
Means

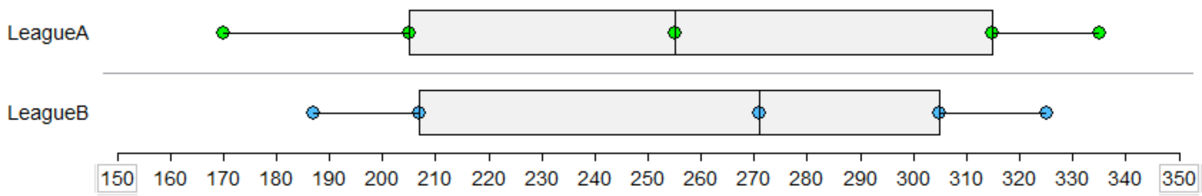


Figure 3: Task adapted from Go Math! (2014)

This task is different from the others as the box-plots are sampling distribution of means, a sophisticated statistical concept that has proven illusive to many tertiary students in introduction to statistics courses. The textbook recommends inferential reasoning with distributions of sample means as a way to reduce variability and make better comparisons, since the means vary less than the original data. Applying the framework to this task, we conclude this task meets the full requirements of an inferential task, as the data are labeled as samples of size 10 and students are asked to generate a conclusion that extends beyond the data at hand. In regard to the task being ill-structured, prior examples in the textbook provide an approach to the problem of comparing the centers of the distributions and looking at the overlapping portions of the inner quartile. Students may or may not understand why this approach works, but it is specified. Hence, we would rank this as low in terms of being ill-structured. Students will note that League B has a higher mean, but the overlapping inner quartiles create ambiguity in terms of which league has higher weight in general. Therefore, the task is close-ended with one correct answer. In addition, the context is not needed for generating the inference. In terms of visual representation, the task ranks medium with a graphic display and no access to the original data. Overall, we would conclude this task does provide some opportunities for students to engage in aspects of informal inferential reasoning, but falls short of requiring all aspects.

Conclusion

With the advent of many new mathematics textbooks claiming to align with national standards and research-based recommendations, tools are needed to assess the fidelity of tasks posed to students. Further, to study the learning effects of first introducing inference through informal approaches followed by formalization, middle school students require authentic experiences with informal inferential reasoning. Without the development and utilization of frameworks based on prior research and educational experiences, we will never know if students have the opportunities to informally generate inferences that later lead to a robust and connected understanding of formal statistics. Finally, we need to hold textbook publishers accountable for providing students with authentic opportunities to sense-make and reason, as outlined by leaders in statistics education.

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THE DEVELOPMENT OF INFORMAL INFERENTIAL REASONING VIA RESAMPLING: ELICITING BOOTSTRAPPING METHODS

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This study focuses on the development of four tertiary introductory statistics students' informal inferential reasoning while engaging in data driven repeated sampling and resampling activities. Through the use of hands-on manipulatives and simulations with technology, the participants constructed empirical sampling distributions in order to investigate the inferences that can be drawn from the data. Students' developing reasoning of sampling and informal inference is reported as they move from repeated sampling methods to resampling methods, along with their reasoning of bootstrapping methods and how this reasoning was applied to make informal inferential claims.

Keywords: Data Analysis and Statistics; Modeling

Introduction

Over the past few decades statistics education has become an integral part of the mathematics curriculum at all levels. Influential documents such as the National Council of Teachers of Mathematics standards documents (NCTM, 1989, 2000), the *Guidelines for Assessment and Instruction in Statistics Education College Report* (Aliaga et al., 2005), and The *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010*)* have emphasized the importance of statistics education at all levels. Prior to these documents, statistics at the K-12 level was often "the mere frosting on any mathematics program if there was time at the end of the school year" (Shaughnessy, 2007, p. 957).

This relatively new emphasis on the learning of statistics brings with it a new emphasis on how statistics is taught. A trend in statistics education is the shift from a focus on theoretical distributions and numerical approximations to an emphasis on data analysis (Cobb, 2007). Cobb asserted that many statistics curricula are outdated and based on how statistics could be learned prior to the computing power of modern times. The use of probability distributions, such as the normal distribution, were once needed since the conceptually simpler approach of simulations by hand was far too tedious to perform. Technology now allows these simulations to be performed nearly instantaneously. New curricula for introductory statistics courses should emphasize the ideas of data creation, exploration and simulation.

This study investigates students' developing informal inferential reasoning while engaging in a data driven instructional unit. Activities in the unit use both hands-on manipulatives and computer simulations to construct empirical sampling distributions from which students made informal inferences.

Related Literature

Informal inferential reasoning has been defined as "the drawing of conclusions from data that is based mainly on looking at, comparing, and reasoning from distributions of data" (Pfannkuch, 2007, p. 149), "the process of making probabilistic generalizations from (evidenced with) data that extend beyond the data collected" (Makar & Rubin, 2007, p.1), and "the way in which students use their informal statistical knowledge to make arguments to support inferences about unknown populations based on observed samples" (Zieffler, Garfield, delMas, & Reading, 2008, p. 44). Synthesizing these definitions, this study examined the claims that students made about populations of data when examining empirical sampling distributions, and how the students used the distributions of data to support these claims. Researchers suggest the use of informal inference before the use of formal inferential procedures (Zieffler, Garfield, Delmas, & Reading, 2008), such as employing the "three R's: randomize data, repeat by simulation, and reject any model that puts your data in its tail" (Cobb, 2007, p.12). This use of simulation to teach informal inferential reasoning can help students build a deep understanding of the abstract statistical concepts (Burrill, 2002; Maxara & Biehler, 2006). College curricula using simulations have indicated modest improvement in students' understanding of inference (Garfield, delMas, & Zieffler, 2012; Tintle, Topliff, Vanderstoep, Holmes, & Swanson, 2012).

There are two forms of simulations in this study, simulations that construct an empirical sampling distribution by: 1) repeatedly sampling from an available population; and 2) resampling from a sample with an unavailable population. Research involving repeated sampling activities have indicated that *some* students develop "a multi-tiered scheme of conceptual operations centered around the images of repeatedly sampling from a population, recording a statistic, and tracking the accumulation of statistics as they distribute themselves along a range of possibilities" (Saldanha & Thompson, 2002, p. 261). However, many students do not focus on empirical sampling distributions for inference and instead compare a single sample statistic with a population parameter. Students have also shown difficulty distinguishing the difference in strength of conclusions made from a small number of samples versus those made with large amounts of samples (Pratt, Johnston-Wilder, Ainley, & Mason, 2008).

The second form of simulation activities aimed to elicit and develop students' ideas about the resampling method of bootstrapping. Efron (1979) introduced the method of bootstrapping and asserted that the bootstrap was more widely applicable and dependable than earlier resampling methods, while also using a simpler procedure. Bootstrapping begins with drawing one sample of data from a population. Bootstrap samples are then constructed by choosing elements from this one sample, with replacement, and creating resamples which are equal in size to the original sample. A statistic from these bootstrap samples is then aggregated to form an empirical bootstrap sampling distribution. If done with hands-on manipulatives, this sampling process using replacement can potentially provide insight into the approach, but it is also very time consuming. However, technology can be used to simulate this procedure in a short period of time, but the use of technology may obscure the underlying sampling process.

While limited research has been done on student learning of statistics with bootstrapping methods (Garfield, delMas, Zieffler, 2012; Pfannkuch & Budgett, 2014; Pfannkuch, Forbes, Harraway, Budgett, & Wild, 2013), researchers have asserted that bootstrapping may promote student learning of the logic of inference (Cobb, 2007; Engel, 2010, Hesterberg, 2006)). The bootstrapping method has already become part of introductory statistics coursework such as the CATALYST curriculum (Catalyst for Change, 2012). Some textbooks (e.g., Lock, Lock, Lock-Morgan, Lock, & Lock, 2013) introduce the method of bootstrapping to define confidence intervals well before discussing confidence intervals with normal approximation methods. Lock et al. claim that the bootstrapping method has become an important tool for statisticians and that it is also intuitive and accessible for introductory statistics students. The authors further state that bootstrapping capitalizes on students' visual learning skills and helps to build students' conceptual understanding of key statistics ideas. There is not yet research evidence to support these claims, which will be explored in this study.

The research questions guiding this study were: 1) What student reasoning develops as they move from repeated sampling methods to resampling methods? 2) How do students develop their reasoning of bootstrapping methods and apply this reasoning to make informal inferential claims.

Theoretical Framework

The focus of analysis for this study was the models of sampling that the students created while engaged in a model development sequence (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Drawing on Lesh and colleagues, in this study, we define models as "conceptual systems … that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)" (Lesh & Doerr, 2003, p. 10). Teaching and learning from a modeling approach shifts the focus of an activity from finding an answer to one particular problem to constructing a system of relationships that is generalized and can be extended to other situations (Doerr & English, 2003). Students' mathematical models are useful for research since they provide a means for investigating students' developing knowledge (Lesh, Hoover, Hole, Kelly, & Post, 2000). Model development sequences consist of three forms of activities: model-eliciting activities that encourage students to generate descriptions, explanations, and constructions in order to reveal how they were interpreting situations; model-exploration activities that focus on the mathematical structure of their models and often use technology in order to develop a powerful representation system; and model-adaptation activities that transform the models created in model-eliciting activities in order to investigate more complex problems (Lesh et. al, 2003). By using a modelling approach to examine student reasoning, we can view reasoning as dynamic and developing over the course of instruction. Student reasoning may not only change from activity to activity, but many times during an activity. This framework allows us to examine the impact of the activities on the development of students' reasoning.

Design and Methodology

In this qualitative case study, the first author collaborated with two introductory statistics instructors to create an instructional unit that consisted of two model development sequences (Figure 1). This study is part of a larger study that examined the development of informal inferential reasoning through simulation activities and the role of hands-on manipulatives versus technological

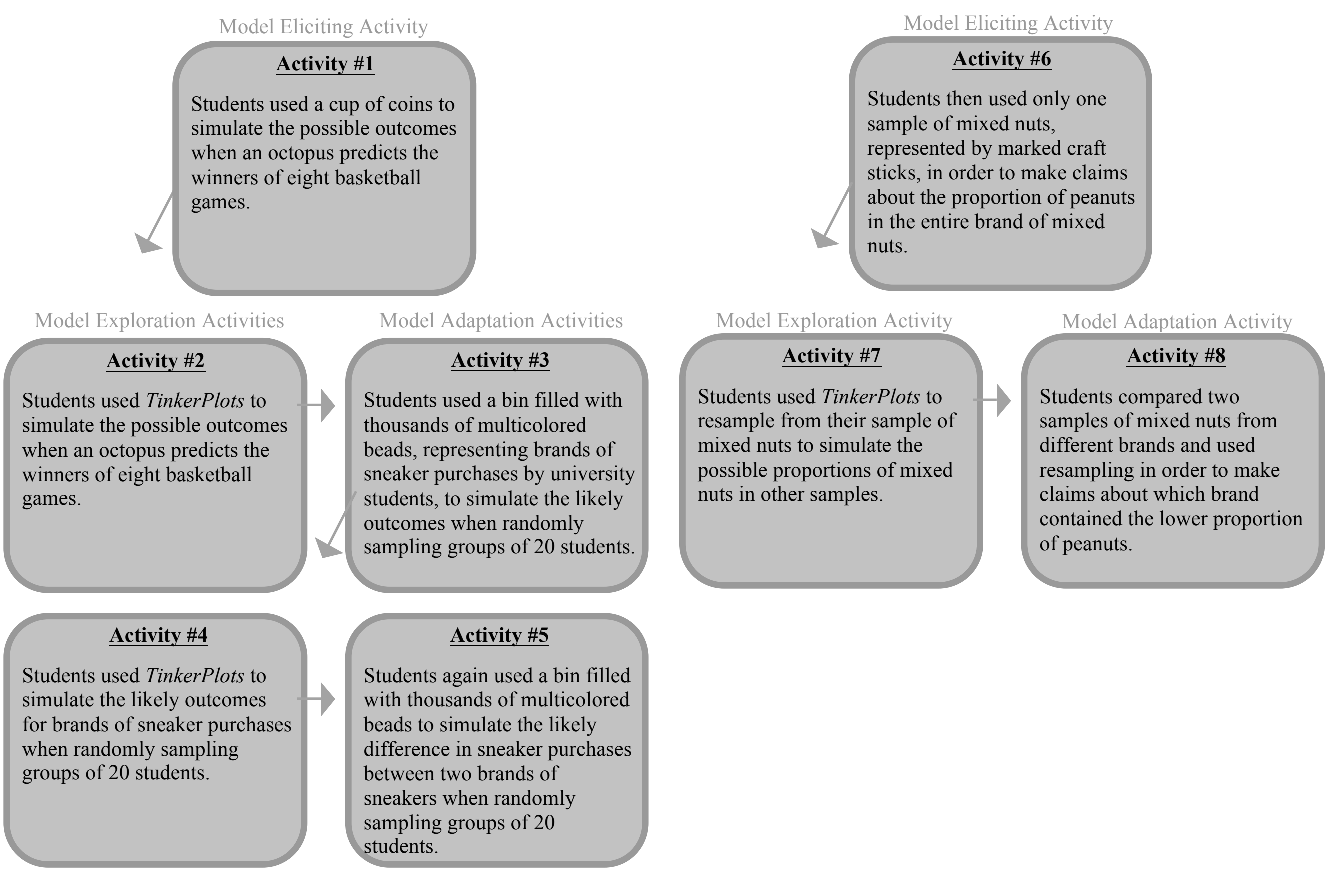

***Figure 1:* Overview of the instructional unit consisting of two model development sequences**

tools with four classes of students at the secondary and tertiary levels. For this study we focus on one group of four students at the tertiary level engaging in the instructional unit. During the unit, the group of students was videotaped and their written work was collected. One student from the group participated in three interviews to discuss her thinking during the instructional unit. The videos, student work, and interview were analyzed with qualitative methods in order to construct the development of the models of sampling used by the participants, and how they were applied to make inferential claims.

The first model development sequence was intended for students to create models that allowed them to draw inferential claims from empirical sampling distributions constructed from repeated sampling from a known population. The second model development sequence no longer had an available population to repeatedly sample from. This put the students in a situation where they needed to extend their models for drawing conclusions by constructing resampling methods to use with the one available sample. Figure 1 provides an overview of the two model development sequences.

Results

We will report the main reasoning, and changes in reasoning, related to the ideas of sampling and informal inferential claims as demonstrated by the four students during the instructional unit. Changes in reasoning occurred both as students progressed through the activities in the unit and also during group and class discussion within the activities.

The first activity in the instructional unit asked the students to determine the likely range of correct predictions made when guessing the winner of eight basketball games. A cup of coins was given to the group as an option to use to draw their conclusions. The group showed an aversion to using the coins to simulate guessing and attempted to calculate the probabilities of each outcome. A class discussion encouraged the group to use eight coins to simulate possible outcomes for the number of correct predictions.

Initial Model of Sampling and Inference

The group simulated the outcomes by flipping eight coins five times. For each group of eight coins they counted the number of heads, which represented a correct prediction. They concluded that because of varying values in their simulations, there is “no definitive answer as to the range of possible outcomes”. This view of one definitive and correct range of possible outcomes was in line with their initial attempts to calculate the probability of each outcome occurring. One student in the group, Megan, was later interviewed to discuss her reasoning in the activity and was still not convinced on the value of using simulations to answer the original question. She had read ahead in the course’s textbook to determine a way of answering the activity through calculations, and constructed a 95% confidence interval with a normal approximation to the binomial distribution.

Second Model of Sampling and Inference

The second activity continued in the same context as the first, but moved on from using coins to simulate outcomes, to the use of *TinkerPlots* (Konold & Miller, 2014). *TinkerPlots* was set up for the students using a spinner with half of the area marked ‘Right’, the other half ‘Wrong’, a window that collected the outcomes simulated by the spinner, and a dotplot that collected the number of correct predictions in each sample. A screenshot of the *TinkerPlots* setup is shown below in Figure 2.

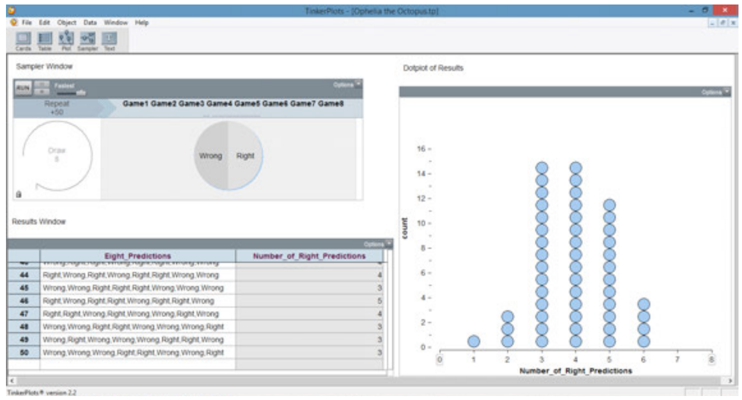

***Figure 2: TinkerPlots* setup for activity #2**

Each group member first used *TinkerPlots* to simulate 10 samples. Together they determined that the outcomes that occurred the most often could constitute a range of likely values. The group discussed how each of their dotplots were slightly different and led them to have varying predicted ranges. After simulating 1000 samples, they discussed how more samples led to each of their dotplots looking very similar and yielding the same intervals of likely values. The group concluded that more samples lead to more accurate predictions. From their dotplots with 1000 samples they determined that between 2 and 7 correct guesses seemed likely. Megan continued to add samples and came to the conclusion that at some point, adding more samples did not change the look of the dotplot. She concluded that the plot was saturated with data.

Third Model of Sampling and Inference

The next two activities involved sampling from a population of university students to determine what number of sneaker purchases out of groups of 20 students were Nikes. This was first done with a bin of thousands of multicolored beads, each color representing the purchase of a certain brand, and then continued in the next activity using *TinkerPlots* to simulate the outcomes. A similar *TinkerPlots* setup was given to the students as before. The spinner was replaced with a mixer containing the same distribution of balls as the beads used as hands-on manipulatives. The students were asked if it was reasonable for Nike to claim that 7 out of 20 student sneaker purchases by the university students were Nikes.

The group used the same methods to construct an empirical sampling distribution as the previous model but the change came with how they drew their conclusions. When using the hands-on manipulatives, all samples except for one showed more than 7 of 20 sneaker purchases were Nikes. A simulation with *TinkerPlots* also showed a majority of the samples falling above 7 out of 20. The students determined an interval of likely outcomes for Nike sneaker purchases and found 7 of 20 to be below the interval. They concluded that Nike should claim that more than 7 out of 20 sneaker purchases at the college were Nikes. They recommended Nike to claim that 10 in 20 purchases were Nikes, which was approximately the center of the distributions.

Fourth Model of Sampling and Inference

The fifth activity was the first time that the students had to deal with the comparison of populations. The context was similar to the previous two activities. Students were asked to investigate if the difference in Nike and Adidas sneaker purchases at a university was the same as the difference in the global sales of 15%, or 3 in 20 students. The same bin of beads was used as in the previous activity, with one color representing Nike purchases and another color, Adidas purchases. The students decided that since they already constructed an empirical sampling distribution of Nike sneaker purchases, they would conduct the same number of samples and count Adidas sneaker purchases. The group chose to calculate the mean values of each distribution and compare them. The means were approximately three purchases apart, which led the students to conclude that the 15% difference in Nike and Adidas global share held true for the college students.

Initial Model of Resampling and Inference

The next change in the students' models occurred during the beginning of the second model development sequence. The first activity put the students in a situation where they needed to extend their models for drawing conclusions by constructing resampling methods to use with the one available sample. The students were told that the manager of bulk food in a grocery store ordered a sample of a new brand's mixed nuts. She plans to order a large shipment of mixed nuts, but has determined that her customers prefer mixed nuts with fewer peanuts. Before she orders, the manager wants to know more information about the percentage of peanuts in this new brand. The students were given a bag of 25 sticks to represent the sample of mixed nuts. Seven sticks were marked with a 'P' to represent peanuts. The remaining sticks were not marked and represented other types of nuts.

The group began by applying a method similar to the previous activities by taking seven samples of five mixed nuts and calculating the average percentage of peanuts. They chose the size of five since it would be easy to take many samples and also to calculate the percentage of peanuts. After some class discussion, they decided to try and take larger samples. Through interactions with the instructor, they found that if they took samples of size 10 in a similar manner to their previous samples, all outcomes were not possible. Since there were only seven peanuts in their original sample, the largest percentage of peanuts in their sample of size 10 would be 70%. After discussing this issue with the class, they decided to take samples of size 10 from their original sample of 25, with replacement. The process of sampling with replacement was more time demanding than sampling without replacement, so the group decided to collect only three samples. They found the average percentage of peanuts in the three samples and concluded that a likely interval for the percentage of peanuts in the population was that average plus or minus an arbitrarily chosen 4%. During an interview with Megan after the activity, we discussed her group's choice of sample sizes of five and 10. She said that larger samples may have provided more accurate results, but would have taken too long to sample with the sticks.

The next activity worked with *TinkerPlots* and gave each group member a different sample of mixed nuts. From these new samples, each member used *TinkerPlots* to collect resamples of 25 mixed nuts (as set up in *TinkerPlots*) and applied similar methods from the first model development sequence to construct a interval of likely values based on the height of outcomes on the dotplot. The group was told the true percentage of peanuts in the brand of mixed nuts, which was a value not included in all of their predicted ranges. Megan chose a much wider range of values than the other students. She concluded that if others did this as well, more of their likely ranges would capture the true percentage of peanuts.

Second Model of Resampling and Inference

The final activity gave the students two samples of mixed nuts from two brands and asked them to conclude which brand had the lower proportion of peanuts. The samples were each of size 25 and contained six and 10 peanuts. Unlike the previous activity comparing sneaker purchases, *TinkerPlots* was available for the students to construct sampling distributions. The group constructed two empirical sampling distributions with 200 samples in each and determined that the first brand likely had 16%-32% peanuts and the second brand had 32%-48% peanuts. These ranges were chosen based on the height of the dotplot for each outcomes. Since the likely interval for the second brand was higher than the other brand's interval (except for the endpoint) they concluded that the second brand is likely to have more peanuts.

During an interview after the activity, Megan and the first author discussed the possibility that both brands had 32% peanuts. Megan concluded that it was unlikely for both brands to have 32% peanuts, but since that was the only value that overlapped between the two likely ranges, she still believed that the second brand was more likely to have more peanuts.

Discussion

The group's model of inference from simulation developed from not drawing a conclusion from simulated data, to using data as evidence to make informal inferences. The first model development sequence was developed as a means for the students to build the necessary tools to draw these informal inferential claims and attempt to apply them to situations in the second model development sequence with an unavailable population. The group constructed some notion of the bootstrapping process by resampling with replacement from their original sample, but did not take resamples that were equal in size to the original sample. The time demanding nature of resampling by hand was noted as one reason for taking smaller sized samples. Immediately after these topics were covered in the class, the course instructor began topics in formal inference. She believed that the students were better prepared for the concepts of confidence intervals and hypothesis testing by participating in the instructional unit. Further research is needed to indicate the connections between the informal inferential models constructed in this study and students' reasoning of formal inference. This study also has implications with the content and design of introductory statistics curricula, and the role of resampling activities on the development of informal inferential reasoning.

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THE INFLUENCE OF A GRADUATE COURSE ON TEACHERS' SELF-EFFICACY TO TEACH STATISTICS

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This paper explores the impact on teachers' self-efficacy to teach statistics from a graduate course aimed to develop teachers' knowledge of inferential statistics through engaging in data analysis using technology. This study uses qualitative and quantitative data from the Self-Efficacy to Teach Statistics Survey (Harrell-Williams et al., 2013) to provide data about teachers' confidence to teach statistical topics. The survey was given to 27 participants from two different institutions before and after the graduate course. We found that participants' self-efficacy to teach statistics increased after participation in the graduate course and references to specific course activities will be identified.

Keywords: Data Analysis and Statistics; Teacher Education-Inservice (Professional Development); Teacher Beliefs

Considerable research has addressed students' statistical thinking (Shaughnessy, 2007), and statistics continues to receive attention in the secondary US mathematics curriculum (NCTM, 2000; Common Core State Standards Initiative, 2010). However, there is a lack of research on secondary teachers' statistical reasoning and beliefs (Batanero, Burrill, & Reading, 2011). In fact, very little is known about teachers' self-efficacy to teach statistics (Harrell-Williams, Sorto, Pierce, Lesser, & Murphy, 2013). This study is situated in self-efficacy for teaching within a graduate course aimed at developing knowledge of the teaching and learning of statistics.

Researchers have investigated the statistical knowledge needed for teaching, using various frameworks (e.g., Groth, 2007). Each of these frameworks has identified teachers' own statistical reasoning as a foundational aspect of their ability to teach statistics. Thompson (1992) argues that researchers should not separate the study of teachers' beliefs from teachers' knowledge since they are intertwined. Thus this study aims to look at self-efficacy to teach as another component of teachers' readiness to teach statistical concepts to their students.

Background and Research Focus

Self-efficacy is often defined as "people's judgments of their capabilities to organize and execute courses of action required to designated types of performance" (Bandura, 1986, p.391). Self-efficacy to teach can be defined as a teacher's "belief to bring about student learning" (Ashton, 1985, p.142). Not only is self-efficacy to teach a central component of a teacher's beliefs (Greshman, 2008; Smith, 1996), it has been has been linked to positive influences on students' learning, the use of more innovative teaching strategies, and time spent teaching certain topics (e.g., Czerniak & Chiarelott, 1990). With these connections, it seems important to improve teachers' self-efficacy to teach. However, it has been suggested that it is hard to impact self-efficacy after teachers enter the classroom (e.g., Smith, 1996).

Bandura (1997) argued that there are four types of sources that may impact one's self-efficacy: mastery experiences, vicarious experiences, verbal persuasion, and physiological responses. For the purpose of this study, the focus is on how mastery experiences impact one's self-efficacy to teach. Mastery experiences are prior experiences in performing a task that are perceived to be a success

(Bandura, 1997). In terms of self-efficacy to teach there are two forms of mastery experiences: classroom teaching experiences and cognitive mastery (Palmer 2011). Arguably, classroom teaching experiences are the most crucial source of self-efficacy to teach because individuals can only assess their ability to teach by participating in the act of teaching (Tschannen-Moran, Hoy, & Hoy, 1998). Cognitive mastery refers to a teacher's perceived success in understanding the content and pedagogy to teach a specific topic (Palmer 2011). The cognitive mastery framework underpins our study to measure the development of self-efficacy to teach statistics from a graduate course aimed at developing subject matter knowledge and pedagogical content knowledge.

Our research is situated within the design and implementation of a graduate course that was largely influenced by Pfannkuch and Ben-Zvi's (2011) recommendations for designing experiences to develop teachers' statistical thinking, as well as the Guidelines for Assessment and Instruction in Statistics Education (GAISE) reports (Franklin et al., 2007; Garfield et al., 2007) and the Mathematical Education of Teachers II report (CBMS, 2012). Over two academic years, a team of four instructors from two institutions designed, and taught, a 15-week course which provided participants opportunities to develop a deeper understanding of a few statistical ideas. Two instructors taught at one university while the other two taught at the other university creating as similar of a course as possible at both institutions through continuous co-planning and reflection.

Throughout the semester-long course, participants implemented the cycle of statistical investigation (Friel, O'Connor, Mamer, 2006) as they engaged in with real data and tasks designed to develop their understandings of variation, distribution, samples and sampling distributions, and inferential statistics, especially randomization approaches using simulations. The course used dynamic software, *Fathom* (Finzer, 2005) and *TinkerPlots* (Konold & Miller, 2011), and online applets such as StatKey (lock5stat.com/statkey/). Assigned readings and discussions centered on (a) the nature of statistical reasoning and how it compares to mathematical reasoning, and (b) students' learning and reasoning related to the aforementioned topics. The software tools, new to most participants, were used to support their learning by allowing them to flexibly explore graphical representations, easily compute statistical measures, compare data sets, and make changes to data to explore conjectures. The software also provided simulation tools necessary to create representations of a population, a sample, and an empirical sampling distribution. This study addresses the following questions: 1) To what extent is teachers' self-efficacy to teach statistics changed from a graduate course focused on teaching and learning statistics? and 2) What learning experiences do teachers identify that influenced their self-efficacy to teach statistics?

Methodology

Participants

Participants came from all the teachers participating in either course across the two institutions. The course served a variety of graduate students (n=27). Participants consisted of one undergraduate pre-service teacher, six pre-service teachers in a masters program; 11 in-service teachers enrolled in a master's program; one full-time master's student in mathematics education; and eight doctoral students in mathematics or mathematics education. Twenty-one participants were female and six were male. Six participants indicated that English was their second language. Most participants had completed the equivalent of an undergraduate degree in mathematics, and all but two had at least one prior course in statistics. Hereafter we refer to course participants as teachers.

Data Collection and Analysis

To examine changes in teachers' self-efficacy to teaching statistics, the Self-Efficacy to Teach Statistics (SETS) survey was administered (Harrell-Williams, Sorto, Pierce, Lesser, & Murphy, 2014). This survey was chosen because it collects both qualitative and quantitative data about teachers' self-efficacy to teach statistics. Researchers argue both data sources are needed within the self-efficacy research (Wyatt, 2014). SETS was administered prior to the first day of class and during the last week of class. The SETS survey contains 44 six-point Likert scale items and six open response items that are aligned with the GAISE framework (Franklin et. al., 2007). An earlier version of this instrument was validated for use in measuring changes in elementary and middle grades preservice teachers' self-efficacy as a result of interventions, such as a course (Harrell-Williams et al., 2013). In addition to an overall score, the instrument provides sub-scale scores that correspond to Levels A-C in the GAISE framework. Although there are not explicit definitions given for each level in the GAISE report, each level is aligned to specific content. The content in level A is considered more concrete and level C is considered the most abstract. For example, in level A students are asked to compare groups without generalization while in level C students answer comparison questions and make generalizations (Franklin, 2007). There were 11 Likert items for level A, 15 items for level B and 18 items for level C. For all Likert items, the stem of the question was "Rate your confidence in teaching high school students the skills necessary to complete successfully the task given by selecting your choice on the following scale: 1 = not at all confident, 2 = only a little confident, 3 = somewhat confident, 4 = confident, 5 = very confident, 6 = completely confident" (Harrell-Williams et al., 2014). For the open-ended portion of SETS, in each GAISE level category, teachers were asked to identify an item which they felt least and most confident to teach to high school students and to explain their reasoning (total of six open-ended items).

The analysis of the SETS data was completed in two phases. The first phase focused on the responses to the 44 six-point Likert scale items. For both the pre- and post-survey, each teacher was given a total score calculated as the sum of his/her Likert scores. Sub-scale scores were also calculated for each teacher. The totals were divided by the number of items, which resulted in a final score that corresponded to the six-point Likert scale. Additionally, a gain score was calculated for each teacher as the difference of pre- and post- scores for each item. Means were computed for pre, post, and gain scores and a Wilcoxon Signed Rank Test for Matched Pairs was conducted to test for the significance from pre to post. Finally, the gain scores were averaged for each teacher and for each item. The item averages were examined for highest average gains in relationship to course content. Teachers with missing values within specific calculations were removed from the sample for that calculation.

The second phase of data analysis focused on analyzing the open-ended responses for ways in which the course influenced teachers' self efficacy to teach statistics. The code "course" identified responses that explicitly referred to specific course activities.

Results

First, we report the extent of the change in self-efficacy to teach statistics through the quantitative data from the pre and post SETS survey. Second, we report our findings from the qualitative responses to identify the course experiences that the teachers identified as influencing their self-efficacy to teach statistics.

Influence of Professional Development on Self-efficacy

We investigated the general influence of the course on self-efficacy to teach statistics by examining the mean scores for the pre and post survey and the mean gains by teacher and by item. Teachers began the course between somewhat confident (score of 3) and confident (score of 4) for each item; however, the teachers finished the course describing their self-efficacy to teach statistics as between confident and very confident (see Table 1). With the exception of one teacher, all teachers showed a positive average item gain in self-efficacy to teach statistics. The highest average item gain was 1.68 Likert points, which was recorded by two teachers. Figure 1 shows the distribution of average item gains by teacher. On average, teachers' self-efficacy for to teach statistics for each item increased one Likert point (0.95 increase). After accounting for missing values and using a Wilcoxon Signed Rank Test for Matched Pairs, the increase in self-efficacy between the pre and post surveys is considered statistically significant (see Table 2).

Table 1: Descriptive statistics of Likert scale items

		Pre	Post	Gain
Overall	N	26	24	23
	Mean	3.62	4.65	0.95
	Standard deviation	0.82	0.64	0.49
Level A topics	N	26	27	26
	Mean	3.95	5.10	1.14
	Standard deviation	0.80	0.51	0.55
Level B topics	N	27	26	26
	Mean	3.75	4.70	0.94
	Standard deviation	0.81	0.71	0.55
Level C topics	N	27	25	25
	Mean	3.39	4.35	0.97
	Standard deviation	0.96	0.74	0.63

Similar results were found for all three GAISE sub-scores. Level A had the highest pre score average of 3.95. This suggests that teachers started out confident in their ability to teach topics within that level. Interestingly, these topics are also the areas were teachers' confidence grew the most with a statistically significant (Wilcoxon signed rank test, $p>0.001$) average gain of 1.14. Level B had a pre survey mean score of 3.75 and a post survey mean of 4.70. Accounting for missing values, the average gain for level B was 0.94, which was also statistically significant. Finally level C started at the lowest confidence at 3.39, implying that most teachers on average feel only somewhat confident in their ability to teach statistics. The post mean score was 4.35, which is a growth in confidence to teach the level C topics in statistics. The average growth for level C was 0.97 points. Similarly, according to a Wilcoxon signed rank test this growth was statistically significant (Table 2). In

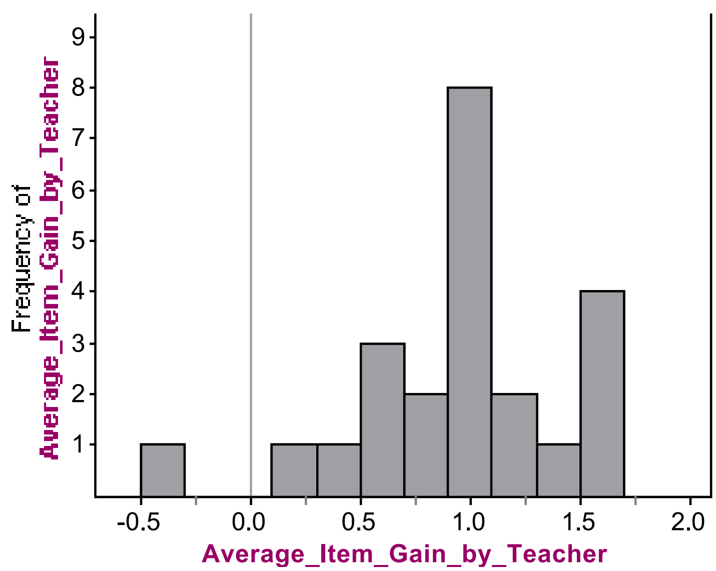

Figure 1: Distribution of Each Teachers' Average Likert Item Gain

Table 2: Wilcoxon Signed Rank Test for Matched Pairs

	Test Statistic	p-Value
Overall	4.14	0.000
Level A topics	4.44	0.000
Level B topics	4.46	0.000
Level C topics	4.07	0.000

addition to average gains at all levels, the standard deviation decrease at all levels indicating a decrease in variability in post confidence ratings.

Examining the average gain by item shed light on the specific content aligned to the teachers' growth in self-efficacy to teach statistics. The items that showed the greatest gain on average across teachers related directly back to the course goals: Item 44 (*Determine if the difference between two population means or proportions is statistically significant using simulations*) had an average gain of 1.85 and Items 9 (*Generalize a statistical result from a small group to a larger group*), 37 (*Evaluate whether a specified model is consistent with data generated from a simulation*), and 43 (*Compare two treatments from a randomized experiment by exploring numerical and graphical summaries of data*) all had an average gain in self-efficacy to teach of 1.42 points. All four of these items address the course foci of inferential statistics using randomization approaches, sampling distributions, and variation. The item with the lowest overall gain (0.48) was Item 33 (*Fit an appropriate model using technology for a scatterplot of two quantitative variables*), which was not a topic explicitly addressed during whole-class activities or discussions within the course.

Teachers' Reflection on Learning Experiences

In the open response items of the SETS instrument used at the end of the course, teachers identified course experiences when describing what, in each level, they felt most confident about. At both institutions, the course began with lengthy discussions on the cycle of statistical investigation (Friel et al., 2006). This cycle became a theme of the course as teachers gained repeated experience with posing statistical questions, collecting data, and data analysis and interpretation. Early in the course, teachers had opportunities to deepen their understanding of distribution through a series of tasks related to interpreting graphical representations. One such task asked teachers to match box plots to corresponding dotplots. This task revealed that a given boxplot could have underlying dotplot distributions that looked somewhat differently. About this activity, one teacher wrote

> "I feel most confident about working with box plots; the [activity] showed both the advantages and disadvantages of boxplots and [how] we can use them to describe data."

Based on research by delMas and Liu (2005), a second task teachers experienced was a game in *Fathom* designed to enhance teachers' conceptualization of the relationship between a distribution and its standard deviation. Teachers remembered this game at the end of the course. For example,

> "After doing the activity of "What Makes the Standard Deviation Larger or Smaller", I noticed a couple of patterns for justifying the characteristics of normal distributions with different centers, shapes, standard deviation, and so on."

As the course progressed, simulations became a means by which teachers developed understanding about variability and sampling distributions. The SETS item (44), which showed the greatest gain in self-efficacy focused on simulations. In the open response items, teachers remembered learning from the simulations with and without technology:

> "[Simulation] is something that we spent a lot of time on in the course. There are a lot of different ways to approach [it] with students such as hands-on simulations or technology simulations."

One hands-on simulation used physical devices, some of which did not have equiprobable outcomes. In the activity, each group had to describe a repeatable action that could produce an unpredictable result and the possible outcomes from this repeatable action. After the event(s) of interest was selected, for which results could be examined from the repeatable actions, each teacher in the group collected a sample of 10 trials. The activity continued with more samples being collected and a sampling distribution being created. While it is a familiar activity for statistics educators, it was not for the participating secondary teachers. One teacher wrote,

> "I liked the activity we did in class of having each person collect data for a sample of 10…I think I have a good conceptual understanding of the relationship between samples, distributions of samples, and populations."

The simulation focus continued as the course ended with randomization techniques. One teacher shared that she

> "already knew about randomization tests, but I feel more confident having multiple pieces of software that can perform the simulation for me. Before I was just using statcrunch and showing my students the output, but now I can actually have them do it!"

A specific course experience referenced in the SETS open responses was the Dolphin Therapy task (Rossman, 2008). This task required a re-randomization technique to test the difference of proportions. Teachers were given index cards to use in the design and simulation of the problem. Eventually, they used StatKey and TinkerPlots for a greater number of samples.

Another course experience that was highlighted by teachers in the open responses of the SETS survey was the course mid-term project. For the assignment, teachers self-assigned themselves to a working group. Each small group examined best practices for teaching learning a specific statistical topic. They applied research literature to create or adapt meaningful tasks and implemented one task with a group of students. Projects were shared through oral presentations and a course wiki. Topics for the midterm project included: Comparing Distributions, Sampling Techniques and Study Design, Sampling Distributions, Hypothesis Testing, Linear Regression/Covariation and Correlation, Using Probability to Make Decisions, Subjective Probability and Bayes Theorem.

The course experiences described above were ones specifically linked by teachers to content in which they felt most confident. In the survey, teachers were also asked to identify particular areas where they felt least confident. Mostly, teachers responded with comments such as *"these items were not specifically discussed in the course" or "I do not think I had a lot of practice … in the course."* Other times, however, teachers provided more insight into particular self-reported deficiencies (e.g. box plots, error, randomization, inference, sampling). Several teachers even suggested that they wanted more time with topics or would continue to refer to course materials to develop a deeper understanding. One teacher wrote,*"Sampling!!! I don't feel very confident teaching it yet. I began to develop a better internal understanding of it in class. I wish I could study it some more in a similar environment as was created in [my course]."* And, another teacher wrote, *"I am confident that randomization is highly important but I still second guess myself...Since I second guess myself, I am somewhat confident because I at least know that I have resources that I can reread."* Despite the fact that all teachers showed gains in self-efficacy overall, the open-ended responses provided details for instructors at each institution regarding potential pivotal experiences for teachers' own development of statistical understanding during the graduate course that seem to influence their statistics teaching efficacy.

Discussion and Conclusions

The results indicate that the course had a statistically significant positive impact on teachers' self-efficacy to teach statistics. These results were seen on the overall level and at all three GAISE levels. This suggests that a graduate course focused on the teaching and learning of statistics can impact a teachers' self-efficacy to teach statistics, and furthermore suggests that teachers can gain in self-efficacy to teach statistics from focusing on content knowledge and pedagogical content knowledge for teaching statistics. Additionally, our data also show that teachers have decreasing confidence to teach from level A to level C. This result holds for both the pre and post surveys and suggests that more abstract material corresponds to lower self-efficacy to teach. This result is similar to results found by Harrell-Williams et al., (2013) with pre-service teachers.

In addition to an overall gain in self-efficacy to teach statistics by the teachers, specific gains related to the course and its objectives were seen. After examining the data by SETS items, large gains were seen on topics related to the course objectives of inferential statistics, sampling distributions, and variation. Additionally, many teachers' mentioned course activities as reasons for their increase in self-efficacy to teach these topics. This speaks positively to the design of these activities and suggests that some course activities can have powerful impacts on teachers' confidence to teach statistics. These seem to be serving as a key mastery experiences.

However, not only did the areas that were emphasized in the course get impacted. There was an average increase on all items on the SETS survey including those that were not specifically stressed in the course. One possible source for this growth could be the course projects that allowed students to investigate topics of their choosing.

In general, these results point to specific activities that work to increase self-efficacy to teach statistics with teachers. Further research needs to be conducted to better understand what type of activities and how these activities are impacting teachers' self-efficacy to teach.

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A METHOD FOR DESCRIBING THE INFORMAL INFERENTIAL REASONING OF MIDDLE SCHOOL STUDENTS

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A primary obstacle in the study of how students' informal understandings of statistical inference can be developed and leveraged into formal statistical understandings has been the lack of a method for describing and categorizing student thinking. In this research an adaptation was made to the Structure of Observed Learning Outcomes (SOLO) taxonomy to focus on Informal Inferential Reasoning (IIR). The adaptation evolved through two iterations of refinement and validation that included classroom observations and task-based interviews as part of a larger project that also developed an assessment for the purpose of eliciting students' IIR.

Keywords: Data Analysis and Statistics; Middle School Education

Focus of the Study

Statistical literacy has become an integral part of being an informed citizen, making informed personal choices, being a professional in the modern workplace, and working in the sciences (Franklin et al., 2005; Zieffler, Garfield, Delmas, & Reading, 2008). In response to this, there has been an increased emphasis on statistics education both in the US (National Council of Teachers of Mathematics, 1989, 2000; National Governors Association Center for Best Practices [NGACBP] & Council of Chief State School Officers [CCSSO], 2010), and internationally (Leavy, 2010; Makar & Rubin, 2009; Shaughnessy, 2007). Informal Inferential Reasoning (IIR), defined by Zieffler et al. (2008) as "the way in which students use their informal statistical knowledge to support inferences about unknown populations based on observed samples" (p. 44), has emerged as a foundation on which students can build their formal statistical understandings. Research has shown that IIR does not develop with maturation or increased content knowledge (Jacob, 2013; Zieffler et al., 2008), but it can be intentionally developed (Jacob, 2013; Langrall, Nisbet, Mooney, & Jansem, 2011; Makar, Bakker, & Ben-Zvi, 2011).

Underlying Framework

Research suggests that statistical literacy is a hierarchical construct (Watson & Callingham, 2003). The SOLO taxonomy is a neo-Piagetian model of looking at hierarchical learning from a cognitive perspective (Shaughnessy, 2007) developed by Biggs and Collis (1982). Because of its strength in describing hierarchical learning it has received a significant amount of attention in statistics education research.

SOLO cycles represent progress in a student's ability to reason with increased complexity about a topic, and are comprised of the following levels: Pre-structural (P), Uni-structural (U), Multi-Structural (M), Relational (R), and Abstract/extended abstract (A). The taxonomy is based on identifying how many methods for accessing a topic a student has. At P, the student has no cognitive structure through which they can access the objective. At U a student has a single relevant way of accessing the objective, and at M the student has more than one (though they are not coordinated). At R the student is able to coordinate different ways of accessing the objective, and at A they can make abstract statements about the objective (Watson & Moritz, 2000). In looking at the development of a concept, P is the beginning point and A is the ending point; however, as the concepts being looked at grow more complex it becomes necessary to use more than one cycle. When this occurs, the A level of one cycle can be considered the U level for a cycle that looks at more complex thought, resulting in the structure of $P\text{-}U_1\text{-}M_1\text{-}R_1\text{-}U_2\text{-}M_2\text{-}R_2\text{-}A$ for what is referred to as a two-cycle model (note that the letter remains the same but a subscript is used to denote which cycle is being referred to). These two-cycles are used widely (see Reading & Reid (2006), Watson & Moritz (2000)), and in response to a question at a conference presentation Reading suggested that a two-cycle SOLO-based framework could be used to describe the development of IIR (Zieffler et al., 2008),.

In order to apply the SOLO taxonomy there was also a need for clearly articulating the types of understandings that would be investigated. Zieffler et al.'s (2008) assessment framework was selected for this purpose. In this framework, IIR is identified as having three components: making judgments or predictions, using or integrating prior knowledge, and articulating evidence-based arguments (Zieffler et al., 2008). It also identifies three types of IIR tasks: Estimating and Drawing a Population Graph (EPG), Comparing Two Samples (CTS) of data, and Choosing Between Two Competing Models (CBM) (Zieffler et al., 2008), which were used in building the assessment and became important in classifying student reasoning.

Methodology

This research was conducted at a rural Midwestern middle school (grades 6-8) with approximately 900 enrolled students, 11% minority students, 16% qualified for the free or reduced-price lunch program. The study involved three cooperating teachers and 188 students. The research progressed through four phases: an initial development and modification phase, two modification phases, and a validation phase.

In the initial development phase a theoretical SOLO adaptation was constructed based off of the frameworks. Eight students were selected from the first teacher's classes using striated random sampling in order to ensure students at various levels of mathematics achievement were represented. The students completed an assessment designed to elicit their IIR, and subsequently participated in task-based semi-structured interviews to further explore their IIR abilities. Open coding was done on the responses from both the interviews and assessments, and the results were used to refine the cycles, cycle definitions, and assessments.

The first modification phase occurred at the beginning of the next school year with a separate group of students and the two remaining teachers. The assessment that was modified during the initial development was administered to 180 students as a pre-test to a unit on statistical inference, and student responses were used to ensure saturation of the codes from the initial development. The results were used to further refine the cycles, cycle definitions, and assessments. The results of the coding were also used to identify key examples that would be included with the framework.

The second modification phase occurred two weeks later, during the unit on inference. Sixteen students were selected using striated random sampling to include students of diverse IIR ability levels as identified by the pre-test. Students were placed into mixed-ability groups and data was collected using LiveScribe pens and by retaining written and digital student work during a six day unit on inference. Student conversations and artifacts were used to ensure saturation of the codes from the first modification. The results were again used to further refine the cycles, cycle definitions, and assessments, and to refine the list of key examples.

The validation phase occurred during the two weeks after the second modification phase, at the end of the unit on inference. The 180 students from the first modification phase took the final version of the assessment, and the 16 students from the second modification phase also participated in task-based semi-structured interviews. The results were used to identify whether any further modifications were needed to the SOLO adaptation or the assessment.

Results

During development a SOLO adaptation was posited that aligned with Reading's recommendations of a two-cycle model with one cycle dealing with naïve IIR and the second with appropriate IIR. Assessment and interview data was collected on eight sixth-grade students, and six of the eight students were classified as P, meaning their IIR ability was insufficient to place them into the Naïve cycle. However, amongst those six students there was a clear pattern; while they were all incapable of reasoning at the Naïve level, structures to their reasoning did emerge in the open coding. Those structures were compatible with the three components of IIR posited by Zieffler et al. (2008), and it became clear that there was value in extending the framework to lower levels of reasoning.

The data from the two modification phases supported the idea that the three question types tested three distinct concepts in IIR that students had developed at different rates. The impact on the SOLO adaptation was that it needed to be flexible enough to be applied to these three distinct concepts within IIR while still having level indicators that were defined clearly enough for inter-rater reliability. The bulk of the modifications in these phases involved balancing the specificity and generalizability of the cycle indicators. The resulting framework is presented in Table 1. Examples from EPG are presented in Table 2, while examples from CTS, and CBM question types were omitted for space but are available on request.

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Table 1: Final SOLO Cycles and Indicators

Construct Cycle	Use of Variability	Use of Context	Certainty and Argumentation
Pre-Stuctural No Cycle		Modifies context to answer question	
Pre-IIR Cycle 1	No concept or use of variability	Answers based solely on context.	Deterministic language
Naïve-IIR Cycle 2	Limited conception or use of variability	Over-use of context over data or vice-versa.	Probabilistic language with minimal impact on conclusions
Appropriate IIR Cycle 3	Appropriate use of variability	Considers both context and data in strategies and solutions	Probabilistic language with uncertainty expressed in solution

Table 2: Solo Taxonomy Examples and Codes for EPG Question Type

SOLO Code	Estimating a Population Graph (EPG) Structure example	Structure Code
Pre-Structural	Expanding/personalizing context to provide answers.	Personal-context
	Student selects numbers from the context and/or problem and combines them arithmetically in an inappropriate or unfocused way	Undirected-arithmetic
U_1	Increasing all data values by a fixed amount.	Linear vertical-only
	Increasing all data values to level off the graph	Leveling-off-data
M_1	Using both strategies in U_1 independently	
R_1	Coordinating both U_1 strategies.	
U_2	Doubling existing data value frequencies	Doubling
	Proportionally increasing existing data values without including data at new values	Proportional vertical-only
M_2	Using both strategies in U_2 independently	
R_2	Coordinating the use of a U_2 strategy with consideration to the context	
U_3	Growing the sample proportionally both vertically and by including appropriate additional data values.	Proportional vertical-horizontal
	Student explanations include appropriate statements of uncertainty that reference (explicitly or implicitly) measures of spread or center.	Appropriate variability
M_3	Using both strategies in U_3 independently	
R_3	Coordinating the use of U_3 strategies.	

PRE-SERVICE TEACHERS' UNDERSTANDING OF ASSOCIATION AND CORRELATION IN AN INQUIRY-BASED INTRODUCTORY STATISTICS COURSE

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To teach statistics at the K-12 levels with success requires that preservice teachers (PSTs) to not only understand the statistics concepts, but also to make connections among them. Statistics educators call for the use of real data, technology, and active learning in an introductory statistics course for PSTs. But these components do not ensure desirable learning outcomes. This presentation provides an exemplar of such an content course, and presents a preliminary framework and qualitative analyses of pre-service teachers' activities related to the concepts of association/correlation in an inquiry-based and technology rich environment.

Keywords: Teacher Education-Preservice; Technology; Teacher Knowledge; Post-Secondary Education

The education community has recognized the importance and challenges of preparing future teachers to teach statistics (Committee on the Undergraduate Program in Mathematics, 2004; Conference Board of the Mathematical Sciences, 2001). Accordingly, the Guidelines for Assessment and Instruction in Statistics Education (GAISE) Project have produced the college report (GAISE, 2005) to provide six recommendations for educators to teach the introductory statistics courses and described 23 desirable student outcomes. This paper presents a snapshot of a content course for pre-service teachers (PSTs) that is designed to echo the call from the GAISE college report: incorporating a variety of data sets that are relatable to teachers' professional life and school life, using an educational software package that is relevant for teaching, and fostering active learning by engaging PSTs in statistical inquiries. By exploring PSTs' activities and their understandings of statistical concepts resulting from using real data and technology in an active learning environment, the realization of this content course and PSTs' learning outcomes in the light of the goals of statistics education as delineated in the GAISE college report is examined, which help fill in the gaps between the standards and their enactment.

In this paper, I present a case study of two groups of PSTs, and scrutinize the ways in which they use technology to engage in a small group activity related to the statistical concepts of association/correlation. *Statistical Association* is about the general relationship between any two or more variables, while *correlation* is specific to measuring the association between two *quantitative* variables to see the strength and direction of the relationship. The former concept is more inclusive than the latter one. In the existing literature, there are more studies on the (mis)conceptions of association in general, with seldom discussion of correlation explicitly.

In the literature, however, the problems used are mostly about finding the association between two or more categorical variables in a contingency table rather than quantitative variables, and the type of variables and its connection to the type of relationship one should look for are not explicit and highlighted. Moreover, these studies only asked the participants to find the statistical association between given variables in different problem settings, rather than asking them to pose a statistical question first and then to select the variables to answer the question. This study can also fill in the gaps of literature on the conceptions of association and correlation by connecting these two concepts, and to other statistical concepts, such as the type of variables.

Based on the literature and the vision of an introductory statistics course described above, I asked two research questions:

- What are the characteristics of PSTs' small group activities with the use of real data and the use of *TinkerPlots* (Konold & Miller, 2005) in an inquiry-based statistics content course?
- How well do PSTs understand and reason about the concepts of correlation/association as a result of learning, given such a context?

Method and Data Analysis

Statistics for Elementary Teacher is a 4-credit course for PSTs in the elementary teacher preparation program at a mid-west research university. The course in this study consisted of 28 sophomore and junior PSTs from social studies, literature, science, mathematics, and special education. The course was a mix of lectures and activities. Data include videos of students' on-screen activities and their talks recorded by a software package installed on the computer, and PSTs' written work. Video recordings were transcribed, and screen-shots were taken to serve as figures in the transcript for illustrative purposes. For this paper, two groups of four PSTs working on a 40-minute task were selected to illustrate the characteristics and patterns of PSTs' activities using technology and a real data set.

The Australia Student Data Task

Pre-task Instruction. The task happened in the second lesson of data analysis. The instructor had introduced graphing and using numerical indicator in regression as two ways to explore the *relationship* between *two* variables. The instructor elaborated on "correlation does not imply causation", and the idea that association can also mean a non-linear relationship between two variables. Also, he discussed other possibilities of the observed *association* such as chance or the existence of a lurking variable.

The Activity. The PSTs were given a data set with 29 attributes of 159 students in south Australia. They were asked to explore the data and formulate three questions that they could answer with these data, including one that compared two or more groups, and one that examines the *associations* between two quantitative variables. They were also encouraged to support their conclusions with the representations that they generated and with descriptions of statistics, and to discuss the possibilities of whether the *correlation* could imply a *causal relationship*.

Theoretical Framework

The characteristics of PSTs' classroom activities with the technology are operationalized by adapting the Model of Learning in Exploratory Data Analysis framework (Fitzallen, 2013). The framework consists of dimensions with *key behaviors* within each dimension. The dimensions are *Generic Knowledge*, *Being Creative with Data*, *Understanding Data*, and *Thinking about Data*, which come from previous research on statistical representation comprehension (Friel, Curcio, & Bright, 2001; Kosslyn, 1989; Moritz, 2004) and statistical thinking and reasoning frameworks (Shaughnessy, 2007; Wild & Pfannkuch, 1999). As a result, this framework is closely aligned to the GAISE report in terms of how the statistics community envisions the statistics education. In the coding process, new and repeatable *key behavior* that exemplifies each dimension was added to the list, and the descriptions of such codes were provided for iterative coding. I used the framework to code the transcripts, and themes related the student outcomes (of the lack thereof) in the GAISE report were surfaced. More analysis is underway for a wider scope, studying more activities across the spectrum of statistical topics. With more activities analyzed, a more comprehensive framework with new codes will be developed to capture the activities of PSTs in a technology-rich and inquiry-based statistical content course.

Results

The two groups of recorded PSTs engaged with the data at many different levels. Most of the time, they were (a) using their generic knowledge of the context and their technology knowledge of the software; and (b) engaging in basic transnumerations, including changing the scale of the axes, and color coding the icons in the plot by the third variable. The table below is a comprehensive account of these characteristics of their interactions:

Table 1. Percentage Of PST's Interactions At Each Dimension And Key Behaviors Within Each Dimension In The Australia Student Data Activity

Dimensions	Counts	Percentage	The two most frequent *Key Behaviors*	Percentage of The Two *Key Behaviors*
Generic Knowledge	N=26	45.61%	Speaking the language of the data and the graph	65.38%
			Understanding how to use the features of software and technology environment	19.23%
Being Creative with the Data	N=16	28.07%	Describing data from graphs	50.00%
			Constructing different form of graph	25.00%
Understanding the Data	N=8	14.04%	Making sense of data and graphs	62.50%
			Answering questions about the data	37.50%
Thinking about the Data	N=7	12.28%	Interpreting the data	57.14%
			Recognizing the limitation of the data	42.86%
	Total = 57			

In addition, patterns emerged from the quality of the salient *key behaviors*. In the next section, some salient characteristics of the activities that are astray from the goals of statistics education (as portrayed by the GAISE reports), and the misconceptions of association/correlation are described.

Anecdotes Mask Data as Opposed to Data Beats Anecdotes

The PSTs brought their own contextual knowledge extensively into their interactions, particularly when the task had open-ended questions (e.g., when a task did not specify which variables to analyze). Some anecdotes put forward by the PSTs were relevant to the discussion. These anecdotes helped the group choose attributes to consider, contributed to explanations of the variables as displayed on the graph, or helped them make sense of the data. But often the PSTs began sharing personal anecdotes irrelevant to the data, and got distracted by the extreme value in the data set without raising any question about the potential mistake in the data collection process. For example, one group spent quite some time to discuss a data point (30 hours of exercises per week) and their own exercise habits. Also, the PSTs did not persist in examining representations when the representations did not conform to their preconceptions of the relationships between the variables based on their contextual knowledge. For example, one group generated 27 different representations without scrutiny, and none of them "can be presented", according to them, in the whole class discussion that followed.

Misuse of Variables in Finding Statistical Relationships

The PSTs showed robust misunderstanding of association/correlation and the type of variables that they should be looking for. Both groups used correlation and association interchangeably in their discussions. As a result, they did not look among quantitative variables consciously to find potential correlation between variable pairs. It was evidenced in their discussion that they had many misconceptions on the variable types, and how types relate to the relationship between variables. Accordingly, even when they were instructed to look for quantitative variables, they would select a qualitative variable, or they tried to convert a qualitative variable to a quantitative one just by modifying the label on the axes to be numeric. Some features of *TinkerPlots* also emerged to contribute to PSTs' confusion about the variable type. For example, regardless of the variable types, the data points were put into bins in the graph by *TinkerPlots* after the PSTs selected the variables, which sometimes let the PSTs see visually how "qualitative" data could be "transformed" into quantitative ones by spreading out the points.

Discussion

The PSTs did not display most of the misconceptions discussed in the literature (e.g., Batanero, Estepa, Godino, & Green, 1996; Batanero, Estepa, & Godino, 1997). But it is evidenced in these two groups of PSTs' discussion that they could not consciously choose the right type of variables to explore the possible relationship, and they seemed to confuse *correlation* and the more inclusive concept *association*, and sometimes even equate them. Continuous variables can be transformed into categorical ones (e.g., age), if needed, but vice versa does not always hold (e.g., race). The lack of experience in collecting and cleaning data may render PSTs less knowledgeable in this statistical concept of variables, and eventually hinder their profound understandings of other related statistical concepts. The presence of technology and statistical inquiry using real data as advocated by the research community does not automatically ensure desirable learning outcomes in PSTs. By studying PSTs' understanding of various topics in statistics in such an environment, we can refine the process of the implementation of an introductory content course by making explicit the necessary connections among certain statistical concepts, and by refining our statistical discourse to make distinctions among different concepts during class discussions.

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MATHEMATICS TEACHERS' APPROACHES TO STATISTICAL INVESTIGATIONS WITH MULTIVARIATE DATA SETS USING TECHNOLOGY

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The purpose of this qualitative case study was to provide rich descriptions of mathematics teachers' approaches to statistical investigations using technology. Twenty participants completed parallel statistical investigations individually at the beginning and end of a course on teaching and learning statistics. Additionally, participants completed two group investigations during the course. The tasks provided different opportunities for participants to engage in statistical investigations using technology. A hybrid qualitative analysis including both deductive coding and inductive coding was used to characterize teachers' approaches.

Keywords: Data Analysis and Statistics; Technology; Teacher Education-Inservice; Teacher Education-Preservice

Introduction

New school mathematics standards in the United States have an increased focus on statistics and data analysis (Common Core State Standards Initiative, 2010), while statistics education researchers have called for an emphasis at the secondary level on statistical investigations (Franklin et al., 2007). Dynamic statistical software developed specifically for teaching and learning statistics exists, but has not yet been widely adopted in schools. There is general consensus regarding the lack of teacher preparation and professional development to teach statistical investigations (MET-II, 2012; ASA & NCTM, 2013) as well as the need to ensure that teachers have adequate statistics content knowledge and opportunities to engage in all stages of a statistical investigation (Franklin et al., 2010). Recommendations included in the MET-II report include a statistics and probability course for teachers emphasizing active learning with hands-on devices (middle school) and technology (high school) focused on a four-step process of statistical investigation.

Since few teachers have had such a course or experienced learning statistics in the ways recommended by the GAISE and MET-II reports, research is needed to understand how to best prepare teachers to teach new statistical standards utilizing the recommended pedagogical practices. Literature in mathematics and statistics education has addressed student and teacher knowledge related to various statistical concepts, but there is a need for a study that examines teachers' approaches to an entire statistical investigation cycle while using technology. The purpose of this study is to describe and characterize teachers' approaches to statistical investigations with multivariate date sets using technology throughout a graduate level course on teaching and learning statistics.

Theoretical Background

There are many descriptions of statistical investigations in the statistics education literature. All agree the process is non-linear and involves going back and forth between various phases. Wild and Pfannkuch's (1999) statistical thinking framework included an investigative cycle adapted from MacKay & Oldford's (1995) PPDAC cycle (Problem, Plan, Data, Analysis, Conclusions). Additionally, Wild and Pfannkuch's (1999) description of five types of statistical thinking, (a) recognition of the need for data, (b) transnumeration, (c) consideration of variation, (d) reasoning with statistical models, and (e) integrating the statistical and contextual, guided the evaluation of teachers' work during a statistical investigation.

Recognition of the need for data to make decisions refers to acknowledging the inadequacy of personal experiences and anecdotes for drawing conclusions and making decisions (Wild & Pfannkuch, 1999). Snee (1999) described recognizing and collecting appropriate data as knowing which data are relevant. Chance (2002) added the key question, "Have we collected the right data?"

Transnumeration is a dynamic process of collecting, forming, and changing data representations to engender better understanding of a process or system (Wild & Pfannkuch, 1999). Transnumeration is involved in selecting appropriate measures, displays, and statistics for presenting and describing data. Chick (2004) described transnumeration as choosing the most appropriate representational technique. Shaughnessy (2007) suggested that different representations of data could reveal new information about the data that were previously hidden.

Consideration of variation means anticipating and recognizing the role of uncertainty and randomness in all processes (Wild & Pfannkuch, 1999). This consideration includes recognizing potential causes and sources of variation as well as knowledge of how to act on variation (Noll, 2007). Shaughnessy (2007) and Pfannkuch and Ben-Zvi (2011) also highlighted the centrality of variation to statistical thinking. Snee (1990) defined statistical thinking as "thought processes, which recognize that variation is all around us and present in everything we do, all work is a series of interconnected processes, and identifying, characterizing, quantifying, controlling, and reducing variation provide opportunities for improvement" (p. 118).

Reasoning with statistical models includes reasoning with tables and graphs as well as more complicated models (Wild & Pfannkuch, 1999). This includes reasoning with distributions or viewing data as an aggregate (Konold & Higgins, 2003). Moore (1999) described the importance of looking for overall patterns and deviations from patterns in data, choosing appropriate numerical descriptions of aspects of the data, and seeking compact mathematical models for sufficiently regular overall patterns in data.

Finally, integrating statistical and contextual reinforces the notion that statistical analysis cannot be divorced from context. Where, why, and how data are collected all play a role in decisions regarding data analysis (Wild & Pfannkuch, 1999). Chance (2002) described this type of thinking as constantly relating data to the context of the problem and interpreting conclusions in non-statistical terms.

Methods

This study examined twenty participants' responses to a pre- and post-task assigned in a graduate level class focused on teaching and learning statistics. The graduate course is considered an intervention because it was specifically designed to improve teachers' knowledge of statistical investigations from posing questions to making inferences. The course engaged teachers as learners in completing statistical investigations, reading relevant research on teaching and learning statistics, and designing statistical tasks for students. The instructor expected that teachers would learn to use dynamic statistical software during the course and assignments provided opportunities to both read about and engage in statistical investigations. As such, it was expected that teachers would demonstrate more evidence of statistical thinking at the end of the semester. This was not an experimental study and there was no control group, therefore no causal claims related to the course and changes in teachers' approaches can be made. However, information about the course provides useful context for understanding possible influences on teachers' approaches.

The participants consisted of one undergraduate in the last year of a secondary mathematics education program, fourteen current or pre-service mathematics teachers enrolled in masters' programs, and five PhD students in Mathematics Education. Sixteen participants were female and four were male. Five of the participants had no teaching experience, two had middle-school teaching experience only, and thirteen had secondary and/or post-secondary teaching experience. Those with teaching experience had between 3 and 15 years of experience with a mean of 6.9 years. Six of the participants had taught a statistics course at the secondary or post-secondary level.

Tasks for Assessment

Two parallel tasks were used for the purpose of describing participants' approaches to statistical investigation with a given multivariate data set using technology. The initial task, completed in the first week of the semester, was modified from Connected Mathematics: Data about Us (Lappan, 2004, p. 40). Participants were given data about sixth grade students' jump roping that included three variables, two of which were categorical (class and gender) and one quantitative (number of consecutive jump ropes). Participants were asked to generate questions, explore one of the questions using technology, describe their exploration and findings, and note any new questions that arose. They were provided with data files for Excel, TinkerPlots, and Fathom. Most participants had used these technologies in earlier courses.

For the final task, given in the last week of the semester, participants were provided with a new data set and asked to complete the same process. The data set was parallel in structure to the initial task including a quantitative variable (students' scores on the PSAT math section), as well as two categorical variables (grade level, and class).

In addition, two group investigations were completed during the course. In the first investigation, groups posed their own questions based on Census at School survey data (http://www.amstat.org/censusatschool/) and downloaded samples of data from the Census at School Random Sampler. The second group investigation involved planning how to collect and analyze data to determine whether different sets of dice were fair. Groups had to evaluate each other's claims and decide which set of dice they would recommend to a game manufacturer.

Coding

All initial and final tasks were blinded and initially coded using deductive, open coding, similar to a grounded theory approach (Strauss & Corbin, 1990). The researcher noted types of questions asked, the order and types of representations and calculations, descriptions of analysis, conclusions, and how technology was used during the investigation. After noting general similarities and differences in approaches, the researcher used inductive coding and identified evidence of Wild and Pfannkuch's (1999) five types of statistical thinking. For example, including multiple representations was considered evidence of Transnumeration when the participant described the new information gleaned from each representation. Multiple representations were also considered evidence of Reasoning with Statistical Models when participants used representations to make decisions about choices of statistical tests and measures. Results of the deductive and inductive coding were synthesized to create rich descriptions of teachers' approaches.

Findings

Participants' approaches to investigations varied more at the beginning of the course than at the end of the course. Most participants were able to ask appropriate statistical questions and draw appropriate conclusions on both the initial and final tasks as well as in the group investigations. On the initial task, participants were more likely to engage in either informal analysis or formal analysis, while on the final task most participants engaged in both informal and formal analysis, using the informal analysis to guide the formal analysis. On the initial task and first group investigation, participants used a wide variety of representations and approaches, while on the second group investigation and final task participants tended to use more similar approaches and representations.

Conclusion & Summary

This study examined teachers' approaches to statistical investigations using technology throughout a graduate level course on teaching and learning statistics. There was evidence of an increase in statistical thinking exhibited between the initial and final tasks. Additionally, various approaches to statistical investigations used by participants throughout the course were documented through an in-depth hybrid qualitative analysis. Detailed descriptions of these approaches and documentation of the statistical thinking evidenced in these approaches should prove valuable for planning experiences designed to engage teachers in statistical investigations.

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EXPLORING REASONING OF MIDDLE SCHOOL STUDENTS ABOUT DATA DISPERSION IN RISK CONTEXTS

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The aim of this research study is to explore students' reasoning concerning variation when they compare groups and have to interpret dispersion in terms of risk. We analyze in this paper the responses to two problems from a questionnaire administered to 82 ninth-grade students. The first one composed of losses and winnings coming from a hypothetical game; the second is about medical treatments. The results show the difficulty students had in interpreting variation in a risk context. Although they identify the data group with more variation, this is not enough for interpreting the variation in terms of risk and making a rational decision. The categories of risk-seeking and risk-aversion are used to explain the behavior of students. As a conclusion, it is suggested that more risk context situations should be studied.

Keywords: Data Analysis and Statistics; Middle School Education

Introduction

Variation is the underlying reason for the existence of statistics (Watson, 2006, p. 217). Moore (1990) emphasized the omnipresence of variation and the importance of modeling and measuring it in statistics; Wild & Pfannkuch (1999) included the perception of variation as part of the fundamental types of statistical reasoning. Burrill and Biehler (2011) proposed a list of seven fundamental statistical ideas in which variation is the second idea after data. Several researchers have explored contexts and problems to encourage students from different scholar levels to perceive, describe or/and measure variation in data. For example, variability in sampling (Watson & Moritz, 2000), chance (Watson & Kelly, 2004), repeated measurement (Petrosino, Lehrer & Schauble, 2003), weather (Reading, 2004). In this paper we propose two problems in risk contexts where dispersion is relevant and we explore the reasoning of middle school students in front of these problems.

Conceptual Framework

Four concepts constitute our conceptual framework: tasks or problems, risk context, reasoning and the SOLO model. An important part of research is to seek problems that promote the capacity to think and reason, about and with the fundamental ideas of the study area. Problems should be formulated in an appropriate context and encourage students to engage with the concept to be learned. Tasks on making decision under uncertainty are common in statistics and have been widely used to promote and analyze important aspects of the statistical reasoning of people. In addition, problems on comparing groups are frequently used to engage students in reasoning about data since in many statistical studies are used this kind of problems. In the present work two problems on decision making and comparing groups of data are proposed, in these problems variation is significant and their solution implies some risk preferences.

The interpretation of dispersion depends on the situation from which the data come. One kind of elemental problems where variation could emerge can be formulated in risk context. These situations appear when there are potential and unwanted results that, as a consequence, lead to losses or damages. A paradigmatic task in risk context consists in making a decision about two games where gains and losses are at stake. Consider the following problem:

The gains of realizations of *n* times the game A and *m* the game B are:
Game A: $x_1, x_2 \ldots ; x_n$

Game B: $y_1, y_2 \cdots; y_m$

Which of the two games would you choose to play in?

The solution is reached by following a flow diagram: 1) Compare $\bar{x}$ and $\bar{y}$, 2) if $\bar{x} \neq \bar{y}$ then chose the Game whose mean is the greatest; 3) if $\bar{x} = \bar{y}$ then there are two options: 3a) Choose any game, 3b) Analyze the dispersion of data in each game and choose according to risk preferences. These preferences can be defined as generalizations of the attitudes to reject or seek the risk identified by psychologists:

> …a preference for a sure outcome over a gamble that has higher or equal expectation is called risk aversion, and the rejection of a sure thing in favor of a gamble of lower or equal expectation is called risk seeking (Kahneman & Tversky, 2000, p. 2).

It is worth noting that in a game the dispersion of gains can be considered a measure of risk. A preference is motivated by risk aversion when an option whose data have less dispersion over another whose data have greater dispersion is preferred. The decision is motivated by risk seeking when the option whose data have greater dispersion is chosen.

This study is located in the area of statistical reasoning; the purpose of the research on it is to understand how people reason with statistical ideas (Garfield & Ben-Zvi, 2008). When students try to justify their responses, elements that they think are important to the situation are revealed; in particular, the data they choose, operations made with these and knowledge and beliefs on which they rest for doing that, are important in reasoning.

The Biggs and Collis (1982) Structure of Observed Learning Outcomes (SOLO) model is based on the assumption that development can be represented in hierarchical structures. Five levels are postulated in SOLO model: Prestructural level, the responses only show that students engage to task but do not use any relevant aspect to its solution. The Unistructural level, responses have one relevant aspect to the task solution. The Multistructural level present more than one relevant aspect but without integrate them. Relational level responses integrate in a coherent way more than one relevant aspect to the task. Finally, at the Extended Abstract level responses would show a higher abstract response. Applying these levels to analyze the answers to the tasks allows us to create a hierarchy that describes levels of students' reasoning of increasing complexity.

Method

The participants were 82 students (aged 14 to 16) belonging to two different ninth grade groups in a private school in Mexico City (last year of middle school).Two problems (which are presented below) were designed to explore the reasoning of the students:

Problem 1. In a fair, the attendees are invited to participate in one of two games, but not in both. In order to know which game to play, John observes, takes note and sorts the results of 10 people playing each game. The cash losses (-) or prizes (+) obtained by the 20 people are shown in the following lists:

Game 1: 15, -21, -4, 50, -2, 11, 13, -25, 16, -4
Game 2: 120, -120, 60, -24, -21, 133, -81, 96, -132, 18

If you had the possibility of playing only one of the two games, which one would you choose? Why?

Problem 2. Consider you must advice a person who suffers from a severe, incurable and deathly illness, which may be treated with a drug that may extend the patient's life for several years. It is possible to choose between three different treatments. People show different effects to the medication: while in some cases the drugs have the desired results, in some others the effects may be more favorable or more adverse. The following lists show the number of years ten patients in each treatment have lived after being treated with one of the different options; each number in the list corresponds to the time in years a patient has survived with the respective treatment. The graphs corresponding to the treatments are shown after.

Treatment 1:	5.2,	5.6,	8.5,	8.5,	7.0,	7.0,	7.0,	7.9,	8.7,	9.1
Treatment 2:	6.8,	6.9,	6.9,	7.0,	7.0,	7.0,	7.1,	7.1,	7.2,	7.4
Treatment 3:	6.8,	6.8,	6.9,	7.0,	7.0,	7.1,	7.1,	7.1,	7.2,	7.4

[There were graphs with values shown above]

What kind of treatment would you prefer (1, 2 ó 3)? Why?

Results

In this section we present examples of responses to questions: If you had the possibility of playing only one of the two games, which one would you choose? Why?, and What kind of treatment would you prefer (1, 2 ó 3)? Why? In order to show answers that were classified at each SOLO level.

Prestructural. In this level one option is chosen but without justification. For example, a student chose game 2, “because you win more”, but there is no evidence of how the data are used. This kind of responses provides no progress in understanding the situation.

Unistructural. In this level the maximum, minimum or mode of each data list are observed and compared to give the response; for example, a student chooses Game 1 “because you can also lose as in game two but fewer and you risk less”. The student provides an indication that shows he compares the minimum of both games. Indeed, it is possible that the way in which the student approaches the problem is influenced by risk aversion, since he skews his attention toward the losses, ignoring the information that provides positive gains.

Multistructural. In this level the sums of the data lists are compared or both the maximum and minimum are considered and, in this case, risk is perceived. For example, a student chose the treatment 3 “Because maybe I will not live nine years but I have secured from 6.8 to 7.4”. Although not mentioned, it appears that the student perceives the risk involved in the first treatment (the possibility to live only 5.2 years) because he gives up the opportunity to live 9, "ensuring" live at least 6.8 years.

Relational. In this level the mode of the data and its range are considered; the final decision is influenced by preferences about risk. Only two cases were included in this level. For example, a student chose treatment 3 "because it is more likely to live 7.1 years or less, but the results are not so far apart and are more likely to live from 6.8 to 7.2 years".

In general, predominate Prestructural responses (76 %); students whose answers are classified at this level understand what is asked them and they make a choice but fail to use the data to support their preferences. However, there are students who see in a single value of each data set (maximum, minimum or mode) a key to make a decision. These responses have been classified in the Unistructural level (15 %); they prefigure the valid scheme of solution. The value chosen is one that students consider a representative of the set. In responses of Multistructural level (7 %), a step forward towards the solution scheme is given, since more than one value of each data is considered. Two main strategies were identified: 1) compare the sum of values of each set data, 2) take into account the maxima and minima. Each of these strategies is an early or primitive form of the two main statistical tools of the case: the mean and dispersion. The second strategy led some students to perceive the risk. Finally, In Relational level responses (2 %), both strategies are used and the decision is made according of attitude towards risk. In these responses a scheme of solution is complete.

Conclusions

Our main result is a SOLO hierarchy which describes the students' reasoning patterns. Such hierarchy is the next: *Prestructural level*, one option is chosen but without justification. *Unistructural level*, the maximum, minimum or mode of each data list is observed and compared. *Multistructural level*s, the sums of the data lists are compared or both the maximum and minimum are considered. *Relational level*, the sums or modes of the data lists are compared and the range is considered; the final decision is influenced by preferences about risk.

In the problems that we have reviewed may be not convincing that the choice of a game or treatment is not completely determined by the behavior of the data, but also depends on the solver attitude towards risk. This relativity can disturb those who believe that science must give absolute and conclusive answers to the problems that arise on it. Relativity of the responses may obscure the main point which is that the analysis of the ranges, and more generally the analysis of variation, that provides information about the risks involved in the situation and therefore helps to make rational decisions. The use in teaching of problems as those treated in this study can help the students to construct schemes for assessing the results of the statistical analysis and help them to retreat from certainty in a profitably way.

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GLIMPSING SECONDARY MATHEMATICS TEACHERS' AFFECT TOWARD STATISTICS: WHICH TEACHING AND LEARNING EXPERIENCES ARE MOST SIGNIFICANT?

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As a result of the recent increase of statistical topics in the school curriculum, there is a need to study the attitudes and beliefs of secondary mathematics teachers with respect to statistics since research has shown that teachers' affect is connected to student learning (Estrada & Batanero, 2008; Shaughnessy, 2007). Because the attitudes, beliefs, and emotions (affect system) of practicing secondary mathematics teachers toward statistics are not known, an initial inventory of these constructs was conducted via a survey instrument. This introductory study explored the connection between affect and statistical teaching and learning experience by examining teachers' responses to several items on the survey. Based on the responses to the survey items, experience teaching a stand-alone statistics course did influence teachers' confidence to teach and learn statistics, while undergraduate courses in statistics had less impact.

Keywords: Teacher Beliefs; Data Analysis and Statistics; Affect and Beliefs

Introduction

Over the last few decades there has been a growing concern that large numbers of secondary mathematics teachers "have backgrounds in mathematics with little or no training in statistics," with no teaching certification available or required for statistics (Gould & Peck, 2004, p. 1). This emphasis on mathematical topics in pre-service teacher programs has contributed to teachers' deficiency in statistical literacy, which may impact student learning of statistical topics (Pierce & Chick, 2008). Because of inadequate formal training in statistics, many teachers may feel anxiety at the prospect of teaching the subject and may elect to exclude statistical topics from their courses altogether. In other cases, teachers' anxiety towards statistics can contribute to a dislike of the subject or a feeling that statistics is not valuable (Estrada, Batanero, & Lancaster, 2011), feelings that may later be transferred to students from their teacher (Estrada & Batanero, 2008).

Secondary mathematics teachers (SMTs) who teach statistics as a stand-alone course gain important experience despite any possible deficiencies in their formal university training in the subject. It is possible that this population of SMTs may have more positive attitudes towards statistics than those teachers who lack any experience teaching statistics. However, the exact nature of any possible differences between the attitudes, beliefs, and emotions (*affect system*) of those teachers who have taught statistics and those who have not is unknown at this time.

Unfortunately, practicing teachers who have no special preparation or education in statistics may exhibit negative attitudes towards statistics (Estrada, et al., 2011). Not surprisingly, in research conducted with prospective teachers it was found that with more courses taken in statistics and more knowledge of statistics, attitudes tended to be more positive (Estrada, et al., 2011). In addition, research with in-service elementary teachers indicated that their earlier education of statistics (and positive learning experiences with statistics) was a major impact on their attitudes (Estrada, et al., 2011). Hence knowledge of statistics alone does not predict one's attitudes towards statistics, but personal experiences that one has with statistics are influential.

In this initial study, teachers' formal educational experiences with statistics were defined by the number of undergraduate statistics courses they completed. The teachers also shared information related to their teaching experiences by responding to survey items related to number of years teaching high school mathematics and whether they had taught statistics as a stand-alone course (or not). In fact, teachers' experience with teaching statistics as a stand-alone course provided a useful framework for considering their responses to the other survey items. Differences in responses between the two groups of teachers (defined by experience teaching statistics versus those without such experience) provided the structure for exploring their formal experiences with statistics, both as a teacher and as a student.

A teacher's affect toward teaching statistics is connected to the teacher's experience: it connects to the teacher's beliefs about statistics, mathematics, statistics and mathematics in relation to one another, and about teaching practice in general. Pierce and Chick (2008) stated that "teachers' beliefs about statistics itself will contribute to both their attitude towards teaching statistics and their [teaching] practice" and "such beliefs will depend on their own experiences of learning and using statistics" (p. 4). Therefore, examining teachers' personalized beliefs about statistics itself was critical, and two survey items were designed to gauge teachers' assessment of their own ability to master statistical content (general versus introductory coursework).

Purpose of the Study

Since the teaching and learning of statistics is influenced by the teacher's affect system toward statistics, this introductory study focused on the teachers' own participation within the teaching and learning cycle. Some of these personal experiences have likely shaped the teachers' affect toward statistics as well as the teachers' views of themselves as students and as teachers. Because of this, the role of the teacher's attitudes, beliefs, and emotions becomes increasingly important as one considers the connection to teaching practice and student learning.

Thus the purpose of this study was to begin to examine the affect system of the population of SMTs toward statistics by focusing on the teachers' experience with teaching and learning statistical topics. The study focused on the following research question: What are the key differences in teaching and learning experiences related to statistical concepts between practicing SMTs who have taught a statistics course versus those who have taught statistical concepts only as part of a regular mathematics class, and are there related differences in affect?

Methods

The SATS-36 (Survey of Attitudes Toward Statistics) was designed to capture the attitudes and beliefs of university students taking an introductory statistics course (Schau, Stevens, Dauphine, & del Vecchio, 1995). Because the attitudes, beliefs, and emotions of SMTs was expected to be similar to this population (Pierce & Chick, 2008), the existing instrument provided the best basis on which to begin studying this new population. Thus, the SATS-36 provided a basis for the survey instrument that was used for the present study. Some additional items were constructed in order to provide demographic information regarding the experience of the teachers with teaching and learning statistical topics.

The teacher response data was collected by using an online survey system which emailed the survey to a random sample of SMTs from a Midwestern state. The survey was initially emailed to 502 teachers. Of the 502 who were emailed the survey, 92 filled out the survey giving a response rate of 18.3%. Based on this response rate, the survey was emailed to an additional random sample of 276. Of the 276 who were emailed the survey, 49 filled out the survey giving a response rate of 17.8% for the second mailing. Thus, the overall response rate was 18.1%. The results for six survey items related to teaching experience and education is found in Table 1; responses were broken down by experience teaching statistics.

Table 1: Statistical Teaching and Learning Experiences

Have you ever taught a stand-alone high school statistics course?		How good at mathematics are you?*	How many years have you taught high school mathematics?	How many undergraduate statistics courses did you complete?	How good at statistics are you?*	How confident are you that you can master introductory statistical content?*
NO n=112	Mean	6.3	3.4	2.4	4.7	5.7
	Std. Deviation	0.6	1.7	0.9	1.1	1.2
YES n=27	Mean	6.5	4.6	2.4	5.7	6.4
	Std. Deviation	0.5	1.7	0.9	1.3	1.2
	P-value for difference between means for two groups	0.108	0.001	0.944	0.000	0.007

*Starred items originated on SATS-36 (Schau, et al., 1995)

Discussion of Results

For each of the five survey items, means and standard deviations were calculated for those teachers who had taught statistics as a stand-alone course and those who had not. Two-sample T-tests were run to determine which items showed significant differences in responses between the two groups of teachers. The resulting p-values from the two-sample T-tests are shown in Table 1. At the 0.05 level, significant differences were found for three items.

There was a significant difference between the two groups of teachers for the total number of years teaching high school mathematics. The mean number of years was 4.6 years (s=1.7) for the group of teachers who had taught statistics as a stand-alone course, and the mean for those without experience teaching a statistics course was 3.4 years (s=1.7) (p-value for difference=0.001). This result suggested meaningful differences in overall mathematical teaching experience and provided a context for evaluating the other survey items.

The differences between the two groups of teachers were also statistically significant for the survey items in the last two columns in Table 1. Both of these items related directly to teachers' confidence in learning and applying statistics, and it was not surprising that the teachers who had taught statistics as a stand – alone course had higher mean responses to these items than teachers with no experience teaching statistics as a stand-alone course. The SMTs who had taught statistics as a stand-alone course generally agreed that they could master introductory statistical content. The difference in level of agreement with this statement was significantly higher for the group who had taught statistics compared with those who had not taught statistics (p-value=0.007). This group of SMTs with experience teaching statistics also reported a higher level of proficiency for their own statistical skills. The difference in level of ability reported by the two groups of teachers on the item "How good at statistics are you?" was significant (p-value=0.000). Based on the responses to these two items, confidence to learn and apply statistical content is strengthened with teaching experience in the population of SMTs.

The SMTs with experience teaching statistics reported higher levels of personal confidence regarding mathematical abilities, but the difference between the two groups of teachers was not significant on this item (p-value=0.108). The teachers who had never taught a stand-alone statistics course had a slightly higher mean value, although the difference between the means on this item were not statistically significant (p-value=0.944) so a difference cannot be assumed for the two populations of teachers on this item. In fact, one could not exclude the possibility that the number of undergraduate statistics courses completed was the same for the two groups.

Based on these introductory results, SMTs who teach statistics as a stand-alone course have been teaching high school mathematics longer than teachers who have not taught statistics as a stand-alone course, their confidence in their ability to employ statistics is generally more positive, and they are more secure in their ability to master introductory statistical content. Taken together these findings are not surprising, but future work involving classroom observations and interviews will delve more deeply into the mechanisms driving the differences.

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EXPLORING UPPER ELEMENTARY STUDENTS' USE OF THE REPRESENTATIVENESS HEURISTIC

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A common theme in the study of math education is that students do not necessarily reason normatively through uncertain probabilistic situations. Instead, they rely upon heuristics as a means of reducing the mental strain necessary for normative probabilistic thinking (Kahneman & Tversky, 1972). This study explores use of heuristical reasoning by fifth grade students on probabilistic tasks related to game spinners. Reasoning classified by the representativeness heuristic was found to be utilized by eleven out of the fourteen students interviewed across four series of questions. Their reasoning, however, was inconsistent and vacillated between normative and heuristic.

Keywords: Probability; Elementary School Education; Cognition

Literature Review

The use of heuristics within statistical reasoning refers to thinking that deviates from the stochastically accepted norm, and is employed in uncertain stochastic situations as a means of mitigating mental strain (Kahneman & Tversky, 1972). Heuristics exist, then, in staunch contrast to normative reasoning, which is considered to be the standard and expected reasoning accepted by stochastic experts. Heuristics are commonly applied in probabilistic situations, and are not exclusively used by students. For example, they have also been found to be employed by seasoned professionals trained in research (Kahneman & Tversky, 1972).

The use of heuristics, although non-normative, can be intermittently successful, as heuristical reasoning will at times provide sufficient information to make reasonable decisions (Shaughnessy, 1992). The sporadic success achieved through non-normative reasoning makes the elimination of heuristical reasoning difficult (Kahneman & Tversky, 1972). These difficulties in eliminating and identifying non-normative student reasoning has become the focus of much research in the teaching and learning of probability (Jones & Thornton, 2005).

The representativeness heuristic is one type of non-normative reasoning which plagues student reasoning. Kahneman and Tversky (1972) identify the application of the representativeness heuristic in the following manner:

> A person who follows this [representativeness] heuristic evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated. (p. 431)

This definition describes two distinct ways in which the representativeness heuristic may be relied upon by those who do not have a deep understanding of probabilistic concepts.

Konold, Pollatsek, Well, Lohmeier, and Lipson (1993) asked students to select which outcome from five coin tosses was most likely, or whether all choices were equally likely. Reasoning that THHTH is more likely than HTHTH because it appears more random represents a limited understanding of randomness, therefore indicating reliance upon the representativeness heuristic. Reasoning that THHTH is more likely than THTTT because it is closer to an equal proportion of heads and tails would demonstrate reasoning characteristic of the representativeness heuristic due to insufficient understanding of sample size. Reasoning characteristic of the representativeness heuristic has additionally been identified in a range of ages beginning in second grade (Davidson, 1995) and continuing through both pre-service (Chernoff, 2012) and in-service (Wilkins, 2007) teachers. This research across various ages indicates that the representativeness heuristic is developed prior to the onset of formal instruction in probability and continues to plague normative reasoning into adulthood.

Additional research has identified non-normative reasoning related to a game spinner. Green (1983) found that students ages 11-16 overwhelmingly struggled to reason about game spinners with noncontiguous sections. This inability to reason normatively may be due to students' use of a subcategory of representativeness known as recency (Jones & Thornton, 2005). Recency may manifest itself visually when students become hyperaware of whether the sections of a noncontiguous spinner are contiguous or noncontiguous (Green, 1983).

The present study specifically seeks to determine: 1) whether upper elementary students engage in use of the representativeness heuristic to reason about game spinners, and if so, 2) whether the representativeness heuristic is consistently applied across questions.

Methods and Procedures

Sample

The participants in this study include 14 fifth grade students in a rural elementary school in the south eastern United States. This age range was selected due to the recommendation of the National Council of Teachers of Mathematics (2000) that formal probabilistic instruction not begin prior to third grade. Students selected were chosen from three ability categories, as determined by the teacher; Level 1 being most advanced, Level 2 being intermediate, and Level 3 being least advanced. One of the participant's responses were uninterpretable in terms of normative and non-normative reasoning, so analyses are based on only 13 of the 14 participants.

Procedures

Participants were engaged in a semi-structured clinical interview during which they manipulated three distinct game spinners and were asked a series of questions about the likelihood of different events. The semi-structured interview was video and audio recorded so as to allow the researcher to retrospectively analyze the students' reasoning with each spinner.

The first two game spinners contained equally proportioned, colored sections, with each color represented once (Figure 1). The third game spinner contains six sections and four colors. Each section covers a different portion of the spinner, and yellow is represented by three sections. The three yellow sections are noncontiguous and the yellow sections represent the smallest overall portion of the spinner.

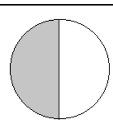	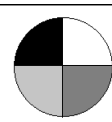	
Spinner 1	Spinner 2	Spinner 3

Figure 1: Spinners used in the clinical interview

Results

In the first of four series of questions, students spun Spinner 1 (Figure 1) and were asked if the resulting color was more likely to be spun again on the next spin, less likely, or equally likely. Seven out of 13 students answered in a manner which suggested use of the representativeness heuristic (see Table 1). One student reasoning normatively indicated understanding that the second spin was independent of the first by remarking that the second spin "might land on either because it's half and half." Many students demonstrating reasoning characteristic of the representativeness heuristic either reasoned that a second spin was more or less likely to result in the same color; this type of reasoning is termed recency and is a subset of the representativeness heuristic (Jones & Thornton, 2005). One student explained that after spinning red the second spin was more likely to be “red again because it’s easier to spin.”

In the second series of questions, students spun two of Spinner 1 at the same time. Prior to spinning, they were asked if it was more likely for both spinners to result in the same color or different colors, or if both outcomes were equally likely (note: it is equally likely to be the same or different). Eight out of 13 students gave responses consistent with the representativeness heuristic by indicating that it was more likely for the spinners to land on different colors (see Table 1). Common responses from students engaging the representativeness heuristic included an explanation that because each color had a 50% chance of being spun, two spinners were more likely to land on different colors.

The third series of questions mirrors the first by asking students to spin Spinner 1 and observe the result, and then spin a second but identical spinner. Only after knowing the result of the first spinner are they asked to determine if the same color is more likely, less likely, or equally likely to be spun on the second spinner. Seven out of 13 students answered in such a way as to indicate their use of the representativeness heuristic (Table 1). Six out of 7 students identified with representative thinking also employed the representativeness heuristic in the second series of questions. Five out of the 7 students identified also employed the representativeness heuristic in the first series of questions.

Table 1: Comparison of Students’ Reasoning in Similar Series of Questions

		Q2		
		R	N	Total
Q1	R	6	0	6
	N	2	5	7
	Total	8	5	13

		Q3		
		R	N	Total
Q1	R	5	1	6
	N	2	5	7
	Total	7	6	13

		Q3		
		R	N	Total
Q2	R	6	2	8
	N	1	4	5
	Total	7	6	13

Series of questions are denoted Q1, Q2, Q3. R denotes representative reasoning and N denotes normative reasoning. All comparisons sum to 13, as one student’s reasoning was not interpretable on the second and third series of questions.*

In the fourth series of questions students predicted the results of spinning each spinner ten times, then were asked whether this represented a sufficient sample to expect results representative of the total population. Four students out of 10 consistently engaged in reasoning consistent with the representativeness heuristic by indicating the results should equate with the theoretical probabilities. One student in particular was unable to predict the outcome of ten spins on Spinner 2. Theoretically, each section of Spinner 2 would be spun 2.5 times out of ten. Practically, this is an impossibility. This student’s trouble with this mismatch was verbalized when she answered, “It… well… I’m not… it can’t.” When this student was instead prompted to make a prediction for spinning Spinner 2 eight times, she readily answered that each color would be spun twice. Students reasoning normatively arbitrarily selected each section to be spun two or three times and did not seem confused when the results did not match their prediction.

Conclusion

The results of this study answered the first research question by indicating that fifth grade students do engage the representativeness heuristic when reasoning about game spinners. Of the 13 students, 10 used the representativeness heuristic predominantly to reason through at least one series of questions. The three remaining students reasoned normatively throughout the interview. In response to the second research question, three groups of students emerged. Students in Group 1 reasoned normatively throughout the interview. Students in Group 2 reasoned both normatively and non-normatively. Students in Group 3 consistently using the representativeness heuristic.

In addition to providing evidence of the use of the representativeness heuristic by fifth grade students, this study further provides evidence that in statistically equivalent situations, students do not necessarily reason in the same way. Series of Questions 1, 2, and 3 were statistically equivalent. However, students did not necessarily reason consistently throughout. Five students engaged the representativeness heuristic in these situations, and conversely, four students reasoned normatively in these situations. The remaining five students reasoned both normatively and non-normatively. This is indicative of the inability of students to view the two situations as identical and further may relate to students' notions of event independence (Konold et al., 1993). This finding lends itself to further investigation of what other statistically identical situations and models exist which students may not be prepared to distinguish between.

The procedural context of situation 2 was not identical to situations 1 and 3, although the probabilities of each outcome are identical. The difference lies in the timing of the spinners, and the inability of the students to know the outcome of the first spinner prior to predicting the outcome of the second. Eight students used the representativeness heuristic as a predominant part of their reasoning on this question, which represents a higher percentage than any other question. Of those eight students, five engaged the representativeness heuristic on Questions 1, 2, and 3.

The current research contributes to the breadth of understanding related to the types of questions to which fifth grade students may apply the representativeness heuristic. This brings relevance to the type of questions which must be considered in upper elementary curricula and in future research.

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HOW STUDENTS NAVIGATE STEM-INTEGRATED DATA ANALYSIS TASKS

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Keywords: Data Analysis and Statistics; Elementary School Education; Measurement

In recent years, the increased attention of policy makers, educators, and researchers on education in science, technology, engineering, and mathematics (STEM) has resulted in various reforms aimed at improving student performance within these disciplines. One approach to this that is gaining in popularity is the integration of the disciplines through rich problem-solving situations. According to a report from the National Research Council's Committee on Integrated STEM Education (2014), research on the impact of STEM integration indicates that this approach has promise both with regard to learning outcomes and attitudes toward the STEM disciplines, but as of yet, the evidence for this is limited, and the benefits are less pronounced with regard to mathematics learning outcomes. More research is needed to identify the mechanisms by which integration can enhance learning, specifically in mathematics. This multiple embedded case study begins to address this by examining the ways students apply statistical concepts and engage with data in integrated settings.

The data analysis techniques used in science and engineering contexts are built on mathematical ideas making data analysis tasks ideal for studying the interactions between these disciplines. In this study, I document the work of eight groups of fifth grade students as they navigate data analysis tasks and attempt to reason from data in science and engineering contexts. The tasks ask students to collect, organize, interpret, and ultimately make decisions or draw conclusions from their data in the context of scientific inquiries or engineering design challenges. Previous literature has argued that tasks such as these that focus on authentic problems with real data and encourage students to generate their own questions (Franklin et al., 2007; Moore, 1998; Watson, 2006) are successful strategies for supporting statistical reasoning, and the results of this study conditionally support those findings. One key observation in this study, however, was that in many cases, students' informal understanding of the phenomenon being tested helped them make sense of the data rather than their analysis of the data allowing them to draw conclusions about the phenomenon. Additionally, students' general number sense and difficulty making sense of measurement error and uncertainty were significant barriers to their success. These findings have implications for the design of data analysis tasks in science and engineering contexts if they are to support statistical reasoning.

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EIGHTH GRADE STUDENTS USE CONTENT SUBSTITUTION WHEN SOLVING PROBABILITY PROBLEMS

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Keywords: Probability; Middle School Education; Problem Solving

Framework. Heuristic and Biases theory (e.g. Tversky & Kahneman, 1974) has been used to study people's cognitive processes by mathematics educators and psychologists (e.g. Fischbein, 1999; Shaughnessy, 1981). The theory was updated to include dual process theory –System 1 (intuition) and System 2 (reasoning)- that may explain biased results when people, unaware, use the *attribute substitution* process linked to System 1 thinking (e.g. Kahneman & Fredrick, 2002).

Purpose. The aim of this paper is to share a particular phenomenon observed from a larger study (Mistele, 2014) that explored eighth grade students' probability problem solving strategies using this updated framework. Focusing on this particular phenomenon required a more targeted research question using the same data. The revised research question is: "In what way was a participant's problem solving strategy influenced by System 1 thinking or slow intuition?"

Participants. There were six eighth grade students from a rural, predominantly white, middle class school located in a mid-Atlantic state. Purposeful sampling (Patton, 1990) was used to identify these students that had strong probability knowledge and prone to using heuristics.

Research design. This was a basic exploratory research study. Data was collected from two, multiple choice, questionnaires and two, tasked based, interview sessions that used the think-aloud protocol (Van Someren, et al., 1994). The questionnaires were analyzed based on the literature. The constant comparison method (Glaser, 1965) was used to analyze the interviews.

Findings/Results. At times, some students used the attribute substitution process as expected (Kahneman & Fredrick, 2002). Another process called, *content substitution* was identified (Mistele, 2014) in which a person replaces probability knowledge with fraction knowledge when using System 1thinking or a slower intuitive process called slow intuition. Fischbein (1999) also noted proportional thinking when exploring his notion of intuition that seems to differ from this.

Conclusion. Some students were observed replacing probability knowledge with fraction knowledge, specifically, they used equivalent fractions procedures. Targeting instructional strategies to off-set the inappropriate use of fraction knowledge requires more research.

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ELEMENTARY PRE-SERVICE TEACHERS' STRATEGIES TO COMPARE VARIABILITY IN DOT PLOTS

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Keywords: Data Analysis and Statistics; Teacher Education-Preservice; Teacher Knowledge

Variability has been recognized as an important concept in K-12 statistics curriculum (Franklin et al., 2007). It is recommended that pre-service teachers be prepared with an adequate knowledge of variability to describe and interpret data, and to use the results to draw conclusions about questions of interest (Conference Board of the Mathematical Sciences [CBMS], 2012). Elementary school teachers are expected to at least possess the concepts of statistics at the level that are expected of middle school students (Franklin, 2013). That is, elementary pre-service teachers (EPTs) should be prepared beyond knowledge about statstical graphs. Prior studies had focused on evaluating EPTs' performance to find the variability in various types of statistical graphs (e.g., Cooper & Shore, 2008; Lem et al., 2013). However we still do not know enough about EPTs' reasoning about variability when they encounter the tasks to compare variability in dot plots. Thus, this study aimed to answer the following research question:

What strategies do elementary pre-service teachers use to compare variability in dot plots?

This work is part of a larger study investigating EPTs' understanding of variability. A task-based interview was used in the study to collect nine EPTs' thinking about variability. Each interview lasted approximately one hour and eight tasks were designed to deeply explore participants' understanding of variability. This study presented nine EPTs' responses in five tasks that asked to compare variability in dot plots. Shapes were deliberately incorporated into the design of the tasks to challenge EPTs' thinking of variability in dot plots. The results showed six strategies identified from the participants' responses and discussed their misunderstanding of variability revealed in their interpretations of the strategies. Implications for learning from elementary pre-service teachers' thinking in comparing variability in dot plots will be presented.

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IN-SERVICE STATISTICS TEACHERS' PROFESSIONAL IDENTITIES

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Keywords: Data Analysis and Statistics; Affect and Beliefs; Teacher Beliefs

In the present US context, statistics represents a substantial component of the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010) in grades 6-12 and generally represents the trend of increasing statistical requirements for both students and their teachers. The expectation is that statistics content will be taught primarily by mathematics teachers, most of whom have limited statistical training. The growth that these teachers are being asked to undertake is complex and may require changes in their beliefs, attitudes, values, and practices germane to the teaching of mathematics and statistics. These and other affective constructs can be profitably viewed as subconstructs of a broader 'identity' construct (Philipp, 2007).

Because statistics is a mathematical science and not a branch of math (Cobb & Moore, 1997; Moore & Cobb, 2000), the professional identity that mathematics teachers are expected to develop (NCTM, 1991) may not be directly applicable to teaching statistics. Instead, a professional identity as a statistics teacher may be required to ensure that students learn statistics at the level expected of them. What precipitates such a statistics identity is not currently known.

The purpose of this study is to understand how exemplary statistics teachers developed their professional identities by answering the following research question: *What learning trajectories help to explain the identity of exemplary statistics teachers?* Exemplary statistics teachers will be identified by experts in the field of statistics education.

The identity framework used in this study synthesizes work from communities of practice (Lave & Wenger, 1991; Wenger, 1998) and Gee's (2000) identity framework. This framework is further informed by Social Cognitive Career Theory (Lent, Brown, & Hackett, 1994) and Philipp's (2007) summary of the various aspects of identity. Because of the multi-faceted nature of identity, multiple data sources will be used, including two semi-structured interviews and surveys that foreground specific aspects of identity (such as attitudes and beliefs). Data collection and analysis will begin in June 2015 with findings to be presented in November 2015.

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Chapter 8

Student Learning and Related Factors

Research Reports

Brief Research Reports

Poster Presentations

MATHEMATICAL AND FINANCIAL LITERACY WITH FAMILIES

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There is a strong link between citizens' quantitative literacy abilities and their financial prosperity. This study applied a social justice perspective to describe families' personal flourishing within the context of numerical, mathematical, and financial literacy (NMFL) education. Four families participated in a weekly community program with mathematical and financial literacy goals. The data showed that participants gained confidence and skills and felt empowered to teach/transfer that knowledge to their children. The author proposes a conceptual framework linking personal flourishing with NMFLs, and suggests the framework be used to investigate and describe quantitative and financial literacy in future empowering pedagogies research.

Keywords: Equity and Diversity; Informal Education; Policy Matters; Affect and Beliefs

Numerical, mathematical and financial literacies (NMFLs) are pressing economic, social, and cultural challenges. There is a strong link between citizens' basic numerical or mathematical abilities and their financial prosperity and civic engagement (Human Resources and Skills Development Canada, 2012). It is becoming evident that NMFLs are increasingly necessary for fully contributing citizens of tomorrow (Steen, 2001). This study applied a qualitative research approach to describe personal flourishing (Grant, 2012) which includes participants making meaning and sense of important aspects in their lives, having a sense of agency, and participating in activities of financial literacy. Personal flourishing was investigated within the context of numerical, mathematical, and financial literacy (NMFL) education.

Four families participated in a weekly evening program (eight consecutive weeks). The Count On Yourself (COY) project was designed to inform participants about NMFLs. Count On Yourself provided a parallel program for adult and child NMFL literacy: while parents were involved in a financial literacy course, their children participated in a *Math Camp* led by teacher candidates from a local university. The goal of the project was for both adult and child participants to become more mathematically empowered. Analysis of the individual and focus group interview data showed that adult participants described a sense of personal flourishing, gained confidence and skills, and felt financially empowered enough to teach/transfer that knowledge to their children. This initiative integrated research and practice and addressed the practical needs of a community.

Based on a research study that sought to analyze how participants would express their ideas of financial literacy and express their personal flourishing, I propose a conceptual framework designed to study programs that promote Numerical, Mathematical and Financial Literacies (NMFLs) and to describe the participants' voices. A robust social justice research vision (Grant, 2012) was applied to offer numerical, mathematical and financial literacies in a community. The research asked: 1) How did the adult participants describe their learning about financial literacy? and 2) How did this impact them and their families, and what, if any action did they take?

On Numerical, Mathematical and Financial Literacies (NMFLs)

Innumeracy, or "an inability to deal comfortably with the fundamental notions of number and chance, plagues far too many otherwise knowledgeable citizens" (Paulos, 2001, pp. 3-4). Numeracy was determined by Smith, McArdle, and Willis (2010) to be "by far the most predictive of wealth among all cognitive variables" (p. 18). Behrman, Mitchell, Soo, and Bravo (2010) worked to isolate the causal effect of financial literacy on wealth accumulation, and found that the largest effect was financial literacy. In these studies, numeracy and financial literacy were both found to be predictors of wealth.

This study is set in a Canadian community, and though North American data was sought and utilized, the issues surrounding Numerical, Mathematical and Financial Literacies are not unique to the North American context. As defined by the Human Resources and Skills Development Branch (2012), quantitative literacy or numeracy is "the knowledge and skills required to effectively manage the mathematical demands of diverse situations". The 2003 study found that over half of respondents (55 %) scored only Level 1 or 2 [of 5 levels] on the numeracy proficiency scale. These low results are significant because numeracy and problem solving are linked to financial prosperity and civic engagement (Behrman et al., 2010; Human Resources and Skills Development Canada, 2012; Smith et al., 2010). Similarly, a Canadian report (Statistics Canada, 2008) found that 49 percent of all adults say they do not engage in household budgeting.

There is also substantial evidence that financial literacy programs "can make an important contribution to the well-being of vulnerable groups" (McFayden, 2012, p. 1). Gutstein (2006) wrote that "[r]eading the mathematical word is equivalent to developing mathematical power" (p. 29) and included that "opportunity to learn, access, and equity all demand that marginalized students get the chance to develop" the tools for mathematical empowerment (p. 30). This study engaged grassroots community resources to support NMFLs, and applied a qualitative research approach for the purpose of improving awareness, access, and quality of programs available to the participating school community.

School Community

The elementary school involved in the Count On Yourself project is located in a smaller urban community adjacent to a larger city. The community elementary school involved had been identified as a low-literacy school. The school Principal was an administrator in the larger community for more than 20 years. The school historically had poor academic performance, and data indicated that students had poor achievement in intermediate grades and were less likely to graduate than their peers (British Columbia Ministry of Education, 2006). The school Principal was highly trusted and a consistent and dedicated member of the community, and I had been a district consultant. Given these predetermined trusting relationships, the Parent Council for the school was quick to approve the program and the research.

The project was announced at a community forum, and four families chose to participate in the Count On Yourself quantitative and financial literacy program. Though all members of the school community were invited to participate, it was fortunate that a fairly small group signed up for the first iteration of this program. A trusting environment could be built with a smaller group. The four families who participated were (aliases assigned) Valerie and Brent, parents of two children in 4th and 5th grade. Ema and Phil, parents of one 2nd grade child and three not yet in school, Anna (whose husband did not participate) with children in 1st and 6th grade, and Carolyn, whose husband and children did not participate.

Theoretical Framework

Reading and Writing the World

Freire wrote about the fallacy of teaching the technical skills of reading as disconnected from social and political contexts. Mathematics is also typically taught as apolitical, and contexts are often imposed rather than the mathematics naturally emerging from situational problems. Freire rejected this technocratic method with respect to teaching literacy, and aimed to teach "adults how to read in relation to the awakening of their consciousness… [he] wanted a literacy program which would be an introduction to the democratization of culture, a program with men *[sic]* as its Subjects rather than as patient recipients" (Freire, 1973, p. 43).

Similarly, this study sought to deepen participants' "understanding of society [leading] to engagement in social movements, at whatever level people are capable of participating given the daily struggles for survival" (Gutstein, 2006, p. 25). Gutstein (2006) viewed "writing the world with mathematics as a developmental process, of beginning to see oneself capable of making change, and…developing a sense of social agency (gradual growth)" (p. 27). The research sought to identify how participants described their gradual growth of understanding and engagement with mathematical and financial literacies.

On Personal Flourishing

Grant (2012) explained how to cultivate a more robust social justice vision of education, and argued for a democratic education for a flourishing and whole life. Brighouse (2006) and Grant (2012) described how, in order to make meaning and sense of important aspects of their lives, individuals should experience personal flourishing and personal autonomy. They must also develop the confidence and skills to contribute to society and to the (broadly defined) economy, democratic competency, and the facility and desire for cooperation (Grant, 2012). Personal flourishing also includes living and doing well, having a positive identity, having family and friends as support mechanisms, financial stability, education, and a commitment to children's flourishing minds.

On Quantitative Literacy

Steen (2001) described elements and expressions of quantitative literacy (QL) that included confidence with mathematics, being able to use mathematics in context, expressing quantitative literacy through citizenship (e.g., understanding data, projections, inferences etc.), the application of mathematics in one's educational trajectory, the application of quantitative literacy in one's personal finance and management, and the ability to make quantitative decisions with respect to one's personal health (e.g., options, dosages, risks, nutrition and exercise data etc.). Quantitative Literacy "empowers people by giving them tools to think for themselves, to ask intelligent questions of experts, and to confront authority confidently. These are skills required to thrive in the modern world" (p. 2).

Table 1: Framework for NMFL Educational Research

Assessing NMFL Educational Research	Description
Theme 1) Financial Knowledge	Data were coded this way when participants indicated a shift in financial knowledge or skills, or described their financial knowledge.
Theme 2) Re-Imagining Self and Possibilities	Data were coded this way when participants examined their assumptions and either realized that things should change, that things could change, or that they were not alone in the struggle for change. This theme represented a shift in beliefs about what was possible and about their agency.
Theme 3) Taking Action	Data were coded in this category when participants felt so strongly about the possibilities for change that they took action. These shifting practices were realizations of their beliefs about their agency.
Theme 4) Impact on Family	Data were coded in this theme when participants described the impact of the COY project on their relationships with other members of their family, including their ability to communicate with their children and impact their children's financial futures.
Theme 5) Features of the COY Program	This theme includes data that described how and why the structures of the COY program were beneficial.

Conceptual Framework

Though Grant (2012) did not write about mathematics education in particular, he advocated for core practices that would help cultivate flourishing lives while describing a robust social justice vision of education. I applied Grant's work to educational research, and in particular, to a Freirean approach and empowering pedagogy, attending to quantitative literacies that I refer to as NMFLs. I invoked some of Steen's *Elements* and *Expressions* of Quantitative Literacy to connect both Grant's and Steen's works together to create a framework through which I could study NMFL educational research. Table 1 shows some of the parallel ideas in both authors' works. This framework was used to analyze the data collected in this study.

Methodology

On Freirean Research and Pedagogy

The theoretical and methodological grounding for this study is decidedly Freirean. Freire's (2000/1970) work originated in an historical place and reality that required emancipatory action in order to free the oppressed from their illiteracy, and his works became symbolic for many marginalized people around the world. Educators and educational researchers have used *Pedagogy of the Oppressed,* and its liberatory practices, in situations that were not necessarily as political, but that arguably required a similar research approach. Freire described the ideas of human agency and empowerment as being essential to worthwhile learning and research practice. Critical social theorist were to take an ethical and political position using theory of education as a "practice of freedom" (Glass, 2001, p. 22) to shape a just and democratic society. Grant (2013) reminded us that social action is "individual or group behavior that involves interaction with other individuals or groups, especially organized action towards social reform" (p. 926). The goal of this project was to act to make financial decision-making and numerical empowerment accessible for the participants in this community of families.

The COY Project – Resources

A series of children's books on Financial Literacy were provided (e.g., Phillips, 2010) as well as adult and teen financial literacy books by Gail Vaz-Oxlade. The adults' Financial Literacy course was based on *Momentum* (see Table 2 below), a curriculum that was developed in Calgary, Canada (2010), and was provided free of charge by a local non-profit community partner. I designed and led this research project; however, it resulted from the collaborative efforts of several groups including the local university's Faculty of Education and four non-profit community partners.

Table 2: Financial Literacy Curriculum Materials – Momentum (2010)

Topic	Content
Assets	set short- and long-term goals for 5 types of assets (human, personal, social, physical, and financial)
Credit	take charge of it, develop an awareness and understanding of advantages and disadvantages of credit, wise use, and consumer credit tools
Budgeting	learn the difference between needs and wants, identify and track spending habits, build a budget and set short- and long-term goals
Consumerism	awareness of effects of consumerism and advertising, and develop ability to control consumerism, and live more simply
Banking	become aware of banking account options and benefits, learn how to use banking services, and access banking tools and resources

The COY project consisted of eight meetings on Wednesday evenings in the school's library. A meal and refreshments was available for participating adults and children. Funds for purchasing and preparing meals were raised by a group of college students from a local institution. The Math Camp curriculum was designed and implemented by the volunteer teacher candidates who participated as Math Camp Leaders. Various mathematics and financial literacy activities were personalized for the school-aged children who attended. Pre-school-aged children attended a child-care service provided free of charge by the school community. Adults participated in a Financial Literacy course; the facilitator was associated with a non-profit agency whose mission was to provide instruction in financial literacy at no charge.

Data Collection and Procedures

The proceedings of the adult financial literacy course were digitally recorded, as were individual pre- and post-interviews. Adults were asked three questions at the beginning of the program in an informal interview setting that was selected by each participant (either at their homes, in the school, or at a coffee shop). They were asked six questions at the end of the program. These were digitally audio-recorded. Each participating adult was interviewed for approximately 35 minutes at the beginning and at the end of the program. The semi-structured interview questions asked early in the program (pre-survey) dealt with the perceived quantitative needs for the families involved and the immediate community, the kind of mathematical learning and support the participants would appreciate, and the predicted benefits of the program. Near the end of the program, similar questions were asked (post-survey), including what kind of support was preferred, what were the benefits and disadvantages, and which specific teaching strategies, advice, structures, or "technologies" were believed to be of most benefit.

Data Analysis

After transcription and multiple readings of the text, the data were coded and classified into the framework's themes, as shown in Table 1. After initial data coding, the results were analyzed with respect to the research questions and the literature using *systematic combining* (Dubois & Gadde, 2014), and the data categories were clarified. Theory building involved the development of a research framework by critically evaluating emerging constructs against ongoing observations refining and re-combining theoretical ideas with empirical data.

Results and Analysis

This research study sought to determine whether participants could express personal flourishing within the context of numerical, mathematical, and financial literacies. As evidenced from the participants' responses, their beliefs, skills, knowledge, and actions could be classified into five themes that are supported by Grant's (2012) description of cultivating flourishing lives, and Steen's (2001) quantitative literacy, and are supported by literature (e.g., Freire, 2000; Gutstein, 2006). The five themes discussed here, and reflected in the participants' voices are 1) Financial Knowledge, 2) Re-Imagining Self and Possibilities, 3) Taking Action, 4) Impact on Family, and 5) Features of the COY Program.

Theme 1 Sample) Financial Knowledge

For this theme, participants made sense of important financial aspects of their lives, and described their financial knowledge:

> Consumerism, everything that you see out there is an ad. You are bombarded on television, media, print, advertising: "You deserve it, it's about you! You're entitled to this. You've worked hard. Enjoy it! Spend it! Here's a way that we can make it more affordable to you. People are sucked into this, and they are really not aware of it until it's too late, and that's for me, what I

> think this community, city, province, country, nation, I'm not going to say the world, it's North America that's got the problem. (Brent's Post-Interview)

Brent described his place in the larger picture of consumerism (one of the topics in the Momentum course). He understood and interpreted consumerism in his world.

Theme 2 Sample) Re-Imagining Self and Possibilities

For this theme, participants described beliefs of agency and confidence. During Valerie's post-interview, she described how she now felt that she had the ability to accomplish something of value:

> I finally felt that I had hope for the future and less of a defeatist attitude, because of what we had learned, we can make as small or as big of a change as you want, and any change would be better than where we were at. (Valerie's Post-Interview)

Theme 3 Sample) Taking Action

For this theme, participants engaged in, participated in, and contributed through personal agency in society. This theme describes adults who take action on their beliefs. During Carolyn's post-interview, she described how the COY program gave her the skills and understanding to run her household budget as well as the family's business budget. She explained:

> I got the financial support that I needed to start and maintain a budget. My household budget – I run my business budget as well. It will have an impact on my husband because I do run the household budget. It will have accountability impact on him. Instead of going with the flow, there will be more proactive planning involved. (Carolyn's Post-Interview)

Theme 4 Sample) Impact on Family

This theme describes adults who have the dispositions and communication skills to make decisions or take social action to contribute to society and participate in the democratic process. During Brent's post-interview, he described his parenting role as also the role of being a teacher for his children:

> I've always believed that I'm not just raising my kids but I'm teaching my kids to be a good parent, so there's a responsibility and there are tools in the program that now I can use to explain finances to them. (Brent's Post-Interview)

Theme 5 Sample) Features of the COY Program

This theme describes features of the Count On Yourself program that the participants felt were beneficial. This included specific topics and the content of the Momentum curriculum, and mostly the structure of the program including the importance of providing a safe, trusting, and inclusive environment for all members of the family:

> The researcher and facilitators were willing to be there, to be involved and share in personal experiences. It was obvious to everyone that you were excited about what you were doing and, because of your enthusiasm it's easier for us to be excited about it when we see your enthusiasm because we can see that it matters to us, and your ability to educate us, and to see the children all excited about learning? That's priceless! (Ema's Post-Interview)

Discussion

The participants were able to make meaning and sense of numerical, mathematical, and financial literacies in context. They developed knowledge from each other and made sense of those ideas throughout the process of the financial literacy course and curriculum. Guba and Lincoln (2005) reminded us that, as critical theorists, we continue to seek the active construction and co-construction of knowledge by human agents. As a group and individually, the participants made sense of financial knowledge and stresses in their lives and in each other's. Guba and Lincoln (2005) also stated that: "Critical theorists, especially those who work in community organizing programs, are painfully aware of the necessity for members of the community, or research participants, to take control of their futures (p. 202)". One of the participants used this very language to describe how she felt that the COY program helped her take control of her live and *give herself a future*.

In other data gathered, participants demonstrated their financial literacy by indicating that they successfully managed their money by paying their bills on time, making sure they didn't spend more than they earned each month, and paid their debts when they owed money. They became more knowledgeable about credit scores, being more in control of finances, and budgeting. Participants also displayed more agency and self-confidence with finances. Being heard was essential for this. The practice of "letting research participants speak for themselves" (Guba & Lincoln, 2005, p. 209) allowed them the voice to tell me they felt empowered themselves, in their ability to speak to their children, and that their children also felt empowered as individuals.

Guba and Lincoln (2005) discussed the validity of a study as *catalytic authenticity* when the researcher creates capacity in participants for positive social change and forms of emancipatory community action. Grant (2012) also spoke of the researcher's ability to encourage social action. In this study, there is evidence that the participants appreciated the community aspect of the project, and took action by visiting banks and credit unions, investment groups, and mortgage brokers. They began budgeting processes online and using software, contacted credit companies to ask for a reduction in their interest rates, opened separate bank accounts to better keep track of their expenses, consolidated their debt, and applied for loans. The exit survey supports this data indicating that all of the participants not only felt comfortable getting help, but also did get help with their finances. Taking action is powerful evidence of emancipation, and by taking action in the ways that they did, the participants showed evidence of practicing democracy.

Perhaps the most significant result of this project was the belief from participants that they had influence upon their children. Though not all families participated as complete units, those who had children in the COY program felt more able, and did discuss financial information with their children when appropriate. All participants agreed that they found value in the program and would recommend it to someone else. The implications for this result is that programs such as Count On Yourself, that involve the whole family, might possibly help break cycles of poverty in communities in which they are implemented. As I continue to consider Grant's (2012) vision, I am struck by his statement that:

> …you learn about how the practice of democracy can be made to work for you or against you and that it is important that you understand the differences as well as you know what you can do to influence an outcome that befits those who are marginalized. (p. 925)

Grant's intent was to clarify what it means to practice democracy. I find that this statement is reflected both in my practice as a researcher and in the evidence that was produced from this study through the voices of the participants. I entered this work to make a difference and to provide a learning space for families who could participate more democratically in society because they chose to learn to be more financially literate. I have learned that I can make a difference, and they have learned that they can make a difference in their own life situations. This most certainly reflects Grant's "practice of democracy" (p. 925).

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DEVELOPING A FRAMEWORK FOR ASSESSING THE IMPACT OF WHITENESS IN MATHEMATICS EDUCATION

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The ideology of whiteness has received little attention in mathematics education. In this paper, we develop a framework for documenting how whiteness shapes mathematics education as a racialized space. Drawing on the sociological concept of "white institutional space" (Feagin, Vera, & Imani, 1996; Moore, 2008), the framework examines mathematics education across institutional, interpersonal, and individual levels of analysis. The authors argue that this framework captures how ideological discourses of whiteness and colorblindness (Lewis, 2004) and racialized hierarchies of mathematics ability (Martin, 2009) are perpetuated through institutional structures and interpersonal relations in mathematics education.

Keywords: Equity and Diversity; Instructional Activities and Practices; Learning Theory

Introduction

Lipsitz (1995) states that "a fictive identity of whiteness" appeared in law as an abstraction and became actualized in everyday life. Much like 'black' is a cultural construction based on skin color, not biology, whiteness developed out of the reality of slavery and segregation, giving groups unequal access to citizenship, immigration, and property. By giving whites a privileged position in relation to the "other", European Americans united into a fictitious community. Whiteness is a constantly shifting boundary separating those who are entitled to certain privileges from those whose exploitation is justified by not being white. However, the boundaries of whiteness have shifted substantially over time (see Brodkin, 1998).

Recently, the ideology of whiteness and its material benefits has been sustained more covertly. Whiteness is supported by a colorblind ideology, a form of maintaining the social order, covertly, institutionally, and with the appearance of not being racial. Bonilla-Silva (2003) connects colorblindness with the resistance to framing, defining, or pathologizing whiteness and the ways that race plays out in the United States since the civil rights movement. While racism often calls forth overt practices such as slavery, the Jim Crow era, and lynchings, but the more recent avoidance of explicit racial discourses signifies colorblind racism, the dominant racial ideology since the civil rights movement (Bonilla-Silva & Forman, 2000).

Under colorblindness, it does not matter whether whites are racially conscious. Whites benefit from an external reading of themselves as white (Lewis, 2004), whether or not they identify as white. In other words, whites benefit not from their own realization of being white, but by others treating them as white. This distinction is important in understanding whiteness as an ideology rather than an identity. Therefore, a felt identity is not a prerequisite to reap unearned privileges. Whiteness functions within structures, deciding how resources, labor, and space will be distributed by means of housing segregation and educational and financial stratification. These structures are in place to benefit future generations, whether those generations adopt an intentional white identity. The point is not that all whites benefit the same, as this would be essentializing a very diverse group of people, but that one's racial position is constructed in relation to a racial history that has distributed space, resources and labor, and reproduced racist discourses (Lewis, 2004).

Connecting Whiteness to Mathematics Education

Whiteness plays out in very real ways through the divvying up of resources such as earnings, homes, wealth, and health (Lipsitz, 1995). Sewell (1992) and Lewis (2004) discuss racism both ideologically and concretely through considering its dual nature: symbolic (ideological) and material (structural resources) (Sewell, 1992; Lewis, 2004). Whiteness and colorblindness produce symbolic and material consequences within mathematics education (Battey, 2013a).

There are common *symbolic* narratives about who is better mathematically – whites and Asians. These perceptions are then made *materially* real in terms of how African American and Latin@s are treated in mathematics classrooms, the forms of instruction available, and course offerings, which in turn lead to different testing outcomes or "achievement gaps." Through impoverished instruction quality, tracking, and reduced funding, society makes the racial ideologies concrete. Therefore, "achievement gaps" in mathematics education reify the idea that whites and Asians are better at math, and African Americans are Latin@s are innately inferior.

Martin (2009) acknowledges the need for further research on race in mathematics education as a social construction shaped by existing sociopolitical contexts. More specifically, Martin (2009, 2013) calls for research on whiteness operating in mathematics education to address forms of racism in relation to achievement, participation, and student learning. Sociological work (Feagin, Vera, & Imani, 1996; Moore, 2008) informs Martin's (2008, 2009, 2013) conceptualization of mathematics education as a *white institutional space* based on four tenets:

> (a) numerical domination by Whites and the exclusion of people of color from positions of power in institutional contexts, (b) the development of a White frame that organizes the logic of the institution or discipline, (c) the historical construction of curricular models based upon the thinking of White elites, and (d) the assertion of knowledge production as neutral and impartial, unconnected to power relations (Martin, 2013, p. 323)

Using these tenets, Martin (2008) highlights how the National Math Advisory Panel is an example of mathematics education policy as white institutional space resulting in "e(race)sure" – the exclusion or ignoring of race – that perpetuate notions of whiteness and white supremacy in mathematics. We similarly draw upon these four tenets in the next section to propose a framework that assesses the extent to which mathematics education is a white institutional space.

Theoretical Framework

Our framework in assessing the impact of whiteness on mathematics education presented in Table 1 considers three dimensions: institutional, interpersonal, and identity. Martin's four tenets cut across these three dimensions. The first tenet of white institutional space, racialized patterns of representation, aligns with parts of both division of labor and physical space. This directly relates to distribution of power, but also the representations of images symbols, and behaviors presented in schools. Therefore, it is not only about the distribution of people, but of the distribution of valuing and devaluing various ways of being as well. The second tenet also aligns with ideology and division of labor, but more in terms of the organizational structure. The organizational structure of the school determines behavioral sanctions and classroom norms that then shape interpersonal interactions and identity construction. The organizational structure also legitimizes certain ideologies over others, such as tracking supporting a fixed notion of mathematical intelligence. The third tenet of historically white curricular models is aligned with the section on history, but as students respond to this history, they take on varying identities in relation to the mathematics and schooling. Finally, the fourth tenet of white institutional space corresponds with how ideological discourses in mathematics differentially shape whites and students of color's mathematics experiences. This, in turn, structures notions of competence and legitimacy in students' negotiations of their mathematics identities including what it means be "good" at mathematics. These four tenets are threaded throughout the framework.

Table 1: Framework to Assess Whiteness in Mathematics Education

Institutional	Ideological Discourses	• Racial hierarchy of math ability • Innateness of mathematics ability • Mathematics as neutral • Abstract individualism • Meritocracy
	History	• Histories of schools • Patterns of inclusion and exclusion • Curricular perspectives • Multiple perspectives
	Organizational Logic	• Distribution of power and work • Organizational structure • Positioning of Stakeholders
	Physical Space	• Physical representations • School messages • Visibility of students • Control of physical expression
Labor	Cognition	• Differential cognitive demand • Distribution of classroom and mathematical authority • Academic expectations
	Emotion	• Management of emotional experiences • Regulation of emotion • Range of emotional experiences allowed
	Behavior	• Discipline • Management of Behavior • Language Norms • Teacher praise/acknowledgment
Identity	Academic (De)Legitimization	• Identification with mathematics • Legitimacy of intellectual ability
	Co-Construction of Meaning	• Hierarchy of mathematics ability • Peer perceptions of each other • Hypervisibility/invisibility
	Agency and Resistance	• Relationship with deficit discourses • Forms of (dis)engagement • Association with peer group

Institutional

Institutional spaces constrain or afford different access to people, resources, and work. In distributing this access, they legitimize certain ideologies through the physical space, positioning of different groups, and presentation of history. The institutional level is responsible for framing the levels of labor and identity since it is responsible for the organization of labor and determines the ideologies and people in which individuals will develop relationships.

Ideological Discourses. As noted in the introduction, broad discourses such as colorblindness and abstract individualism often accompany whiteness. Within mathematics education, whiteness takes the form of racial hierarchies of mathematics ability (Martin, 2009) as well as the innateness of mathematics ability (Ernest, 1991). The racial hierarchy of mathematics ability benefits the identities that white and Asian American students can construct with the domain, but accompanying discourse around the innateness of ability makes the racial hierarchy stable. Evidence for these discourses come in teachers' and schools' stable notions of high and low mathematics students that are then institutionalized in forms of tracking and subsequent differential access to cognitive demand. In terms of privilege, the discourses are evidenced by the automatic attribution of Asian Americans and whites as being good at mathematics and/or surprise when these student struggles.

History. Schools have histories that are inseparable from issues of exclusion, segregation, and differential resources in the United States. These historical issues contribute to current educational inequality. For instance, a school may have been segregated, then bussed in African American students, only to see white flight result in home prices dropping and the tax base that determines school funding collapse. Therefore, a history of inclusion or exclusion has an impact on teacher retention, school demographics, and school funding. Curricula also present who has been involved in constructing history. The inclusion or exclusion of groups within curricula communicates to students whose perspectives matter and who is important. Finally, the perspective within curricula communicates notions of exclusion, assimilation, resistance, or valuing regarding different cultures and values. Martin (2008), for example, describes how the National Math Advisory Panel's curricula recommendations focused on algebra and other mathematical content to advance white elites' agenda of international competitiveness.

Organizational Logic. Schools are organizations that situate people in different ways and distribute power accordingly. How that power is distributed and who it is distributed to matters. The power distribution between administrators, teachers, parents, and students says a lot about who is included and valued within schools. For instance, parents who are viewed as over-involved with the influence to determine curriculum, positions them as having power in contrast to those framed as oppositional in defending their children, uninvolved, or not caring. In these differing logics, parents are granted varied power. The same can be true for teachers and students. Organizational logic is what determines who has power, who does what work, and who evaluates whom. In this distribution of power, there is the potential to have different races in more privileged and more subservient roles leading to inequitable racial representation in positions of power. This distribution determines different forms of labor including the labor that is required of students and the extent to which this prepares them for future success.

Physical Space. Images, charts, symbols, and objects are concrete representations that communicate central aspects of institutions. Pictures that designate notable people in history, student recognition, and school history pass on messages about who is accepted, welcomed, and who can excel academically. Images, histories, and perspectives of African American and Latin@ students can be invisible at times (Moore, 2008). This can contrast with the hypervisibility (Higginbotham, 2001) when students are asked to speak for their race or teachers hyper focus on the misbehavior of students of color. Aligned with this, charts about acceptable behavior can be ways of controlling students. For instance, behavioral norms that promote militaristic rules of order or student "uniforms" are clear messages that the school sees students as needing to be controlled. Repeated school slogans in schools such as "I'm smart! I know that I'm smart" found in Kozol's (2005, p. 36) work communicate just the opposite. If students were assumed to be smart, there would be no need to repeat these types of mantras. Similarly, the lack of these messages in predominantly white contexts is an implicit transmission that students are expected to be intelligent, under control, or that these students do not need to see representations of current and historical figures that do not look like them. This is also a way to perpetuate whiteness, by communicating that there are such a limited number of significant African Americans or Latin@s that white students do not need to know about them.

Labor

How labor is divided in classrooms can reflect the presence of whiteness. Normative expectations of emotional and behavioral work can restrict students to being certain types of students - controlling them to fit unquestioned cultural expectations. When forms of labor are restricted in such a way that students of color's contributions and behaviors are not seen as valid, it can be a sign that whiteness is operating in a context. We use three dimensions of labor to detail how whiteness can operate within classrooms: cognition, behavior, and emotion.

Cognition. Cognition is interpersonal in the sense that the kinds of mathematical work students are asked to do sends messages about what students are capable of. A number of researchers have documented the lower levels of work that African American and Latin@ students are asked to do in classrooms. Classroom settings that only ask students to replicate procedures, follow worksheets page by page, and lack the opportunity to engage in cognitive depth permeate the literature for African American and Latin@ students (Ladson-Billings, 1997; Lubienski, 2002). Additionally, how authority is distributed, both for classroom procedures and the mathematics, also speaks to whether a teacher holds expectations that students can self-monitor their behavior and gain command of the mathematics. If these ways of parsing classroom cognition are coupled with ideologies of a racialized hierarchy of mathematical ability, then they are signs that whiteness is at play. But it is also more complex than this. For instance, even in a mostly African American classroom, some students may have more access to content and authority than others. If the students who are seen as more capable fit norms for white behavior, then whiteness is still at play. Patterns as to which students have access to which cognitive tasks can be quite telling.

Emotional. Coping with discrimination and racism in everyday experience requires significant emotional labor in terms of sadness, frustration, and anger (Moore, 2008). However, schools and classrooms often do not provide the time, space, or support for students to process these experiences and emotions. When students do process or exhibit these emotions, they can be seen as angry, aggressive, or violent rather than struggling with a complex and unfair world. Moore (2008) discusses how law schools continue to ignore and undervalue this emotional labor:

> Coping with everyday racism in the law school frequently produces frustration, anger, or sadness, but the institutional logic of the law school does not recognize expressions of these emotions as legitimate. Students are thus forced to manage their emotion in order to avoid further marginalization… This demands that students of color perform invisible and emotional labor that their white counterparts are not required to perform. Both in the law school and in the profession of law, this labor is expected of law students of color, yet it goes unrecognized and unrewarded (p. 31).

Additionally, students must manage the ways in which they express emotions to avoid deficit discourses about being perceived as argumentative, angry, aggressive, and a multiple of other negative associations. When students of color are expected to relate experiences they consider unfair in a calm, dispassionate, and disconnected way, then whiteness is restricting acceptable ways of grappling with the emotions of discrimination and racism (Moore, 2008). Finally, this emotional labor places an undue cognitive burden on students as well. Dovidio and Gaertner (2008) found that when solving mathematics problems, African American students within groups that made them process emotions related to discrimination more, performed work slower when compared to those who did not. Steele and Aronson's (1995) work on stereotype threat can also be seen as the result of the added emotional labor due to priming race during cognitive tasks.

Behavior. One way in which labor is handled is by deeming certain student behaviors appropriate and others not. This has immense consequences in classrooms as harsh and frequent discipline has been found to frequently lead to missed instructional time and expulsion from school for African American and Latin@ males in particular (Gregory, Noguera, & Skiba, 2010). Within mathematics, this can take the form of deeming certain ways of language use as inappropriate for mathematical argumentation or by requiring students to sit still in seats in regimented ways (Battey, 2013b). Further, whiteness can function by valuing behaviors of white students over others in subtle ways of how language and behavior are perceived to align with understandings of appropriate classroom actions. When students align with white ideals about behavior, their actions will likely be praised or sanctioned. When students do not align, maybe through being too argumentative, too quiet, too excited, or abrasive, we would expect to see behavior to be called out, positive behaviors to go unnoticed and a hyper-focus on misbehavior leading to increasing discipline and eventually suspensions and expulsions. When teachers employ such behavioral control despite substantive mathematical contributions in classrooms (see Battey, 2013b), it is evidence that a broader ideology is at play.

Identity

Martin (2009) defines *mathematical identities* as "dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the condition of their lives" (p. 326). The construction of mathematical identities, however, is not a strictly personal, internal process as it is constantly negotiated with institutional influences and interpersonal encounters. More specifically, the organizing white frame relegates African Americans and Latin@s as mathematically incapable and innumerate and thus grants unquestioned legitimacy to whites in mathematics education spaces. This aligns with Martin's (2009) notion of mathematics as a racialized such that the social construction of whiteness is maintained in mathematics classrooms through the inequitable learning opportunities and academic de-legitimization experienced of marginalized students.

Academic (De)Legititmization. Mathematics classrooms that function as white institutional spaces require students to negotiate academic legitimacy across a racialized hierarchy of ability based on white norms and values. Understanding mathematical identities, therefore, can only be attained by detailing processes of negotiation within racialized discourses as opposed to traditional analyses of achievement gaps between different races (Martin, 2009). With whites and Asian Americans – considered "honorary whites" (see Bonilla-Silva, 2003) – at the top of the hierarchy of mathematics ability, whiteness in mathematics classrooms operates in ways that they are assumed or assume themselves that they are mathematically intelligent. Conversely, students of color's legitimacy is always under question so that they need to prove themselves mathematically capable by subscribing to white views of success that structure the academic spaces. Deficit perspectives on students of color's mathematics ability stem from these ideological discourses and in turn position these students as illegitimate members of mathematics classrooms resulting in poor relationships with teachers, lower-quality instructional experiences, and expressed disidentification with the mathematics subject (Spencer, 2009). Therefore, whiteness can be seen in students' stable identification or dissociations with the mathematics domain, consistent with racial hierarchies.

Co-Construction of Meaning. Students construct mathematical identities in relation to the people and the institutions in which they participate. Therefore, the explicit and implicit ways in which people and institutions pass on messages are critical for how students develop mathematical identities. For instance, ability grouping or tracking along racial lines send messages to students about the racial hierarchy of ability (Lewis, 2004). Teacher comments about low and high students or needing to learn the "basics" pass on messages more overtly (Battey & Fanke, 2015). Within school contexts, students construct what being good at mathematics means. Maybe being good means the student who finishes first, but cannot explain their thinking. Additionally, Moore (2008) discusses peer perceptions of academic support programs in law schools such that some students think that students of color in the programs were admitted to the school based on race rather than earning it. This anti-historical view ignores the reasons for programs that remedy institutional racism. This view also perpetuates whiteness by not recognizing the material racism that produced and continues to produce differential access to educational quality. However, as institutions leave these perspectives, programs, and racism unaddressed, they participate in limiting spaces for students to construct identities that counter the racial hierarchies contained by whiteness.

Agency and Resistance. Despite racial oppression, it is important to also consider African Americans and Latin@s agency in negotiating their racial identities and mathematics success. Although Martin (2009) uses African Americans' experiences to illustrate racial struggles in mathematics, his discussion can be extended to other marginalized student populations as they "negotiate and resist the racialization processes that attempt to position and confine [them] within an existing racial hierarchy" (Martin, 2009, p. 325). This illustrates the importance of inclusion of the voices and experiences of those marginalized. Martin (2009, p. 315) states:

Moreover, because little attention has been given to resistance, contestation, and negotiation of these meanings, disparities in mathematics achievement and persistence are often inadequately framed as reflecting race effects rather than as the consequences of the *racialized* nature of students' mathematical experiences [emphasis in original].

Therefore, examining unchallenged racialized discourses in mathematics classrooms is making plain whiteness as taken for granted. Unchallenged racial discourses keep individual experiences of race internal for both whites and students of color. However, for students of color, this is more detrimental because unchallenged, they may either disassociate from their race, community, and history to succeed mathematically, or internalize the discourse. For mathematically successful African American students then, they may disassociate from peers or downplay their success. For mathematically unsuccessful Latin@ students, they may disassociate from mathematics or schooling through resistance by active challenging of educators views of students, purposeful disengagement, or dropping out.

Conclusion

Whiteness is a widespread ideology in society. While it is getting more attention in the broader education literature, mathematics educators have been slow to research it's impact on African American and Latin@ students (Battey, 2013a). However, its impact on white students is just as important in making unearned privileges visible to the field. We hope this framework supports the field in identifying the effects of whiteness at different levels of the educational system. The goal for us it to support the development of a mathematics space that builds collective consciousness of racism in order to prevent students of color from internalizing deficit ideologies (Feagin, 2006; Moore, 2008). This in turn would open more space for student identities that challenge existing racial hierarchies.

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THE RELATIONSHIP BETWEEN MATHEMATICS IDENTITY AND PERSONALITY ATTRIBUTES WITH STUDENTS' CAREER GOALS

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In this study, we surveyed 570 Calculus I and Calculus II students at two large public universities in the Northeastern region of the United States. We explored the relationship between these students' career goals in mathematics and other STEM fields, with their mathematics identity and self-identified personality attributes. Our findings suggest that mathematics identity can be used as a way of explaining persistence in mathematics and other STEM fields. We also found certain personality attributes to be correlated with persistence in these fields and these personality attributes varied based on gender. We conclude with a detailed analysis of our findings and some implications.

Keywords: Affect and Beliefs; Gender; Post-Secondary Education

Introduction

There is a shortage of individuals entering STEM (Science, Technology, Engineering, and Mathematics) careers in the United States (Langdon, McKittrick, Beede, Khan & Doms, 2011; National Academy of Sciences, 2010). For the past 50 years, there has been a notable decline in the number of college students choosing STEM majors, and college students are more likely to switch from a STEM major to a non-STEM major than the other way around (Seymour & Hewitt, 1997). Furthermore, data suggest that there has not been notable increases in the percentage of students receiving STEM degrees in recent years, with only a 1% increase of bachelor's degrees awarded in STEM fields between 2002 and 2012 (NSF, 2015). Given that the growth of STEM jobs was three times that of non-STEM jobs during the past decade or so, this shortage of citizens interested and educated in the STEM fields is of national concern (Langdon, McKittrick, Beede, Khan, & Doms, 2011).

In addition, there is evidence of a continued underrepresentation of women choosing a career in STEM fields. For instance, based on data reported by the National Science Foundation, in 2013 although women made up 46.1% of the entire U.S. workforce, they were only 14.8% of those employed as engineers, 11.8% of those employed as physicists and astronomers, and only 25.4% of those employed as mathematical or computer scientists (NSF, 2015). Furthermore, the percentage of women earning degrees in STEM fields has been declining. For example, the percentage of women earning bachelor's degrees in the mathematical sciences fell from 48% in 2001 to 43% in 2009. Other fields, such as engineering and computer science, have seen similar declines in women earning such degrees during the same timeframe (NSF, 2015). Though these results inform us of the underrepresentation of women in these fields, they do not provide insight as to why these trends are occurring. It is important to explore reasons why this underrepresentation is still persisting and why gender gaps in participation are increasing in some cases.

Prior research has shown a connection between individuals' mathematics identity and students' persistence and commitment to mathematics and other STEM fields (Boaler & Greeno, 2000; Cass, Hazari, Cribbs, Sadler, & Sonnert, 2011). Furthermore, in previous work, Piatek-Jimenez (2015) noted specific self-identified personality attributes common amongst women mathematics majors.

For this study we are combining these two constructs to determine what relationships exist between students' mathematics identities and self-identified personality attributes with their intended choice of career.

The following research questions were used to guide this study: 1) What is the relationship between students' career goals in a STEM field with their mathematics identity and self-identified personality attributes?; 2) What is the relationship between students' career goals in a mathematics field (mathematician and mathematics teachers) with their mathematics identity and self-identified personality attributes?; and 3) How do these models vary between females and males?

Theoretical Framework

Identity is a construct that is becoming increasingly utilized in mathematics education research when exploring students' persistence and attrition in mathematics (Boaler & Greeno, 2000; Martin, 2000; Piatek-Jimenez, 2008; Piatek-Jimenez, 2015). Though different authors have conceptualized the construct in slightly different ways, we view "mathematics identity" as how individuals see themselves in relation to mathematics based on their perceptions and navigation of everyday experiences with mathematics (Enyedy, Goldberg & Welsh, 2006). We view this as being a part of an individual's "core identity", which is a more enduring sense of who an individual is and who he or she wants to become (Cobb & Hodge, 2011).

Our framework for mathematics identity draws from prior work in the field of mathematics and science (Cribbs, Hazari, Sadler, Sonnert, 2012; Carlone & Johnson, 2007; Hazari, Sonnert, Sadler & Shanahan, 2010). In our framework, mathematics identity is comprised of the sub-constructs interest, recognition, and competence/performance. Interest refers to an individual's desire or curiosity to think and learn about mathematics. Recognition refers to how an individual perceives others view him or her in relation to mathematics. Competence/performance refers to an individual's beliefs about his or her ability to understand and perform in mathematics. It is the inclusion of these three factors that provide a better picture of an individual's mathematics identity.

While mathematics identity is a strong predictor for students' career goals in certain STEM fields, such as engineering (Cass et al., 2011) and mathematics (Cribbs et al., 2012), other factors influence an individual's choice whether or not to pursue a STEM career. Furthermore, Gee's (2001) work related to identity indicates that an individual may have many different identities. For example, a woman mathematics major may identify as both a woman and a mathematics major. Potentially conflicting expectations about what it means to be a woman and what it means be a mathematics major may inform her decision whether or not to enter a career in mathematics.

In particular, our study is concerned with the role that gender plays in the expectations and choices an individual makes. For example, gender stereotyping could play a role in how individuals view themselves and subsequently the choices they make. Research notes that parents (Furnham, Reeves & Budhani, 2002; Frome & Eccles, 1998) and teachers (Helwig, Anderson & Tindal, 2001; Li, 1999) hold different beliefs about males' and females' abilities in mathematics. In addition, many scholars have noted connections between individuals' personality attributes and career choice (Ackerman & Beier, 2003; Buddeberg-Fischer, Klaghofer, Abel & Buddeberg, 2006; Schaub & Tokar, 2004). Furthermore, Luyckx, Soenens, and Goossens (2006) found correlations between an individual's identity and personality traits. By exploring mathematics identity and students' self-identified personality attributes, we might develop a better understanding of how expectations and other identities interact to influence students' career goals.

Methods

This study collected data from two different universities in the northeast region of the United States by administering surveys in the fall of 2014 with students enrolled in Calculus I and Calculus

II courses, yielding 570 completed surveys. The survey was developed to collect student demographic information, career goals, perceptions related to mathematics identity, and self-identified personality attributes. Content validity was insured through: 1) pulling from literature related to personality attributes, gender, and mathematics identity (Cribbs et al., 2012; Piatek-Jimenez, 2015; Ely, 1995; Jones & Myhill, 2002; Luhtanen & Crocker, 1992), 2) conducting a pilot test with survey items related to personality attributes and gender stereotyping in the spring of 2014, and 3) conducting a pilot test with the completed survey in the summer of 2014. The initial pilot test also included a series of focus group sessions with participants to further refine the survey items. Both pilot tests were done with college students enrolled in mathematics and mathematics education courses at the universities where the study was conducted.

Logistic regression was used to address each research question because the outcome variable (career goals) was dichotomous. One regression model was created to examine students' career goals in a STEM field and another was created to explore students' career goals specifically in a mathematics field (mathematician or mathematics teacher). Table 1 details the possible choices available on the survey and corresponding number of responses. We can see from the table that of the 53 participants with career goals in a mathematics field, just under 60% of them were interested in becoming K-12 mathematics teachers.

Table 1: Number of Students Selecting each Career Goal

Career Goal	Number	Percent of Sample
Non-STEM (e.g. lawyer, business person)	197	35
STEM		
Mathematician	22	4
Math teacher	31	6
Life/earth/environmental scientist	52	9
Physical scientist	26	5
Engineer	176	31
Computer scientist, IT	53	9
Science teacher	8	1
STEM Total	368	65

Because we were looking at mathematics identity in particular (and not STEM identity), we were also interested in how the results might differ when comparing students who have chosen a mathematics field to those choosing a non-STEM field. Therefore, in conducting the regression analysis for research question 2, all other STEM participants were excluded in order create a model that compared how participants selecting mathematician or mathematics teacher related to those who selected a non-STEM field. A proxy for mathematics identity was used based on results from a previous analysis (Cribbs et al., 2012). In addition, the following control variables were considered for the model: gender, age, class standing in college, and mathematics course enrollment.

Results

Table 2 details the results for research question 1: What is the relationship between students' career goals in a STEM field with their mathematics identity and self-identified personality attributes? Only significant control variables, with a significance level of $p < 0.05$, were included in the final models shown in this section.

Table 2 shows that the only control variable remaining in the model was "current math class." The results indicate that mathematics identity significantly predicts students' career goals in a STEM field with an odds ratio of 1.5. In other words, a one unit increase in the mathematics identity proxy increases the odds of a student selecting a STEM career field by one and a half times. The results also indicate that several self-identified personality attributes are correlated with students' career goals STEM. The positively correlated items are "I am able to be 'one of the guys'" with an odds ratio of 1.2, "I am inquisitive" with an odds ratio of 1.3, and "I am passionate about my major" with an odds ratio of 1.4. The negatively correlated items include "I am feminine" with an odds ratio of 0.9, "I am forceful with my opinions" with an odds ratio of 0.8, "I earn good grades" with an odds ratio of 0.6, "I am academically motivated" with an odds ratio of 0.8, and "I show concern for people's well-being" with an odds ratio of 0.7.

Table 2: How do mathematics identity and personality attributes predict STEM career goals

(N=570)	Estimate	SE	Odds Ratio	Sig.
Intercept	1.258	0.837		
Controls				
Current math class	0.280	0.081	1.323	***
Mathematics Identity	0.406	0.087	1.501	***
Self-Identified Characteristics				
I am feminine	-0.143	0.051	0.867	**
I am forceful with my opinions	-0.226	0.072	0.798	**
I earn good grades	-0.543	0.121	0.581	***
I am able to be "one of the guys"	0.185	0.076	1.204	*
I am academically motivated	-0.250	0.113	0.779	*
I show concern for people's well-being	-0.308	0.106	0.735	**
I am inquisitive	0.268	0.098	1.307	**
I am passionate about my major	0.315	0.087	1.371	***

*p<0.05 **p<0.01 ***p<0.001

Table 3 details the results for research question 2: What is the relationship between students' career goals in a mathematics field and their mathematics identity and self-identified personality attributes?

Table 3: How do mathematics identity and personality attributes predict career goals in a mathematics field (mathematician or mathematics teacher)

(N=250)	Estimate	SE	Odds Ratio	Sig.
Intercept	-2.684	1.355		*
Mathematics Identity	1.340	0.248	3.819	***
Self-Identified Characteristics				
I am self-sufficient	-0.375	0.176	0.687	*
I am concerned about future career obligations	-0.385	0.139	0.681	**
I cry easily when I am angry/upset	0.217	0.103	1.242	***
I have high career aspirations	-0.931	0.240	0.397	*
I am concerned about future family obligations	0.343	0.141	1.409	**
I do not mind sacrificing my personal time for my studies	-0.441	0.153	0.643	*
I have a strong work ethic	0.217	0.103	1.763	*

*p<0.05 **p<0.01 ***p<0.001

Table 3 shows that mathematics identity significantly predicts students' career goals as a mathematician or mathematics teacher with an odds ratio of 3.8. The results also indicate that several self-identified personality attributes are correlated with students' career choice as a mathematician or mathematics teacher. The positively correlated items are "I cry easily when I am angry/upset" with an odds ratio of 1.2, "I am concerned about future family obligations" with an odds ratio of 1.4, and "I have a strong work ethic" with an odds ratio of 1.8. The negatively correlated items include "I am self-sufficient" with an odds ratio of 0.7, "I am concerned about future career obligations" with an odds ratio of 0.7, "I have high career aspirations" with an odds ratio of 0.4, and "I do not mind sacrificing my personal time for my studies" with an odds ratio of 0.6.

Table 4 and 5 detail results for research question 3: How do these models vary between females and males?

Table 4: How do mathematics identity and personality characteristics predict career goals in a STEM field for females

(N=191)	Estimate	SE	Odds Ratio	Sig.
Intercept	5.155	1.499		***
Mathematics Identity	0.731	0.152	2.088	***
Self-Identified Characteristics				
I am spontaneous	-0.298	0.136	0.742	*
I am forceful with my opinions	-0.221	0.109	0.802	*
I earn good grades	-0.710	0.213	0.491	***
I am able to be "one of the guys"	0.292	0.128	1.340	*
I am academically motivated	-0.600	0.206	0.549	**
I am passionate about my major	0.352	0.167	1.422	*
I care about my appearance	-0.404	0.147	0.668	**

*p<0.05 **p<0.01 ***p<0.001

Table 4 indicates that mathematics identity significantly predicts female students' career goals in a STEM field with an odds ratio of 2.1. The results also indicate that several self-identified personality attributes are correlated with female students' career goals in STEM. The positively correlated items are "I am able to be 'one of the guys'" with an odds ratio of 1.3 and "I am passionate about my major" with an odds ratio of 1.4. The negatively correlated items include "I am spontaneous" with an odds ratio of 0.7, "I am forceful with my opinions" with an odds ratio of 0.8, "I earn good grades" with an odds ratio of .5, "I am academically motivated" with an odds ratio of 0.5, and "I care about my appearance" with an odds ratio of 0.7.

Table 5: How do mathematics identity and personality characteristics predict career goals in a STEM field for males

(N=378)	Estimate	SE	Odds Ratio	Sig.
Intercept	-0.751	0.915		
Controls				
Current math class	0.312	0.103	1.366	**
Mathematics Identity	0.270	0.115	1.310	*
Self-Identified Characteristics				
I am forceful with my opinions	-0.198	0.100	0.821	*
I earn good grades	-0.358	0.139	0.699	*
I do not mind sacrificing my personal time for my studies	-0.281	0.107	0.755	**
I am inquisitive	0.303	0.120	1.354	*
I am passionate about my major	0.378	0.104	1.459	***

*p<0.05 **p<0.01 ***p<0.001

Table 5 indicates that mathematics identity significantly predicts male students' career goals in a STEM field with an odds ratio of 2.1. The results also indicate that several self-identified personality attributes are correlated with male students' career goals in STEM. The positively correlated items are "I am inquisitive" with an odds ratio of 1.4 and "I am passionate about my major" with an odds ratio of 1.5. The negatively correlated items include "I am forceful with my opinions" with an odds ratio of 0.8, "I earn good grades" with an odds ratio of 0.7, and "I do not mind sacrificing my personal time for my studies" with an odds ratio of 0.8.

Discussion

Our findings provide evidence that mathematics identity can be used as a way of explaining student persistence in STEM fields. Specifically, a shift in the mathematics identity proxy of one standard deviation corresponds to a 1.5 higher odds of having career goals in a STEM field and a 3.8 higher odds of having career goals as a mathematician or mathematics teacher. The odds ratio is much higher for career goals as a mathematician or mathematics teacher than for a general STEM field, as might be expected.

Our results also suggest that certain self-identified personality attributes are positively correlated to students' career goals in STEM. In particular, we found that "I am able to be 'one of the guys'" is positively correlated with having career goals in a STEM field. This finding might be a result of society's stereotypical belief that scientists and mathematicians are generally male (Picker & Berry, 2000). Therefore, individuals who believe that they can fit in as "one of the guys" may be more likely to choose a career in STEM. We also found that the attribute "I am passionate about my major" was positively correlated with choosing a STEM field. This, too, many be related to the belief that scientists and mathematicians are obsessed with their field (Piatek-Jimenez, 2008; Picker & Berry, 2000). Furthermore, we found that "I am inquisitive" was positively correlated with choosing a STEM field. We believe that this finding could relate to the fact that the field of science is often affiliated with inquiry, and other STEM fields, such as engineering and mathematics, are often related to problem solving.

There were also five negatively correlated self-identified personality attributes with choosing a STEM field. One of these attributes was "I am feminine". This may not be a surprising result given that, as previously mentioned, STEM fields are often believed to be masculine fields. This finding is consistent with that of Piatek-Jimenez (2015) who found that women mathematics majors often do not identify with being highly feminine. Another attribute that negatively correlated with wanting a STEM career was "I show concern for other people's well-being". This finding is consistent with Morgan, Isaac, and Sansone's (2001) work that shows college students believe that the physical and mathematical sciences are less likely to have "people-oriented" careers. Therefore, individuals who are highly concerned for other people's well-being may be less likely to choose a STEM career. Two items we found more difficult to interpret were "I earn good grades" and "I am academically motivated" as being negatively correlated with choosing a STEM field. Initially these both seemed counter-intuitive to us, given that STEM fields tend to be viewed as more rigorous fields. However, we believe this is the exact reason that "I earn good grades" did show up as a negative correlation. Because STEM fields are often seen as more rigorous, students in these fields may be used to earning lower grades than in other non-STEM fields. As of yet, we are still unable to determine why "I am academically motivated" was negatively correlated with STEM. In future work, we intend to do qualitative interviews with students to learn more about this finding.

In addressing our second research question, our results shown in Table 2 suggest that certain self-identified personality attributes are correlated specifically with career goals in a mathematics field, and that these are slightly different than for STEM careers, in general. In particular, we found that "I am concerned about future family obligations" was positively correlated with wanting a career in mathematics while "I am concerned about future career obligations" and "I have high career aspirations" were negatively correlated with wanting a career in mathematics. We believe that these findings may be a result of the fact that approximately three-fifths of the participants wanting a career in mathematics were intending to become a K-12 mathematics teacher. Given that the societal perception of teaching is that it provides a schedule more conducive to fulfilling family obligations (Anthony & Ord, 2008) than possibly other STEM fields may be a reason for this finding.

In order to address our third research question, we ran the model with regards to choice of STEM career separately for females and males. There were three personality attributes that remained consistent in all three models. We found that "I am passionate about my major" was positively correlated and "I am forceful with my opinions" and "I earn good grades" were negatively correlated in all three models. This finding shows that these personality attributes in predicting STEM career goals were not dependent upon gender. However, there were personality attributes that did depend on gender. Interestingly enough, the personality attribute of "I am able to be 'one of the guys'" was found to be statistically significant for female students wanting STEM careers but not for male students. We find this result to be important because it demonstrates the need for women to feel like they fit in as "one of the guys" when choosing a career in a STEM field. This result suggests that there may still be an embedded societal belief that the STEM fields remain male-dominated and male-driven and that in order to enter these fields, one must be able to be "one of the guys".

While having a strong mathematics identity is an important factor in choosing mathematics or other STEM career goals, it certainly is not the only factor influencing such decisions. Many factors play a role in career choice, and our work suggests that personality attributes may be one of those factors. Furthermore, we found that the personality attributes that correlate to choosing STEM careers differ slightly between males and females. If our goal is to encourage more students, and specifically more women, to enter STEM careers, a better understanding of the role that these personality attributes play in such decisions will allow us to better recruit talented students into these fields.

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WHAT IS EQUITY? WAYS OF SEEING

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Prospective teachers must be prepared for their role in providing equitable access for learning high quality mathematics. Therefore, it is imperative that mathematics teacher educators provide opportunities to develop an equity-centered orientation in teacher preparation courses. In this study, we begin to address this issue by identifying what prospective teachers attend to in a classroom vignette of an African American male student who is above grade level in mathematics and exhibits disruptive behavior during instruction. The results of the study indicate that while participants are beginning to attend to cultural influences, most responses are focused on classroom management strategies.

Keywords: Equity and Diversity; Teacher Education-Preservice

Introduction and Background

The National Council of Teachers of Mathematics (NCTM) asserts that in order to teach in an equitable manner, teachers and schools must maintain "high expectations and strong support for all students" (NCTM, 2000, p. 11), meaning mathematics teachers must provide opportunities for students to learn challenging mathematics regardless of their students' "personal characteristics, backgrounds, or physical challenges" (p. 12). For the past two decades, mathematics educators conceptualized what it means to teach mathematics for equity (Gutiérrez, 2002; Gutstein, 2003; Hart, 2003; Matthews, 2005); yet only in recent years have mathematics teacher educators documented efforts to prepare prospective teachers (PTs) to teach mathematics while considering matters of equity (Bartell, 2010; Freitas, 2008; Wager, 2014). Unfortunately, many PTs and practicing teachers do not know how to make these necessary connections, especially with students who are different from their own culture and background (Futrell, Gomez, & Bedden, 2003; Turner, Drake, Roth McDuffie, Aguirre, Gau Bartell, & Foote, 2012). Therefore, within the current educational system, students from non-dominant backgrounds are often denied equitable opportunities to learn (Wager, 2014).

To further complicate the issue, many mathematics teachers dismiss issues of equity as relevant factors in the mathematics classroom because they view mathematics as a universal, culture-free subject (Rousseau & Tate, 2003). However, there is a growing body of mathematics education researchers who understand that mathematics and mathematical knowledge are neither universal nor culturally neutral, but are situated in a sociocultural framework (Ukpokodu, 2011). Moreover, Gay (2000) argues that if we "decontextualiz[e] teaching and learning from the ethnicities and cultures of students [it] minimizes the chances that their achievement potential will ever be fully realized" (p. 23). Reducing the opportunity gap in mathematics education is possible by transitioning to an equity-centered paradigm. With this goal in mind researchers, practitioners, and teacher educators in the mathematics education community must learn how to "value the cultural and lived experiences of all children…[and emphasize] the belief that all children possess strong intellectual capacity and bring a wealth of informal, out-of-school knowledge to the teaching and learning process" (Lemons-Smith, 2008, p. 913). It is imperative that mathematics teacher educators begin this process by encouraging prospective and practicing teachers to critically examine their current beliefs while explicitly addressing the elements of teaching mathematics through an equitable lens. Attending to these practices in teacher preparation programs can help all teachers observe the actions that occur in the classroom, and determine effective strategies that will enhance all students' access to high quality mathematics instruction.

The extant literature in mathematics education provides an initial glance into equitable mathematics pedagogy, yet there is a need to better prepare PTs for their role in creating opportunities that provide equitable access for learning high quality mathematics. It is imperative PTs are given the opportunity to develop an equity-centered orientation toward mathematics teaching and learning to effectively instruct *all* students. In order to accomplish this goal, we must first recognize what PTs attend to as they direct their attention to various classroom events and how they relate the events to broader principles of teaching and learning, including cultural contexts that contribute to students' learning. In this study, we begin to address Wager's (2012) call and Hand's (2012) recommendation to assess how equity issues are perceived in classroom episodes by reporting what PTs notice about students' mathematical thinking and its relation to culture, home, and community.

Theoretical Framing for the Study

Equity may be viewed as a process or as a product (Crenshaw, 1988; Gutiérrez, 2002; Martin, 2003; Rousseau & Tate, 2003). Seeing equity as a process means treating all students equally, without regard to race, ethnicity, or economic background. On the other hand, seeing equity as a product means differentiating instruction based upon students' needs in order to promote equal learning outcomes. We adopt the view of equity as a product, and define teaching mathematics for equitable outcomes as approaches that are respectful of students' ethnic, racial, and economic background and promote equal learning outcomes.

Classroom episodes are complex; therefore, it is inevitable that individuals choose, consciously or subconsciously, what they attend to and then use these events to make instructional decisions. Before defining an approach that develops PTs' orientations toward teaching mathematics through an equity lens, we must first attend to how PTs perceive classroom situations and identify what they notice—or attend to—during the teaching and learning process. This process is referred to as the discipline of noticing. The ability to notice noteworthy classroom interactions or events among the plethora that occur in a complex classroom environment is a key component of teaching expertise (van Es & Sherin, 2006).

van Es and Sherin (2010) emphasize the key in the process of noticing is identifying what is significant, and then reasoning about the situation to effectively contribute to mathematics learning. Furthermore, Wager (2006) argues that mathematics teachers must not only attend to students' achievement, but also explore access, identity, and culture in order to provide students with equitable opportunities. Turner et. al. (2012) expand upon this sentiment in the context of noticing as they emphasize noticing critical elements within a classroom environment includes not only analyses of mathematical thinking, but also considers the impact of cultural knowledge as a foundation of students' mathematical knowledge base. The process of noticing generally involves three components: (a) identifying noteworthy instances within a classroom situation, (b) relating classroom interactions or events to principles of teaching and learning, and (c) using the cultural context to interpret the events and make instructional decisions. In this study, we incorporate a noticing lens to explore what PTs attend to as they reflect on a teaching vignette and how the events they notice correspond to equitable practice.

Method

In this paper, we describe an activity used in our elementary mathematics methods course that was designed to encourage PTs to face existing (and often hidden) biases in order to alter unproductive beliefs and consequently broaden their *ways of seeing*. We view this as an essential activity to help PTs: (a) develop an awareness of equity, (b) define what equity means in classroom instruction, and (c) implement equity practices within the mathematics classroom. Participants in the study were PTs from three different universities in the U.S. who were within a year of their student teaching experience. The demographics of the PTs reflected demographic patterns of elementary education majors at our universities and included 91.4% white females, 4.3% white males, 1.4% Asian female, and 2.9% African American females. A less diverse population of PTs is facing an ever increasingly diverse population of students in our public school systems.

To access the PTs' thoughts and ideas on issues related to equity in the mathematics classroom, we used classroom vignettes, each representing a different scenario that potentially challenges learners' equitable access to high quality mathematics. The five authentic topics included one of the following teachers: (a) one who exhibits gender bias for class participation, (b) one with preconceived biases that cause him to withdraw from students who are from different backgrounds than himself, (c) an instructor who does not take time to develop relationships with her racially diverse students, (d) one who recommends a new English Learner with limited English proficiency for special education services without adequately assessing the child's content knowledge, and (e) a teacher who is frustrated with a student who is above grade level in mathematics and exhibits disruptive classroom behavior.

To help ensure all PTs fully participated in the discussion of the equity cases, each PT independently read and responded to corresponding reflection questions for each case (Data Source 1 [DS 1]). Using a modified version of "jigsaw," a cooperative learning structure, the PTs were divided into five groups, where each group was randomly assigned a specific case and asked to discuss their responses (DS 2), which were audio-recorded and transcribed. The PTs recorded their thoughts to the guiding questions that accompanied their case on chart paper (DS 3), and an expert was selected from each small group to share their results with other members in the class. Everyone from each group, except the expert, rotated from case to case and listened to the "expert" report the group's analysis of the situation before the PTs shared their individual thoughts pertaining to the events in the vignette. At the conclusion of the discussion, the PTs recorded their thoughts and ideas on post-it notes that were placed on the chart paper (DS 4). Once the PTs finished rotating to each of the five groups, they returned to their original group where the expert shared what he/she learned from the discussions with the other groups. Finally, the PTs shared their thoughts and ideas on each case within a large group discussion.

For the purpose of this paper, we focus our results on one case, *Eric*, in particular because of the dichotomy the PTs perceived between Eric's mathematical knowledge and his exhibited behavior during a mathematics lesson. See figure 1 for the stated case and corresponding reflection questions.

Case: Eric

Eric, an African American third grader, is a good-looking nine-year old boy who was retained in first grade. He is below grade level in reading and language, but above grade level in mathematics. He has been removed from his home because of abuse and is being raised by a caring grandmother. During mathematics lessons, Eric harasses the other children. He closes their books while they are working, knocks their books and pencils off their desks, writes on their papers, and crumples up their work. He does his best to disrupt the lessons by making gestures and deliberately making noises. Ms. Maben, his teacher, is frustrated.

Reflection Questions:

1. If you were the teacher in this situation, how would you respond?
2. What are the implications for elementary teachers?
3. Additional comments or thoughts?

Figure 1: The case of Eric and corresponding reflection questions.

Data Collection and Data Analysis

The first part of the classroom assignment with Eric's case served as an opportunity to gain insight into what individual PTs highlighted as significant. When students read the vignette and addressed the reflection prompts we could ascertain the three elements in the noticing process. First we identified what the PTs found noteworthy in the classroom situation with Eric. Secondly, we determined how the PTs related classroom interactions or events to principles of teaching and learning from their responses to the first, and particularly second, question. Finally, we thoroughly examined their responses for evidence of using the cultural context in the given situation to interpret the events and make instructional decisions.

As PTs continued to work through the jigsaw activity discussing the scenario in small groups and with a follow-up large group discussion, the mathematics teacher educator in each of the three methods courses guided the conversations to further engage PTs in the second and third characteristics of noticing: (a) using contextual information to reason about witnessed events, and (b) making sense of the events through a contextual lens and using the knowledge to inform instructional decisions.

The data generated from the individual reflections, group poster notes, and transcriptions from small group discussions served as evidence of PTs' conceptions related to issues of equity. The data analysis process began by first making notes to identify PTs' comments in the transcripts of the small group discussions and all written work. These notes served as indications of what the PTs noticed from the vignette. Then the margin notes were summarized into short phrases that emphasized the nature of the comment and were coded independently by three researchers for common themes. Any differences were resolved through refining the common themes. Finally, using Barnes and Solomon's 2013 observational categories as an initial analysis template, the common themes were clustered into the following four categories: (a) classroom management (i.e., classroom events—including disruptive events, pace changes, routines or procedures), (b) classroom environment (i.e., physical setting, equipment, demographics, grade level), (c) task selection (i.e., activities students do during the teaching episode), and (d) communication (i.e., interactions or dialogue between or among students, the classroom teacher, school personnel, or the family).

Results

Eric's case provided an opportunity for mathematics teacher educators to engage PTs in a discussion on issues related to equity, which brought an awareness to and challenged their stereotypes, hidden biases, and unproductive beliefs about students from diverse backgrounds. The results across the three classes (see Table 1) provided an interesting perspective on what the PTs noticed in the case.

While the mathematics teacher educator guided follow-up discussions to the activity that emphasized principles of equity, throughout the three classes the PTs primarily attended to classroom management with 39.9% of the referring responses pertaining to strategies such as reward systems, punishments, or behavior plans. About one-third of the classroom management responses encouraged a classroom teacher in this situation to provide students like Eric with more leadership responsibilities. Specifically, one PT expressed,

> I would make Eric the classroom leader and give him jobs to do around the classroom. Like having him turn the lights on and off and having him sharpen the pencils. This will make him feel more like he is a part of the classroom and hopefully it will make him more of a helper than someone who disrupts other student's work.

Approximately 34% of the PTs' classroom management responses focused on classroom behavior systems, with some students indicating that Eric was in need of a behavior intervention plan or an Individual Education Plan (IEP). In fact, one PT stated, "If he was put on an IEP for his

Table 1: Summary of what prospective teachers noticed from the case of Eric

BROAD CATEGORY	CLUSTERED THEMES	CODE CATEGORIES
CLASSROOM MANAGEMENT 39.9%	Positive Reinforcement 11.8%	Reward system (e.g., praise when good) Positive rewards for class Special role (e.g., leader, peer tutor)
	Negative Consequences 14.6%	Punish Eric for his behavior Remove Eric from group Hold Eric back
	Behavioral Assessment 13.5%	Document behavior Individualized Education Plan (IEP) Firm, Clear Rules Set Goals
COMMUNICATION 29.7%	Collaboration with Family 21.3%	Contact Grandmother for ideas Speak with Student 1-1
	Collaboration with School 8.4%	Contact Principal Refer to Guidance Counselor Refer to Special Education Teacher
TASK SELECTION 22.5%	Increase Rigor 12.9%	Provide Enrichment Activities Challenge Eric More
	Increase Amount of Work 9.6%	Provide "Busy Work"
CLASSROOM ENVIRONMENT 7.9%	Culturally Relevant Pedagogy 7.3%	Home learning/Background Connect with Student
	Positive Influence/Support System 0.6%	Male Role Model

behavior, then maybe he could get instruction in a special education setting. This way he would not be disruptive towards his peers." Another PT expressed her belief that the behavior was such an obstacle that Eric should not continue with his peers to the next grade. She articulated, "I believe that Eric would be a great student that could really benefit from being held back this year so that he could grow and learn to practice age appropriate behaviors." These responses seem to imply that the student is at fault for the behavioral issues. The PTs who responded to the first reflection prompt whose solutions to the situation included behavior plans, IEPs, or retention were focused on Eric's behavior without critically reflecting on how the teacher's actions may be contributing to his outbursts.

While the PTs did indicate positive solutions to address Eric's behavior, several identified classroom management strategies focused on negative consequences for Eric's actions such as removing him from the group. One PT even noted, "Eric would be on an island far away from the rest of the class." From these responses, it was evident the majority of PTs first noticed the behaviors, and initially directed concern to "deal with" Eric's actions rather than attend to additional information that could provide insight to the source of these behaviors. These responses reflect a common perception or stereotype of black males as a menace to society, which can negatively impact the child and provides an obstacle for him (or her) in receiving opportunities for high quality instruction.

Communication was the second most common category that the PTs responses aligned to with 29.7% of the referring responses coded under this category. For example, in order to help subside Eric's disruptive behavior, several PTs suggested speaking to Eric one-on-one or having Eric speak to a counselor, principal, or some other positive male role model. While collaboration with families, the school, and constructive influences in the community are integral to students' successes, the PTs again first noticed Eric's behavior and directed their contacts and references as resources for fixing the behaviors they found to be inappropriate for school. These actions seemed to indicate that the PTs did not feel comfortable or confident in effectively working with children who exhibit such behaviors, and also that they do not initially consider the whole child and how the classroom environment or teachers' instructional methods may be a key factor in transforming behavior.

One of the most interesting results under the task selection category was that in each of the three courses many of the PTs focused on giving Eric more work. Some PTs across the three groups, focused on providing more challenging work, with one even emphasizing the job of the classroom teacher is to "make sure that a student is being challenged and engaged on a level that is appropriate for them." Several of the responses (12.9% of 22.5% of the task selection responses) PTs generated seemed to support the need to select appropriately challenging tasks for Eric and indicated that a strategy to quell his behavior could be to stimulate his thinking. One PT wrote, "I would gather different math problems that challenge Eric and meet his learning needs. If need be I would have an associate take Eric out during math to work on more challenging problems to keep him engaged." However, others suggested Eric should be given additional work to keep him busy so he would not "distract" the other students. These students' responses (9.6% of 22.5% of the task selection responses) seemed to overlook the possibility that Eric was bored with the current assignments, and rather than exploring opportunities to stimulate his thinking through high quality tasks that appropriately challenge Eric, the PTs merely indicated the need to increase the quantity of work rather than the rigor. Decisions such as these again serve as additional obstacles in providing access to high quality mathematics instruction.

While many PTs did attend solely to Eric's disruptive behavior, approximately 7.9% of the responses reflected an attempt to identify the source of the issue by focusing on ways they could establish a positive, caring classroom environment where students felt accepted as individuals. One PT claimed, "Students just want love and attention. This is why I think it is so important to create a caring community in the classroom. I truly believe that when students trust and feel loved and accepted they will preform much better in school." The PTs noticed the former abuse in the fictional child's past and his current living situation being with his grandmother. Recognizing these events can negatively impact a child and can be exhibited through defiant or inappropriate behavior. Consequently, several PTs saw the need to create a place in which Eric could feel safe and secure. One PT articulated, "I think that the student is acting up because of his home life and I have seen this many times. Yes, you should have clear, set rules in class, but as an educator you should take the time to see where the student is coming from and maybe provide resources such as counseling." One of the participants who commented on classroom environment recognized the need for a supportive environment and made connections to her field placement,

> This similar situation happened in my practicum classroom. The boy was African American and was constantly causing problems. His home life was very busy and his single mother had him and 7 other kids with another on the way so he did not get much attention at home. I do not think that he was abused, but he definitely was disruptive in the classroom. He did not respect me and when I taught, he would do all of those things mentioned above with messing with other people's papers and making noises during lessons…Eventually while I started to build a relationship with [him] he was really sad to see me go. It really is about making a relationship with the child and then showing them that you want to help them learn.

Although classroom environment was the least referred response in the reflection assignment, it was reassuring that some PTs did notice the need to develop a positive classroom culture by connecting with their students and providing support systems. The attention to classroom environment in the large group discussion provoked PTs' thinking on issues related to equity. In subsequent discussions, PTs began to uncover some of their stereotypes and unproductive beliefs related to Eric's scenario. They recognized that initially many beliefs focused on Eric and his family situation as the root of the problem, rather than noticing that behavioral issues may stem from inappropriate instructional strategies. While more efforts are needed, these discussions seem to contribute to moving PTs into the other dimensions of noticing.

Conclusion

To provide opportunities for *all* students that will challenge the status quo, we must first identify what is salient to PTs by understanding what they attend to in complex classroom situations. With this knowledge, mathematics teacher educators can more effectively address PT's philosophies and beliefs. We understand teachers cannot attend to everything that occurs in a classroom; thus they must make choices—whether consciously or subconsciously—about what they notice. Barnes and Solomon (2013) found that novices are often attracted to whatever is most salient or personally intriguing, such as evidenced in this study with classroom management strategies to address a student's behavior.

While classroom management is at the forefront of our PT's thoughts, we must help them move beyond this surface layer of noticing. NCTM (2014) argues that effective mathematics instruction builds from students' culture—their values, beliefs, language, and experiences. Therefore, the teaching and learning of mathematics is not void of cultural influence, but positioned within a social context. Noticing involves three components. From the results discussed above it is evident that PTs are attending to noteworthy instances in a classroom situation. However, mathematics teacher educators need to provide more explicit support and attention to the second and third aspects of noticing: (a) using contextual information to reason about witnessed events, and (b) making sense of the events through a contextual lens and using the knowledge to inform instructional decisions. Aligning teachers' beliefs about equity and equitable practices in mathematics education with productive beliefs (NCTM, 2014) and research-proven attitudes and practices (Jackson, 2010) is vital. Engaging PTs in classroom activities—such as the one described with Eric's scenario—helps foster reflective discussions, which can positively contribute to developing an equity-centered orientation toward mathematics teaching and learning.

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MAINTAINING CONVENTIONS AND CONSTRAINING ABSTRACTION

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Conventions play an important communicative role in mathematics. Likely due to the complex relationship between conventions and school mathematics, few education researchers have questioned or investigated the consequences of instruction and curricula that primarily, if not unquestionably, maintain conventions. Drawing on Piagetian notions of abstraction and our work with students and teachers, we argue that students' repeated experiences with instruction and curricula that maintain conventions likely constrain students' learning opportunities. We hypothesize that by 'breaking' conventions, educators could better support students in differentiating those aspects of their activity essential to a concept from those that are unessential. We characterize student work on two tasks to illustrate potential relationships between the nature of students' abstractions and what we perceive to be conventions.

Keywords: Abstraction; Radical constructivism; Student cognition; Multiple representations

…la mathématique est l'art de donner le même nom à des choses différentes. (Poincaré, 1908, p. 29)

Mathematicians and mathematics educators widely hold the study of mathematics as dependent upon abstraction. Said simply, abstraction is the process of coming to understand some sense of invariance among seemingly different activities, situations, or objects so that this understanding is not tied to particular features of any one activity, situation, or object; translating the Poincaré quote above, "mathematics is the art of giving the same name to different things." Despite the central role of abstraction in the study of mathematics, students' abstraction processes and how educators can support students in constructing productive (in the short- and long-term) meanings through abstraction remain pressing areas of research (Oehrtman, 2008; Simon et al., 2010). We agree with Thompson (2013), who argued that making fundamental improvements to U.S. mathematics education requires that educators at all levels take the meanings that students abstract more seriously.

We draw on theoretical accounts of abstraction and our work with students (and teachers) to clarify students' abstracted meanings. Namely, we argue that students' actions and opportunities to abstract productive meanings for mathematical concepts are likely unintentionally constrained by educators who maintain conventions common to school mathematics (e.g., using the Cartesian horizontal axis to represent a function's input). After discussing our motivation and background, we provide a theoretical framing of abstraction and examples of student activity that clarify how educators maintaining particular conventions might constrain students' experiences and hence abstracted meanings. We also illustrate students operating productively in non-canonical situations so that we can clarify how experiences with such situations might influence the meanings that students abstract. We close with connections to related areas of research and ideas about lines of inquiry that can contribute insights into students' abstraction processes.

Background and Motivation

The present work emerged from a collection of studies (e.g., Moore, 2014a, 2014c; Moore, Paoletti, & Musgrave, 2013; Thompson & Silverman, 2007). Our goal in each study was to understand students' and teachers' *quantitative and covariational reasoning* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson, 1993)–how individuals conceive of a situation as composed of measureable attributes and relationships between these attributes–

including how teachers support said reasoning. Our research settings have included teaching experiments (Steffe & Thompson, 2000) and task-based clinical interviews (Goldin, 2000), with these settings sometimes occurring in the larger context of professional development projects. Our data analyses efforts have followed a combination of conceptual analysis techniques (Thompson, 2008) and open/axial methods (Strauss & Corbin, 1998). Such efforts entail fine-grained, iterative analyses for the purpose of constructing viable models of individuals' thinking. We direct the reader to our prior studies for more detailed explanations of the work that informed this theoretically oriented paper.

Motivating our present focus, it was in trying to support students' and teachers' quantitative reasoning that we noticed their difficulty assimilating non-canonical situations in quantitative ways (Moore, Silverman, Paoletti, & LaForest, 2014; Moore, Silverman, et al., 2013). We found this outcome noteworthy for two reasons. First, the students' difficulties assimilating situations that we had designed to be non-canonical often led to their (consciously or subconsciously) imposing conventions on situations. For instance, in cases that we had designed tasks to explicitly define–in spoken or written text–a Cartesian axis oriented vertically as representing a function's input values (see the following section for task examples), students persisted in conceiving the axis oriented horizontally as a function's input (Moore, Silverman, et al., 2013). Second, the students and teachers often exhibited actions within canonical situations that they were unable to relate to and re-present in non-canonical situations. As an example, students would describe a graph of a function in terms of how two quantities varied in tandem, but then be unable to describe a non-canonical graph of that same (to us) function in an equivalent way (Moore et al., 2014). Due to the frequency of these outcomes, we inferred from the students' actions that what we perceived to be conventions were instead inherent aspects of students' meanings (i.e., not conventions) (Moore et al., 2014). This inference has led us to question the relationships between students' and teachers' abstracted meanings and what we, as educators and researchers, perceive to be conventions. In addition to discussing our theoretical framing, we use student work on two tasks to clarify important differences with respect to these relationships.

Abstraction and Student Illustrations

A detailed comparison of theories of abstraction is beyond the scope of this work given space constraints, and thus we discuss those constructs most relevant to our focus: Piaget's constructs of (pseudo-)empirical and reflective abstractions. The former type of abstraction, whether empirical or pseudo-empirical, concerns itself with the results of activity, whereas reflective abstraction concerns itself with internalized coordinations (Chapman, 1988; Piaget, 1980; von Glasersfeld, 1991). Empirical abstractions involve characteristics of experiential objects (e.g., color) including the results of sensorimotor activity on those objects. Empirical abstractions support repeated actions and motor patterns based in sensorimotor experience. Pseudo-empirical abstractions are similar to empirical abstractions in that they foreground abstraction from objects and results of activity, but they differ from empirical abstractions due to the individual introducing properties of these results into objects at the level of mental actions (Dubinsky, 1991). For instance, after working several problems graphing linear functions, a student might abstract rate of change or slope as an indicator of direction (e.g., all lines with a positive rate of change means slope upward left-to-right). Such an abstraction is not constrained solely to sensorimotor operations and observables (e.g., empirical abstractions), but the generalization does stem from patterns tied to the product of an activity (e.g., graphing linear functions) and conditions for this activity (e.g., having a rate of change and graph).

Reflective abstractions involve re-presentation, symbolization, and the coordination of (mental) actions so that the locus of abstraction is on activity itself, as opposed to activity results (Chapman, 1988; Piaget, 2001; von Glasersfeld, 1991). Relative to graphing linear functions, a student might conceive rate of change as entailing the imagery of coordinating the relative change in two quantities. The student might also coordinate this process with graphs in several orientations or coordinate systems; the linear function and rate of change become a coordinated system so that each is a "pointer" to coordinating relative changes, but the process itself need not be carried out. In the case of pseudo-empirical abstraction, the rate of change value or slope signals executing an activity and obtaining a particular graph-as-picture; the linear function or rate of change symbolizes a structure of mental actions and associated representations all at once. In what follows, we provide student responses to two non-canonical tasks and then synthesize their activities with respect to these notions of abstraction.

Student Illustrations 1

We presented the graph in Figure 1 to undergraduate (secondary) mathematics education students (*Student Quotes for Figure 1*) with the claim that a hypothetical student deemed it a graph of the inverse sine function. We also provided a statement by the hypothetical student: "Well, because we are graphing the inverse of the sine function, we just think about *x* as the output and *y* as the input." Among other considerations, we designed the task to incorporate a non-canonical representation of the inverse sine function and to capture fundamental aspects of the inverse relationship, namely the understanding that if $y = \sin(x)$, then $x = \sin^{-1}(y)$ with appropriate restrictions on *x*.

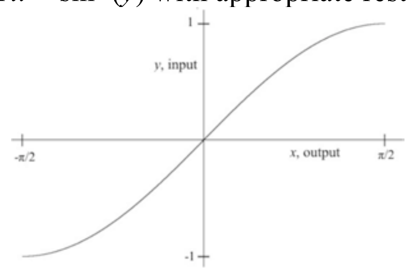

Figure 1: A Non-canonical Graph of the Inverse Sine Function

Student Quotes for Figure 1

Molly: I feel like he's missing the whole concept of a graph…I know you can call whatever axis you know if you are doing time and weight or volume or whatever. You can flip-flop those and be OK. But not necessarily with the sine graph. Like a sine graph's like a, it's a graph like everyone knows about, you know.

Rowena: I'm thinking this just kind of looks like the sine graph, like the plain sine graph [*laughs*]. Which is going to be different…I don't know if, or like an inverse function, like the graph of an inverse function, like, can't be the same as the original graph.

Ariana: You could just like disregard the *y* and *x* for a minute, and just look at, like, angle measures. So it's like here [*referring to graph of* $\sin^{-1}(x) = y$*, see Figure 2, left*], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]. And then show them here [*referring to graph of* $\sin^{-1}(y) = x$*, see Figure 2, right*] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]. So even though the curves, like, this one looks like it's concave up [*referring to graph of* $\sin^{-1}(x) = y$ from $0 < x < 1$] and this one concave down [*referring to graph of* $\sin^{-1}(y) = x$ from $0 < x < \pi/2$], it's still showing the same thing [*denotes equivalent changes on graphs, see Figure 2*].

We contend that each student *understood* the given graph, and we interpret their actions to suggest differences in their meanings. Molly and Rowena understood the sine function to be uniquely associated or in a one-to-one relationship with the given graph. An implication of this understanding is that "the sine function" or "inverse sine function" were, in the moment of their assimilating the graph, as much about associating a function name to a unique shape as about associating a function name and graph to a particular relationship. In contrast, Ariana understood "the sine function," "inverse sine function," and graphs in terms of a relationship between covarying quantities. Moreover, she was not constrained to reasoning about this relationship in a particular coordinate orientation. Hence, when we presented Ariana an alternative graph, she understood differences in shape did not imply a difference in the represented relationship; to Ariana, both graphs represented an invariant relationship associated with "the sine function," written as $y = \sin(x)$, and "the arcsine function," written as $x = \sin^{-1}(y)$.

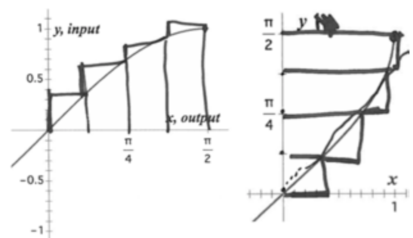

Figure 2: Two Graphs, One Relationship

Student Illustrations 2

We presented a graph (Figure 3) to undergraduate (secondary) mathematics education students (*Student Quotes for Figure 3*) with the claim that a hypothetical student produced the graph to represent $y = 3x$. We designed the task to explore rate of change (or slope as rate of change) in non-canonical axes orientations. The following student responses are in reply to our asking them to comment on the student's solution including their interpretation of its correctness.

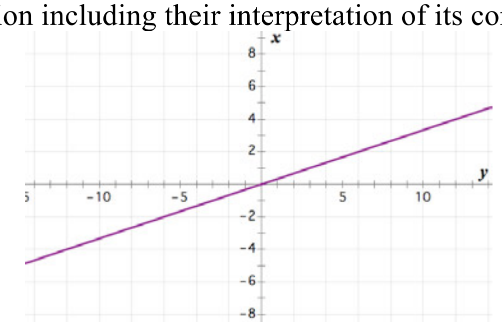

Figure 3: A Non-canonical Graph of $y = 3x$

Student Quotes for Figure 3

Rowena: Because if you turn it this way [referring to Figure 3 rotated 90-degrees counterclockwise] then this [traces left to right along the x-axis which is now oriented horizontally] and this [traces top to bottom along the y-axis] and it would be still not right though…this [laying the marker on the line which is sloping downward left-to-right] is negative slope. So I would…show them like the difference between positive and negative slopes also. Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero [making hand motions to indicate a direction of line for each]. Like where the slopes were…it's important to know which direction they're going…

Rubeus: They messed up the placement of x and y…They are looking at it like this [rotating graph 90-degress counterclockwise]…If you are looking at it this way, it's a negative slope [tracing graph] and it should be a positive slope [tracing imagined graph upward left-to-right]…slope is wrong.

Amelia: I think it demonstrates understanding of the relationship…I don't care what axis you put it on…[interview rotates graph and says a student claims it has negative slope] [The way we teach slope] is just a very visual thing and no understanding of like what slope means or where it came from…your negatives are over here and your positives are over here [referring to horizontal axis value orientations]…So if we look for like a change of x of one [identifies change with a segment], zero to one, we see that y changes by positive three [identifies change with a segment]…positive slope because you are looking at change…If you're so obsessed with convention and the way things are supposed to be, you're going to take more work to get it to the way that you are comfortable with…than to just interpret the graph.

Rowena and Rubeus drew on meanings for slope rooted in visual queues including the direction of a line (e.g., "where the slopes are"). When they rotated the graph, which we interpret to be for the purpose of maintaining conventional x-y axes orientations, the students understood the "slope" to be different and hence they understood the line to represent a different relationship once rotated. In contrast, Amelia assimilated the graph in terms of reasoning about how two quantities vary in tandem while remaining attentive to how these quantities were represented with respect to the axes. No matter the rotation, Amelia understood the graph and slope in terms of a relationship such that the rate of change of y with respect to x is 3 (e.g., $y = 3x$). Amelia also distinguished between someone constrained to a conventional understanding of slope (e.g., "visual" understandings) and someone who is focused on interpreting the graph as a relationship.

Looking Across the Illustrations

Despite differences in the tasks and student responses, we interpret an underlying commonality to Molly, Rowena, and Rubeus' actions and meanings. Likewise, we interpret there to be an underlying commonality to Ariana and Amelia's actions and meanings. In the former case, the students primarily focused on observables or perceptual features of the given graphs in relation to particular topics (e.g., function name or slope). In the latter case, the students operated beyond the level of observables and perceptual features. They understood the graphs in terms of interiorized covariation schemes, and they drew on these schemes to make sense of the non-canonical representations and conceive invariance among perceptually different graphs.

Returning to theories of abstraction, we interpret Molly, Rowena, and Rubeus' actions to be compatible with meanings stemming from pseudo-empirical abstractions due to their focus on observables and the products or results of activity (e.g., produced shapes and associations with these shapes). One explanation for this result is that the students had experiences constrained to associating a function name with *one particular* graph or associating slope values with lines in *one particular* coordinate system orientation. Through repeatedly and habitually assimilating experiences to these meanings, the students abstracted function names and slope as essentially facts of perceptual shape. We interpret Ariana and Amelia's actions to be compatible with meanings stemming from reflective abstractions. That is, their understandings of "the sine function" and slope entailed internalized coordinated actions that they re-presented to make sense of the non-canonical graphs. One explanation for this understanding, which we expand upon in the next section as a future line of inquiry, is that the students had sustained opportunities to make sense of functions, function properties, and function graphs (including those that are non-canonical) in terms of covarying quantities. Hence, the students came to understand slope (or rate of change) and function names in ways not restricted to visual shape, but instead as properties of internalized covariation schemes.

Connections and Moving Forward

Mamolo and Zazkis (Mamolo & Zazkis, 2012; Zazkis, 2008) hypothesized that individuals face difficulties constructing sophisticated understandings of mathematical ideas when they only experience instruction that maintains particular conventions. Zazkis (2008) explained, "Of course, conventions are to be respected…but there is a need to become aware of them" (p. 138). Zazkis (2008) proposed that by becoming aware of conventions and structure specific to a representational system, individuals have an opportunity to develop "richer or more abstract schema" (p. 154). We agree with Mamolo and Zazkis, and we contend that students have said opportunities due to the nature of abstractions that can occur when students have experiences with both canonical and non-canonical representations. These experiences give students the opportunity to abstract meanings that differentiate what is essential to a mathematical idea from what is a convention or structure of the representational system. Students who only have experiences with conventional representations (e.g., *one* graph in a particular axes orientation) do not have the same collection of experiences or activity to reflect upon and coordinate in order to differentiate aspects of their activity that are critical to an idea from those that are not.

One area of research that the relationship between conventions and abstraction is most relevant to is that of multiple representations. Educators often frame a focus on multiple representations in terms of graphical, tabular, analytic, etc. representational contexts. We argue that it is productive to extend the 'multiple representations' perspective to incorporate the use of canonical and non-canonical representations within these representational contexts. Without offering students an opportunity to work with both canonical and non-canonical representations, we likely unintentionally enable students' construction of meanings that take what we perceive to be conventions of a representational context to be essential or unquestionable facts of mathematical ideas. On the other hand, offering students experiences with both canonical and non-canonical representations in a multitude of representational contexts enable them to reflect on and coordinate their activity both *within* and *among* representational contexts. In doing so, students have a collection of experiences to reflect upon in order to come to understand an idea in terms of what is invariant among their activity within and among representational contexts; we envision that these experiences might address Thompson's (1994) call for a focus on "representations of something that, from the students' perspective, is *representable*" (p. 40).

We also conceive relationships between abstraction and conventions to be relevant to researchers who explore students' quantitative and covariational reasoning. Specifically, we have found it necessary to be more careful and nuanced in our claims about students' covariational reasoning, especially in distinguishing between a student who is able to carry out a particular activity in one coordinate orientation versus a student who has internalized and coordinated their actions in a way that is not constrained to one coordinate orientation. For instance, despite students exhibiting actions in one coordinate orientation compatible with that of Ariana (e.g., drawing and comparing segments within a canonical representation of a function in ways that suggest covariation mental actions outlined by Carlson et al. (2002)), we have found that students encounter much difficulty extending their activity to other coordinate orientations and systems. We interpret such difficulties to highlight the importance of researchers identifying the extent that students' meanings are tied to their activity and its results (e.g., pseudo-empirical abstractions versus reflected abstractions). With respect to characterizing students' quantitative reasoning, researchers might explain students' learning through constructing an *increasingly abstract quantitative structure*. By increasingly abstract quantitative structure, we mean a student constructing and reconstructing a quantitative structure that becomes so internalized and operational that the student is increasingly able to assimilate novel representational contexts and situations to that structure, as opposed to being dependent on activity tied to particular representational contexts and conventions.

We close by noting that we do not intend the reader to interpret that pseudo-empirical abstractions are not desirable, nor are we arguing for the dismissal of conventions. Pseudo-empirical abstractions are a critical part of the learning process and are the source material for reflective abstractions (Dubinsky, 1991). For instance, in the context of graphing, students abstracting shape-based associations is a natural outcome (Weber, 2012). An issue arises when a student repeatedly assimilates their experiences to these shape-based associations so that the shape associations become nearly the entirety of her or his operating meanings. This outcome stands in contrast to a student coming to know graphical representations so well that they can operate at the level of shape associations while anticipating these associations in terms of internalized processes that can be carried out and adjusted if necessary. Despite the critical role of pseudo-empirical abstractions to the learning process, as educators we must look to support students in pushing beyond pseudo-empirical abstractions, lest the results of those abstractions become the entirety of their meanings. Simon et al. (2010) argued that gaining insights into nuances of students' abstraction processes with respect to their activity is a promising area of research. In accomplishing this goal in the context of using canonical and non-canonical representations, we might come to better understand how to support students in experiencing mathematics as "l'art de donner le même nom à des choses différentes" (Poincaré, 1908, p. 29).

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BILINGUALS' NON-LINGUISTIC COMMUNICATION: GESTURES AND TOUCHSCREEN DRAGGING IN CALCULUS

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This paper discusses the importance of considering bilingual learners' non-linguistic forms of communication for understanding their mathematical thinking. In particular, I provide an analysis of communication involving a pair of high school calculus bilingual learners, who interacted with a touchscreen-based DGE (dynamic geometry environment). The paper focuses on the word-use, gestures, and touchscreen dragging actions in student-pair communication. Findings suggest that the students relied on gestures and dragging as multimodal features of the mathematical discourse to communicate dynamic aspects of calculus. This paper raises questions about new forms of communication mobilized in dynamic, touchscreen environments, particularly for bilingual learners.

Keywords: Classroom Discourse; Equity and Diversity; Technology

Introduction

In British Columbia, Canada, "In 2011–12, one in four (23.8 %) of public school students spoke a primary language at home other than English. Almost double the number of [English language learners] (135,651) live in families where the primary language spoken at home is other than English […]" (BCTF, 2012, p.11-12). Speaking from my own experience teaching mathematics in Canada, the home languages spoken in a typical mathematics classroom are very diverse, ranging from five to ten in any given classroom. This context is one result of globalisation and rapidly changing student demographics not only locally but worldwide.

Currently, research focusing on bilingual learners' mathematical communication has provided tremendous insight into the *complexities* of teaching and learning mathematics in multilingual contexts: the language dilemmas of teaching mathematics (Adler, 1999), the role of code switching in learning mathematics (Clarkson, 2007) as well as associating mathematics learning with socio-economic and epistemological access (Setati, 2005). These studies, however, had not critically examined bilingual learners' communication patterns and, in particular, addressed their competence in mathematical communication. As argued by Moschkovich (2010), future studies on bilingual learners must consider broader linguistic frameworks for understanding bilingual learning.

Research on multimodality can shed light on bilingual learners' communication as a multimodal activity that includes the use of language, gestures and interactions with diagrams (Arzarello, 2006; de Freitas & Sinclair, 2012; Radford, 2009). Aligned with the idea of multimodality in mathematical thinking, a small number of research studies have drawn on bilingual learners' non-linguistic forms of communication such as gestures and diagrams (Moschkovich, 2009). Moreover, although numerous studies have discussed the effect of the DGE-mediated learning of calculus concepts (Yerushalmy & Swidan, 2012), research on the effects of touchscreen-based DGE is limited. It is hypothesized that a touchscreen-based DGE may offer additional affordances by providing tactile and kinesthetic modes of interaction—hence, further facilitate bilingual learners' communication in calculus.

In a previous study working with bilingual learners, I showed that certain dragging actions on a touchscreen-based DGE constitute a form of communication (Ng, 2014). Using a Sfard's (2008, 2009) communicational approach, my analysis showed that some dragging actions were not merely dragging, but also instances of gestural communication—to communicate *dynamic* features and properties in the sketch as obtained by dragging. The touchscreen-dragging modality allows the dragging with one finger on the touchscreen and the gesturing with the index finger to blend together as one action. The importance here is that the *dragging-gesturing* action subsumes both dragging and gesturing characteristics, in that it allows the point to be moved on the screen (dragging), and it fulfills a communicational function (Sfard's definition of gesturing).

Building on this work, the current study examines the communication patterns that arise as bilingual learners interact with a touchscreen-based DGE. In particular, I investigate bilingual learners' language, gestures, and dragging in DGE as they communicate a given calculus concept in their non-native language. My study concerns the non-linguistic resources utilized by bilingual learners in communication as they work with a DGE. Further, it is the goal of the study that this analysis will identify bilingual learners' competence in mathematical communications.

Theoretical Framework

Sfard's communicational framework (2008) is based upon the social dimensions of learning, which suggests that learning is located neither in the head nor outside of the individual, but in the relationship between a person and a social world. It provides a suitable theoretical lens for highlighting the communicative aspects of thinking and learning. For Sfard, *thinking* is part and parcel of the process of *communicating*. This non-dualistic approach, which disobjectifies thinking as a purely cognitive phenomenon, is helpful for examining the relationships between talking, gesturing and mathematical thinking. Sfard (2009) defines *language* in unrestrictive terms, as any symbolic system used in communications, and *gestures* as bodily movements fulfilling communicational function: "Language is a tool for communication, whereas gesture… is an actual communicational action" (p.194). In this sense, a gesture can be performed to communicate with others (interpersonal) or with oneself (intrapersonal). Sfard's approach highlights the way in which thinking and communicating (for Sfard, this includes talking and gesturing) stop being "expressions" of thinking and become the process of thinking in itself.

Sfard (2008) proposes four features of the mathematical discourse, *word use, visual mediators, routines, and narratives*, which could be used to analyze mathematical thinking and changes in thinking. For the purpose of this paper, the first three features will be used for analyzing the use of language, gestures, and dragging in one's mathematical discourse. *Word use* is a main feature in mathematical discourse; it is "an-all important matter because […] it is what the user is able to say about (and thus to see in) the world" (p. 133). However, as a student engages in a mathematical problem, her mathematical discourse is not limited to the vocabulary she uses. For example, her hand-drawn diagrams and gestures can be taken as a form of *visual mediator* to complement word use. According to Sfard (2009), *utterances* and *gestures* inhabit different modalities that serve different functions in communication. Gestural communication ensures all interlocutors "speak about the same mathematical object" (p.197). Gestures are essential for effective mathematical communication: "Using gestures to make interlocutors' realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects" (p.198). Gestures can be realized *actually* when the signifier is present, or *virtually* when the signifier is imagined. Sfard (2009) illustrates how a student uses "cutting", "splitting", and "slicing" gestures to realize the signifier "fraction". Since these gestures were performed in the air, where the signifier "fraction" is imagined, they provide an instance of virtual realization. Therefore, the same signifier "fraction" may be realized differently with different kinds of gesture or word use.

Routines are meta-rules defining a discursive pattern that repeats itself in certain types of situations. In learning situations, learners may use certain words or gestures repeatedly to model a discursive pattern, such as looking for patterns and what it means to be "the same". Drawing on Ng (2014), dragging is taken as a significant form of communication in this study; it is taken as both a routine for defining a discursive pattern that repeats itself in activities with touchscreen-based DGE, and a visual mediator as a multimodal feature of the students' discourse. A student may use dragging to explore (routine) and signify (visual mediator) the variation of the tangent slope. Using this notion, it is possible for students to incorporate *dragging-gesturing* to respond to each other in communications. Indeed, it was found that as one student suggested that the secant line will get "closer" to the tangent line, another student seemed to have responded by her *dragging-gesturing* to bring the lines "together". These gesture-utterance correspondences were also noted in the analysis of other pairs of bilingual learners' routine involving dynamic sketches (Ng, 2014).

Methodology

The participants of the study were three pairs of 12th grade students (aged 17 to 18) enrolled in a calculus class in a culturally diverse high school in Western Canada. All participants were bilingual learners and self-volunteered to participate from a class of 25, in which roughly half of the class were also bilingual learners. They were regular partners during assigned pair-work activities and were described by their teacher-researcher (also the author) as motivated and comfortable working with each other. The study took place at the end of the first trimester of the school year in the participants' regular calculus classroom, outside of school hours. At the time, the participants have just finished learning key concepts in differential calculus where the iPad-based DGE, *Sketchpad Explorer* (Jackiw, 2011) was consistently incorporated into the lessons. During these lessons, students were invited to explore dynamic sketches in pairs for roughly ten minutes followed by teacher-led classroom discussions about the activities. Therefore, the participants were experienced with exploring and discussing, in pairs, concepts in differential calculus with dynamic sketches.

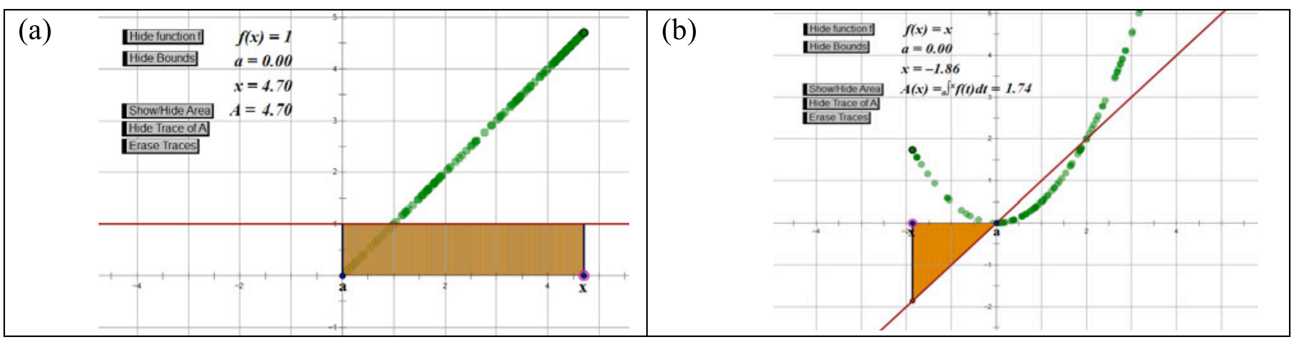

Figure 1(a-b): Screenshots of the DGE used in the study (with all buttons, "*Show function*", "*Show bounds*", "*Show Area under f*", and "*Show Trace of A*" activated). The bounds "a" and "x" are draggable; the green traces represent $A(x) = \int_a^x f(t)dt$.

The task used in this study invited the students to explore and discuss a sketch in *Sketchpad Explorer* that they had not previously seen. The sketch contains five pages all related to the concept of area-accumulating functions (Figure 1a, b). The participants were not told what concept the sketch was related to, but they were told that this concept was new to them (at the time of study, they had just spent one lesson related to integral calculus, that of indefinite integrals in their regular classroom). The participants were asked to "explore the pages, talk about what you see, what concepts may be involved" on all pages of the sketch before reporting back to the teacher as a pair. Before they began, they were ensured that their teacher would check in with them from time to time in order to make sure that they understood what they were expected to do and could ask questions that were related to the sketch. In total, 70 minutes of video data were collected in the study.

Analysis of Data

In this section, I provide detailed analyses of one participant pair, Sam and Mario's communications over the aforementioned task. Sam and Mario, whose home languages were Mandarin and Cantonese respectively, do not share a common home language. A 10-minute episode is chosen from the 30 minute discussion and analyzed in detail. The episode was chosen to highlight the different resources: language, dragging, and gesturing used by the student pair within the 10-minute discussion.

At the start, the students seemed unsure about what to do with the points *"a"*, *"x"*, and the green point in the sketch. They questioned the functions of the DGE with question markers "what" (eight times) and "how" (once) in the first two minutes of the episode. Most of these questions were formulated as Sam used the dragging modality to investigate the behavior of different points. For example, Sam asked "what" repeatedly as he tried to drag the green point which was not draggable. Although he had acknowledged that he had previously pressed a button which showed "an area", he had not realized that the green point had plotted the area in terms of *"x"*, evident in his questions "what's trace of A" and "how do we drag this trace". Then, upon dragging "*x*" around, he finally concluded "oh, oh, it's this one. Ok, make sense." His confusion seemed to have resolved perhaps because his dragging of *"x"* made the green point move; hence he realized the green point was a dependent and non-draggable object. However, it appeared that he remained unsure about what the green point meant, stating that he did not yet "understand this".

At 00:53, Sam and Mario took turns *dragging-gesturing* in a conversation-like manner, beginning with Sam's *dragging-gesturing*, which spanned 30 seconds (Figure 2a). During this occurrence, Sam dragged *"x"*, then *"a"*, and finally *"x"* again. Observing the students' word use and dragging actions, it seemed that both students made some progress in their learning of area-accumulating functions during this span. For example, as Sam was dragging *"x"*, he uttered, "*as* we drag this, the area *becomes*…" This utterance-dragging combination suggests that Sam was thinking about area as having dynamic qualities. It shows how dragging mediated the way Sam thought of the area as a becoming. The use of "become" implies something is happening, in particular, the area was changing *as "x"* was dragged. Furthermore, Sam's statement structure resembled an "if… then…" statement structure which calls upon a causal or functional relationship between *"x"* and the area. It was interesting to note that Sam never finished his sentence after uttering "become", perhaps because he had yet to *realize*, in a Sfardian sense, the simultaneous change in the variables despite noticing the area is changing. Similarly, Mario used a hedge word in his utterance, "it's *like* the area," suggesting a degree of uncertainty about whether or not the green traces meant the area.

Following Sam's prolonged dragging, different draggers and speakers were observed in the episode. For example, as Mario dragged *"x"* back and forth, Sam was responding verbally and simultaneously, "You see how this one moves? So it's like the area." A similar exchange was also noted earlier, where Sam was the dragger, as Mario spoke, "is that the area?" These two instances where the dragger and speaker were different people seemed effective for having the students communicate mutually and simultaneously. Although it may seem impolite and unconventional for one student to "talk over" another student, the presence of "talking over someone else's dragging" was not an issue here. Indeed, Sam's utterance did not interfere with Mario's dragging and vice versa; rather, from the way one talked about area while the other was dragging, they seemed to have made significant progress as a result of this concurrent communications.

Also observed in the first two minutes was the consistent use of gestures by the students in mathematical communications. Namely, Sam used three types of gestures, which in Sfard's terms, functioned quite differently in each usage. In the beginning, Sam used a pointing gesture as he talked about the bounds to make sure both interlocutors *spoke about the same mathematical object* (Figure 2b). Later, he used his hand to signify the linear pattern of the green traces, an instance of *actual* *realization* (Figure 2c). Finally, he flipped his right index finger left and right while uttering, "No you can't. You can only go like", which was another actual realization of the possible movement of the green point (Figure 2d). Moreover, this gesture was not accompanied by any speech, which suggests that Sam relied on gestures as a visual mediator, or a multimodal feature of his mathematical discourse, to communicate in the absence of word use.

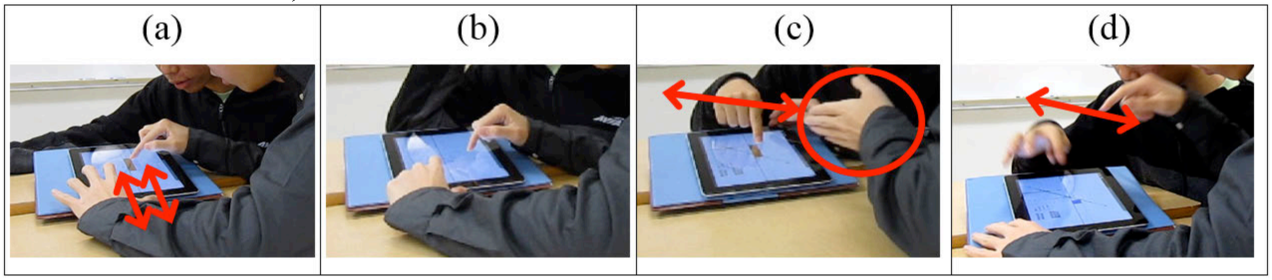

Figure 2: Screenshots of Sam's (a) dragging actions and (b-d) gesturing in the excerpt.

From 02:00 to 05:00, Sam and Mario consistently utilized dragging and gesturing to communicate mathematically. For example, as Sam took on the role of dragging and asked the question, "is that how the area is changing?" Unlike earlier, Sam was able to describe exactly that *the area is changing* with no hedge words at the 02:02 mark in the episode. He continued to drag for a span of 9 seconds without speech before letting Mario also tried dragging for another 6 seconds without speech. The switching of draggers suggest that both students were communicating mathematically while dragging. If speech was analyzed alone, some important analyses about the students' thinking in between speech would have been missed.

At 03:02, Sam and Mario performed a series of hand gestures. Initiated by his own dragging of *"x"*, Sam suggested that "oh, wait a sec. This is actually, the derivative of the graph, function," while he used a hand gesture to signify the shape of a linear function. To restate what he had said, he then used his index finger and traced a "U" shape in the air as he continued to conjecture that the line was "probably the derivative of x, x-squared". Mario responded with a similar "U" shape gesture as he asked "is this x-squared", suggesting that they verified Sam's conjecture. These gestures and word use pairings, which provide strong evidence that the students were engaging in *conjecturing* about the shape of the green traces, help identify Sam and Mario's competence in the mathematical activity.

It was noted that the deictic pointing word "this" was used extensively, appearing five times in this part of the episode. Using deictic words, the speakers no longer need to refer to the mathematical objects by describing them verbally, but they can use deictic words along with different gestures to replace the descriptions completely. This was found in Sam's "*this* is actually, the derivative", "no matter how you move, *this* one always", and "*this* is probably x, x-squared". As Sfard explains, gestures help ensure that the interlocutors speak about the same mathematical objects. Significantly for Sam and Mario, gestures served a complementary function to language in communication. The two students were able to use a combination of utterances and gestures to communicate about the mathematical objects effectively.

The students' realizations of the area-accumulating function could be observed in their discourse after the 05:00 mark of the episode. First, the questions posed hereafter were markedly different from before 05:00. Recall that previously, Sam had asked repeatedly, "what", at times without finishing his questions. It is possible that Sam's "what" questions reflected his uncertainty of what each point or button meant in the sketch. In contrast, having explored the sketch for some time, Sam asked three questions that began with "why". He asked, "why is there something to do with area" twice, and "why is it". By asking these "why" questions, it seemed that Sam was looking for the reason as to why the relationship of the two graphs were related to the area under a function. Considering Sam has only learned the topic of indefinite integral at the time of study, this was a valid question because Sam had yet to learn the idea of "definite integral as area" in his class. Regardless, asking "why" implies investigating the reasons of something that is clearly existential. In this case, Sam seemed to be investigating the reason why the *area* under a function had to do with its antiderivative.

At 05:17, a prolonged dragging action was performed by Sam, while the two students exchanged comments verbally back and forth. In particular, by far the longest spoken sentence was said by Sam within his own dragging action: "Ya. So the graph we have *here* is *specifically* the derivative of the… what we just graphed *here*, like the function *here* is *basically* the derivative of what we just graphed *here*." The sentence was very rich in a multimodal sense because it was spoken while the speaker was dragging, and gestures were used simultaneously as the speaker uttered "the function here". Some interesting word use was also observed. For instance, the word "here" was used four times, and the words "specifically" and "basically" each once. In line with a previous analysis of the use of deictic words, the use of locative noun "here" accompanied by gestures allowed the speakers to talk about the same mathematical object. Although Sam used the same word "here" four times, he actually meant to refer to two different mathematical objects, the function and its derivative. This could be why Sam used different gestures to specify which object he was talking about as he said "the function here". Secondly, the contrasting use between "specifically" and "basically" by Sam was also fascinating. Since Sam used the word "basically" quite frequently throughout the task, his word use "specifically" as opposed to "basically" in this sentence drew attention to the analysis. Consistent with his usage of "basically" in other parts of the transcript, it seemed that Sam used the word to suggest a generality or invariance that exists outside of the sketch. In contrast, it is speculated that he used "specifically" in the context of the specific page of the sketch to refer to the particular "graph" that was the derivative of another. According to this speculation, Sam was able to talk about area-accumulating functions both in its generality and particularity, which is a highly valued practice in the mathematics community.

Discussion

The analysis provides strong evidence that Sam and Mario, both bilingual learners, utilized a variety of resources in communication, with visual mediators in the form of gestures and dragging taking on a prevalent role. These included gestures accompanying deictic words and gestures for communicating geometrical notions of calculus. Moreover, the *dragging-gesturing* action emerged in the touchscreen dragging action and fulfilled the dual function of dragging and gesturing. These actions were repeatedly demonstrated by both students for communicating temporal relationships in calculus as well as in their routine of developing the mathematical discourse. In the presence of a dynamic visual mediator, the students' routine evolved from typical utterance-utterance sequences. Gestures-gestures and gestures-utterances sequences were observed in the conversation. Related to this, I observed one student *dragging-gesturing* simultaneously as the other spoke; this allowed two students to communicate simultaneously without interfering with each other. These observations support the claim that bilingual learners make use of gestures and dragging as important forms of communication. Using Sfard's communicational framework to define gestures as communicational acts is especially useful for understanding the mutual communications involved in these new communication routines.

The analysis shows that dragging and gestures transformed the students' word use. Initially, the students seemed unsure of what to make of the sketch; they used dragging to formulate their questions about the behavior of the sketch. Then, they began to explore and conjecture the relationship of the two functions in both geometrical and algebraic terms through dragging and gesturing. Sam and Mario made extensive use of verbs such as "become", "move" and "go," which imply change or motion while they used the dragging modality to change the area under a function. Moreover, gestures in the form of actual realizations were accompanied by the use of locative nouns "here" and deictic word "this". These gestures and word use pairings could potentially reduce the number of words to be spoken in a sentence.

The results of the study were encouraging not only in that the pair of bilingual learners was able to grasp quickly the functions offered by the touchscreen and DGE, but also in the way they communicated significant calculus ideas effectively incorporating linguistic and non-linguistic features in communications. These results have important implications towards the mathematics teaching and learning for all learners at large and bilingual learners in particular. For example, the study points to the use of DGE in pair-work activities for facilitating students' communications. At the beginning of their explorations, Sam and Mario did not use the functions of the DGE purposefully. As they began to learn to use the functions of the DGE, they began to engage in calculus ideas, and the DGE began to take on an important role in the communications. This implies that the role of the DGE is not a static one, but rather *dynamic* that is constantly evolving during the activity.

Also, it could be said that the design of the dynamic sketches played a significant role in facilitating the students' mathematical communications. The *Hide/Show* buttons allowed the students to talk about their ideas gradually, one button at a time, while the *dragging* affordance enabled them to attend to dynamic relationships and connect algebraic with geometric representations of calculus. In tune with previous studies on DGE-mediated student thinking (Falcade, Laborde & Mariotti, 2007), the students may have exploited these functionalities offered by the sketch and hence communicated about area and derivatives geometrically and dynamically as a result. Furthermore, the touchscreen-based DGE seemed to offer a haptic environment for learners to interact with dynamic relationships, where the nature of gestures and dragging is re-conceptualized (see Sinclair & de Freitas, to appear).

This study argues for an expanded view of bilingual learners' communication that includes utterances, gestures, dragging and diagrams. Due to their complementary functions, these elements must not be accounted for in isolation but as a full set of resources in mathematical communication. Although Sfard has not specifically addressed the distinction between dynamic and static visual mediators, the distinction is important for this study because of the potential for the dynamic visual mediators such as gestures and DGEs to evoke temporal and mathematical relations (Ng and Sinclair, 2013), particular for the study of calculus (Núñez, 2006). As shown in my analysis, *dragging-gesturing* emerged as a significant form of communication, and this was facilitated by the dynamic visual mediator presented on the touchscreen-based DGE. Future studies should consider extending the notion of visual mediators and routines to include gestures and dragging on touchscreen-based DGE. In particular, this paper calls for more studies in the area of DGE-mediated learning to investigate the role of *dragging-gesturing* in other types of mathematical activities and within other branches of mathematics.

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PAWNSHOPS TO TEACH PERCENT AND PERCENT TO TEACH PAWNSHOPS

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This paper focuses on student learning in the context of a curricular module on pawnshops piloted with 15 students in an urban high school. The paper describes pedagogical frameworks guiding the development of the module and summarizes key features of its curriculum. Analysis focuses on student growth with respect to mathematical understanding of percent and opinions about pawnshops. Findings include student adoption of ratio strategies indicative of conceptual understanding of percent, and development of critical opinions about pawnshops as a lending system.

Keywords: Equity and Diversity

Introduction

Percent is a concept fundamental to the middle school mathematics curriculum. The Common Core State Standards indicate that by 8th grade, students should be able to solve multistep percent problems and apply proficiency with percents to any array of marketplace applications. In addition to being a focal point of middle grades mathematics, the concept of percent undergirds a variety of topics in secondary school mathematics, like scaling, probability, statistics, and modeling.

High school students are known to have difficulties with problem-solving related to percent (Lembke & Reys, 1994; Moss & Case, 1999; Parker & Leinhardt, 1995). Common errors include ignoring the percent sign; difficulties translating between decimals, fractions and percents; and confusion about multiplicative and additive relationships (e.g., Moss & Case, 1999; Parker & Leinhardt, 1995). Finding a given percent of a specific amount is considered the most straightforward for students; however, even at age 17, most students have difficulty with, for instance, finding 4% of 75 (Parker & Leinhardt, 1995).

Percent is prominent across an array of common marketplace transactions (Parker & Leinhardt, 1995). Banking transactions routinely communicate percents, which is essential for any consumer. Beyond its importance in terms of functional literacy for borrowers, percent is necessary for critical literacy (Apple, 1992), enabling citizens to evaluate quantitative information or analyze quantitative relationships. For example, percent figures prominently and broadly as a mathematical tool in teaching mathematics for social justice (e.g., Gutstein, 2005).

Despite its ubiquity in the marketplace, adults are known to have difficulty with percents. In an investigation of adult learners, while most knew that 100% represents the whole and 50% means half, their understanding of 25% was limited; fewer than a third of the adults could find the sale price of an $80 coat on a 25% off sale (Ginsburg, Gal, & Schuh, 1995). Pre-service teachers were successful at answering a multi-step word problem involving percents but had difficulty explaining why particular operations made sense in the task's solution (Lo & Ko, 2013).

An explanation for difficulty with percent tasks is that it has traditionally been taught strictly using procedural methods. Even though students with pictorial representations of percent are known to be more successful at problem-solving on percent tasks (Lembke & Reys, 1994), classroom instruction is traditionally unaccompanied by mathematical representations that can support conceptual understanding. Representations that can link procedures for solving percent problems to conceptual understanding of percent are area models (e.g., Haubner, 1992), halving or doubling models (e.g., Moss & Case, 1999), 100-board models (e.g., Wiebe, 1986) dual-scale number lines (e.g., Dole, 2000), or ratio tables (e.g., Middleton & Van den Heuvel-Panhuizen, 1995).

This paper explores the teaching and learning of percent within the context of a larger project. The larger project adopts perspectives that integrate culturally relevant pedagogy (Ladson-Billings, 1994) and place-based education (Gruenewald, 2003). The next section briefly describes these frameworks and their correspondence with the project's curriculum.

Guiding Frameworks

Culturally relevant mathematics pedagogy (CureMap, Rubel & Chu, 2012) is based on Ladson-Billings' (1994) theory of culturally relevant pedagogy and Gutstein, Lipman, Hernandez, & de los Reyes' (1997) application of culturally relevant pedagogy to mathematics. CureMap is comprised of teaching mathematics for understanding, centering instruction on students, and providing students with opportunities to think critically about and with mathematics. The three dimensions are interrelated and guide this project's curriculum design. The larger project, of which this study is a part, takes pawnshops as a familiar phenomenon from the city streetscape and uses it as a guiding theme for the investigation. Mathematics helps students to understand how pawnshops, as local and visible features of a spatial landscape, work.

Teaching for understanding denotes an emphasis on connections between mathematical concepts, procedures, and facts. Classroom instruction that supports teaching for understanding includes rigorous tasks, representational tools, and norms for participation (Carpenter & Lehrer, 1999). Understanding the concept of percent is fundamental to interest and to being able to model a pawnshop loan. To build this understanding, 4% simple interest was organized as a \$4 amount for every \$100 dollars borrowed in a ratio table (e.g., Middleton & Van den Heuvel-Panhuizen, 1995). Figure 1 shows an example of tables used by the teacher in our project. The table sets up students to conceptualize percent as a ratio and use a ratio table to calculate 4% of various amounts. The table has multiplicative properties (i.e., 4% of 700 is 7 times 4) and additive properties (i.e., 4% of 150 is 4+2). A ratio table can be used to calculate any percent of any quantity but is especially efficient for benchmark values.

Organizing lessons for students to be central participants is an essential component of culturally relevant pedagogy. This module emphasized student participation, particularly emphasizing whole-class discussions to build shared understanding. For example, in one of the beginning lessons, the teacher performed a skit to clarify with students the vocabulary and elements of a pawnshop transaction, and collectively defined the transaction as a collateral loan with a monthly, simple interest rate.

Centering instruction on students can be accomplished in terms of content that connects to students' lives (e.g., Moll, Amanti, Neff, & Gonzalez, 1992). There is an additional, place-based interpretation of centering: taking a spatially local phenomenon and studying it, at various levels of scale, in students' local places. Focusing on a specific context, in place, and using mathematics to make sense of that context is an explicit stance that differs from focusing on percent as a theme with applications across multiple contexts.

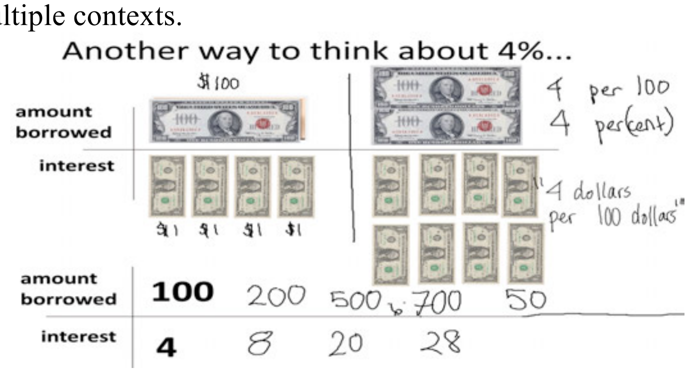

Figure 1: Ratio table used in classroom (11/6/2014)

Real-world contexts with social justice ramifications can motivate mathematics and mathematics enables better understanding of the real-world context (e.g., Gutstein, 2005). In this case, percent is a tool with which to model and compare loans toward exploring critical notions of pawnshop loans as a form of predatory lending or dominant notions of alternative financial institutions (AFIs) as important financial resources for low-income people. Drawing again on notions of local space and place, to investigate spatial patterns in the distribution of financial institutions in the local city, one lesson invited students to consider how to normalize the distribution to analyze which neighborhoods have more AFIs, since numbers of AFIs could be compared to households, land area, or to numbers of banks. Students explored data-rich spatial distributions on interactive, digital maps in the context of understanding the relative cost of different types of loans, set in terms of spatial representations of demographic statistics. Students conducted field research in the school neighborhood to explore the distribution of financial institutions and to interview pedestrians about their experiences as customers of these businesses.

An assumption is that students will develop critical opinions about pawnshops and their role in students' lives, the local community, and the broader city, and that this development is predicated on an understanding of the concept of percent. This paper explores that assumption by asking two research questions:

- Did students develop conceptual understanding of percent through their participation in this project?
- Did students develop critical thinking about pawnshops through their participation in this project?

Research Context

This module was piloted in a high school in a large Northeastern city in the United States. The school is located in one of the city's lowest income neighborhoods and provides free lunch to 100% of its students. Students identify as "Hispanic" (75%) and "Black or African American" (25%). Despite an appearance of uniformity across just two broad categories, the student body includes recent immigrants from the Dominican Republic, second or third generation of immigrants from Puerto Rico or the West Indies, and African Americans. About 20% of the students are classified by the school district as English language learners, and about 20% as needing special education (data was obtained from the school district about the previous school year).

Students are required to pass through metal detectors staffed by police personnel to enter school because of its district classification as "persistently dangerous." Incoming students test scores, on average, are "below proficient" and below city average, and the school has a six-year graduation rate just below the city average. The school suffers from low attendance; about half of its students are categorized as "chronically absent." Despite the statistics that portray the students and school as struggling, the school consistently receives high marks on its district progress report that factors parent and student surveys heavily in its metrics because parents and students express enthusiasm and positivity about the school and its teachers.

The classroom teacher in this study was in her 8th year of teaching, all at this school, and identifies as a white woman. She collaborated on classroom implementation of another curricular module as part of the larger project and had been previously involved in two cohorts of professional learning communities around culturally relevant mathematics teaching. The teacher participated in a 4-day summer training institute and collaborated with the design team to pilot the curricular module in the fall of 2014 in her advisory class, comprised of 16 tenth-grade students (who are in her geometry class). Fifteen of these students participated in the research; one was present in the sessions but could not consent to participate in the research because of disability status. Seven of 11 students took and passed the state's entry-level algebra test the previous year, but only three students' scores exceeded the "college-ready" threshold.

Data Sources and Methods

All 10 class sessions of the module were observed and audiotaped by three researchers, and student written work was collected. Fieldnotes were taken during the observations and were coordinated with corresponding audio to enable detailed memos about each session. Engagement was rated independently by three researchers using a 5-point rubric from Kitchen, Depree, Celedon-Pattichis, & Brinkerhoff (2006) and averaged. A post-focus group session was conducted with five student volunteers from the class and audiotaped.

Analytical memos were produced for every class session and the focus group session. To answer the first research question, students' written responses on assessment items were categorized in terms of correctness and strategy type. Three lessons that focused on percent as function to determine monthly interest and a 15-minute section of the post-focus group that focused on students' self-reflections about the mathematics they learned in this project were included in this analysis.

To answer the second research question, fieldnotes and written work from an introductory class session and audio of whole-class discussion portions from six lessons, totaling 132 minutes, was used to create transcripts. A coding tree was created and used to code transcripts line by line in terms of speaker and reference to pawnshops. Dedoose, a software for coding qualitative data, was used to aggregate students' ideas about pawnshops by speaker, date and in terms of whether the perspective expressed critical or dominant perspectives about pawn shops.

Results

Development of Conceptual Understanding of Percent

On the written pre-assessment, nine of 14 students computed 4% of 150 correctly, but they were limited to decimal-based strategies, mostly with a calculator. During the class sessions, students who had computed 4% of 150 correctly using a decimal strategy demonstrated difficulty articulating a conceptual understanding of 4%. For example, several students who had used a decimal strategy and arrived at a correct answer on the pre-assessment conjectured in class that 4% might mean one-fourth, or a quarter.

Students readily took up the ratio-based strategy initiated by the teacher in the class session that introduced the ratio table. In written work during the lesson in which the table was introduced, eight of 12 students present showed that they could use ratio tables to compute given percents of multiples of 100. Some students could use ratio tables using doubling and halving to compute 8% of multiples of 50 (i.e., 8% of 50, 8% of 150), and few students computed 8% of 75, a slightly more challenging example.

The next day, empowered with a conceptual understanding of percent, students were tasked with using mathematics to model a pawnshop loan in a story problem that required them to extend the monthly interest over 4 months and to factor in the one-time fees. This lesson's engagement was rated at 4.33 (serious/widespread), one of only two lessons with highest student engagement. Our interpretation of this engagement is that students had been carefully introduced to the elements defining the transaction, possessed conceptual tools with which to approach the problem-solving task, and were interested in being able to interpret the mathematics of this pawnshop scenario.

On the post-assessment with 11 students present, six correctly computed 6% of 800 and four demonstrated a ratio approach. Nine students attempted to answer a second question, which asked them to model a pawnshop loan, and five had the correct answer. On this problem, five students used a ratio strategy toward computing 4% of 250. These results on the written assessments might seem underwhelming to readers, but should be considered in light of the context of struggling students. One student in the focus group described how her participation impacted her mental mathematics strategies and her confidence about those strategies. In response to the question, “Did you learn any math?”, Tacee declared:

> I learned a lot. Because, like, before, when I was finding percent, I would only know how to find 50% like, oh -I’d be like- oh it’s half. But then like, now, I know how to find, like for any number (11/25/14).

Use of ratio-table strategies became more efficient and sophisticated for some students by the end of the project. When asked to demonstrate how to compute 6% of 250, four of the five students in the focus group quickly arrived at a correct answer, in some cases without writing anything down. When asked how to accommodate this strategy to a problem like 6% of 280, two of the students in the focus group (the two students with the highest scores on the previous year’s state algebra test), demonstrated strategies that extended beyond the halving, doubling, and combining that had been worked on in class with the teacher. One student scaled 100 to 280 by a factor of 2.8, and correspondingly scaled 6 by a factor of 2.8. A second student first found 6% of 250 using doubling and halving, and then used the entry for 50 to scale to 5 by dividing by 10 and reasoned that “as the money increases by five dollars, the interest also increases by 0.3.” He then used this rate to scale up to 30 dollars toward finding the correct answer (see Figure 2).

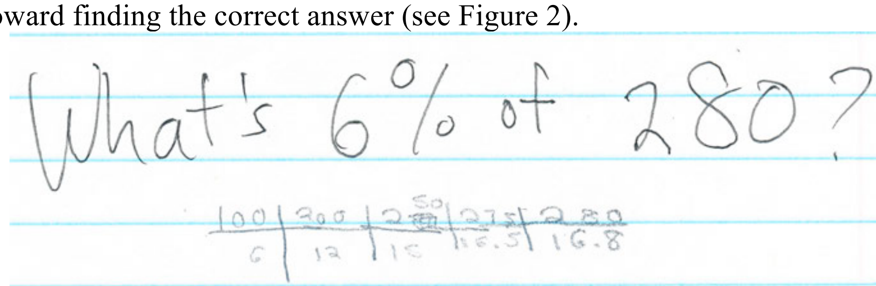

Figure 2: Sample ratio used by student to compute 6% of 280

These examples demonstrate how the ratio table approach was effective as a remedial tool and functioned as a springboard for students with stronger mathematics backgrounds toward more sophisticated strategies.

Development of Critical Thinking About Pawnshops

Critical thinking with mathematics about pawnshops can refer to the phenomenon at increasing levels of complexity (see Figure 3). At the scale of individual transactions, it is possible to use mathematics to analyze a transaction in terms of its high interest rates compared to other loans, or the way that its appraisal system might undervalue items from the perspective of the borrower. Dominant perspectives might dictate that pawnshops are legal enterprises, with published interest rates and, like any business, need to make sufficient profit.

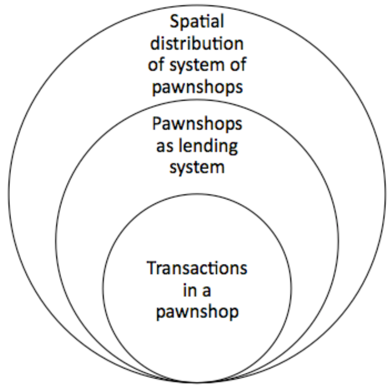

Figure 3: Levels of complexity for thinking about pawnshops

Beyond individual transactions, pawnshops can be considered as a system within the financial landscape. A critical perspective might examine that system as one that, by virtue of its extremely high rates relative to other options and the quick access it provides to small loans, is a system that preys on low-income people. A dominant perspective is that pawnshops form one choice among many options and are a financial resource for those who have poor credit or need small sums quickly.

A spatial perspective on pawnshops considers distribution across places. Patterns in that spatial distribution of mainstream and alternative financial institutions might provide evidence to suggest that the AFIs target geographical neighborhoods that have more unemployed people, immigrants, or low-income families. A dominant perspective about the spatial distribution is that pawnshops, with banks, are located in heavy shopping areas.

On the first day with the module, students were asked to name associations with pawnshops, and the teacher penned contributions on a word wall. Forty-nine words or phrases were named; 34 were related to the pawnshop transaction, with most referring to the appraisal process (25). Students less frequently (12) offered associations that might be indicative of critical perspectives about pawnshops as a lending system, with statements like "advantage," "power," and "they get over on you." One comment hinted at a spatial, critical perspective: "the 'hood."

At the end of the module, in a whole-class discussion organized to elicit student opinions, students made 30 references to pawnshops. These 31 references (made by 8 students) mostly (20) were oriented toward pawnshops as a lending system, though some (8) referred to aspects of individual transactions, and few (3) referred to the spatial distribution. This demonstrates greater sophistication in the students' thinking about pawnshops; prior to the module, their references were concentrated tightly around aspects of the transaction, and by the end of the module, their references had shifted largely to focus on pawnshops as a lending system.

For example, when the class was asked to agree or disagree with the statement, "Pawnshops prey on our community by charging much higher annual interest rates than banks," one student, Lina, disagreed, providing a dominant perspective about pawnshops as a system, explaining, "*I don't think they [pawnshops] do it intentionally, I think they just do it as a choice. Like, if you don't have the credit then you, like, it's their, like it's someone's fault if they have bad credit 'cause it's you're fault if you didn't pay all your debts or whatever… they're helping you out by at least letting you borrow money…*" Another student, Bo, disagreed, offering a critical perspective: "*I only see pawnshops inside, like, poor neighborhoods. So I feel like they, like, preyin' on the poor neighborhoods, because, like, they don't have nothin' so if they come, then they gonna take they money*." Bo continued, recalling examining the posted interest rates and fees inside a check-cashing store during the field research session, "*since it's so expensive, right, then like the people who can't afford it have* *to, like, do that because if they really need the money then they gonna have to raise all they money and then, to get that, then they gonna be broke"* (11/20/14). In that summative whole-class discussion in general, more statements had critical orientations (18) than dominant (13).

Discussions and Further Questions

Findings include student interest in and adoption of ratio models for percent calculations. For struggling students, with weak scores on standardized tests, this topic remediated a central focus of middle school mathematics. At many points in the module, students pointed out that it is important for a consumer to be savvy in terms of the individual transaction, because "pawnshops get over on you." Students felt that understanding mathematics would help them to better navigate an appraisal or loan process. One student, Sheeda, shared in the focus group that she found learning about the mathematics to be more useful to her than she had at first expected:

> But then when we started like, going more in deep into the project we was like oh that's what they really be doing, so like we really have to have that skill of math to, you know, to, pursue… like, that. Like, if we ever went to a pawnshop. (11/25/14)

In other words, mathematics would enable a position of strength instead of weakness for students as consumers. In response to a follow-up question as to whether the mathematics they learned was new, she elaborated:

> *Sheeda*: cuz basically we learned it, but basically we, didn't, like pursue more into it. Like, we just like, oh that's something that we learned in school, like, we-we like…
> *Lina:* to learn it, just to learn it…
> *Sheeda*: Yeah, it felt like, we felt like…
> *Lina:* ...we did, applied learning.
> *Sheeda:* yeah, it felt like — exactly, that's like what I was saying — like, it felt like we really needed to, like, pursue it*."*

These two students made the point here that the context was not only motivating for them, but that learning this mathematics felt necessary.

By the end of the module, students expressed points of view that demonstrated that the module had opened up space for thinking more broadly than about individual transactions in pawnshops. Those points of view became more critical in orientation, and it is our interpretation that that students' learning how pawnshops work, an element of which is how to use percents to calculate interest, was a key factor in this growth. Although the module included a focus on using maps and data to make spatial arguments, such arguments were rarely assimilated into students' opinions. An interpretation of this finding is that the students' map and data analysis was done individually or in pairs at laptops computers, without time for interpreting or layering those observations in whole-class discussions. Collectively building understanding in whole-class discussion competed for time with individual investigations that focus on single-user technologies.

This investigation of pawnshops used mathematics of percent in different ways. This paper has a central focus of understanding how to use percent as a function and as a proportion to model a pawnshop loan and to compare loans. Yet percent was used in other parts of the module to represent demographic data as a statistic. Using percent as a function to model loan interest typically scales *up* from $100, while using percent as a statistic typically scales *down* to 100 people. Future research could focus on relationships between conceptual understanding of percent as a function, proportion, and statistic, in mathematical investigations that draw on data that pertain to demographics and place.

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STUDENTING IN THE SECONDARY MATHEMATICS CLASSROOM: MARIA

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Studenting is comprised of the behaviors that students perform or exhibit in a learning situation, such as the mathematics classroom. These may or may not be intended to help themselves learn, and may or may not conform to the teacher's goals and expected actions. In this paper I report on the case of a particular student, Maria. Observations, together with excerpts of informal interviews, are used to provide a preliminary analysis using activity theory as a theoretical lens. Initial results reveal that 'good' studenting behaviors do not necessarily result in mathematics learning, nor indicate a primary motive of learning. Analysis indicates Maria's primary motive was to get a good grade and be seen as a good student.

Keywords: Teacher Beliefs; High School Education; Affect and Beliefs

Studenting

The term 'studenting' was coined by Gary Fenstermacher in 1986. Initially, he describes this concept in terms of a cohort of student behaviors including "*getting along with one's teachers, coping with one's peers, dealing with one's parents about begin a student, and handling the non-academic aspects of school life*" (p. 39). In essence, Fenstermacher describes studenting as what students do to help themselves learn. A later definition encompasses other behaviors such as "*'psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments*" (Fenstermacher, 1994, p. 1) and other similar practices.

Analyses have shown that the process of schooling produces a number of unintended consequences, some desirable, but also many that are patently objectionable (Engeström, 1991) and counterproductive to the goal of student learning. Preliminary studies have shown that across the board students are finding ways to subvert the expectations of the teacher in ways that the teacher is not aware of (Liljedahl & Allan, 2013a; 2013b).

Theoretical Framework – Activity Theory

Considerable research can be found looking at students' motives and behaviors, but what is lacking is a catalogue of student behavior across activity settings in the mathematics classroom. This finding drives the first research question.

Activity Theory was chosen as a framework for analysis for its ability to describe what *is* rather than what *is ideal*. Rather than categorizing students by their behaviors (e.g., on-task or off-task), presuming students behave rationally or ideally (e.g., game theory), or assuming the students' primary goal is learning (e.g., didactic contract (Brousseau, 1997)), activity theory allows for a description of what is observed and said without overlaying preexisting assumptions or judgments. These observations, taken together, can then be used to develop a hypothesis for what is driving student action: their motive, which may be something other than a desire to learn. Individual behavior and motive within the classroom collective is best viewed through a theoretical lens primarily comprised of Leontiev's Activity Theory (1978).

For Leontiev, "*[a]ctivity does not exist without a motive; 'non-motivated' activity is not activity without a motive but activity with a subjectively and objectively hidden motive*" (1978, p. 99). The object of an activity is its motive, and is something that can meet a need of the subject. Motives arise from needs, which are the ultimate cause of human activity. Figure 1, below, illustrates the relationship between the elements in Leontiev's development of activity theory.

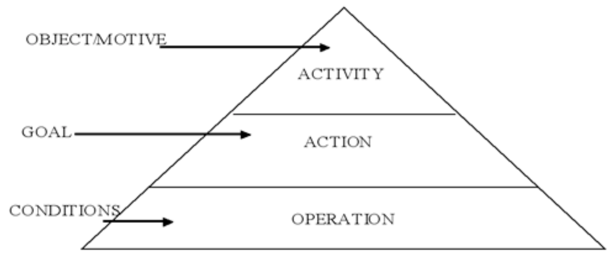

Figure 1: Pyramid Representation of Leontiev's Three-Level Model of Activity (1974)

Motives drive activity, and activities are directed at goals. People have many goals, which shift in importance and in content on the basis of both contextual and intrapersonal factors. At the very top of this hierarchy is the motive for activity. At any time an individual has a hierarchy of these motives, the order of which is determined through and as a result of one's activity.

Actions are the many steps that comprise an activity, although not all are immediately related to the motive (Kaptelinin & Nardi, 2012). Actions are directed towards specific targets, called goals. Goals are conscious; in contrast to motives, of which a subject is not usually aware.

The fact that motive is often hidden from the subject suggests difficulty in determining the ultimate motive. This obstacle can be overcome by utilizing an "*actions first*" strategy (Kaptelinin & Nardi, 2012). The strategy begins at the level of goals, which people are generally aware of and can express, and the analysis is subsequently expanded up to higher goals and ultimately the motive. Given the primary interest of this study is the students' actions/goals and activity/motive only the top two levels of Leontiev's pyramid will be considered in this paper.

Cataloguing the observable aspects of studenting (actions) and analyzing these together with students' goals through the lens of activity theory can offer new insights into student motives and provide researchers and educators with evidence to better understand student behavior.

Methodology

The nature of the research questions requires a particular approach – an ethnographic study, though the study itself is not ethnography. In accordance with this approach, I spent significant time immersed in the classes under study observing and interacting with students, taking fieldnotes, and asking questions. Analysis occurred throughout the process of data collection whereby what was observed and recorded in one lesson provoked questions and shifted focus for the next observation/interviews. Both an etic and emic perspective were utilized.

Participants

The data for analysis are taken from a larger study conducted in three secondary school mathematics classes in British Columbia. Maria was in the Foundations 11 class.

Data Collection and Analysis

Data were collected during the 2013-2014 school year. Throughout the fall semester the class was observed for twelve periods, each period ranging from 60 to 75 minutes. Classroom lessons and informal interviews were audio recorded and transcribed for later analysis and comparison with field notes taken during the class. The data discussed here has been subjected to an initial analysis using Leontiev's activity theory in order to determine the likely primary motive underlying Maria's behavior.

Results and Analysis

The analysis is influenced by all observations, interviews, and interactions with Maria, not all of which are provided here. A selection of observations and interview transcripts are provided to render a sketch of Maria's behavior and justify the hypothesized motive.

What Maria Did (Actions)

Over the five-month period the class was visited the researcher observed Maria in many activity settings. Maria was always present, on time, and prepared for class. She tried every homework question and asked for help from neighbors or Mr. Matthews when it was needed. She was often the first to begin a new activity and remained focused for the duration of the task. She took notes during lessons and for example problems. She also stayed after class on occasion to do extra review and finish homework. Maria could often be relied on to supply answers (usually correct) if Mr. Matthews asked a question of the class during a lesson. She would follow the teacher's explanations and, if she was able, provided explanations when asked 'why'. These explanations were largely procedural. She asked questions when Mr. Matthews was doing example questions during a lesson, such as "*If you gave us those two roots, how would you graph it?*". During group work she did not actively take a leadership role but often ended up leading group presentations.

What Maria Said (Goals)

The following is a summary of some Maria's written responses to some prompts given to all students in the class regarding note-taking habits and review for tests.

She takes notes when there is a lesson, even if she has done the topic previously, and writes them in her own words, rather than exactly what the teacher says because it "*helps me remember the notes*". She wrote that she takes notes during class because it helps her "*pay attention to the lesson"* and it helps her "*to study later on when there is going to be a test*". Maria thinks her teachers expect her to take notes because they know it will be helpful to students in the future.

To review for a chapter test Maria wrote that she uses both her notes and her textbook to ensure she "*fully understands the subject/topic*". She "*looks over*" her notes both at home and in class, reading them "*to see if I can follow all the steps*". She does all the questions in the assigned textbook homework that "*I know I'll have trouble with*" and also does "*some of each type of question*". During class on the day before a test she'll work on the assigned review and if she has difficulty "*I'll ask someone who might understand*". After a particular test she said she "*should have done more practice on converting equations with the word problems*".

Discussion and Conclusion

Maria demonstrates the activity of a conscientious student. According to Mr. Matthews', Maria "*should*" be achieving a high 'A' in the class, but when it comes to the test she doesn't seem to have fully understood the material. This perplexing situation could be understood by examining Maria's actions to determine what is driving her behavior.

Figure 2 depicts the top two levels of Leontiev's (1974) triangle as applied to Maria's actions, her voiced and indicated goals and, extrapolated from these, her motive.

In all aspects Maria demonstrates qualities of what she perceives to be a 'good' student. Maria's actions demonstrate that her understanding of a what a 'good' mathematics student does includes: doing and submitting all the homework, asking for help (and being seen to do so), answering questions in class, participating, taking notes, doing practice for procedural fluency.

Maria's studenting is so successful that she seems to have fooled Mr. Matthews into thinking she is a good mathematics student – yet she seems to struggle with non-typical problems and relies significantly on 'remembering' how to do questions. Maria is very good at studenting, just not mathematics. Since her actions align with the features that Mr. Matthews appears to consider a part of being a good mathematics student, Maria's performance on assessments is puzzling.

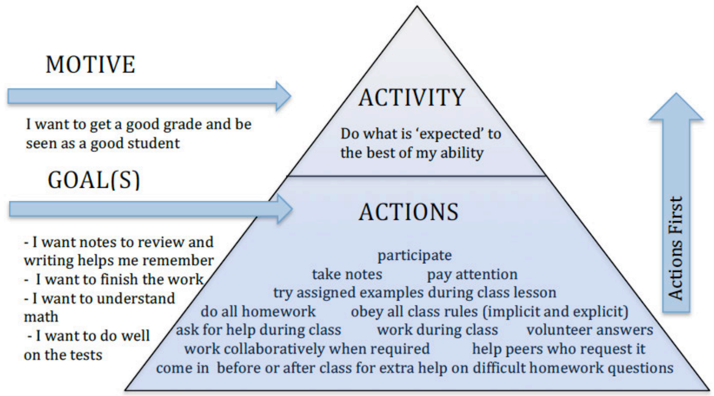

Figure 2: Using an 'actions first' approach to determine Maria's motive.

Although the teacher saw all of Maria's actions as indications of learning, they were, in fact, just proxies for learning. Thus even when students are conforming to teacher expectations, as in the case of Maria, there appears to be a gap between actual student activity and learning, the ideal outcome associated with a primary motive of learning. Frameworks like the didactic contract (Brousseau, 1997) may apply when the desire to learn drives student activity but when learning *is not* the primary motive, other approaches must be explored, such as activity theory.

Other preliminary results show that are a variety of goals that manifest in similar actions, suggesting that considered alone, a students' actions are insufficient to deduce their motive. Also, even when students do not conform to the teacher's expected behaviors, from the student's perspective there is a certain rationality to their actions. It is anticipated that extended analysis will provide further evidence to support this claim.

What the case of Maria suggests is that a deeper understanding of the perspective of the student within the classroom unit could serve to provide teachers with cause for reflective thought on their policies and practices that influence student learning.

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CONNECTING INDIGENOUS AND WESTERN WAYS OF KNOWING: ALGONQUIN LOOMING IN A GRADE 6 MATH CLASS

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In this study we explored the connection between Algonquin ways of knowing and Western mathematics found in current math curricula. We used a cyclical research design of consultation, co-planning, co-teaching, and co-reflection to explore the potential of grounding mathematical instruction in the traditional Algonquin activity of looming. Results suggest the activity supported students' complex mathematical thinking including patterning and algebraic reasoning, proportional reasoning and spatial reasoning. The experience also resulted in a reconceptualization of mathematics learning for both Algonquin and non-Native students.

Keywords: Equity and Diversity; Elementary School Education

Objectives

The activities in this paper come from an ongoing long-term project in which we collaborate with Elders and community members from different Indigenous communities across Ontario to explore connections between Indigenous and Western ways of knowing mathematics. The goal of this project is to explore connections between the mathematical content knowledge based on the Ontario curriculum expectations and the mathematics inherent in Indigenous cultural practices. We are also exploring connections among evidence-based teaching pedagogies founded on the principles of reform math instruction, Indigenous ways of teaching and learning, and contextual knowledge based on students' lived experiences. The work outlined here was conducted in a Grade 6 classroom at a small public school near Pembroke, Ontario. The school population comprises approximately 20% Algonquin students from the nearby Algonquins of Pikwàkanagàn First Nation, and 80% non-Native students.

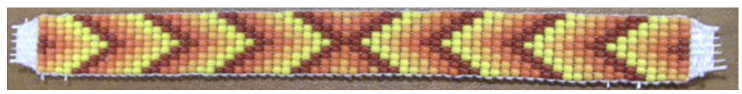
Theoretical Framework

Ethnomathematics is a theoretical framework that has generated a growing area of research of how math curricula can and should respond to local culture (D'Ambrosio, 2006). From the ethnomathematics perspective, school mathematics, which is derived from 15th century Western European traditions, is one of many diverse mathematical practices and is no more or less important than mathematical practices that have originated in other cultures (Mukhopadhyay et al., 2009). Ethnomathematics provides a foundation for developing culturally responsive education, which refers to efforts to make education more meaningful by aligning instruction with the cultural paradigms and lived experience of students (Castagno & Brayboy, 2008). Research has shown that creating connections between math instruction and Indigenous culture has had beneficial effects on students' abilities to learn mathematics (Cajete, 1994). Long-term studies by Lipka (2002, 2007) found that culturally responsive education in mathematics had statistically significant results in terms of student achievement. Recent researchers have also been exploring the insights Indigenous epistemologies and practices can provide into understanding diverse mathematical pedagogies (Barta et al., 2003).

Methods

Research Design

Our core research team for this project included two Algonquin teachers, and the operations manager of the Algonquins of Pikwàkanagàn Cultural Centre, Christina Ruddy, who is an expert loomer. The team also included three non-Native teachers from the school, including the Grade 6 teacher, Mike Fitzmaurice, in whose classroom the study took place. We followed a cyclical approach (consult, plan, teach, reflect, share) as a framework for the project, which ensured that we continually cycled back to members of the Algonquins of Pikwàkanagàn community for guidance. The cycle includes a consultation phase during which we met with community leaders from Pikwàkanagàn. A key theme raised by community members was the importance of revitalizing Algonquin culture. One focus of this cultural revitalization has been beadwork, including loomwork, which has always had significance for this community. For this study we co-planned a unit of instruction based on Algonquin looming – a type of beading that is done on a loom, and involves stringing beads onto weft threads and weaving them through warp threads. Once the lesson sequence had been designed by the team, Mike and Christina co-taught the lessons over the course of two weeks in the grade 6 classroom. All lessons were videotaped and specific classroom incidents were transcribed. The team viewed video clips to co-analyze students' mathematical thinking and cultural connections. Finally, the results of the lessons were shared with community members though parent and community meetings. In this paper, we share some of the mathematical thinking from week 1 of the sequence, which focused on pattern design.



Findings

Looming in the Classroom

Christina taught the students that looming pre-dates the arrival of Europeans to North America. Historically sinew and porcupine quills would have been used but currently plastic and sometimes glass beads and nylon thread is used. The pattern for each looming project is created on graph paper (see Fig. 1). The columns represent the weft threads, and the number of columns corresponds to the horizontal length of the beadwork. The rows represent the warp threads, and the number of rows corresponds to the width of the beadwork. The rows and columns of the design space are numbered. Below each column, the number of each colour of bead is entered (with the number representing the colour of the beads, in this case blue numbers for blue beads and red numbers for red beads). This helps the beader to know the order for stringing beads for each column, or line of beads on the weft thread. Each column should add to the total number of beads on each weft thread (so in the example in Fig. 1, each column should add up to 7).

Patterning and Algebra

Students were introduced to a pattern of chevrons that were two-beads wide and created using two alternating colours; a 2-colour 2-bead Chevron pattern (Fig. 1). To identify the unit of repeat (also referred to as the "pattern core"), some students focused on the numeric pattern and noted that even though the numbers repeated after the 2nd column, the colours were different

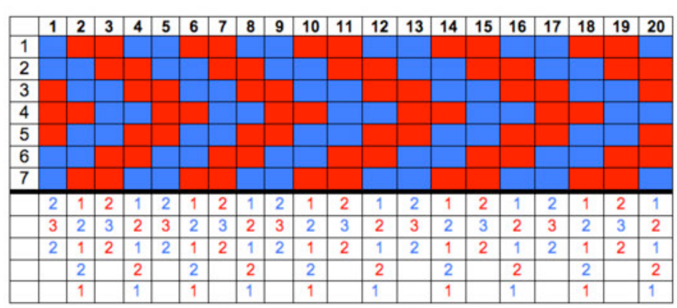


Fig. 1: 2-Colour 2-Bead

so the first 4 columns made up the core. Others identified the visual pattern of the first 4 columns and described the central blue chevron surrounded by parts of red chevrons on either side.

Students then created a 3-colour 3-bead chevron (Fig. 2) and found that column 10 was identical to column 1, and so reasoned that the core comprised columns 1-9. Students described imagining superimposing the first 9 columns onto the next 9 columns to see if they “matched.” Noticing that the pattern seemed to begin with a partial chevron, a few students wondered what would happen if they added a column “before” (to the left of) column 1 on the pattern (which they referred to as “column 0”). This inquiry was explored by the rest of the class. They discovered that the core “moved” to the left by one column, and comprised columns 0 – 8 (Fig. 3). They continued to add columns to the left of column 1, and noticed the core shifted by as many columns as were added. However, they also noticed that the number of columns in the core did not vary. For a 3-colour, 3-bead Chevron pattern the core was always made up of 9 columns.

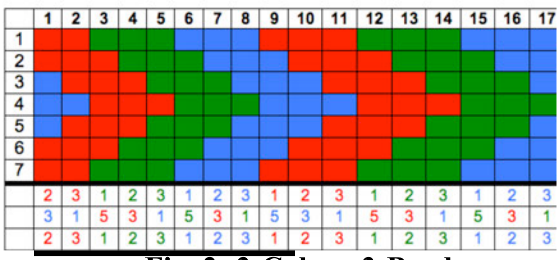


Fig. 2: 3-Colour 3-Bead

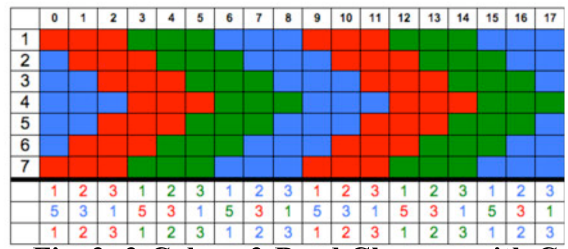


Fig. 3: 3-Colour 3-Bead Chevron with Column

Based on these experiences, the students generated an algebraic generalization to predict the number of columns in the pattern core of any Chevron pattern. The width of the core is determined by a multiplicative relationship between the number of beads that made up the chevron and the number of colours (chevron width x colour = core width). The idea of a “moving” pattern core was new to these students, as previously they had never considered that a pattern can extend to the left as well as the right of the given elements. This allowed them to recognize that the core of a pattern is not necessarily defined by the first element, however what remains constant is size of the core (in this case, the number of columns).

The students were then asked to make predictions about columns in the pattern further down the sequence, for example, what the 35th column would look like. The students identified the multiplicative relationship between the number of columns in the core, and the numbered columns of the design template. For example, if the core of a pattern comprises 4 columns, the 35th column would be the same as the 3rd column, since the core would be repeated 9 times to the 36th column (which would be identical to the 4th column) and the column before the 36th column would be identical to the 3rd column.

Proportional Reasoning

At the end of the first week, the students designed their own pattern to use as the basis of their bead creations. They learned that 5 columns on the pattern template equated to 1cm of beadwork. Students were asked to measure their wrists and calculate how many columns wide their bracelets would need to be, how many beads of each colour and in total they would need.

Some students used proportional reasoning to estimate the total length of their finished bracelets using a fixed ratio of 1 cm = 5 columns. We also saw evidence of students using the pattern core as a unit, for example, considering the core as simultaneously one core and 4 columns. For example, Wyhatt with a wrist of 17cm worked with the relationship of 5 core units = 20 columns (or 4 cm), 10 core units = 40 columns (or 8 cm), 20 core units = 80 columns (or 16cm) and one more core unit plus a single column (first column of the core unit) for 85 columns (17 cm). This is an example of using a composite unit (Lamon, 1996) for more sophisticated reasoning – the 20-column template represented 5 pattern core units and 4 cm simultaneously. Students exhibited an ability to use single or composite units as the basis for multiplicative thinking (e.g., using the unit of 5 columns for 1 cm, or using the unit of the pattern core of 4 columns with an understanding that 20 columns = 5 cores *and* 4 cm).

Spatial Reasoning. Numerous studies suggest that spatial reasoning skills, including mental manipulation and spatial visualization, are linked to mathematical achievement (e.g., Gunderson et al., 2012). We found that designing 2-dimensional patterns on a grid, and identifying components of the pattern (like the pattern core) provided an opportunity for students to engage in visuospatial thinking. As they worked to isolate the columns that made up the core of the pattern, the students were able to mentally visualize isolating the pattern core superimposing one core onto the next to determine whether it "matched".

Conclusion

Ethnomathematics, and culturally responsive teaching specifies working within the mathematical systems established within a particular culture, in this case, the Algonquin culture of Pikwàkanagàn. The connections that we made between Western and Algonquin mathematics took the form of creating a "third space" (Haig-Brown, 2008; Lipka et al., 2007) in the classroom by exploring the potential of Algonquin activities for mathematics, and bringing Algonquin culture into the classroom. This third space, merging Algonquin and Western ways of knowing, is created when the knowledge and perspectives of traditionally excluded communities are privileged alongside the dominant society's pedagogy and content.

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TEACHER PERSPECTIVES ON MATHEMATICS EDUCATION FOR LANGUAGE LEARNERS: ADAPTING ELL EDUCATION MODELS

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Teachers in Korea have recently confronted new challenges as the population of linguistically and culturally diverse students increases. Ethnically diverse students as well as Korea-born students returning from long residences in foreign countries have great difficulties orienting themselves within Korean schools due to lack of Korean language proficiency and cultural differences. We named this student group as Korean language learners (KLLs) and investigated teacher perspectives on effective mathematics pedagogy for KLLs. Using two educational models for English language learners (ELLs)—Cummins' Quadrant Model and the Sheltered Instruction Observation Protocol (SIOP) Model—we examined survey results of a small group of elementary teachers in South Korea in terms of effectiveness and feasibility of the ELL models.

Keywords: Equity and Diversity; Teacher Beliefs; Elementary School Education

Research Objectives

The classroom culture of Korean schools has recently changed as the population of linguistically and culturally diverse students increases. We associate immigrant students and Korea-born students, returning from long residences in foreign countries, in terms of their linguistic difficulty, and name them as *Korean language learners* (KLLs). Recent studies have found that students from international marriages and immigrant families tend to academically perform at lower levels than general Korean students (Cho et al., 2006; Song et al., 2010) and mathematics is one of the subjects they have the most difficulty with (Cho & Lee, 2010; Jang & Choi-Koh, 2009). Moreover, Korean public schools have failed to provide multicultural programs or linguistic supports tailored to the needs of KLLs (Kim & Kim, 2012), and teacher preparation programs have not prepared teachers for implementing multiculturalism in their instruction (Mo & Hwang, 2007).

We investigated South Korean elementary teachers' perspectives on the use of effective mathematical pedagogies for language learners, especially KLLs. We specified several pedagogies within two educational models designed for English language learners (ELLs) and examined survey results of how teachers considered implementing the ELL pedagogies in Korean school contexts.

Frameworks

One trend of ELL research has been to highlight the distinction between academic language and everyday language (Moschkovich, 2007) although they are not exclusively separated. Two ELL education models were chosen to address the importance of considering both language development and academic context: Cummins' Quadrant Model (Cummins, 2000) and the Sheltered Instruction Observation Protocol Model (SIOP Model: Echevarria, Vogt, & Short, 2004). We adapted and connected these models in a mathematical context.

Cummins' Quadrant Model

Cummins (2000) considered a degree of context and a degree of cognitive demand in language tasks or activities to create a two-dimensional model (see Figure 1). Quadrants A and B include instructions that may be appropriate for language learners because contextual clues help them overcome language barriers. In the context-embedded instruction, students actively negotiate meaning through scaffolding and feedback provided by their teacher (Cummins, 2000). Further, students need to be challenged cognitively as well as provided with appropriate contextual and

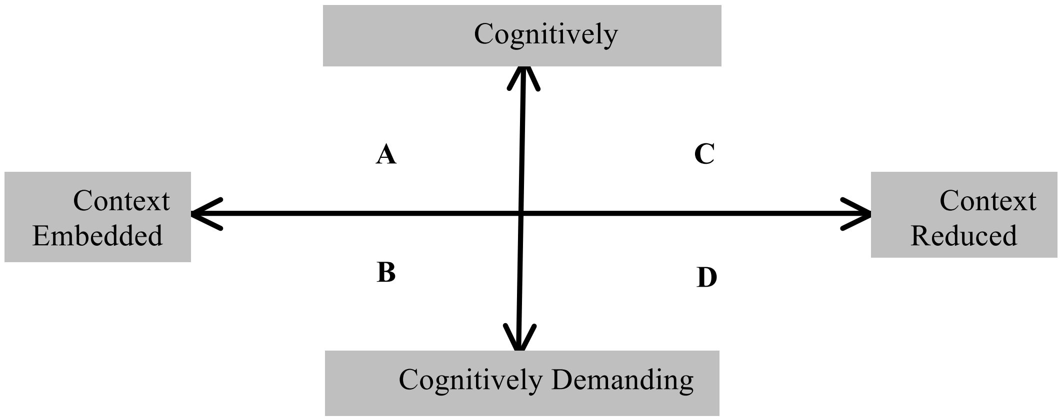

Figure 1: Four - Quadrant Model (adapted from Cummins 2000, p. 68)

linguistic supports as emphasized in Quadrant B (Gibbons, 1998; Vincent, 1996).

The SIOP Model

We chose the SIOP Model as one of the approaches to be placed in Quadrant B. The SIOP Model was designed to teach content subjects for ELLs. The eight components of SIOP Model are: lesson preparation, building background, comprehensible input, strategies, interaction, practice/application, lesson delivery, and indicators of review/assessment (Echevarria et al., 2004). Because we were interested in the strategy component, we chose five strategies that have potential for teaching mathematics to KLLs: Higher-order thinking questions, visual/physical activity, scaffolding, graphic organizer, and group activity.

Methods

Participants and Setting

We conducted a survey with 27 Korean elementary teachers in a large urban area where the population of multicultural students was relatively high compared with other places in South Korea. Due to the location, some participants had taken multiculturalism courses in various forms, although none of the trainings specialized in mathematics education. The survey asked them to evaluate a lesson plan designed by the SIOP Model developers (Echevarria et al., 2010) to teach geometric figures for second grade ELLs. Because the lesson included all five SIOP strategies mentioned above, we examined how Korean teachers identified and valued those strategies for teaching KLLs.

Data Sources and Analysis

The objectives of the survey were (1) to examine if teachers identify the strategies designed for language learners, (2) to examine whether they consider the cognitive demands and contextual aspects, and (3) to see how they would choose to implement the lesson for their students. The first part of the survey asked about previous experiences and beliefs on KLLs, and the second part included open-ended questions about teachers' insights on the SIOP lesson plan. To analyze the data, we used the constant comparative analysis method (Fram, 2013), which employed open coding, axial coding, and selective coding (Strauss & Corbin, 1990). Finally, we built a map of all categories and found themes that emerged in and across categories.

Results

Teacher Beliefs on KLLs

Differentiation of mathematics instruction. More than 90% of respondents agreed that they should differentiate their mathematics instruction for KLLs. The more experiences teachers had with KLLs, the deeper and richer the insights in their responses were. For example, one teacher who had a long-term experience teaching KLLs pointed out the difficulty of teaching conceptual knowledge to KLLs rather than teaching procedural knowledge. In addition, there was no relation between the beliefs on differentiating instruction and the multicultural trainings they received. One reason for the small influence of training might be that the multiculturalism courses were not specialized in mathematics.

Appropriateness of the storytelling mathematics textbooks. Recently, the Korean government issued storytelling mathematics textbooks, which included sufficient pictures and contextual clues. One of the survey questions asked how appropriate the storytelling textbook was for KLLs. The responses were prevalently negative. The teachers disagreed that the storytelling textbook was suitable for KLLs because it required teachers to use complicated discourse in order to make sense of the story contexts in the textbook. This reveals that teachers perceive language as a barrier for KLLs to understand mathematics. Even if pictures and contexts support word problems, teachers believe more sentences yield more difficulties rather than more contextual clues.

Evaluation on the SIOP Lesson

Most teachers responded that the SIOP lesson would be effective for KLLs. They identified various types of linguistic supports: linguistic objectives, graphic organizers, or practicing sentence patterns. They thought these strategies were different from their typical mathematics lessons. One teacher articulated that linguistic supports would be useful, not only for KLLs, but all students, as follows:

> I have seen many students who are capable in mathematics have great difficulty expressing their ideas in language. Particularly lower-grade boys have trouble with verbalizing an image in their head. This lesson is impressive in the sense that it guides students to learn how to express their understanding in sentences in concrete ways.

Among the five strategies, teachers identified visual/physical activities, group activities, (verbal) scaffoldings, and graphic organizers. However, none of them noticed the higher-order thinking question, although it was written at the top of the lesson plan. The series of teacher questions that explicitly appeared in the lesson plan multiple times were not mentioned either. In short, they were able to identify the SIOP strategies and generally agreed that the lesson would be effective for teaching KLLs. However, little recognition was found on cognitively demanding aspects such as the higher-order thinking question. Moreover, it should be noted that the Korean education system or Korean teachers' pedagogy has its own features and structures that could be an obstacle when implementing the ELL strategies in the classrooms in South Korea.

Discussion

Although the teachers in Korea had not received any official training to teach mathematics for KLLs, it was a positive sign that they could identify important strategies that support language learners. However, the results imply that their understanding is limited because they did not pay attention to the aspects of cognitive demand and they heavily focused on language difficulty rather than providing contextual clues or scaffoldings. This view might be related to the conception that language learners are deficit learners (Moschkovich, 2007).

This study provides an international lens on teacher perspectives and openness towards multicultural education. Diversity is becoming prevalent internationally, so it is important to recognize other countries' situations and efforts in educating diverse learners because we can learn from their experience and also find some possible collaboration. This study contributes to teacher educators' understanding of the needs of language learners in mathematics education and teachers' adaptability of a linguistic education model in content areas. Based on the results of this study, the efforts of applying ELL education models can give a meaningful indicator for future education models for other language learners and teachers of language learners if there are adequate considerations about cultural and political situations.

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FOSTERING LEARNING-BASED CONVERSATIONS IN MATHEMATICS

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Conversations can be communicative moments in which there is an interpersonal and intimate nature of turning round ideas for the purposes of growth. Interactions among grade 12 students demonstrates the possibility of learning to learn mathematics through conversation. In exploring opportunities for learning-based conversations, constructivist grounded theory sponsored the development of a Model for Fostering Learning-based Conversations in Mathematics. The four features of preparation, presence, mode, and pace represent a space where students engaged in talking about their learning through conversational moments. This report will use student data to demonstrate how conversations which focus on ways of learning mathematics empower students.

Keywords: Classroom Discourse; Learning Theory; High School Education; Metacognition

Purpose of the Study

Mathematics education reforms (NCTM, 2000, 2014; Western and Northern Canadian Protocol, 2008) have emphasized students' personal development of mathematical ideas through conversation (e.g., Chronaki & Christiansen, 2005; Herbel-Eisenmann & Cirillo, 2009). As an enduring challenge, absent from these reforms and from many students' experiences is explicit discourse about processes of learning–identifying strategies students use to learn (e.g., homework, taking notes) and how to adapt strategies. Current scholarship could respond by addressing how students engage in conversation about their processes of learning and how learning-based conversations support improved mathematical learning.

The empirical research presented in this report is part of a larger study that addressed the question: *What is the nature of students' learning when they engage in conversations to shape their personal processes of learning mathematics?* The purpose of the study was to understand how high school students learn to learn mathematics (McFeetors, 2014). This report focuses on one facet of the study: how students engaged in conversations about their ways of learning mathematics. The conversations were "learning-based" because they foregrounded *learning* in the students' awareness (Polanyi, 1964/1969) and students improved their learning through discourse. In the presentation, I examine conversational features which afforded students opportunities to experience growth in awareness of their learning processes, improve their approaches to learning mathematics, and view themselves as capable learners of mathematics.

Mode of Inquiry

Constructivist grounded theory (Charmaz, 2006) returns to the symbolic interactionist root of grounded theory while looking through a constructivist lens as an interpretive process for inquiring into dynamic phenomena. Theory is constructed on a provisional basis and contingent to the context. The researcher moves from rich empirical data through abstraction toward developing a mid-range interpretive theory through guiding processes like coding, memoing, categorizing, theoretical sampling, saturation, and sorting. The researcher's reflexivity results in theorizing as both process and product enabling other researchers to apply and extend the work.

The study was situated in an academically-focused suburban school in a Western Canadian city. Thirteen grade 12 students who were taking a pure mathematics course volunteered to participate in the study. Students were enrolled concurrently in a course, *Mathematics Learning Skills*, that provided support for their mathematical learning. Within the *Learning Skills* course, I assisted the teacher in coaching students to improve their approaches to learning mathematics while simultaneously collecting data.

I recorded field notes after observing each class over four months. Students took part in: 1) bi-weekly interactive journal writings (Mason & McFeetors, 2002) about how they were learning mathematics; 2) one of three small groups where they developed a learning strategy that supported understanding (three to five sessions of approximately 30 minutes each); and, 3) two informal interviews focusing on approaches to learning mathematics for understanding (each approximately 30 minutes long). In addition to generating data, the interactions also afforded students moments to be in conversation about their approaches to learning mathematics. Using line-by-line coding and the constant comparative method (Glaser & Strauss, 1967), data analysis involved developing codes for characteristics of student conversations. Categories of analysis were constructed by grouping codes related by conceptual qualities of conversation. Category names, like "pace", are descriptive of conversational features students valued and are abstracted from the data. I constructed the Model for Fostering Learning-based Conversations in Mathematics by exploring the interrelationship of the categories.

Perspective

Rather than using an interpretive framework, I adopted Blumer's (1954) notion of sensitizing concepts to "merely suggest direction along which to look" (pp. 7-8). *Conversation* was used as a sensitizing concept in this study. Surveying the uses of *conversation* in curriculum inquiry (Bakhtin, 1986; Belenky, Clinchy, Goldberger, Tarule, 1986; Gadamar, 1965/1975; Gee, 1996; Noddings, 1994; Varela, Thompson, & Rosch, 1991) and mathematics education (Bauersfeld, 1995; Cobb, Boufi, McClain, & Whitenack, 1997; Davis, 1996; Ernest, 1993; Gordon Calvert, 2001; Sfard, Nesher, Streefland, Cobb & Mason, 1998) pointed to a particular form of communication in which there is an interpersonal and intimate nature of turning round ideas for the purposes of growth, as well as a way of being in the world. The related literature served to sensitize me as a researcher to five particular conversational features – including *withness*, *listening*, *dynamic*, *uncertainty*, and *form* – which guided my attention to the ways in which students were in conversation about their processes of learning and informed generating a model.

Results

The four features of fostering learning-based conversations include: preparation, presence, mode, and pace. The features represent qualities of providing opportunities for students to talk about and improve their learning strategies. It is more explanatory of the occasioning of learning-based conversations, rather than of the qualities of conversations themselves. As such, it demonstrates ways in which teachers could provide opportunities for students to talk about how they learn. The features were created by looking at the range of examples from the study and attending to what the students emphasized when they identified conversations about their learning and what I noticed in their conversations through interpretation of data that the students did not explicitly identify as conversational moments. Students' quotes are included below.

The *preparation* feature points to the varying degrees of advanced planning that took place in providing opportunities for the students to attend to their learning. This feature has a temporal dimension, from spontaneous to deliberate interactions. In offering help in class, I recorded a field note where Teresa "asked me if it was like a question in her notes … I encouraged her that she had used a great strategy" for getting unstuck—an example of being alert to a spontaneous moment to shift the focus to learning. When I deliberately showed students a list of learning strategies they used during an interview, Grace exclaimed, "That's a lot! … I thought I only had two or three ways to learn math." The deliberate planning for learning-based conversations allowed for deep exploration of the processes and meaning of learning for the students.

The *presence* feature refers to the composition of members of a learning-based conversation, ranging from internal dialogue to including teachers and peers. While self-talk often focused on mathematical thinking, Danielle described "sitting on the bus, and I was thinking … how would I be able to separate my ideas and stuff" as a catalyst for a new learning process. In small group sessions, students suggested and considered different ways of learning mathematics content. Students valued different perspectives on learning processes, not looking for experts to inform them but rather a responsiveness to turning round ideas in conversation with fellow inquirers.

The *mode* highlights the dialogic nature of learning-based conversation through various forms, including spoken interactions, textual artifacts, and a hybridity of these two modes. Spoken discourse occurred mainly through one-on-one interactions and small groups. It required moving students from noticing content to identifying how learning occurred. Interactive journal writing, as a textual artifact, gave students time to pause and consider deeply their approaches to learning. It was a safe space to share emergent thoughts about learning and as I wrote back I could draw learning into view. Upon returning a journal, Kylee exclaimed, "This idea for the cue card is great! I'm going to try it tomorrow"—a hybrid of student, me, and text.

The *pace* feature indicates a shift in classroom rhythm that allowed for a suspension of time from content to explore issues of learning. When Shane explained that "sometimes I just think about how I learn", it was within a relaxed feeling of learning-based conversations contrasted with the rapidity of mathematics content. Opening up brief moments mattered to students. For instance, Ashley identified "the [small] groups that we're doing, it's mostly concentrated there" for conversations where learning strategies were developed that helped her succeed in mathematics. The different intensity fostered students' choice to engage in learning-based conversations where they inquired into ways of learning mathematics.

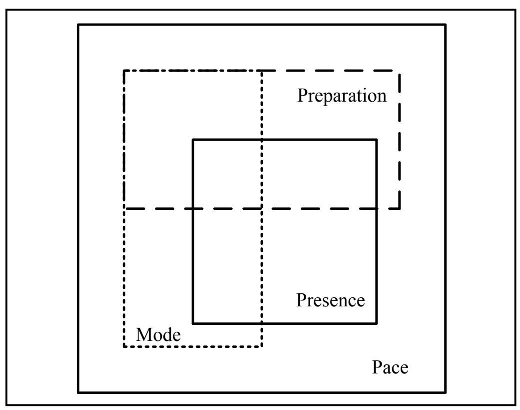

Figure 1: Model for Fostering Learning-based Conversations in Mathematics

Discussion: Integrating the Features into a Model

The Model for Fostering Learning-based Conversations in Mathematics, in Figure 1, represents a space highlighting the complexity of students' opportunities to be in conversation about their learning. Pace, diagrammed as the exterior square, situates the other three features of preparation, presence and mode, in a particular moment. Each of the conversational moments within the study could be placed within the space created by the rectangles. The placing of the conversational moments is what creates the space in which the students were talking about their learning and shaping their learning strategies through conversation. By explicating features which identify ways to foster learning-based conversations with students, I hope to invigorate mathematics education research to draw learning processes into view to empower students to improve and succeed in learning mathematics.

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LEARNING FROM FAILURE: A CASE STUDY OF REPEATING A MATHEMATICS COURSE FOR PRESERVICE ELEMENTARY TEACHERS

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The following paper is a description of a qualitative case study of a student as she attempted a course for the second time. This study took place at a four-year university in the first of a three-course sequence of mathematical content courses for preservice elementary teachers. Data was collected over the course of a single semester using a combination of interviews and short-response, reflection questionnaires and was analyzed using an open-coding process. A rich description of the student's experiences includes a discussion of how the course curriculum, instructor feedback, ability to seek help, and having "the right team" contributed to the student's negative and positive experiences in the course.

Keywords: Teacher Education-Preservice; Affect and Beliefs

Introduction

The National Council of Teachers of Mathematics, in their *Principles and Standard for School Mathematics*, argued that preservice teacher preparation programs have a significant impact on school mathematics through their instruction of future teachers (NCTM, 2000). Echoing that sentiment, Campbell et al. (2014) claimed that elementary teacher preparation programs have "a core responsibility for enhancing the content and pedagogical knowledge of prospective and practicing teachers as well as influencing their beliefs regarding mathematics teaching and learning" (p. 455). As a result, Clark et al. (2014) called for increased research to better understand potential influences on teachers' beliefs and how experiences impact mathematical views.

The research study presented in this paper took place a four-year university in the Rocky Mountain region. At this particular university, all elementary education majors are required to complete a three-course sequence of mathematical content courses. This research study focused on the first course of that three-course sequence. To continue to the next course in the sequence, students must achieve a grade of 70% or higher in their current course. Some students (henceforth called *repeaters*) fall short of this expectation and find it necessary to repeat at least one of the courses. In order to explore ways to better serve these repeaters, the purpose and goal of this research study was to explore the experiences of current repeaters. To achieve this goal, I sought to answer the following research questions: from the student's perspective (1) what circumstances contributed to the student needing to retake the course? and (2) what circumstances contributed to the student being successful in their current attempt at the course?

Methodology

The purpose of this case study was to develop a holistic view of the experiences and circumstances surrounding students as they attempted to repeat the entry-level mathematics content course for preservice elementary teachers.

The Setting and Course

The research study presented in this paper took place during the 2014 fall semester at a four-year university in the Rocky Mountain region. The participants were enrolled in an entry-level mathematics content course for preservice elementary teachers and all were repeating for the first time. According to the course description, the purpose of this course was to explore the real number system and basic arithmetic operations. Specifically, the course focused on mathematical structures,

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

patterns, and properties through a format of problem solving and exploration. All classes utilized a combination of direct instruction, whole-class discussion, and small-group exploration.

Participants

I administered a demographic questionnaire to approximately 230 students enrolled in multiple sections of the course. From those surveys, I invited six repeaters to participate in the research study, three of which agreed to participate. The case study presented in this paper will focus on one of the student's experiences as she attempted to complete the course for a second time.

Data Collection and Analysis

I conducted semi-structured interviews and collected short-response reflection surveys over the course of the semester during which the students were repeating the course. During the interviews, I asked the students questions about their experiences in their previous attempt to pass the course as well as their experiences in their current attempt. To gather information for the interviews, an online, short-response self-reflection questionnaire was administered. Each questionnaire included questions about the student's current experiences in the course. Administration of these questionnaires occurred the week prior to the second and third interviews. I used an open coding process to separate the interview data into key parts: (1) description of the case and (2) emergent themes.

Results and Discussion

In the following section, I will present a detailed description of the experiences of a single student, Quinn, as she attempted to repeat the course. The choice to focus on this student was a result of the rich, illustrative nature of her case record.

About Quinn

During the semester of this research study, Quinn was a sophomore majoring in Special Education with an emphasis in English as a Second Language (ESL). She moved to the United States approximately 8 years ago and attended an American junior and senior high school prior to attending the university. While English was not her first language, she expressed enjoyment from studying the subject. She also admitted that mathematics was her weakest subject. She believed that she learned best when she had a clear procedure in front of her, that included clear pictures and diagrams, emphasizing that "the steps [are] really important to [her]." When asked about questions that do not have a clear procedure, she argued that there is always a procedure to solve the problem. However, she blamed her language barrier for her difficulties in solving such problems. She appeared to have a very procedural view of what it means to study mathematics.

At the time of Quinn's first attempt, the fall semester of her freshman year, the expectation to pass the course was that all students needed to achieve a 75% or higher. Quinn achieved a low 70% which indicates that she almost passed the course the first time. In a study focusing on a course similar to the course in this study, Harkness, D'Ambrosio, and Morrone (2007) found that students, who struggled with the content of the course, "struggled to make sense of the mathematics, to value group work as an opportunity to socially construct knowledge, to understand the role the teacher played in shaping the learning through her enactment of teaching, and to change their self-concept and self-efficacy with mathematics" (2007, p. 251). This did not appear to be the case for Quinn. She claimed that it was not the mathematics content that caused her problems, but rather the process of explaining of her reasoning. She also valued group work and had a strong belief that her instructor and group members should help her learn the material. However, she found that the situation in her first attempt at the course, was not a supportive learning environment and, therefore, not conducive to this type of learning.

Course Curriculum

A key portion of the course curriculum is an expectation that students explain their reasoning when solving mathematics problems. Additionally, students are expected to analyze examples of incorrect student thinking in order to practice dealing with student misconceptions. As a result of her previous mathematical experiences, Quinn argued that she was at a disadvantage since she had never been asked to explain and justify her reasoning. In addition, her language barrier resulted in a greater challenge to achieve the expectations of the course.

After initially struggling, she eventually changed her perspective on the importance of explaining her reasoning as well as her role as a future elementary teacher. She argued that there should be an emphasis on explanations in elementary mathematics curriculums. She was motivated to emphasize this practice when she becomes a teacher. Research has shown that the perceptions of mathematical learning held by teachers have strong influences on their students' views of learning mathematics as well as their academic achievement (Campbell et al., 2014; Hadfield & McNeil, 1994; Bekdemir, 2010; Clark et al., 2014). Campbell et al (2014) found that, when teachers believed that mathematics learning consisted of focusing on procedural skills and the teacher had low mathematical abilities, student mathematical achievement decreased. Therefore, while it is difficult for students to achieve, it is important the course curriculum continues to place high importance on student explanations of their reasoning.

Instructor Feedback

During her first attempt, Quinn claimed that she received very little feedback beyond simply right or wrong, on her work. However, the feedback she received during her current attempt helped in several ways: (1) it helped her understand *what* she did right and wrong, (2) it helped her better understand what she needed to work on, and (3) it helped her build her confidence with the material she did understand. As a result, Quinn believed that she was better able to learn from her work. This confirms the findings of Harks, Rakocsy, Hattie, Besser, and Klieme (2014) who found that students perceived process-oriented feedback positively contributed to student understanding and motivation.

While receiving helpful feedback appeared to have a positive impact on Quinn's success, she seemed to make a connection between receiving feedback on a problem with the importance of completing that problem. She argued that if a problem was not going to be graded, then it was unfair to ask the student to complete the problem. There appears to be a disconnect between Quinn's belief that she obtained most of her understanding from completing the assignments and a lack of importance of a problem if no feedback was to be received.

Ability to Seek Help

During both attempts at the course, Quinn sought help from multiple places with mixed results. She admitted that she should have done a better job of seeking help the first time she took the course. However, she described a feeling of fear about having to ask for that help. This perception of asking for help is common among students (Butler, 1998). Butler suggested that students avoid asking for help because "they perceive help seeking as evidence of incompetence and thus as threatening to their perceptions of ability" (p. 630). When she did ask for help, she did not find the type of help she needed to be successful. For example, Quinn sought help by going to the on-campus, mathematics tutoring center. She encountered a tutor who, according to Quinn, treated her poorly and made her feel like helping her was not worth their time. She identified this as a key moment in her failed attempt at the course. Quinn expressed fear when asked if she would ever return to the center. She claimed that the experience rattled her confidence so significantly that she would never return. This suggests that a single event outside of the classroom may have a significant impact on a student's ability to seek help as well as on their mathematical confidence.

Having "the Right Team"

According to Quinn, her cooperative classroom structure consisted of "the right team." Her first "team" consisted of members that she perceived as not supportive of her learning. She expressed feelings of discomfort in asking them for help as well as decreased motivation and confidence in completing her mathematics as a result of interacting with her "team." This was in significant contrast to her second "team." During her second attempt she felt she had a supportive relationship with her instructor, small group members, as well as outside study partners. Each of these "team" members played a key role in contributing to Quinn's success.

Conclusion

Swars (2005) claimed that the previous mathematical experiences of preservice elementary teachers play an important role in shaping their current views of mathematics teaching and learning. As such, Swars argued that teacher preparation programs must provide preservice elementary teachers opportunities to reflect on and become more self-aware of negative mathematical experiences. As demonstrated by Quinn's story, it is clear that a course as well as outside experiences can have significant impacts on preservice teachers' attitudes and motivation towards mathematics. This is an important consideration for preservice elementary programs as they design and implement their programs.

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WIDENING THE VISION OF MATHEMATICS: CHALLENGES, NEGOTIATIONS, AND POSSIBILITIES

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Teachers' views about mathematics greatly impact the way they teach. Research indicates that many teachers project mathematics as a linear subject, and view mathematics teaching as about teaching a set of formulated procedures concerned with developing skills. Teachers' traditionally-held views of mathematics may constrain their ability to perceive and present mathematics as a human activity. Limited modes for curricular resources, professional development, or models exist to help support teachers to attend to the social, cultural, and political dimensions of mathematics education. In this paper, we draw upon the theoretical fields of ethnomathematics and its subset culturally responsive mathematics education to describe curricular efforts that were aimed to promote meaningful connections between mathematics content, context, culture, and the society.

Keywords: Equity and Diversity; Teacher Education-Preservice; Curriculum; Classroom Discourse

Introduction

In a traditional school setting, mathematics is mostly perceived and presented as an elite body of knowledge stripped of its rich social, cultural, and historical connections. We oppose this view and argue for countering the narrow vision of mathematics that confines it to the school walls. When thinking about mathematics, seldom does one consider culture, context, diversity, history, or politics. Many individuals, including educators believe that there is no place for such constructs in mathematics classrooms. Thus, it is necessary to institute a change at the teacher education level to challenge teachers' traditionally-held perceptions of mathematics and its pedagogy. We, the authors, are involved in a broader research project aimed to identify more meaningful ways to promote connections between ethnomathematics theory and practice as it applies to the preservice teacher education (PST) context. In this paper, we present part of the findings - in particular, we describe curricular efforts specific to two mathematics teacher education courses that were helpful in addressing the broader research goal.

Theoretical Underpinnings

The research field of ethnomathematics (D'Ambrosio, 1985) contributes to the theoretical base. The foundation of ethnomathematics rests in its "openness to acknowledging as mathematical knowledge and mathematical practices elements of people's lives outside the academy" (Mukhopadhyay, Powell and Frankenstien, 2009, p. 75). An ethnomathematics curriculum with a critical perspective will help generate a meaningful dialogue centered on a culturally responsive mathematics education (CRME) (Gay, 2000), which is aimed to empower and transform learners. The key tenets include: helping students connect academic mathematics to other forms of mathematics, connecting school mathematics to the socio-cultural-ethnical aspects of home culture, enabling teachers to practice equitable practices that cater to all learners, and allowing both students and teachers to acknowledge and celebrate their own and each other's cultural background (Gay). A critical ethnomathematics curriculum draws on the principles of a CRME and addresses the challenges that it poses to a traditional mathematics curriculum. It challenges the "Eurocentric narrative", confronts "what counts as knowledge in school mathematics", and attends to the disconnect between "mathematics education and social and political change" (Mukhopadhyay et al., 2009, p. 72).

Methodology

We chose narrative inquiry as a methodology to frame and present our research findings. It is our work as teacher educators and researchers that we find interesting and this is the story we are compelled to tell. From this perspective, our work shifts from a mere analysis of concrete data to an introspection and deep reflection of the complexities of our practices. In our narratives, we consciously choose and present episodes from our professional and practical knowledge base that will best help us attend to the central theme of this paper. The episodes are drawn from two distinct professional experiences aimed to challenge and broaden PSTs views of mathematics and teaching through a) Content-focused coursework offered in an on-campus setting and b) an immersive early field experience facilitated away from campus.

Content-Focused Coursework

The contextual setting for university A is a mathematics content course *Patterns and Structures through Inquiry (PSI*) for PSTs. This is a three credit-hour course, which a PST completes near the end of their program of study. PSTs pursuing licensure to teach middle school mathematics typically enrolled in this capstone course in their fourth year at the university. The PSI course is designed to foster critical thinking, solving complex problems, engaging with other learners, and communicating mathematical ideas. The course was designed in line with the principles highlighted in Presmeg's (1998) ethnomathematics course "mathematics of cultural practices". Course content was delivered using practical and research components. As part of the practical component, each week, PSTs participated in at least one content exploration activity set in a social/cultural/historical/political context. The research component required participants to identify and investigate a personally meaningful practice, which emphasizes that mathematics is a human activity.

Practical Component (Content Explorations)

For content analysis, participants (re) examined mathematical topics (while accessible to school students and teachers) that typically lie outside the focus of the traditional K- 8 mathematics curriculum. An overview of the content exploration tasks is available at http://bit.ly/17H2DiA. As part of content explorations on geometry, we studied the Kolam geometry of the women of South India (Siromoney, 1978). Kolams are interesting and intricate art forms created in the threshold of their homes by women in South India. PSTs engaged in the art of Kolam drawing and investigated the potential it held for teaching and learning mathematics. Some of the student-identified teaching ideas include the following: a) Geometry– Polygons & Transformations (e.g., Identify and name polygons in a given Kolam, Identify the symmetries and transformation in a given Kolam) b) Algebra – patterns & sequences (e.g., Determine the patterns & sequences in a series of Kolams, c) Combinatorics (e.g., Decorate Kolams using a set of colors subject to specific constraints), and d) Discrete Mathematics (Investigating graphs, paths, circuits in a given Kolam). Prospective teachers enjoyed learning about (and doing) the mathematics inherent in the Kolams and socio-cultural activities of other cultural groups (e.g., Islamic Star patterns, Sona drawings). In light of their exposure to such mathematical investigations, PSTs were more willing to look for and acknowledge forms of mathematics that exist outside the realms of the academia. Evidence to this fact was exemplified in the enactment of the research component of the course.

Research Component (Course Project Investigations)

All participants completed a course project that required them to identify and investigate a personally meaningful practice, which highlights the fact that mathematics is a human activity. Each participant developed a mathematical activity and engaged in both content and pedagogical explorations. Here is a quick overview of one of the projects: PST Heather's late grandmother taught her the art of cross-stitching. Heather developed this activity as a tribute to her grandmother. She incorporated technology to (e.g., geogebra) to replicate some of the patterns and discussed underlying mathematical ideas. She used this cultural art form to enhance students' understanding of graphing on a Cartesian plane. This activity enabled students to create colorful and intricate patterns on a Cartesian grid and uncover the connections between art and mathematics.

Early Field Experience in an International Setting

Currently, teaching in the United States remains a homogeneous profession; the majority of teachers and PSTs are of European American descent; PSTs tend to be cross-culturally inexperienced, live within 100 miles of where they were born, and desire to teach in schools similar to those they attended with fewer than 10% hoping to teach in either an urban or multi-cultural setting (Cushner and Mahon, 2009). Heyl and McCarthy (2003) suggest that experiences in international education have the potential to be the most influential factor in developing cultural awareness in PST education. In order to gain a more global perspective and culturally responsive teaching of mathematics, an early field experience was developed at University B to address the PSTs inexperience relative to a CRME.

Setting and Background

At University B mathematics majors seeking teaching certification are able to fulfill their senior capstone requirement through a month long field experience in Arusha, Tanzania. (The fieldwork is completed in one month, while the coursework is completed over 8 months, 2 months prior to travel and 6 months post travel). Prior to departure, students attended six orientation meetings in which they read and discussed articles about Tanzanian culture, history, and education, and learned basic Kiswahili in preparation for the program. During their month long experience in Tanzania the PSTs taught mathematics in elementary and secondary English medium schools for 60-75 hours. The PSTs were responsible for developing the lessons, teaching the lessons, and designing assessments, often without textbooks, materials or resources. In the following discussion we provide an example of how this experience addressed some of the key tenets of a CRME.

Parallel Lines and Kangas

Over the course of the month long experience the PSTs had numerous opportunities to infuse African culture and mathematical context. However, at the onset of this experience the PSTs in an attempt to contextualize the mathematics they were teaching, provided contexts which were unfamiliar or irrelevant to the Tanzanian students. For example, one PST taught a lesson about parallel and perpendicular lines. She provided the context of roads "that run parallel to one another" and roads "that are perpendicular to each other". It was evident to the PST that the students were confused by her explanation, and she attempted to provide another context which did not provide clarification for her students. When she debriefed with her peers at the conclusion of the day, she came to realization that her attempt to contextualize the mathematics was a meaningless (and western) context. In order to make the concepts of parallel and perpendicular lines meaningful to her students, she decided that it would be necessary to provide a context that was familiar and relevant to her students. She accomplished this by situating parallel and perpendicular lines with kanga fabric. (Kangas are brightly colored rectangular fabrics often with geometric patterns that African women wear as dresses, skirts, and headpiece). Using a familiar context to the students, they were easily able to see the meaning she was trying to convey of parallel and perpendicular lines. The PST then realized the power of situating her instruction of rotations, translations, and reflections using kanga fabric, rather than the pen and paper demonstration she originally planned. In this way, the PST connected school mathematics to the socio-cultural-ethnical aspects of her students' home culture.

Discussion

In both settings (on campus and away from campus courses), PSTs were provided ample opportunities to learn about the social constructions of mathematics that highlighted the contributions of people who were not necessarily from the mainstream academia, mathematical ideas of people from all walks of life. In their weekly journal entries and debriefing sessions PSTs noted that such activities greatly "challenged their perceptions of mathematics", "enhanced mathematical understandings", "offered a glimpse into cultures and societies and the mathematical activities that live and thrive in such contexts".

During class discussions and field experiences our PSTs examined questions such as: What is mathematics? If and how informal mathematics is different from formal mathematics? How are mathematical ideas exemplified in the activities of just plain folks? From a dominant Eurocentric view, which forms of mathematics is acknowledged and valued? Why might this be the case? What can we do about this? Many of the course activities were geared towards helping them think deeper about plausible responses to such questions. We readily acknowledge that some of these responses are at best emergent, yet powerful; we hope that, in their quest for deeper answers to these questions, PSTs will further broaden their perceptions of mathematics and its teaching. Consequently, we hope that they will be able to engage in a "two-way dialogue in which [the different forms of] knowledge (community knowledge, school knowledge) and their associated values are brought into the open for scrutiny" (Civil, 2002, p. 146).

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NON-DOMINANT STUDENTS' AND THEIR PARENTS' MATHEMATICAL PRACTICES AT HOME

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In response to the global mobility of populations, there have been growing international interests in mathematics learning and teaching in linguistically and ethnically diverse classrooms. In order to better support mathematics learning at school for non-dominant students, I examined informal mathematical knowledge they accessed at home. Focusing on Filipino immigrant mothers and their children in Japanese schools, this paper highlights how these mothers' informal mathematics knowledge attached to everyday practices can influence their children's development of mathematical literacy at school.

Keywords: Equity and Diversity; Informal Education

Purpose of the Study

There have been growing international interests in mathematics learning and teaching in linguistically and ethnically diverse classrooms (Barwell, 2009). Supporting linguistically and culturally non-dominant students in the current globalized mathematics classrooms is a multi-layered process to address languages, ways of knowing, and cultural practices. One of the central research agendas is to challenge the prevailing deficit views towards non-dominant students including English language learners (Moschkovich, 2007). This paper examines everyday mathematical practices unique to transnational families, who live across and beyond national borders (Kitchen & Civil, 2011). Drawing from perspectives of parents and children, this paper considers whether and how participants' everyday mathematical practices are connected with children's development of mathematical literacy.

Internationally, the Organisation for Economic Co-operation and Development (OECD) promotes *mathematics literacy* as "reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain and predict phenomena" (OECD, 2013, p. 17). Providing access to mathematics literacy for all students is a pressing issue, especially in linguistically and ethnically diverse mathematics classrooms.

Theoretical Framework

Sociocultural theory maintains that learning and development are fundamentally social and cannot be reduced to the phenomenon closed within an individual (Vygotsky, 1978). One of the central tenets of sociocultural theory is to highlight capabilities and competencies that people exhibit in the cultural practices they engage in everyday lives (Saxe, 2012; Scribner & Cole, 1981). These competences are often neglected in the school contexts, especially for non-dominant students and families. One of the key concepts to highlight their resources is "funds of knowledge," which is defined as "historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being" (Moll, Amanti, Neff, & Gonzalez, 1992, p. 133). Drawing from the *funds of knowledge* perspective, Civil (2007) demonstrated ways in which Latino/a parents' and children's mathematical knowledge exhibited through their everyday practices (i.e., gardening) could facilitate academic mathematic learning. This paper builds on the sociocultural theory while carefully paying attention to the aspect of power dynamics, which is essential to interrogate the legitimacy of knowledge exchanged in school contexts (Nasir & Hand, 2006). Examining the aspect of power is also essential in rethinking the relationship between school knowledge and knowledge gained through non-dominant students' family practices.

Methodology

Based on Vygotsky's idea, Newman and Holzman maintain that method is simultaneously both a tool and a result and propose the *tool-and-result methodology*. Tool-and-result methodology conceptualizes the method as an activity or a search that simultaneously explores and generates tools and results that are "elements of a dialectical unity/totality/whole" (Holzman, 2009, p. 9). In this study, interview data with 12 Filipino mothers and 10 of their school-aged children was used to gauge the needs of Filipino mothers and their children (all the names used are pseudonyms). Workshops were organized subsequent to the interviews with mothers, as an attempt to address some of the issues raised in the interviews.

In the interview, parents undervalued their mathematical knowledge, compared to the mathematical knowledge taught at schools. Parents expressed their feeling of powerlessness, to support their children for their school learning, due to a language barrier and also a perceived hierarchy between Japanese mathematics curriculum and Filipino mathematics curriculum (please see Takeuchi, in press). In order to address this issue, one of the workshop themes was set out to collectively reflect on their everyday practices involving mathematics. The workshop was video recorded, content-logged, and transcribed in part. For this paper, I will introduce narrative descriptions generated based on video-recorded interactions and interviews.

In order to examine how children understood the concepts learned at home, in the children's interview, I included a modified word problem used for PISA 2009 assessment on the international currency, which asked to convert 35,000 Japanese yen to the Philippine peso (at the exchange rate: 1 YEN=0.52PHP).

Results

In this section, I will first introduce transnational families' mathematical practices focusing on calculating international currency conversion. I will then show how children applied the concepts learned at home, in order to solve a word problem.

Parents' Everyday Mathematical Practices: Calculating International Currency Conversion

During the workshops with parents, the participants and I explored some of the mathematical practices that they engaged in their everyday lives. One of the commonly reported practices was calculating international currency conversions. Five Filipino mothers attended the first workshop and all of them stated that they would constantly engage in computing international currency conversions. All the Filipino mothers I interviewed said that they were from a big family of lower SES and financially supporting their family was the main reason they initially came to Japan. Many of them were still sending money back to their family members in the Philippines.

When converting between the Japanese yen and the Philippine peso, participants had a strategy of doubling the Philippine pesos (to convert to yen) and halving yen (to convert to Filipino pesos), to get an approximate value. The following narrative description of the workshop shows this point.

> A researcher asked an average price of a picture book in the Philippines and participants answered "About 300 pesos. They are cheaper." A researcher asked, "Then, how much would that be in Japanese yen?" Evelyn and Michelle in unison said "double it." Iren said "600 yen." A researcher said, "Then, can we say 1 yen = 0.5 peso?" Multiple participants immediately responded, "That was before." Janice said "It's now 1 yen = 0.45 peso."

The subsequent conversation addressed the meaning of this fluctuation. Filipino mothers used common sense reasoning, which is strongly connected to their everyday experiences. The conversation described in the following narrative depicts some of the ways in which Filipino mothers answered the question, "what does this change in conversion rate mean to you?"

> Irene said, "For us, if the peso is 0.5, it's good for us and it's good for my family in the Philippines too." Everlyn said, "In my case, I send 100,000 yen (note: approximately $1,000 USD) but I need to add 10,000 yen to fill the loss. So, for my family in the Philippines, the price is the same."

As seen in this narrative description, participants presented the informal knowledge of doubling and halving to convert between the Philippine peso and Japanese yen. They also demonstrated an understanding of the concept of fluctuating currencies and the consequences of this fluctuation.

Children's Mathematical Literacy: International Currency Conversion

Half of the child participants reported that they learned about calculating international currency conversions and talking about time differences since when they were young (as young as Grade 2), because they are the practices specific to transnational families. For example, one parent, Grace, explained, "When we went abroad, my son asked me how to calculate the currency. Because he's very good at managing money, so he's interested." On the other hand, some parent participants said that they would not teach children these practices. A parent, Christine, said, "At home, I don't teach my kids anything about the Philippines."

When given a word problem that requires understanding of international currencies, two Grade 6 participants accurately solved the word problem. Four other participants reported that they knew the doubling and halving strategy to covert Japanese yen to the Philippine peso but could not complete the word problem. The majority of students (five out of six students) who demonstrated partial or complete problem solving were those who learned about international currencies at home, from their mothers.

Grade 6 students who solved the problem performed beyond the grade level expectation, as the original PISA problem targeted for Grade 9 students. The following excerpt describes how a Grade 6 student, Mary, engaged in this problem.

> Looking at the word problem, Mary was mumbling, "I know about the exchange rate but I don't know how to solve this." She looked at the problem for a while and suddenly said, "I got it." She then wrote down vertically "35000 yen = ? 1 yen=0.52." She stared at what she wrote for a while and said, "Got it." She wrote "35000 x 0.52=" She then identified the answer, 182,000 pesos.

This example implies that the informal knowledge of doubling and halving for converting two currencies could be used to solve a word problem used at school. In contrast, those who reported not having learned about international currencies at home tended not to be able to interpret the meaning of the word problem, even when they were able to decode its texts.

Discussion and Conclusion

This paper highlighted Filipino mothers' mathematical understanding, attached to their everyday practices. By drawing from children's interviews, I explored the possibility of connecting non-dominant students' mathematical practices at home with their development of mathematical literacy in school. This finding is particularly significant to the discussion on how to design educational practices which can meaningfully bridge out-of-school resources to in-school learning by considering non-dominant students' and their parents' positionality (Gutiérrez, 2013). These Filipino mothers raising children in Japan perceived their mathematical resources to be less valuable, compared to the school knowledge. This positionality was internalized to their children, who also perceived their mothers' mathematical knowledge to be less valuable.

As previously discussed, countering the deficit views toward non-dominant students will be an essential challenge to create more equitable learning opportunities in mathematics education

(Gutiérrez, 2007; Moschkovich, 2007). This study highlights the mathematical understandings of non-dominant parents and students, focusing on everyday mathematical practices that are familiar to them. As indicated through children's interviews, there is a potential that informal mathematical knowledge can facilitate mathematical literacy, for instance, in PISA assessments. To further draw from non-dominant parents' and children's expertise identified in this paper, designing a school-based lesson to bridge in and out-of-school practices will be necessary as global mobility increases worldwide.

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EXPLORING THE CULTURE OF SCHOOL MATHEMATICS THROUGH STUDENTS' IMAGES OF MATHEMATICS

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In this paper, we discuss students' images of mathematics constructed through their experiences of school mathematics learning. We draw from data collected within our project, which investigates students' experiences learning mathematics in Canadian schools and the ways that these experiences contribute to students' images of mathematics and their mathematical identities. In doing so, we point to the ways in which schools engender a particular image of mathematics for students. We conclude by positioning the students' images of mathematics with respect to the mathematics curriculum and teaching practices in schools.

Keywords: Affect and Beliefs; Curriculum; Elementary School Education; Middle School Education

Purpose of the Study

In this paper, we explore images of mathematics held by Kindergarten to Grade 9 students in Canada. We examine data collected in one Canadian province (Alberta) and relate our findings to different groups' images of mathematics, as documented in the literature, and to the local curriculum guiding teaching practice in the participants' jurisdiction. From this analysis, we examine ways in which particular images of mathematics are developed through students' experiences in schools.

Theoretical Framework

The theoretical frame underlying this research is enactivism—a theory of embodied cognition that emphasizes the interrelationship of cognition and emotion in learning and that troubles the positioning of self and identity as purely individual phenomena (Kieren & Sookochoff, 1999; Maturana & Varela, 1992). Learning in this frame is seen as reciprocal activity—the teacher brings forth a world of significance with the learners within a cultural milieu (Maturana & Varela, 1992). Students' mathematical images and understandings are therefore not determined solely by the teacher or by the learner; rather, they are dependent on the kind of teaching experienced and the (mathematical) culture within which students are immersed. Educators have focused on better and better ways to transmit content, all the while forgetting that the content is not primarily what the students have been learning. Instead, we suggest that they have been learning a culture of mathematics—a particular vision of, and way to be with, mathematics. Our frame, then, prompts us to seek to understand how students come to have particular relationships with mathematics and what mathematics means to them.

Review of the Literature: Images of Mathematics

Images of Mathematics in the Media

Western media tend to portray mathematics as a difficult subject only understood by a select few who are often portrayed in negative, stereotypical ways, such as being socially inept, unattractive, and mentally ill related to their obsession with mathematics (Mendick, Epstein, & Moreau, 2007). Mathematics is usually portrayed narrowly, typically as numbers and arithmetic (e.g., basic operations, counting) or as financial mathematics (e.g., money, consumerism) (Hall, 2013). Other researchers' explorations of popular culture representations of mathematics and mathematicians have found mathematics portrayed as "a secret language, possibly a code, which is difficult to 'crack'" (Epstein, Mendick, & Moreau, 2010, p. 49) and only occasionally as beautiful or utilitarian (e.g., used to solve crimes).

Mathematicians' Images of Mathematics

Many mathematicians have discussed their views of mathematics. For instance, Furinghetti (1993) argues that people's views of mathematics are almost wholly based on their in-school experiences, and that in-school experiences must therefore portray mathematics as a set of human activities that are highly related to "real world" experiences. The subtitle of Lockhart's (2009) book, *A Mathematician's Lament,* declares mathematics to be "our most fascinating and imaginative art-form." Similarly, Taylor (2014) notes that mathematics is that which is "eternal and has beauty and structure to die for" (p. 34).

Elementary and Secondary Students' Images of Mathematics

Research regarding elementary and secondary school students' views tends to focus on students' feelings about mathematics, as opposed to their views on what mathematics is. Perkkilä and Aarnos (2009) addressed the latter topic by exploring the views of Finnish children, aged six to eight, by asking them to draw themselves in the "land of mathematics." The findings indicate that students view mathematics as a solitary pursuit involving only certain types of mathematics. Young-Loveridge, Taylor, Sharma, and Hawera (2006) found that students tended to view mathematics as being related to number and/or operations, as opposed to other topics.

Methodology and Research Design

The data on which we focus for this paper were gathered in the province of Alberta, which is located in Western Canada. The study's participants are Kindergarten to Grade 12 students, post-secondary students, and members of the general public, but we focus here on data collected in the first phase of the study, which includes students from Kindergarten to Grade 9. Forms of data include semi-structured interviews, drawings (that represent participants' ideas about what mathematics is and their feelings when doing mathematics), and written and oral mathematics autobiographies (accounts of participants' histories of learning mathematics).

To date, 94 interviews with Kindergarten to Grade 9 students (41 girls and 53 boys) have been conducted. In order to reveal students' images of mathematics, in this paper we focus primarily on data related to one of the interview questions: "When you hear the word mathematics, what images come to your mind?" These data are considered with supplementary data that further explain and explore participants' responses. Elsewhere (e.g., Hall, Towers, & Martin, 2015), we explore other aspects of the data.

All of the interviews were transcribed verbatim. To answer the aforementioned question, the data were analyzed through emergent coding, primarily using in vivo coding to stay as close as possible to participants' own words. Then, codes with more than one response were entered into a Wordle, a visual representation of response frequencies (i.e., a larger font size indicates a more frequent response). To explain and contextualize this interpretation of the data, we supplemented our analysis by drawing on additional information from the student interviews.

Findings

As shown in the Wordle in Figure 1, the participants' images of mathematics were narrowly focused on number sense and numeration.

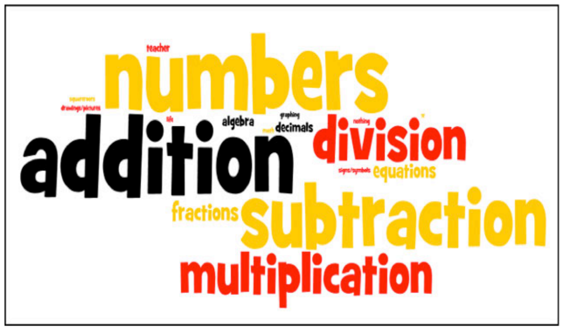

Figure 1: Wordle Representation of the Students' Responses

Responses aligning with the top five categories of response—numbers and the four basic operations—were ten- to twenty-fold more common than most of the other responses. Even the next most common responses, such as "fractions" and "equations," still align with this mathematical topic area. This pattern was observed from Kindergarten through to Grade 9 students. For example, Kindergarten students described mathematics as: "Numbers, five and six and stuff," or "How many numbers makes more numbers." Primary grade elementary students described mathematics as: "Adding and subtracting," and "Counting." Similarly, upper-grade elementary students depicted mathematics as: "Like, numbers and letters and, like, the adding and subtracting signs and, like, the multiplying and dividing signs and all that stuff," and "Numbers and additions, subtraction, multiplication, division." This tendency to associate mathematics with numbers and the basic operations was common across all grade levels.

To investigate this pattern of restricted images of mathematics further, we considered participants' responses to other questions about their earliest mathematics experiences. These recollections also tended to focus on learning about number operations, and in school settings. The majority of students described their earliest memory of learning mathematics in number contexts, such as: "Two plus two. It was one of the first things I was asked in math" and "I remember doing the math worksheets in I think it was Grade 1 about the simple addition." A Grade 8 student recollected his earliest mathematics learning as "we were doing adding like one plus one equals two and stuff." Other students commonly described reciting a multiplication table by repeatedly practicing the table or singing a song. Overall, for the majority of students, their first encounter with mathematics learning was described in relation to basic operations.

Discussion

We are concerned that students are developing, through their engagement in the culture of school mathematics, a very narrow vision of the nature of mathematics, one that does not include the full territory of mathematical ideas and topics that are described by mathematicians (and others, such as mathematics educators). While we acknowledge the ongoing scholarly debate about the relationship between school mathematics and the mathematics practiced by mathematicians (e.g., Watson, 2008) and agree that the aim of school mathematics teaching ought not to be entirely commensurate with generating the kind of images of mathematics that mathematicians hold, we do think that the artificially narrow view of mathematics being generated in schools is problematic. Recent research (Rapke, 2012) has shown the synergies that exist among creating and learning, mathematics, suggesting that the nature of school mathematics can (and should) be more like that of research mathematics, including creative exploration in the full scope of mathematical topics.

In the Alberta Program of Studies (curriculum) document for mathematics, from Kindergarten to Grade 5, between 40% and 50% of the expectations align with the Number strand (Alberta Education, 2014). We believe that teachers are guided strongly by the mandated curriculum documents for their jurisdiction and consequently (over)emphasize number concepts in their teaching of mathematics, particularly in the early years. This (over)emphasis on number and operations may be contributing to students' skewed images of mathematics. The mathematics education community has proposed alternative views of mathematics compared to the widespread notion of mathematics as number and computation. For example, Boaler (2008) suggests that, "mathematics is all about illuminating relationships such as those found in shapes and in nature" (p. 18). Such enhanced conceptualizations of mathematics have significant implications for mathematics pedagogy, implying consideration of a wider breadth of mathematical topics as being "basic" to mathematics education. The findings of our study demonstrate the significance of attending to the nature of students' images of mathematics in relation to the culture of school mathematics in Canada.

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MATHEMATICS LEARNING AMONG UNDERGRADUATES ON THE AUTISM SPECTRUM

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This study examines the mathematical learning of an undergraduate student on the autism spectrum. I aim to expand on previous research, which often focuses on younger students in the K-12 school system. I have conducted a series of interviews with one student, recording hour-long sessions each week. The interviews involved a combination of asking for the interviewee's views on learning mathematics, self-reports of experiences (both directly related to courses and not), and some particular mathematical tasks. I present some preliminary findings from these interviews and ideas for further research.

Keywords: Equity and Diversity; Geometry, Post-Secondary Education

Background on Autism-Related Research

The Autistic Self Advocacy Network (2014) states that autism is a neurological difference with certain characteristics (which are not necessarily present in any given individual on the autism spectrum), among them differences in sensory sensitivity and experience, different ways of learning, particular focused interests (often referred to as 'special interests'), atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. Over the past few decades, there have been many research studies about learning in students on the autism spectrum, such as those reviewed by Chiang and Lin (2007). A large portion of these studies focus on K-12 students, and particularly elementary students, but some of the ideas and procedures in those studies lend themselves to use in a post-secondary context.

While most mathematics-related research on people on the autism spectrum also takes place among younger children, there have been multiple reports, such as those from James (2010) and Iuculano et al. (2014) which indicate an association between autism and heightened mathematical interest and ability. Due to the young age of the students, however, it remains to be seen whether those tendencies extend to university-level mathematics content.

Interview Procedures

I started out with the intent to find students who were on the autism spectrum and currently taking one or more mathematics courses at SFU. I found help from a center for students with disabilities to recruit for students who would be qualified for the study, and was put in touch with one student who was willing and able to do the interviews. It should be noted that this method will only identify those students who are both able to seek assistance from such a center and see the need to do so, and this constitutes only a portion of the fairly wide autism spectrum.

The student I interviewed, Joshua (a pseudonym) was studying integral calculus and linear algebra. I conducted interviews every week for this term. These were scheduled for one hour, but were sometimes continued for a short time past the scheduled hour. I typically started by asking the student to share any particular thoughts on the week's course materials. I also asked various questions and assigned tasks related to the covered course material. Some of these were tasks that have been used with typical student populations in the literature, such as the example-generation tasks used by Bogomolny (2006) and the Magic Carpet Ride sequence used by Wawro et al. (2012). I have also given other mathematical tasks not directly related to the material covered in the courses being taken, such as the paradoxes examined by Mamolo and Zazkis (2008); one reason for this was the interplay between visual and algebraic explanations seen in some student responses to these paradoxes.

Theoretical Framework

There were several reported characteristics of people on the autism spectrum which I thought could be promising for mathematics education research. In particular, I was interested in details of prototype formation, special interests, and geometric approaches. I will detail each of these with a comparison to the particular findings relevant to them in Joshua's case.

Prototype Formation

I started looking into prototype formation after reading a study by Klinger and Dawson (2001). It suggested that people on the autism spectrum did not form prototypes of objects when given tasks asking about group membership, instead taking an approach based on lists of rules. Although this is presented as a problem, like many other autism-related studies, I suspected that this approach could be helpful for more abstract or proof-based mathematics. I have found many other students having trouble with mathematical questions that appear to result from a prototype-based approach, and this is particularly true when the course focuses on mathematical proof. In fact, I found a very similar division reported in mathematics education research by Edwards and Ward (2004), phrased as lexical or extracted definitions versus stipulative definitions. This did not appear to be the case for Joshua; he reported having this kind of thinking in the past, but was quite focused on “big picture” ideas today (this was, in fact, a recurring phrase in the interviews).

Special Interests and Learning

I found the idea of 'special interests' (variously known as circumscribed interests, splinter skills, savant skills, and a variety of other names with varying connotations within the autistic and research communities) to be applicable to some of my findings. These are intense, focused interests occurring in people on the autism spectrum (often studied in children, like much autism-related research). Some of these are mathematically related, such as a focus on particular facets of arithmetic, prime numbers, or aspects of geometric shapes, as explored by Klin et al. (2007). The development of these interests in later life is something that does not appear to have been studied much in previous research, however. Unfortunately, research in this area (particularly from an Applied Behavioral Analysis framework) can discount these skills or even view them as detrimental to learning, as seen in Dawson, Mottron, and Gernsbacher (2008), and may even attempt to eliminate these skills. From what I have seen, however, such interests can be very helpful in imparting motivation when viewed from the right perspective. Joshua has reported a strong interest in chemistry, which he often used in analogies for mathematical concepts in our interviews. There have been several instances where he has reported more enthusiasm, better understanding, and better performance when able to see chemical applications to the topics in the courses, and has sought out additional information outside the course materials in order to make these connections.

Geometric Focus and Visualization

Particularly due to the work of Temple Grandin, one of the most famous people on the autism spectrum, there is often an association between the spectrum and visualization or spatial reasoning (Grandin, Peterson, and Shaw, 1998). While I would caution against being too broad with an association like that, I did find a strong preference for visual, spatial, or geometric reasoning in the interviews I conducted. This was particularly successful with integral calculus, where the student independently thought about what three-dimensional integrals might be like. The correct conclusion was reported for a 'flat' extension (of multiplying by a constant length), and the student did realize that this would not work for more complex three-dimensional shapes (although not to the point of developing multiple integrals). Reports of classroom progress continually reflected higher performance in areas that could be viewed in a geometric or otherwise physical way. Comparing topics across the two courses, this leads to some surprising results, such as reported satisfaction with washer and shell rotations, but issues with algebraic formulas like dot products.

I found the solution Joshua gave in one interview for the first Magic Carpet Ride task to be particularly notable. I showed the problem setup from the paper by Wawro et al. (2012), giving the two modes of travel with vectors (3,1) and (1,2), and asking for a way to get to the house at (107,64). Since I asked this relatively late, Joshua had already seen the vector material that shows the 'standard' way to do this. However, the solution he gave was instead done by drawing the vectors out on paper. He measured (30,10) and (10,20), plotted (107,64), shifted one of the vectors so that it would end at (107,64), and extended the vectors in order to find via measurement their point of intersection. (I checked this, and it was accurate enough to provide the correct solution.) When asked for an algebraic solution, he calculated the equations of the lines corresponding to those vectors and found their point of intersection, still clearly based on the same visualization.

Paradoxes

I have also presented several paradox tasks during my interviews. I gave the Hilbert Hotel and Ping-Pong ball paradoxes from the paper by Mamolo and Zazkis (2008), as well as the Painter's Paradox (involving the volume and area of Gabriel's Horn obtained by rotating 1/x) and an 'infinitesimal staircase' paradox. The Ping-Pong ball paradox involves an infinite set of Ping-Pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are inserted and the '2' ball is removed, and so on. The respondents are then asked how many Ping-Pong balls remain in the barrel at the end of the minute. The last one involves a staircase being divided up into finer and finer steps (at each stage having perimeter 2) to approach an incline (of length $\sqrt{2}$). Like many of the students in previous studies, the student I interviewed found these to be strange and paradoxical. However, they also appeared to inspire a great deal of enthusiasm; through having a series of interviews, I was able to see that these paradoxes had inspired independent thinking outside of the interview sessions. The response provided to the Ping-Pong ball paradox was something I also found particularly notable. Joshua first said that there should be infinitely many, then decided that there should be none after being asked what the numbers of the remaining balls were. However, the analogy he provided here was of slowing down molecules at extremely low temperatures in order to study them; it appeared as if this mental image was being used to accommodate the presence of the balls at times prior to the conclusion of the experiment. I also found it notable that I did not see any tendency toward rejecting the mathematical facts after they had been presented, unlike in many of the students in the prior studies.

Limitations and Suggestions

So far, I have only been able to conduct interviews with one research subject, and thus it is important to be careful not to overly generalize the results seen in the interviews. There are also considerations about the content that would apply to the courses that Joshua took, although I think that linear algebra is a fortunate course to have an interviewee from. Some questions that have occurred to me: Would people on the spectrum with less clearly related special interests still use them for mathematical analogies? Is there a tendency for either (more specifically) a bias toward geometric processing or (more generally) a tendency to strongly prefer one type of processing? Are people on the spectrum generally more inclined to accept conclusions that are viewed as paradoxical?

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THE MICRO-POLITICS OF STUDENTS' LANGUAGE REPERTOIRES IN COUNTING CONTEXTS

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Counting is mediated by language – the language used for counting reflects people's meaning and experience of the process, and their experience is impacted by the way language is used. We investigate children's language repertoires for counting in English. With this, we aim to understand better the political nature of counting at the most basic levels. This is to extend the literature, which already identifies political aspects of counting on the macro scale. We theorize the politics of language in mathematics learning as applied to this situation and, as a way of setting up our investigation, we illustrate how counting at the micro-level can be political.

Keywords: Classroom Discourse; Equity and Diversity; Number Concepts and Operations

Introduction

The understanding and experience of counting, like the understanding and experience of any mathematical idea, is mediated by language. Certain language repertoires are necessary to convey the ideas and perhaps even to perform counting action. At the same time, the language used to describe these ideas and enact the processes shapes the way people conceptualize them.

This recursive nature of language compelled us to develop a research project to investigate children's language repertoires in relation to conjecture. We began with contexts involving risk and prediction because we had noticed similarity in language repertoires between prediction and conjecture. We found political implications of the ambiguity in meaning of words that are used for conjecture, prediction, assessment of risk, and establishing authority (Wagner, Dicks, Kristmanson, 2015). Now we extend this work to explore students' language repertoires in contexts involving counting. Data collection will be complete in June 2015, and so we present here our theoretical and conceptual approaches along with some hypotheses that come from our experiences as mathematics teachers and mathematics education researchers. In the conference presentation, we will also present from our empirical results.

The Politics of (Mathematical) Language

Our interest in the experience of counting includes the way counting positions people in relation to each other. Positioning theory (Harré et al, 2009) points us to the distribution of rights and duties, or, in other words, the politics of counting, just as it did in our work that revealed the politics of prediction language (Wagner, Dicks, Kristmanson, 2015). Careful language analysis helps us produce warranted claims about the politics within our research contexts.

> [Politics] is about how to distribute social goods in a society: who gets what in terms of money, status, power and acceptance on a variety of different terms, all social goods. Since, when we use language, social goods and their distribution are always at stake, language is always "political" in a deep sense (Gee, 2011, p. 7).

In this project we understand mathematical education as a number of created and re-created practices within social and cultural contexts. These practices are networked with other practices outside the mathematics classrooms (Valero, 2007). Thus they are political, indicating that power is distributed between the different networking practices. In line with Valero (2004), we understand power as situational, relational, and in constant transformation. Power works between these practices in the network as macro-level processes. However, power also works at the micro-level in the immediate situational contexts between participants and (un)available materials. These micro-level actions are the focus in this project.

The relations between the macro-level practices and participants' micro-level actions are dialectical. Macro-level practices give meaning to micro-level actions, offering participants subject positions. However, the participants' actions also give meaning to the mathematics practices and position themselves in ways that are reflexive, relational and contextual in relation to the discipline and to other individuals in their learning contexts (Wagner & Herbel-Eisenmann, 2009). Thus, participants are implicated in the construction and circulation of power within mathematical practices (Gutiérrez, 2013). We emphasize the importance of power relations between the macro- and micro-levels, through positionings and discourses – in other words how we fluently relate to each other (Gutiérrez, 2013; Wagner & Herbel-Eisenmann, 2009). In this project, the negotiation and distribution of power on the micro-level are foregrounded, while we acknowledge and connect to the related macro-level power distributions, discourses and negotiations (Morgan, 2006). Discourses are about "negotiating and maintaining relationships among its participants" (Morgan, 2012, p. 181) and hence establishing relations and positionings. These relationships also imply a need for us researchers to critically reflect on ethical questions about how we position ourselves and write about the Other, as suggested by Andersson and Le Roux (2015).

We use a systemic functional linguistics framework (SFL) to help us identify the qualities of interpersonal interaction as they appear in the distinctions made through grammar and lexicon. SFL is built on the recognition that language involves the interconnectedness among construction of experience (*ideational* metafunction), relationships with others (*interpersonal* metafunction), and connection with other circulating text (*textual* metafunction) (Halliday, 1973).

Number and Power

At the macro-level, number is often associated with power. School curriculum is positioned as equipping children to be powerful in and outside of school, and number skills are generally taken as central to such numeracy. Bishop (1990) has gone so far as to show how advanced counting systems – exponential based number systems, in particular – made colonialism possible. With rudimentary counting it is hard to organize and hold control over vast resources. Historically, technologies of counting (quantification) are associated with certain political structures (Porter, 1995).

We have not found research that focuses on the micro-level politics of number. When children count, who "gets what in terms of […] status, power and acceptance on a variety of different terms" (Gee, 2011, p. 7)? To illustrate the micro-level interactions in counting contexts, we suggest that, before you read further, you choose some friends and decide amongst yourselves who has been in the most countries. You will have to count the countries you visited of course. After engaging in this activity, the next paragraphs will be more meaningful.

Counting countries seems quite straightforward at first, but it doesn't take long to find controversy. For example, both of us have visited Yugoslavia before it was divided into smaller countries. Shall we count one for Yugoslavia because it was one country when each of us was there? Or can we count three or four (different for the two authors) for the current countries represented by parts we visited? We may decide it counts as one, because it was only one country when each of us was there; a clearer question would be how many national political entities we have visited. However, Germany complicates such reasoning. We have both spent time in East Germany, West Germany and modern, unified Germany. Shall we count all three of these entities that we have visited? Furthermore, one of us travelled through some countries by train. Does it count to travel through a country on a train if one doesn't get off the train? What about flying over a country? Or landing in a country to refuel but staying on the plane? What about a one-hour stop that includes a passport stamp? Or crossing the border and being escorted out by police? And then there are disputed territories, like the West Bank (of the Jordan River), or First Nations (Aboriginal lands never conceded to colonialist governments). These are just some of our political controversies when counting countries. You probably have your own.

When we count, we have to decide what counts and what does not count. For example, what counts as being "in a country", and what counts as "a country"? This is political because different people will have unique reasons for wanting the counting to be done in certain ways. Furthermore, it is possible that you have not travelled outside your country, which highlights yet another political aspect. The question of how many countries privileges people for whom travel has been possible, and thus excludes others. Someone has to decide what to count and that decision sets up certain people's experiences as normative. For example, who decided that counting countries is a worthwhile endeavor (it was us, but you may have made the question your own as you started dialoguing about it with your friends)? Maybe it would be better to count our meaningful interactions with diverse people, or it may be even more appropriate to reflect on the qualities of those interactions instead of quantifying them. Indeed it is possible to travel to many countries and remain insular.

Methodology

As with our earlier work in this project, focusing on the language of prediction, we have students work in groups in class and subsequently interview them to extend the group work. We work with students with relatively limited repertoires and present them with situations that we expect to push them to the limits of their language resources. Thus we begin this work with 4- to 7-year olds, and engage them in counting in increasingly challenging ways. For these tasks, we draw on the tasks suggested by Wagner and Davis (2010), in their article distinguishing between quantity and number sense. We consider the micro-politics of quantification. We consider how participants construct and negotiate roles and responsibilities as they decide what counts (For example, how big does a tree have to be to be a tree? What kind of plant counts as a tree?), how to talk about a quantity when the numbers exceed individuals' quantity sense, and what benchmarks they use for communicating their sense of quantity. Selecting benchmarks requires identifying something that has a taken-as-shared meaning by one's interlocutors. We identify how these micro-political moves are manifested in language. At the end of the complementary interviews with the children we draw attention to things the participants have said and ask them to describe what they mean by these things.

In the interactions with participants, we avoid using specialized mathematical language ourselves and refrain from suggesting to participant students how they might perform the tasks and/or communicate their ideas. We want to identify their strategies for communicating their counting, and consider what these language strategies say about the process of counting and the related politics. Thus, we avoid a deficit perspective that would rate the students on the basis of which skills and language they know. We have already noticed the problems with such deficit approaches – for example, we have evidence that a participant not using a language skill does not indicate inability (Wagner, Dicks and Kristmanson, 2015).

Language is used to make distinctions that are relevant to the people in an interaction. For example, the prevalence of gender distinctions in personal pronouns in many languages signifies that people in those cultures consider it important to make such distinctions. An individual may find a way to avoid making such a distinction and find this a challenge because our language does not have some gender-inclusive personal pronouns – e.g. using 'they' instead of 'he' or 'she'. Such practices may become acceptable to others and enter into a culture's language repertoire. Just as we invent ways to avoid a distinction, we can invent ways to make distinction when no language strategy is established for that distinction. This is a phenomenon at work in mathematical problem solving contexts (Wagner, 2009), and also in scholarship. Thus, in our research, we look for language strategies that enable the process of deciding what to count and what not to count (the process of establishing boundaries or categories). And we consider how these processes are political acts.

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ESTABLISHING MATHEMATICAL CARING RELATIONSHIPS WITH UNDERREPRESENTED STUDENTS IN A COLLEGIATE STUDENT SUPPORT SERVICES PROGRAM

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Keywords: Post-Secondary Education; Equity and Diversity; Affect and Beliefs; Teacher Beliefs

While access to higher education in the United States continues to improve, the success of historically underserved students in college, as measured by persistent degree completion, has not progressed in recent decades (Brock, 2010). Moreover, these students remain underrepresented in fields that depend on mathematics (Treisman, 1992; Engle & Tinto, 2008). As such, many universities offer federally funded Student Support Services (SSS) programs to address student success for historically underserved students including minority students, first-generation students, and low-income students. At the University of Maryland, the Academic Achievement Program (AAP) provides administrative and academic support including mathematics instruction and tutoring for historically underserved students. While quantitative research on these programs is common, our study uses qualitative methods including ethnographic interviews and in-depth classroom observation to explore affective responses of both students and teachers during mathematical learning in a SSS program.

Interview participants included eight students and three AAP staff members. Each of the field notes from the interviews and observations were analyzed and coded using grounded theory techniques. Following this work, we considered Hackenberg's (2005) definition for developing mathematical caring relations (MCRs): " Establishing a MCR entails aiming for mathematical learning while attending to affective responses of both student and teacher" (p. 1). This conceptual framework provides a potential lens for examining if and how SSS programs develop mathematical caring relationships and whether or not they play a role in student success. Our study approaches this problem from both the student perspective and the perspective of mathematics instructors and staff in the SSS program to provide a more holistic picture.

Our initial findings indicate students found positive relationships with their mathematics professor or AAP instructor as influential to their success. Students also bring in strong beliefs and attitudes about mathematics from past schooling experiences including experiences with former teachers. Finally, SSS programs may provide additional opportunities for students to form MCRs because of individualized and small-group instruction, but this is only the case if the instructors and students are invested in creating mathematical caring relationships.

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INVESTIGATING THE EFFECTS OF CLASSROOM CLIMATE ON MATH SELF-EFFICACY

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Keywords: Affect and Beliefs (and Emotion and Attitudes); Post-Secondary Education

Students' self-efficacy influences the kinds of activities they are likely to engage in, how much effort they are willing to invest in a task, as well as how much they will persevere in the face of challenges and disappointments (Bandura, 1977). This means that even if a student has the appropriate skills and incentive, they may still not be academically successful due to low self-efficacy. This makes self-efficacy an important research construct. The focus of this study is to examine how the classroom environment that the teacher creates and the strategies the teacher uses can influence a student's mathematics self-efficacy.

Previous research suggests that there is a relationship between classroom environment and self-efficacy. For example, Ebert-May, Brewer and Allred (1997) and Fencl and Scheel (2005) found that college students' beliefs in their own ability to be successful in a college science class could be improved by the use of a variety of cooperative learning techniques. However, the relationship between classroom climate and improvements in student self-efficacy is not well understood and little work has been done on this in the area of mathematics.

In order to better understand the relationship between classroom climate and mathematics self-efficacy, we conducted a qualitative study where we interviewed seven college students twice. These college students came from education, pure mathematics, and a Calculus course in order to examine the development of self-efficacy in different populations. Our aim was to look for factors within the classroom which may have influenced the development of a student's math self-efficacy.

Initial results suggest that students with a high self-efficacy in mathematics in all three categories describe their mathematics confidence being influenced by the perceived support and encouragement from their teacher. Some characteristics that were mentioned as a way to create a supportive environment in the classroom included vocal support of the student, flexibility from the teacher, and opportunities for group work. Results have implications for how we design mathematics classrooms that support and encourage productive mathematics self-efficacy.

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THE PROFILE OF STUDENTS' BELIEFS: THE COLOMBIAN CASE

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Keywords: Attitudes and Beliefs; Gender; High School Education

Theoretical Background

Beliefs are a powerful force in the behavior of people. If a person believed firmly that can do something, he or she will do it, and if instead believes it is impossible to do, nothing will change his mind and convince that it is possible. The beliefs that people have about themselves and what is or is not possible in the world around them have a great effect on the daily efficiency. All people have in turn beliefs that serve as resources and beliefs that limit them considerably in different areas of their lives (Dilts, 2004). As stated Kober (2015), students of all ages have understandings, skills and beliefs that significantly influence the way in what they remember, reason, solve problems and acquire new knowledge. Only the factor related to personal competence in mathematics is shown.

Methodology and Results

In the data gathering it was used as an expanded and contextualized version of the questionnaire on beliefs about mathematics (*Mathematics-Related Belief Questionnaire, MRBQ*) developed at the University of Leuven (Op 't Eynde & De Corte, 2003) and originally implemented in Belgium (Op't Eynde & De Corte, 2003, 2004). An additional element in this work was that some qualitative questions were asked, which allowed to obtain additional information about the beliefs in a more natural and spontaneous way from students. The sample consisted of 950 students (509 males and 439 females, 2 unanswered) between 14 and 17 years of age of six public institutions. For all final responses (950 students) a reliability analysis using Cronbach's alpha was computed, obtaining an appropriate value of $\alpha = 0.933$.Some of the responses to the qualitative and quantitative questions are shown in table 1.

Table 1: Some responses to qualitative questions by gender

Qualitative questions				Quantitative items	
	M	W	Total (%)		**Total mean**
Math is my favorite subject	85	45	13.7	Mathematics requires more memory than imagination and creativity	2.45
Career choice (engineering or related)	97	20	12.3	If I have had any difficulty with math in school, I will surely have in college	2.29
Do you think your math skills could improve at some point? Yes	250	183	45.6		

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AN EQUITABLE APPROACH TO THE STUDY OF UTILITY IN MATHEMATICS EDUCATION

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Keywords: Middle School Education; Equity and Diversity; Affect and Beliefs

As students continue to question teachers about the usefulness of mathematics, and the Common Core State Standards urges teachers to help students view math as useful, it becomes increasingly important to interrogate our conceptualization of usefulness. Currently, utility research primarily focuses on the usefulness of mathematics for everyday life now or in the future and for one's future career, with emphasis on usefulness for the individual. However, I propose a more equitable approach that incorporates students' developmental needs and values, as well as sociocultural context. This might be particularly important for groups of students that have been historically underserved in math education. For example, U.S. working-class contexts often promote interdependence, or connectedness and the needs of others (Grossmann & Varnum, 2011), rather than independence. Thus, students from working-class backgrounds might be more motivated by uses of mathematics for interdependent purposes, as some social justice research promotes (Gutstein, 2006). Similarly, considering adolescents' developmental needs also suggests the importance of one's relation to others: Social responsibility is a main goal for early adolescents in the school setting (Wentzel, 1993). Since negative consequences can result from a mismatch between an adolescent's developmental needs and features of her environment (Eccles et al., 1993), such values are especially important to consider.

Preliminary findings presented are part of a larger project with four 7th grade math classes in a predominately Latino/a working-class suburb of a large Midwestern city. Survey results reveal that students have strong interdependent reasons for wanting to do well in school. Of twelve items, "make my family proud," "help my family out after I graduate," and "provide a better life for my own children" received the highest mean ratings of importance. Qualitative data from student interviews also corroborates this finding. However, students reported perceiving mathematics as more useful for accomplishing individual goals than for helping their families. This highlights a potential mismatch and suggests one way we might attack the broader enduring challenges of a) finding ways to make mathematics instruction more equitable and b) improving student motivation in mathematics. By broadening our conception of usefulness and leveraging students' values and experiences, we can highlight uses of mathematics for valued domains and help underserved students feel more connected to the mathematics they are learning.

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RACIAL IDENTITY AND MATHEMATICS LEARNING AND PARTICIPATION WITH MIDDLE GRADES STUDENTS

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Keywords: Cognition; Equity and Diversity; Learning Theory; Middle School Education

Despite calls for equity in school mathematics, major inequities along racial lines persist (e.g., Gutierrez, 2008). Although these inequities are well documented, race and racial identity have been under theorized in relation to mathematics learning and participation (Martin, 2009), and consequently mathematics is often framed as a neutral curricular domain (Battey, 2013). This is problematic because of the potential impact racial-mathematical socialization, mathematical identity, and racial identity have on mathematical learning and participation (English-Clark, Slaughter-Defoe, & Martin, 2012). While numerous studies have investigated mathematical learning, there is a dearth of research that investigates the intersection between racial identity development and mathematical learning. Addressing this issue is particularly important for middle grades students because identity formation is actively occurring then (Way, Hernández, Rogers, & Hughes, 2013).

This poster outlines one way the intersection of racial identity and mathematical learning and participation can be studied with middle grades students. We use a radical constructivist framework to study mathematical learning (Von Glasersfeld, 1995) and draw upon racial identity development (e.g., Helms, 1994) and colorblindness (Bonilla-Silva, 2010) literature. The study included 18 racially and mathematically diverse 7th and 8th graders from an urban public school and, consisted of four interviews—three on mathematical generalizations, and one on racial identity and student experiences in mathematics classes. These interviews will be analyzed for mathematical generalizations (Ellis, 2007) and the ways in which students' and teachers' racial identities impact these interactions. The following research questions guide this study: 1) what are themes regarding race and racial identity among middle grades students, particularly related to mathematics? 2) how might the racial identities of students and their teachers impact interactions aimed at supporting students to make mathematical generalizations? 3) what connections might be drawn between students' racial mathematics identities and their willingness to participate in challenging mathematical problem solving situations?

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JUSTIFICATION IN THE CONTEXT OF LINEAR FUNCTIONS: GESTURING AS SUPPORT FOR STUDENTS WITH LEARNING DISABILITIES

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Educational policy mandates access to challenging mathematics for students with learning disabilities (LD) (Confrey et al., 2012; No Child Left Behind, 2002). Rather than spending a disproportionate amount of time receiving procedural instruction, mathematics and special education researchers have recommended providing students with LD more opportunities to engage in solving challenging, conceptual mathematics tasks and justify their problem solving processes (National Council of Teachers of Mathematics, 2000; Woodward & Montague, 2002).

To meet these expectations, students with LD will likely need key supports and strategies to minimize difficulties with memory and processing (e.g., working memory) while engaging with challenging mathematics (Keeler & Swanson, 2001). During the challenging task of justifying problem solving processes, gesturing can be a key support for students with LD as they manage the difficulty of this process (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001).

The researchers in this study conducted a qualitative case study to analyze the justification processes of ten students with LD enrolled in an Algebra I course working in the context of linear functions. We transcribed approximately eight teaching sessions for each participant, coded the data, and organized the data into emerging themes (Brantlinger et al., 2005). An independent rater monitored interpretive validity during data analysis (Maxwell, 1992).

The findings indicated that gesturing often served as a support for memory and processing as the students with LD developed their justification skills. The students often used gestures to point to equations and coordinate planes during mathematical conversations. They also often used their forearms to demonstrate the slope of a line while describing their thinking processes. As the learners became more sophisticated at justifying their problem solving processes (as well as more knowledgeable of linear functions concepts), they relied less on gesturing and informal language and become more frequent users of formal mathematical language. More research is needed to describe how gesturing may provide support for students with LD as they engage in discussions about challenging mathematical topics.

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REMATH: BLACK STUDENTS' LEARNING EXPERIENCES IN NON CREDIT BEARING UNIVERSITY MATHEMATICS COURSES

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Remediation is a long standing and growing phenomena in mathematics education and in the transition to higher education, but there has been limited attention to remedial classrooms and students' learning experiences in these settings. Remedial mathematics courses offered in four-year universities "provide beginning college students with another chance to learn or (relearn) the mathematics supposedly taught to them in high school." According to the National Center for Education Statistics (NCES), remedial mathematics courses are listed in 80% of four-year university course catalogs. Nearly 40% of all "traditional undergraduates" take at least one remedial course in reading, writing, mathematics or some other content area; this figure has risen considerably since the NCES report of 1989 (Lesik, 2006).

The goal of this research project is to explore the effects of remedial mathematics courses on students. More specifically, we expect to uncover how students in mathematics remediation courses experience persistence in STEM related fields.

The sampling procedures will ensure that the participant pool includes African American students pursuing a diverse array of potential major concentrations. The selected participants will be African American students enrolled in remedial courses offered at a Midwestern university.

Using a phenomenological approach, this study employs classroom observations, student surveys and semi structured interviews of selected students enrolled in remedial courses. This research centers on theoretical frameworks developed to study mathematics learning as a narrative construct (Martin, 2000; Sfard & Prusak, 2005). Data analysis of interview transcripts will consist of coding and categorizing emergent themes to be used to recommend interventions and suggestions for the improvement of course administration.

We expect to uncover students' attitudes and beliefs concerning the effects of these courses on their persistence in their academic programs and offer suggestions for program improvement policies.

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PARENTAL EXPECTATIONS FOR HIGH SCHOOL STUDENTS IN MATHEMATICS

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This study builds on both expectancy-value theory and Martin's (2000) concept of sociohistorical contexts. Expectancy-value model suggests that parents' expectations for their children serve as strong predictors of children's expectations to succeed in their studies (Froiland, Peterson, & Davison, 2012). Froiland et al (2012) indicate that early parental expectations for children's post-secondary educational goals has a stronger effect on 8^{th} grade achievement than assisting in doing homework. Also, Martin (2000) indicates that African American parents draw on sociohistorical accounts such as their own mathematics learning experiences in socializing their children into the subject and setting expectations for them.

African American students suffer from the effects of having 'spoiled identities' namely; low quality instruction and overrepresentation in mathematics courses which lead to nowhere academically (Inzlicht & Good, 2006). This is due to a general belief, promoted by dominant narratives about African American students, that their parents do not value education. While some of these narratives may be true, Martin cautions that the school experiences of African American parents is much more nuanced than often reported. In order to account for the influence of African American parents on their children's mathematics achievement, this study focused on parents' mathematics learning experiences and their expectations for their children regarding the study of mathematics. The following research question guided the study; what is the nature of African American parents' mathematics learning experiences and how do they draw on their learning experiences to set mathematics goals for their children? Using a case study design study, we report on two African Americans parents whose children were enrolled in mathematics tutoring and mentoring program based in a large Midwestern University. Both parents were interviewed lasting about 30 minutes. The interviews were transcribed by both authors and coded for themes using a grounded theory approach. Results indicate that both parents did not identify themselves as 'math persons' in school. A typical comment is: "I am not good at Math, I am English major (Grace: parent)." However, these parents expressed regret at not taking mathematics serious in school because they are unable to assist their children in doing homework. Contrary popular narratives that African American parents have low value for education, we found that these parents had high levels of expectations for their children. Both of them did not expect their children to obtain grades lower than B in mathematics. Grace, whose daughter is in an Advanced Placement class comments:" I expect her to be challenged so if her math becomes too easy I expect her to take math courses to challenge her." Implications from this study regarding parent-teacher relationship will be shared during presentation.

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STUDENTS' IMAGES OF MATHEMATICS EXPLORED THROUGH DRAWINGS

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In this poster, we explore images of mathematics held by Kindergarten to Grade 9 students. We draw on data collected within our project, which investigates students' experiences learning mathematics in Canadian K-12 schools and post-secondary institutions, and the ways that these experiences contribute to students' images of mathematics and their mathematical identities. To date, 94 interviews have been conducted. In this poster we present examples of students' images of what mathematics is, as explored through the drawings they created in response to prompts about what mathematics means to them. The theoretical frame underlying this research is enactivism—a theory of embodied cognition that views learning as reciprocal activity in which the teacher brings forth a world of significance with the learners within a cultural milieu (Maturana & Varela, 1992). Our frame, then, prompts us to seek to understand *how* students come to have particular relationships with mathematics and *what* mathematics means to them.

Few studies have explored students' drawings as a means to understand their perspectives on mathematics, however, in this genre, Perkkilä & Aarnos (2009) suggest that students view mathematics as a solitary pursuit involving only certain types of mathematics. These findings are also reflected in research involving the general public, which tends to show narrow, stereotyped views of mathematics as a subject area, a perspective also reflected in the popular media, which tends to portray mathematics narrowly as numbers and arithmetic or as financial mathematics (Hall, 2013). This narrow view of mathematics contrasts with the breadth of perspective on mathematics discussed by mathematicians and mathematics educators. For example, French mathematician, Claire Voisin, who specializes in algebraic geometry, has described mathematics as "movement trying to express itself" while Peter Taylor (2014), a mathematician at Queen's University in Ontario, Canada, notes that mathematics is that which is "eternal and has beauty and structure to die for" (p. 34).

Our findings, developed through thematic analysis of the drawings, overwhelmingly show that participants' images of mathematics are narrowly focused on number and basic operations. We are concerned that students are developing, through their engagement in the culture of school mathematics, a very narrow vision of the nature of mathematics—one that does not include the full territory of mathematical ideas and topics. Our poster presents examples of drawings contributed by K-9 students, examines schooling structures that contribute to these images (e.g., narrow, number-focused curricula in the early years), and discusses implications for teaching.

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THE INVISIBLE HAND OF WHITENESS AND THE COMMON SENSE OF MATHEMATICS EDUCATION REFORM

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Background and Theoretical Perspective: This poster presents the components of a critical analytic essay on the state of mathematics education research and reform. Agreeing with Martin (2013) that mathematics education research "increasingly purports to be committed to equity for all children" (p. 323), despite the absence of clear evidence of progress made toward this goal within the past 30 years of reform (Martin, 2003), I consider what and whose other goals this rhetoric has served. As the mathematics education research community has been impressively successful in realizing what Wolfmeyer (2014) refers to as "national mathematics"—culminating in the Common Core State Standards—I take a critical look at this historical moment.

This paper argues that equity rhetoric and increased curricular centralization are linked through the "common sense" (Gramsci, 1971) of mathematics education research and reform. Gramsci has used the concept of "common sense" to describe how *ideas* support an "elite" in winning and rewinning consent to rule. With this as a framework, I provide an analysis of processes by which the prevailing common sense of mathematics education research is informed by the set of racial (and racist) representations oft referred to as whiteness (Mills, 1997), as well as geared toward profitable, top-down measures of social control in corporate capitalism.

Conclusions: I present and interrogate three ideas that are common sense to mathematics education research and reform. The first is the raced, classed, and spatially-located imaginary of the "urban child" apart from the imagined "white social body" and deemed in need of reform by way of school mathematics (Martin, 2013; Popkewitz, 2006; Leonardo & Hunter, 2007). Second is the scientific and moral authority of mathematics education researchers to decide what types of pedagogy benefit the "urban child". This authority is constructed through theoretical paradigm shifts that appear to afford a more favorable view of "the child" through the gradual inclusion of additional dimensions of "context." It is also constructed in juxtaposition to stereotyped representations of opponents to reform, which have included mathematicians and cultural conservatives but erased teachers and academics of color making critiques of process-oriented pedagogy (Delpit, 2006). The third common sense idea is that mathematics education reform should be linked to broader and profitable structures for top-down administration. Curricular standards are stamped with the values of reform pedagogy (e.g. problem-solving and mathematical practices), and thus enjoy a common sense association with equity. In turn, centralized standards to train teachers in, develop curriculum for, and assess understanding of expand both markets for these services and possibilities for "urban" school surveillance.

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IMMIGRANT STUDENTS' MATHEMATICS LEARNING EXPERIENCES IN CANADIAN SCHOOLS

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Keywords: Equity and Diversity; Elementary School Education; Middle School Education

Because of global mobility of populations, mathematics classrooms in many parts of the world are becoming linguistically and ethnically diverse. In the school district in which we conducted the research reported here, more than 25% of students are considered to be English language learners. Previous studies have found that teachers tend to assume that mathematics learning is universal, while immigrant students and parents can bring different expectations and norms (e.g., Gorgorió & Abreu, 2009). In order to better understand immigrant students' trajectories of mathematics learning, we examine their lived experiences of learning mathematics as revealed through autobiographical interviews and artistic renderings.

The broader study of which this is a part investigates students' experiences of learning mathematics in Canadian schools, and the ways that these experiences contribute to students' images of mathematics and their mathematical identities. To date, 94 autobiographical interviews have been conducted with Kindergarten to Grade 9 students. This project also utilized drawings in order to reveal students' experiences that cannot necessarily be verbalized. In this poster, we focus only on students who had prior schooling in countries other than Canada. To interpret this element of our data, we draw from sociocultural theory, which seeks to understand how cultural practices and artifacts mediate human learning (Cole, 1996).

Our results indicate that the majority of immigrant students in our study described differences in mathematics learning experiences between Canadian schools and schools in other countries. Differences were commonly identified in the following areas: practices of homework, individual learning and group work, the way teachers reward "good mathematics students," and support and help from teachers. These differences tended to contribute to some immigrant students' confusions, struggles, and frustrations in school mathematics learning in Canada. For example, some students reported how they were negotiating multiple norms regarding group work. Another significant common thread was pressure from parents to succeed in mathematics. Many immigrant students in our study were learning additional mathematics at home with their parents or supplementing school learning by afterschool mathematics learning programs. Some of the student drawings and narratives revealed how students conceptualized mathematics as a competitive, performance-oriented discipline, an orientation that seemed to be at odds with the educational cultures of the Canadian schools in which they found themselves.

Despite the common teacher belief that mathematics learning is universal, our study unveils ways in which immigrant students are navigating different norms and practices of mathematics learning. Our research implies that language learning is not the only support needed, but negotiation of meaning attached to school mathematics practices is also necessary to better support immigrant students.

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TEACHER-STUDENT RAPPORT AND MATHEMATICS ACHIEVEMENT: AN EXPLORATORY STUDY

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Purpose. It is well documented that relationships have profound effects on our well-being and quality of life (Landsford et al., 2005). The teacher-student relationship is no exception. While there is no consensus on effective teaching, it is clear that effective teachers focus both on content and caring (McEwan, 2002). However, in the field of mathematics education, we have little empirical evidence to show exactly how teacher-student rapport and students' mathematics achievement are related. The purpose of this study is to use a large sample size to examine the relatedness of teacher-student rapport and students' mathematics achievement. The central research questions to be addressed in this study are: Are teacher-student rapport and students' mathematics achievement related? If so, what is the magnitude?

Method. A total of 6,916 third graders from Central China participated in the study. This study is part of a large research project at Beijing Normal University. Each of the third graders was asked to complete an affect questionnaire and a mathematics assessment. Four of the questions in the affect questionnaire involve teacher-student rapport: (1) When I have different views from my teacher or textbooks, I have an opportunity to explain my thinking; (2) When I face challenges, I am not willing to ask my teacher for help because I am worried about any criticism from the teacher; (3) When I make mistakes, the teacher encourages me to find the mistakes by myself and correct them; and (4) When I improve my mathematics achievement, the teacher gives me praise. The mathematics assessment involves items on number and algebra, geometry and spatial reasoning, and probability and statistics. These are three of the content strands in the Chinese curriculum standards. The data was analyzed by correlation analysis and regression analysis.

Results. The results of the study showed that there are moderate correlations between teacher-student rapport and mathematics achievement in each of the content strands. In particular, the correlation coefficient between teacher-student rapport and student achievement on number and algebra is .221, .159 between teacher-student rapport and student achievement on geometry and spatial reasoning, and .097 between teacher-student rapport and student achievement on probability and statistics. Regression analyses showed that, although the four items on teacher-student rapport can significantly predict students' achievement in number and algebra, geometry and spatial reasoning, and probability and statistics, only less than 5% of the variance can be explained.

Conclusions. This study shows that a good teacher-student rapport plays an important role in improving students' academic ability in terms of number and algebra. We will explore the reasons why a good teacher-student rapport can promote students' academic ability in the future.

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Chapter 9

Teacher Education and Knowledge

Research Reports

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Brief Research Reports

Poster Presentations

PROFESSIONAL NOTICING DURING PRESERVICE MATHEMATICS LESSON STUDY

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The purpose of this research was to understand how lesson study participants in a teacher education program professionally noticed as they engaged in meetings as a component of the professional development cycle. Specifically, the focus was on how the context of lesson study provided opportunity for professional noticing, defined as attending to students' mathematical thinking, analysing students' mathematical thinking, and deciding how to respond when teaching on the basis of students' thinking. Results indicated that the structured format of lesson study in the teacher education program both afforded and constrained the incidences of verbalizing professional noticing in a group setting. Findings provide perspective for structuring lesson study for use in teacher education programs to support professional noticing.

Keywords: Elementary School Education; Teacher Education-Preservice

Statement of Purpose

Recent findings in mathematics education highlight the importance of focusing on students' mathematical constructions for knowing how to support future learning and development (Norton & McCloskey, 2008). Professional noticing is a specialized practice that encompasses: a) attending to students' thinking, b) analyzing students' thinking, and c) deciding how to respond on the basis of thinking (Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008). From this point forward we refer to this type of professional noticing as *noticing*. Lesson study is a professional development process that provides context for noticing, but has most commonly been used with inservice teachers. While the use of lesson study with preservice teachers is limited, teacher education programs implementing this practice can provide opportunities for preservice teachers to notice through reflection (Murata, 2011). Knowing more about supporting preservice teachers' noticing through lesson study is important for structuring teacher education programs to focus attention on students' mathematical understandings. This study seeks to answer the following research question: How do preservice teachers (teacher candidates who are presently in a teacher education program) professionally notice as they engage in the lesson study process?

Theoretical framework

In recent years, research studies have examined methods for teacher preparation related to teacher knowledge, effective teaching, and the relationship between knowledge and teaching (Cochran-Smith & Zeichner, 2005). Providing preservice teachers with opportunities to understand student-centered teaching and develop the pedagogical content knowledge necessary for effective instruction is essential for promoting high leverage teaching practices (Hill, Ball, & Schilling, 2008;). One method for developing this type of knowledge is through cultivating the ability to notice (Jacobs, et al., 2010). The notion requires attention to student thinking through "both observation and the medium through which observation takes place" (Mason, 2011, p. 45). During the process of noticing, effective teachers become aware of students' thinking and are able to construct tasks that direct attention to relevant learning opportunities (Mason, 2011).

The meaning of noticing has developed through literature on professional vision, which has explored nuances between varying stakeholder perspectives on classroom events and influencing factors that shape those viewpoints (Goodwin, 1994). Studies on professional vision commonly focus on teachers recording and studying their practice and researchers analyzing the noticing and reasoning of the teachers involved. Seidel et al. (2011) describe professional vision as the intersection between noticing and knowledge-based reasoning with the assertion that this type of reasoning involves the ability to describe what was noticed, link observations to prior knowledge about teaching and learning, and connect to theory to predict future classroom events. A distinguishing factor between noticing (Jacobs et al., 2010) and professional vision is the difference with foreseeing future practices (Seidel et al., 2011). Considering the roots in professional vision, we focus on noticing as the guiding framework in our study because of the preservice teacher context and importance of basing teaching decisions on evidence.

Related Literature

The following describes lesson study as traditionally practiced with inservice teachers, followed by a review of the literature on lesson study with preservice teachers.

Lesson Study Overview

Teacher knowledge can be fostered through the process of lesson study, leading to improvement in instructional practice (Murata, 2011). Lesson study, originally termed *Jugyokenkyu* in Japan, means lesson and study or research (Fernandez, 2010). During the lesson study process, participants, usually a small group of 5-7, begin by setting a goal for the lesson study process and then working collaboratively to plan a lesson to teach in one of their classrooms. To plan the lesson, participants study research materials, curriculum guides, and other artifacts to decide on appropriate instructional methods and content for a mathematics lesson. Next, a teacher in the group teaches the lesson to his or her students while the other participants observe the lesson. Following the lesson, the participants meet together to reflect on the lesson and to revise the lesson for teaching in another classroom. Once the lesson is revised, the modified version is taught in another teacher's classroom. This student-focused process affords participants the opportunity to notice different aspects of the lesson (Murata, 2011).

Lesson Study with Preservice Teachers

As lesson study has expanded beyond Japan to other countries and contexts, the process has begun to infiltrate teacher education programs (Fernandez, 2010). Recent research findings indicate that lesson study provides a context for preservice teachers to make connections between theory and practice as they take part in multiple components of the teaching cycle, including designing lessons and teaching (Murata, 2011). As a result, there is increased interest in knowing more about supporting preservice teachers' professional practices in lesson study.

Methodology

This research focuses on the use of lesson study with preservice teachers as a component of a teacher education program at a public university in the Midwestern portion of the United States. Case study methodology was employed to analyze how these participants noticed during seven iterations of the mathematics lesson study process (Yin, 2009).

Participants and Context

Participants in the case study included one team, comprised of six preservice teachers, one classroom teacher, and a university facilitator who taught the university course. Consistent with case study research, we focus on this one team of participants, as opposed to a larger number of participants, to provide opportunity for an in-depth examination of the practices and understandings of these individuals (Yin, 2009).

The format for the lesson study cycle was predetermined based on a university protocol. To begin the first cycle, the team met together to plan a mathematics lesson that would be taught by preservice teachers in the classroom teacher's first grade classroom. Weekly, two of the preservice teachers taught a whole-class mathematics lesson while the other participants (four other preservice teachers, one classroom teacher, and the university facilitator) observed the lesson and took detailed field notes. Following the teaching, the group met to reflect on the teaching with an aim toward discussing students' mathematical thinking. The university facilitator led these meetings and followed a protocol that provided opportunities for all participants to share their ideas. Following these reflective discussions, the team made plans for the next lesson that would be taught and the iterative cycle repeated. For the purposes of this research, we focus on the reflective meetings that occurred in this cycle following the teaching. We refer to these meetings as the *lesson study analysis meetings*; each of the seven iterations of the process included one of these meetings. Therefore, data for this study are video recordings of these seven meetings and the accompanying transcripts.

Data Analysis

The purpose of data analysis was to: 1) Identify instances of noticing during the lesson study analysis meetings, and 2) Understand how preservice teachers were noticing based on conversational interactions with the other team members. Data were analyzed in two phases: Phase One and Phase Two.

In Phase One, all data were coded by talk turn, signified by a change in speaker. Data were analyzed based on the degree of noticing as defined by Jacobs et al. (2010): attending (describing students' actions or words), analyzing (interpreting student actions or words), or responding (describing pedagogical suggestions for future teaching). Across the seven transcripts, 809 talk turns were coded by each of two researchers. The researchers met to reconcile differences in codes.

In Phase Two, data identified as noticing from Phase One were coded a second time to understand how preservice teachers were noticing based on conversational interactions. These instances were open coded (Corbin & Strauss, 2008). Identified themes included talk turns that were: based on a previous comment, initiated response, narrowed conversation, provided generalization, negotiated, asserting expertise, making a connection, addressed another issue, prompting participants, or provided context. These themes provided an explanation for how and why the noticing occurred within the context of the lesson study analysis meetings.

Statement of Results

Findings from this study indicate: 1) Noticing from the preservice teachers occurred in response to a prompt by the university facilitator, 2) When responding to this prompt, the preservice teachers commonly answered independently, meaning their conversation did not build on what others said, instead they reported one by one without continuity in the conversation topic, and 3) Preservice teachers noticed by attending and analyzing student thinking, but they rarely attended, analyzed, and decided how to respond on the basis of students' thinking about the same subject. The following provides short excerpts of data to provide evidence for these themes. Further data will be presented if accepted for presentation.

Response to Prompt

The preservice teachers noticed in the lesson study analysis meetings when the facilitator gave a prompt eliciting a discussion on students' mathematical thinking. The prompts encouraged the preservice teachers to respond about how students were thinking about mathematics, which resulted in noticing. Prompts included, "Let's focus on evidence of students' thinking," and, "What were specific ways students were thinking?" Following these prompts, it was common for the preservice teachers to engage in conversation to answer the facilitator's questions, during which they attended to students' thinking or analyzed evidence of student understanding. As this occurred, patterns of initiating and responding took place, meaning the facilitator would initiate a conversation, and the preservice teachers would respond. The following provides an example of a prompt, followed by noticing:

Facilitator: Alright, what are some specific ways that student work, students were thinking that you saw, observed as you were walking around?

Preservice 3: One thing I saw, Meagan, it was on the worksheet, and she constantly referred to the number line and I thought that was pretty neat because she's using her tools, the tools around. And like I said, I saw major improvement with Meagan, so I can tell that she's able to, able to, now being able to use tools around her without saying hey, look at the number line if you get lost!

Preservice 2: And I saw that with a lot of kids, actually while they were doing their worksheet, I know this is six oranges but how do I write six? Have them look at their number line and then would count up to six, they're like oh! And then they would write the six down, like they knew the objects.

In this instance, the facilitator asked a specific question, and even encouraged the preservice teachers to discuss students' thinking in the prompt. Following the prompt, preservice teachers responded by specifically discussing how students were making connections between figures of pieces of fruit and writing the corresponding numeral. Thus, the preservice teachers responded to the prompt by providing evidence about how students were thinking as they were learning to count and abstractly represent quantity with numerals.

Individual Response

As the facilitator gave a prompt and the preservice teachers responded, it was rare for the preservice teachers to relate their responses to other responses (as was done in the aforementioned example when Preservice Teacher 2 continued discussing the same topic Preservice Teacher 3 had initiated). Instead, it was typical that the preservice teachers would independently report on what they had noticed instead of trying to make connections across the different types of students' thinking that were noticed. Essentially, each preservice teacher responded to the prompt as if the other preservice teachers had not already responded, which resulted in a defragmented conversation in which the conversation did not build, but rather preservice teachers operated as individuals all responding on their own to the same prompt. The following provides a small excerpt to demonstrate this pattern:

Facilitator: Right we're just going around and saying what went well.

Preservice 4: I thought the book was a really good way to launch off your lesson. I think since they were familiar with the book that also helped. You know they knew what the book was about and were excited about it.

Preservice 5: Um, I think… I liked that they were able to take something home. The kids were really excited that they could keep this so you know, they were proud of what they did, which I thought was nice.

Preservice 6: I liked how you guys did the counting from like all the way up to twenty because they were doing five, ten and twenty. There were some kids having trouble with the higher numbers so it was like as a group they got to explore the numbers.

Preservice 2: Um, I liked the little wrap up at the end and kind of liked the flashcard thing where you held up the numbers and they tell you.

The preservice teachers each responded on their own to the facilitator's prompt; their comments did not clearly connect to each other to create a cohesive conversation. Instead, each preservice teacher highlighted something she noticed and contributed to the discussion by responding to the prompt, but not necessarily responding to the other preservice teachers' responses.

Attended and Analyzed

Across the seven transcripts, the preservice teachers engaged in noticing by attending to students' thinking, analyzing their mathematical thinking, and deciding how to respond on this basis; however, these three types of noticing did not commonly occur together. Instead, the preservice teachers engaged in dialogue that included attending to students' thinking and analyzing students' thinking, but did not make clear connections based on this evidence to verbalize noticing about how to respond. This does not necessarily indicate that they did not consider this connection mentally, however verbal evidence was rarely present in the lesson study analysis meetings. When the preservice teachers did make comments about how they would respond, they were often disconnected from conversation that attended to or analyzed student thinking. In other words, attending and analyzing occurred at points in the meetings and deciding how to respond occurred at separate times in the conversations. Therefore, incidences when all three identified types of noticing took place based on the same idea were rare. Within each meeting, it was common for attending and analysis to occur at the beginning of the meeting and for noticing that focused on responding to occur at the end; there was essentially a shift mid-way through the lesson study analysis meetings. Possible explanations for this are included in the discussion section.

Discussion of Results

The intent of this research was to describe how preservice teachers noticed as they engaged in the lesson study process. This information is critical for teacher educators to consider as it provides insight for knowing how to better support preservice teachers' focus on students' mathematical thinking through the act of noticing.

Importance of Prompts

As the preservice teachers engaged in the lesson study process, the prompts the facilitator asked were paramount in eliciting responses in which the preservice teachers noticed. The facilitator asked very pointed questions about students' thinking, to which the preservice teachers responded. In this way, the facilitator encouraged routine noticing by providing a context that required preservice teachers to be reflective in their responses and consider students' thinking (Miller, 2011). In doing this, the facilitator provided scaffolded support that is necessary for teaching others to notice (Star, Lynch, & Perova, 2011). Thus, the facilitator's prompts supported noticing by scaffolding preservice teachers' thoughts and directing them toward students' thinking. This is an important finding for teacher education programs looking to provide a similar context for the development of professional practices. In this instance, it was the facilitator's questions and prompts that often elicited noticing from the preservice teachers.

Attending and Analyzing

When noticing occurred, the preservice teachers commonly attended to students' thinking and analyzed their thinking (Jacobs et al., 2010). This required the preservice teachers to observe classroom events and make interpretations based on these events (van Es & Sherin, 2008). Hence, the lesson study analysis meetings provided a focused and specific context in which this type of analysis could occur, indicating the plausible benefits of using lesson study in a teacher education program. Within the context of lesson study, the preservice teachers engaged in a structure that was focused on students' thinking: as they taught or observed lessons, they were cognizant that they were going to be asked to report back on the basis of students' understanding. Likewise, as they planned lessons, they were mindful that they would be held responsible for teaching and or observing the lesson and then reflecting on the lesson—this process provided them the opportunity to complete a cycle of formative teaching by engaging in the three components of noticing. These findings indicate that teacher education programs should consider the possible benefits of lesson study for promoting the noticing practices of preservice teachers.

Lesson Study Structure

Despite the positive outcomes based on the noticing that occurred in the lesson study analysis meetings, findings identified segmented noticing responses in the meetings. Specifically, attending and analyzing student thinking was often separated from discussions about deciding how to respond on the basis of students' thinking (Jacobs et al., 2010) One possible explanation for this findings is the structure of the lesson study analysis meetings, as designed by the teacher education course. With the course, the facilitator followed a protocol in which the group reflected on the previously taught lesson, then discussed changes they could have made to the lesson to improve student thinking, and concluded by planning the next lesson. Perhaps the structure constrained the inclusion of further discussion of implications of analysis of students' thinking because preservice teachers and the facilitator understood the structure of the lesson study analysis meeting to progress along those three steps. This also could be related to the finding that the preservice teachers did not necessarily build their comments based on the aforementioned comments. These results indicate that a more fluid structure to the lesson study analysis meetings may provide for increased incidence of all three types of noticing occurring at the same time. For teacher education programs, this structure within lesson study may initially be helpful for encouraging noticing; however, facilitators or programs should be aware of this disconnect among the three types of noticing and perhaps reduce the stringency of a similar protocol for the meetings over time. Nevertheless, findings show promise for the use of lesson study in supporting preservice teachers to professionally notice.

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PRE-SERVICE SECONDARY TEACHERS LEARNING TO ENGAGE IN MATHEMATICAL PRACTICES

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The Common Core Standards require students to learn content and mathematical practices, and so teachers must have content knowledge and be able to engage in practices themselves. This raises the question of how novice teachers learn to engage in mathematical practices. I investigate pre-service secondary teacher learning of mathematical practices following participation in a mathematics content course for teachers using a pre/post design. Four participants completed think-aloud interviews solving algebra tasks. All participants increased their engagement in mathematical practices and began to engage in them in more nuanced ways. Many changes in participants' practice engagement were related to opportunities to learn in the content course around making sense of problems, justification, and attention to precision. These results have important implications for teacher preparation and research on teacher learning.

Keywords: Teacher Education-Preservice; Mathematical Knowledge for Teaching; Teacher Knowledge; Problem Solving

What mathematical understanding is necessary for high school teaching, and when and how do teachers develop it? Recent research answers this question by considering mathematical knowledge for teaching (MKT) –mathematical knowledge entailed by the profession of teaching (Ball, Thames, & Phelps, 2008). This view of professional knowledge requires considering knowledge of school mathematics, knowledge of mathematics beyond the school curriculum, knowledge of how to unpack the mathematics of the school curriculum, and pedagogical content knowledge. Implicit in the definition of MKT is the relationship between knowledge *of* mathematics and engagement *in* mathematics. One way to describe the act of engaging in mathematics is through the lens of the mathematical practices. Thus, practices and mathematical knowledge for teaching must be considered jointly when investigating teachers' mathematical preparation. In this paper, I explore pre-service teacher learning of mathematical practices.

Teacher Engagement in Mathematical Practices

Mathematical practices describe the tools needed to *do* mathematics. They include making a conjecture, justification, and attention to precision(Common Core State Standards, 2010). Because mathematical practices are a key part of the Common Core, a focus on teacher engagement in them has become a particularly salient issue for current research. Despite robust literature on teacher knowledge, teacher engagement in mathematical practices has not been explicitly incorporated into commonly used definitions of MKT. "Conceptions of teacher knowledge have seldom considered the kinds of mathematical practices that are central to teaching. For example, rarely do teachers have opportunities to learn about notions of definitions, generalization, or mathematical reasoning" (RAND, 2003, p.21). Attention to teachers' engagement in mathematical practices matters for two reasons. First, teacher engagement in practices helps demonstrate what teachers *do* with the mathematical content that they *know*. Second, the ways in which teachers engage in practices themselves may affect how they teach students to engage in practices.

I draw on the mathematical practices identified in the Common Core Standards to describe ways in which secondary novice teachers *do* mathematics. I conceptualize teacher engagement in mathematical practices as being intertwined with MKT, just as the standards of mathematical practice are necessarily embedded within mathematical content (McCallum, 2014). Teacher engagement in practices as mathematical problem solvers themselves is connected with subject-matter knowledge. Similarly, understanding how to teach students to engage in practices is connected to pedagogical content knowledge.

Much of the existing literature on teacher engagement in mathematical practices emphasizes proof. Overall, research on teachers and proof has endeavored to (1) understand teacher beliefs about proof and its role in the classroom (e.g., Knuth, 2002; Staples, Bartlo, & Thanheiser, 2012) and (2) investigate teacher knowledge related to the analysis of specific types of proof (e.g., G. Stylianides, A. Stylianides, & Philippou, 2007). However, there remains a gap in the literature about how teachers actually engage in proof themselves. Given the emphasis on mathematical practices in the Common Core Standards(2010), it is critical to investigate teachers' engagement in proof, in addition to their knowledge about it.

Equally important is research that investigates practices more broadly, looking beyond formal proof alone. Mathematical practices rarely occur independently of one another, making it critical to look at them in concert with each other and within a variety of mathematical content domains. Investigating teacher engagement in mathematical practices more broadly will contribute a great deal to understanding teachers' MKT and its link to mathematical practices. Just as pedagogies of enactment support novice teachers in learning to teach (Grossman et al., 2009), teacher engagement in mathematics can support student engagement in mathematics. This makes it essential to conduct research that focuses on teacher engagement in mathematical practices connected with teacher content knowledge. If we argue that teachers need to develop MKT and need to engage in mathematical practices, what then does it look like for teachers to be learning to engage in mathematical practices?

Teacher Learning

Following (Lave & Wenger, 1991),I take a situative perspective on teacher learning; that is, learning is described as a change in the way a teacher participates in a community of practice. In the context of teachers learning math content and practices, evidence of learning can come from changes in the way teachers interact with one another in the context of solving a math problem, but it can also come from changes in the way they individually reason about a math task (Cobb & Bowers, 1999). Focusing on mathematical practices in particular, learning means looking at the way teachers take on, or appropriate (Moschkovich, 2013; Rogoff, 1990), the mathematical practices and how they transform their engagement in those practices within a community of mathematics teachers. For example, a pre-service teacher might appropriate the practice of justification by utilizing more mathematically appropriate proofs or explanations (such as using examples to motivate a generalized proof, rather than using examples as proof).

In this study, I consider the following question: What did participants in an abstract algebra course for future secondary teachers learn about mathematical practices? In particular, I examine the extent to which their engagement in mathematical practices changed from the beginning of the course to the end of the course.

Methods

To answer this question, I conducted a case study of an abstract algebra course designed for future secondary teachers. The course took place at the beginning of a yearlong preparation program. I selected this site for my case study largely because of the program's commitment to the deep mathematical preparation of future teachers. A mathematician taught the course; he tailored the course to attend to the needs of the teachers he was preparing. I observed all sessions of the abstract algebra course. The course met for ten weeks for three hours each session.

Participants and Data Sources

Four pre-service teachers participated in the study. Though these participants are not a representative sample of all future secondary math teachers, their mathematical preparation is consistent with that of typical pre-service secondary math teachers. The participants each had different trajectories into teaching and as such represent different specific features of novice teachers. It is valuable to treat each participant as an individual case study. Daniel entered the teacher preparation program after a long career in engineering and business. He had an undergraduate degree in engineering. Laura was a paraprofessional for several years before pursuing her math credential. She had a math major, but expressed a lack of confidence in her math ability. Sam served in the military after high school and then got an associate's degree in engineering. He later returned to school to finish his undergraduate degree in math and then complete his credential. Tim entered the teacher preparation program immediately after completing his undergraduate degree in math and physics.

At the beginning and end of the abstract algebra course, participants completed in-depth task-based interviews. Participants were asked to think aloud as they solved (Ericsson & Simon, 1980), and they engaged in "free problem solving" (Goldin, 1997). Though the interview tasks addressed high school content, the problems were non-familiar, that is, participants were unlikely to have seen any of the particular problems before. I investigate participant learning by comparing participant engagement in mathematical practices at both time points. Because of the think aloud structure, participants' solutions were both oral and written.

Data Analysis

I coded each interview for engagement in mathematical practices, using the Common Core practice standards as a framework and dividing each standard into sub-codes. For example, during the pre-interview algebra task, Laura decided to test some specific values to make sense of a more general algebraic statement. She said, "I'm just going to put real numbers in this for a minute". I coded this as trying a special case (MP1), because it was evidence trying a particular example while solving a general problem. Table 1 shows additional mathematical practice codes.

Table 1: Sample mathematical practice codes

Code name	Code description	Example
Connect representations (MP1)	Explain correspondences between equations, verbal descriptions, tables, and graphs	Sam (pre): Your slope is going to be somewhere in the middle, and then when you add these together [*(b + d) in the equation* $(f + g)(x) = (a + c)x + (b + d)$] the intersect is going to be somewhere in the middle [*indicates origin of the graph*].
Test conjectures (MP3)	Make conjectures and build a logical progression of statements to explore the truth of their conjectures.	Daniel (post), after conjecturing that the slope must always be -1: So I will pick a point that's at (1, 2) [*plots (1, 2)*]. That's one of the points. The other point should be at (2, 1) [*plots (2, 1)*]. Nicely, we see this slope is now going to be negative [*connects points (1, 2) and (2, 1)*].

Using a multiple case approach (Miles & Huberman, 1994), I describe participants' individual learning as well as investigate trends across participants. I used analytic memos to create problem-solving cases for each task. I looked across the four participants at each time point to identify similarities and differences in their content knowledge and practice engagement. Finally, I looked for change over time by comparing the pre and post interviews.

Interview Tasks

The algebra tasks used during both think aloud interviews required participants to prove a statement about linear functions. This makes them excellent sites to consider participant learning around proof and justification along with other mathematical practices. Both tasks focused on high school level content, though the particular tasks themselves were unfamiliar to participants. Both dealt with linear functions, a standard part of school algebra. Both tasks also required participants to prove a general statement about linear functions was true. The focus on the pre-task was on the sum of two functions at a particular point. The focus on the post-task was on the *x*- and *y*-intercepts of a particular line. Table 2 shows the pre and post algebra tasks.

Table 2: Interview algebra tasks and key features

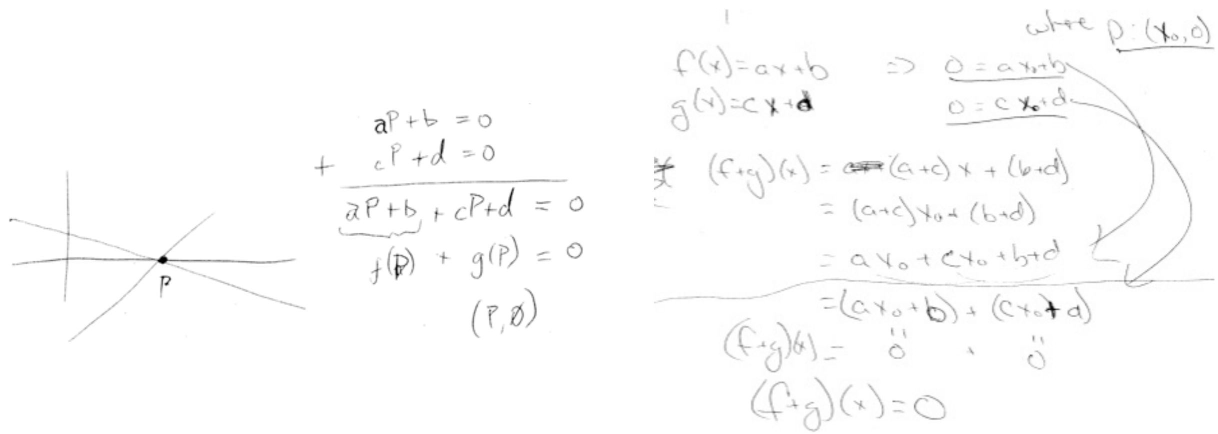
	Pre-Interview Algebra Task	**Post-Interview Algebra Task**
Key features	High school level content Linear functions Prove a general statement is true Focus on the sum of two functions at a particular point	High school level content Linear functions Prove a general statement is true Focus on the *x* and *y*-intercepts of a particular line
Tasks	Prove the following statement: If the graphs of linear functions $f(x) = ax + b$ and $g(x) = cx + d$ intersect at a point P on the x-axis, the graph of their sum function $(f+g)(x)$ must also go through P. (TEDS-M International Study Center, 2010)	Take a point (p, q) on the Cartesian plane. Reverse the coordinates to obtain a second point (q, p). Prove that on the line between these two points, the x-intercept and the y-intercept are the sum of the coordinates.

Findings

As a group, participants engaged in many of the mathematical practices, and the practices they engaged in were directly related to the nature of the tasks. All participants worked to make sense of the problems (MP1), create representations (MP2), construct arguments (MP3), and attend to precision (MP6); three participants engaged in making use of structure (MP7). Nobody engaged in modeling with mathematics (MP4), for instance, because the tasks did not entail mathematical modeling. All four participants showed changes in their engagement in mathematical practices, but the nature of the changes varied across participants. In this section, I explore these changes in detail. Additionally, I connect some of the observed changes in math practice engagement back to the opportunities to learn present in the abstract algebra class.

Learning to Justify and Attend to Precision: Daniel and Tim

Daniel and Tim both did very well on the pre-tasks. Daniel produced a complete and correct algebraic argument, providing his rationale aloud as he talked through his solution. He supported his argument with graphical examples (see Figure 1a). Tim produced a complete algebraic argument that was nearly correct except for an imprecise use of mathematical notation (see Figure 1b). Tim did not show substituting a point P into the equation, though that seemed to be his intention based on what he said. So his final line reads as though the result were true for any x value, rather than for a particular value of P (he wrote $(f+g)(x) = 0$, rather than $(f+g)(x_0) = 0$). Based on their performance, it seemed as though there would be little opportunity to see growth on the post task. However, both showed growth across several mathematical practices.



(1a) Daniel's work (1b) Tim's work

Figure 1: Daniel and Tim's written pre-task work

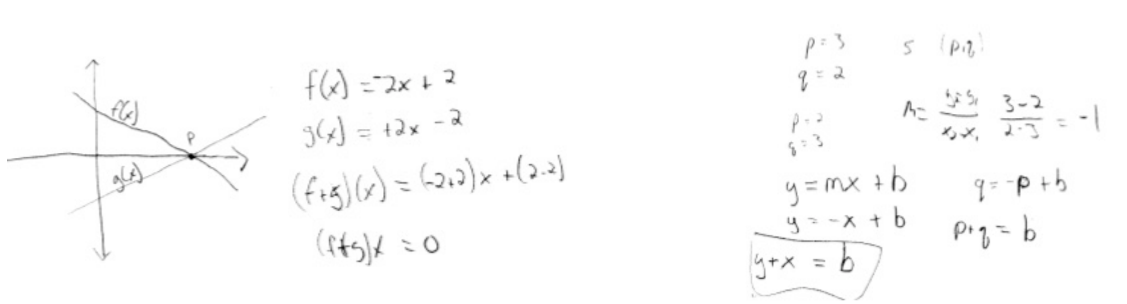
Daniel showed growth through more substantial and meaningful connections between his algebraic and graphical representations, and through his attention to precision. He took the same basic approach to the two problems, using a mostly algebraic approach with graphical examples on both. On the post-task, Daniel made stronger connections between the two representations (MP1) than he had on the pre-task in order to overcome an initial error. On the post-task, Daniel also carefully noted which variables were free and which were fixed. This attention to precision (MP6) enhanced the rigor of his final proof, giving him a more complete and detailed argument (MP3) than his work on the pre-task had been, even though both were correct results.

Tim also showed growth in his attention to precision (MP6), along with attending to the domain of his argument (MP3) and the way in which he communicated his conclusions. In particular, he distinguished between his preliminary scratch work and how he would develop a more formal proof. These changes allowed Tim, like Daniel, to produce a rigorous and detailed proof on the post-task that showed growth over his performance in the pre-task.

Learning to Make Sense, Persevere, and Justify: Sam and Laura

Sam and Laura struggled with the pre-tasks, though in different ways. Sam attempted to use an algebraic representation, but misinterpreted the problem and was unable to construct a complete argument. Laura tried using a special case, but due to not attending to all the conditions of the problem, chose an example that led her to believe the statement she had to prove was false, even though participants were told to prove the statement was true. After conjecturing that the statement was false, Laura did not attempt to justify her conjecture in any way.

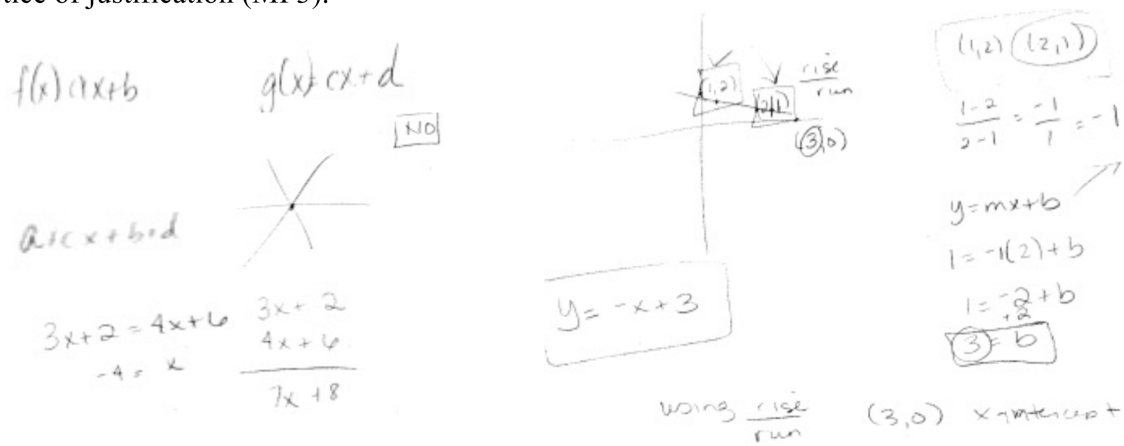
In the post-task, Sam showed tremendous growth. He was able to accurately analyze the given information, monitor his progress and develop a solution plan, and choose a generative special case (MP1). One difficulty he had with the pre-task was choosing a special case that was too specific, and obscured some of the generality of the problem. In the post-task he chose a more appropriate special case (see Figure 2). Then he was able to generalize from the special case, something he had been unable to do in the pre-task. He also compared his special case argument to his more general argument and was able to evaluate them (MP3). Sam went from being unable to complete an argument to having a full, nearly complete proof. His work on the post-interview task was limited only by not attending to the meaning of his variables.



(2a) Sam’s pre-task special case (2b) Sam’s post-task special case

Figure 2: Sam’s pre and post-task special cases

Laura too demonstrated substantial change through her work on the post- task (see Figure 3). In this case, she correctly selected a special case (MP1) and was able to construct a complete and correct argument for that special case (MP3). She did so through increased work connecting representations (MP1). She also engaged in communicating her conclusions and evaluating her arguments (MP3), two practices not visible in her pre-task interview due to her early incorrect conjecture. Though she did not attempt to generalize her special case result, Laura commented that she knew that was the next step. She showed important changes in the way in which she engaged in the practice of justification (MP3).



(3a) Laura’s pre-task written work (3b) Laura’s post-task written work

Figure 3: Laura’s pre and post-task written work

Discussion: Connections to Opportunities to Learn

Overall, participants changed their engagement in mathematical practices, particularly making sense of problems (MP1), constructing arguments (MP3), and attending to precision (MP6). The results hint at the possibility of a learning trajectory around practice engagement. Laura and Sam improved in how they made sense of the problems, and this supported them to better construct reasonable arguments. Tim and Daniel demonstrated proficiency in making sense of the problems in the pre-task, but showed growth in communicating their conclusions and through their attention to precision.These particular practices were also a major focus of the abstract algebra class participants were taking(Baldinger, 2014).

Participants regularly had opportunities to make sense of problems during the abstract algebra course. For example, the professor emphasized how participants could use a special case to help them discover a more general solution. This is the approach Sam took on the post-task, and the approach Laura knew she should take.Participants also had opportunities to construct arguments and justifications. One such opportunity to learn occurred when the professor talked about communicating conclusions and the differences between scratch work and formal proof. This echoes the distinction Tim described in his work on the post-task.A third focus of the class was attention to precision. For example, the professor explicitly discussed the importance of defining the meaning of variables in a problem. Daniel and Tim both built on that in their work on the post-task. Participants' performance on the post-task reveals some important connections to the opportunities to learn in the abstract algebra course. This suggests the potential value of the content course as a site for learning about mathematical practices. Participants also reflected that they had learned how to engage in mathematical practices in the course (Baldinger, 2014), providing further evidence to support the idea that the improvements in performance on the post-task might be related to the learning opportunities in the abstract algebra course.

Implications and Future Directions

This case study illustrates what four pre-service secondary teachers learned around engagement in mathematical practices at the beginning of their teacher preparation. Though their learning was related to their experiences in the abstract algebra course, this is not necessarily a causal relationship; participants had numerous other learning experiences during this time. Additionally, these four participants are not representative of all pre-service teachers, and their learning trajectories are not necessarily "typical". However, each unique case provides insight into the variety of learning trajectories experienced by pre-service teachers.

There are several implications based on the results of this study. First, not surprisingly, these pre-service teachers exhibited distinctive learning trajectories for mathematical practices. In this case, despite differences in their starting places, all four participants showed changes in their engagement in mathematical practices. This suggests that teacher preparation programs can be sites for learning to engage in mathematical practices, just as the programs can be sites for developing other knowledge necessary for teaching. Additionally, such diverse learning trajectories need to be accounted for in the design of teacher preparation programs.

The abstract algebra course was clearly an important site for participants to learn to engage in mathematical practices. Incorporating more opportunities of this nature might support teacher engagement in a wider range of practices. Building on that, the practices proved portable across content levels. The participants learned to engage in these practices addressing college-level mathematics, but demonstrated their engagement on secondary-level tasks. This emphasizes the value of mathematics content courses beyond developing content knowledge.

Through engaging with multiple practices in an interconnected way, participants were more mathematically productive on the post-task than on the pre-task. Laura's post-task highlights the way engagement in communicating her conclusions (MP3) depended on her ability to connect her algebraic and graphical representations (MP1). Daniel and Tim's solutions show how attention to precision (MP6) can improve the quality of an argument. The interconnectedness suggests the value of learning about practices in conjunction with one another. The implication for teacher preparation is to provide multiple opportunities for pre-service teachers to engage in a variety of practices, rather than focusing on a single practice. Furthermore, this suggests the value of looking more holistically at practice engagement in research.

This study raises a question about how to measure engagement in mathematical practices in a way that accounts for varied learning trajectories.The interview tasks used in this study did this by focusing on accessible high school level content. However, the choice of tasks also limited the practices that might have been assessed. Additionally, the in-depth interviews conducted for this study were exceptionally illuminating but would be inefficient to implement across large teacher preparation programs.Given the importance of understanding teachers' engagement in mathematical practices, it is important to identify alternative measurement strategies.

The standards of mathematical practice in the Common Core (2010) provide a practical motivation for understanding how practices are learned and how they can be taught. It is reasonable to imagine that if teachers have not had opportunities to engage in mathematical practices themselves, it will be difficult for them to create opportunities for their students to engage in mathematical practices. Learning to engage in mathematical practices can be seen as a first step toward learning to teach others to do so, creating an imperative to more fully understand teacher learning around mathematical practices.Future research must consider other contexts for learning along with other content areas. That will help develop a more complete picture of teacher learning around practices and provide greater insight into how to structure relevant learning opportunities. Additionally, it will be valuable to consider the relationship between a teacher's ability to engage in mathematical practices and the strategies that teacher uses to support student engagement in mathematical practices. Teachers must be able to support students in all aspects of their mathematical learning, and should have opportunities to learn to do this as part of their preparation. Understanding teacher learning around mathematical practices is a crucial part of supporting teachers in their work with students.

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AN INVESTIGATION OF PREK–8 PRESERVICE TEACHERS' CONSTRUCTION OF FRACTION SCHEMES AND OPERATIONS

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Supporting students to build robust fraction schemes and operations is an enduring challenge in mathematics education. Recent research has explored a developmental trajectory of fractions schemes and operations constructed by upper-elementary and middle school students in an effort to support student learning. This study broadened the existing research by investigating PreK–8 preservice teachers' construction of fractions schemes and operations. This paper specifically explores data from PreK–8 preservice teachers regarding one scheme (Iterative Fraction Scheme) and way of operating (Three Level Units Coordination). Results focus on 13 special cases that disagreed with current conceptions of the developmental trajectory.

Keywords: Learning Trajectories (or Progressions); Mathematical Knowledge for Teaching; Rational Numbers; Teacher Education-Preservice

Introduction and Objectives

In light of the enduring challenge that understanding fractions concepts places on PreK–8 preservice teachers (PSTs), our research project sought to validate a developmental trajectory of fractions schemes and operations specifically for PSTs. In particular, this study extends current research that has worked to establish this trajectory for elementary and middle school students (Hackenberg, 2007; Norton & Wilkins, 2012; Steffe & Olive, 2010). For this study PSTs responded to tasks designed to determine the fractions schemes and operations with which each participating PST seemed to operate.

The initial goal was to validate the established trajectory with a new population. Upon analysis of the data, the trajectory seemed consistent with previous research; that is, each lower level of fractional understanding appeared to be a prerequisite to higher levels of understanding. However, upon further analysis, 13 special cases emerged that deviated from this trend. In these 13 cases, PSTs demonstrated ways of operating that aligned with an Iterative Fraction Scheme (IFS) while simultaneously lacking the prerequisite operation of Three Level Units Coordination (3UC). This paper focuses on our work examining these special cases. We worked to answer the research question "Must PSTs interiorize 3UC before constructing an IFS?" Although studies have shown that, without 3UC, elementary and middle school-aged students cannot develop an IFS (e.g. Hackenberg, 2007; 2010), it appeared possible for PSTs to do so.

Background and Theoretical Framework

PSTs and Fractions

Many studies document PSTs' difficulties with making sense of fraction concepts and fraction computation, particularly related to fraction division (e.g., Ball, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Newton, 2008; Tirosh, 2000; Van Steenbrugge, Lesage, Valcke, &

Desoete, 2014). One possible cause for this enduring challenge is the prevalence of part-whole thinking in U.S. mathematics curriculum (Newton, 2008; Watanabe, 2007; Yang, Reys, & Wu, 2010). Part-whole thinking is a common way to define fractions: The fraction m/n is defined as m equal-sized parts out of n, where a total of n parts make up the whole (where m and n are positive integers). Using this fraction scheme, 3/4 would be thought of as 3 equal-sized parts out of 4.

Previous studies have established that PSTs rely primarily on part-whole thinking (Newton, 2008; Sowder, Bedzuk, & Sowder, 1993; Tirosh et al., 1998; Zhou, Peverly, & Xin, 2006). Although part-whole reasoning can provide an initial understanding of fractions, research has documented the limitations of part-whole reasoning (Mack 2001; Olive & Vomvoridi, 2006; Saenz-Ludlow 1994). One of the major limitations of part-whole reasoning relates to understanding improper fractions. Students typically struggle to reason with fractions greater than 1 when the only way they know to think about fractions is part-of-a-whole (Thompson & Saldanha, 2003). When 3/5 only makes sense as three parts out of five, it is difficult to make sense of 7/5, or seven parts out of five.

Besides the reliance on part-whole thinking, another possible cause for PSTs' issues with thinking about improper fractions as numbers is difficulty with coordinating multiple levels of units (Hackenberg, 2007). Construction of improper fractions as numbers requires the *interiorization* of coordinating of three levels of units (Hackenberg, 2007, 2010; Norton & Wilkins, 2012). *Interiorization* is the reorganization of internalized actions as an assimilatory structure; students who have interiorized actions (as operations) do not need to carry them out in activity (i.e., they can be taken as given prior to activity) (Olive, 2001).

Table 1: Fraction Schemes

Schemes	Associated Mental Actions
Part-Whole Fraction Scheme (PWS)	Producing m/n by partitioning a whole into n pieces and disembedding m of those pieces.
Partitive Unit Fraction Scheme (PUFS)	Determining the size of a unit fraction relative to a given unpartitioned whole by *iterating* the unit fraction to produce a continuous partitioned whole.
Partitive Fraction Scheme (PFS)	Determining the size of a proper fraction relative to a given unpartitioned whole by *partitioning* the proper fraction to produce a unit fraction and *iterating* the unit fraction to reproduce the proper fraction and the whole.
Reversible Partitive Fraction (RPFS)	Reproducing the whole from a proper fraction of it by *partitioning* the fraction to produce a unit fraction and *iterating* that unit fraction the appropriate number of times. Note that the action of partitioning implicitly involves splitting because partitioning is used to reverse the iterations of a unit fraction (e.g., 3/5 as three iterations of 1/5).
Iterative Fraction Scheme (IFS)	Reproducing the whole from an improper fraction of it by partitioning the fraction to produce a unit fraction and iterating that fraction unit fraction the appropriate number of times. Note that, in addition to splitting, this way of operating implicitly involves coordinating three levels of units: the unit fraction, the improper fraction, and the proper fraction contained within it.

Fractions Schemes and Operations

Steffe and Olive (2010) suggest a learning trajectory for students' development of fractions knowledge in terms of schemes and operations. The hierarchy of schemes is found in Table 1. In addition, two ways of operating also essential to this framework: 1) Splitting, which is the mental action of simultaneous *partitioning* and *iterating*, and 2) mental actions associated with Three Level Units Coordination (3UC).

Methods

Context and Participants

The participants were 109 undergraduates enrolled in the first of three required mathematics courses for PreK–8 PSTs. The focus of this first course was number concepts and operations, including fractions. The 109 students include only those taking the course for the first time. Of the 109 participants, about one-half were freshman (52%), about one-third were sophomores (31%), and the rest were upperclassmen (17%).

Data Collection

The participants were given a fractions schemes and operations assessment at the beginning of the semester, specifically before any instruction related to fractions or fraction computations. The assessment contained four items associated with each of the seven fractions schemes and operations (see Table 1), resulting in a total of 28 items. Each item was designed to provoke a response that would indicate whether or not the student had constructed that particular scheme or operation. These items were developed for and previously used with upper elementary and middle school students (Norton & Wilkins, 2012; Wilkins & Norton, 2011). However, PSTs have and tend to use knowledge that upper elementary and middle school students do not necessarily automatically use (e.g., fraction division algorithms). Therefore, to obtain a better understanding of the PSTs' ways of operating, the participants were also asked to provide a brief written explanation for their responses.

Two coders independently rated the responses using both the written work on the assessment and the brief explanations provided, according to Norton and Wilkins (2009, p. 156). Each item was given a score of 0, 0.4, 0.6, or 1 based on the amount of evidence observed for a given scheme or operation (see Table 2).

Table 2: Scoring Rubric

Score	Evidence Shown
0	Strong counterindication that the PST could operate in a manner compatible with that particular scheme or operation
0.4	Weak counterindication that the PST could operate in a manner compatible with that particular scheme or operation
0.6	Weak indication that the PST could operate in a manner compatible with that particular scheme or operation
1	Strong indication that the PST could operate in a manner compatible with that particular scheme or operation

Each coder summed the four individual item scores, resulting in an overall raw score between 0 and 4for each scheme or operation. The overall raw scores were then used to infer whether or not the PST had constructed that particular scheme or operation. Overall raw scores greater than or equal to 3 indicated that the PST had constructed that particular scheme or operation. Overall raw scores less than or equal to 2 indicated that the PST had not constructed that particular scheme or operation.

Overall raw scores between 2 and 3 required each coder to infer from all the given information whether or not the PST had constructed that particular scheme or operation. If a disagreement occurred between the two raters, the raters reexamined all the relevant information together to reach a consensus. (The average kappa statistic for measuring inter-rater reliability for 3UC was .81 and for IFS was .93.)

Data Analysis

First, descriptive statistics were calculated in order to compare the percentages of PSTs that had constructed each of the different schemes and operations. For this paper, we focus in particular on Three Level Units Coordination (3UC) and the Iterative Fraction Scheme (IFS).

Second, data were entered into 2 × 2 contingency table to examine the hypothesized association between 3UC and IFS. The gamma statistic, G, was used to test the magnitude of the association (Siegel & Castellan, 1988). We were specifically interested in testing whether the interiorization of 3UC preceded the construction of an IFS. Based on prior research (Hackenberg, 2007; 2010), we predicted a direct or positive association between the interiorization of 3UC and the construction of an IFS, specifically that the interiorization of 3UC occurs prior to the construction of an IFS. If the data associated with 3UC and IFS are consistent with this hypothesis, then we would find a positive direct association (i.e., $G > 0$) and a *weak monotonic relationship*. A weak monotonic relationship is characterized by a staircase pattern in the contingency table, with data falling predominantly in the diagonal and lower left cell. Because the G statistic is a symmetrical measure of association it does not by itself provide evidence of developmental order among the schemes and operations.

Following procedures outlined by Wilkins and Norton (2011) we tested for developmental order among the schemes and operations by first visually examining the table for evidence of a staircase pattern. Empirically, using a binomial test, we tested whether the difference in the number of cases in the off-diagonal cells was in the hypothesized direction and different from what would be expected by chance. We hypothesized a direct (positive) association between 3UC and IFS and also hypothesized a developmental order. For this hypothesis we used a one-tailed gamma and binomial test.

Results

In Table 3, descriptive statistics for 3UC and IFS are presented. Less than half of the PSTs had interiorized the coordination of three levels of units (47%). About a quarter of the PSTs had constructed an IFS (27%).

Table 3: Descriptive Statistics

Scheme/Operation	Percentage	*SD*
3UC	47%	0.50
IFS	27%	0.44

Based on our hypothesis, PreK–8 PSTs should interiorize 3UC prior to constructing an IFS. Table 4 presents the frequencies of PSTs' construction of an IFS and coordination of three levels of units. Overall, no association between the coordination of three levels of units and the construction of an IFS was found ($G = .23$, $p = .15$, one-tailed). An examination of the off-diagonal cases ($n = 48$) found 13 PSTs who had constructed an IFS prior to interiorizing the coordination of three levels of units. This is a relatively large number of cases that counter the hypothesis. However, the distribution of the off-diagonal cases was statistically beyond chance; exact binomial, $p = .001$ (one-tailed).

Table 4: Frequency of 3UC and IFS Scores

	IFS		
3UC	0	1	Total
0	45	13	58
1	35	16	51
Total	80	29	109

Note. $G = .23$, $p = .15$, one-tailed; Exact Binomial $p = .001$ (one-tailed).

Because the result from the gamma statistic is inconsistent with the theory relating 3UC and an IFS (Hackenberg, 2007; 2010), it is important to further examine the 13 students found to have constructed an iterative fraction scheme prior to interiorizing the coordination of three levels of units. One point of concern is that the fractions tasks used in the assessment were designed for upper elementary and middle school students. As previously discussed, PSTs have and use knowledge that upper elementary and middle school students do not automatically employ. For example, by using algorithms for dividing fractions or finding equivalent fractions, PSTs' procedural responses to the 3UC and IFS tasks, as well as their written explanations, may actually mask evidence providing an indication for (or evidence providing a counterindication against) interiorizing 3UC or constructing an IFS. For both the 3UC and IFS tasks, the 13 PSTs responses were re-examined to find patterns in their thinking and representations.

One observation found was that many of the 13 PSTs used either fraction division or equivalent fractions to answer the 3UC tasks. For example, Figure 1 shows how one PST used fraction division to answer a 3UC task. In her explanation, the PST described her thinking: "The pizza shows 3/4 of a pizza and each person wants 1/8 so I divided 3/4 by 1/8. I found the reciprocal making it 3/4 ÷ [sic] 8/1 and found that 6 people could get 1/8 of the pizza." Although her answer is correct, her procedural work and written description do not provide clear evidence that this PST is actually coordinating three levels of units.

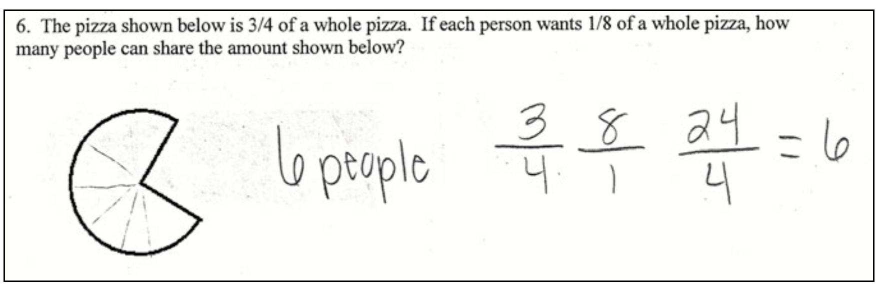

Figure 1. PST used fraction division to answer a 3UC task.

Figure 2 exhibits how another PST solved a 3UC task, this time using equivalent fractions. In her written work, she explained that she wanted to find out "how many eighths are in 3/4 so I multiplied top and bottom by 2 to reach eighths and I got 6/8. Therefore 6 people can have 1/8 of the 3/4 pizza." Again, the procedural work and written explanation do not provide evidence for or against this PST having interiorized 3UC.

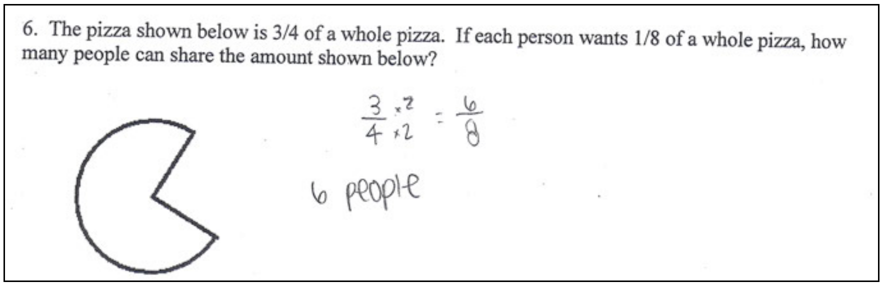

Figure 2. PST used equivalent fractions to answer a 3UC task.

Another observation found in the responses to IFS items was that over half of the 13 PSTs changed the given improper fraction to a mixed number to find an answer. Although these PSTs were able to determine the correct answer, it seemed that they would have been unable to do so without converting to a mixed number based on their written explanations. As an example, one PST (see Figure 3) wrote in her explanation, "I changed 7/3 to a mixed fraction and saw it was roughly twice the size as the candy bar so I split it into thirds and counted 3/3 to make one." If this PST had constructed an IFS, she would likely respond by partitioning the given stick into seven equal pieces and taking three of those pieces to represent the whole candy bar—a more efficient way of operating. In this illustrative work, it seems as though the PST relied on the mixed number of 2 1/3 to solve the task instead of considering the improper fraction of 7/3 as a number in its own right.

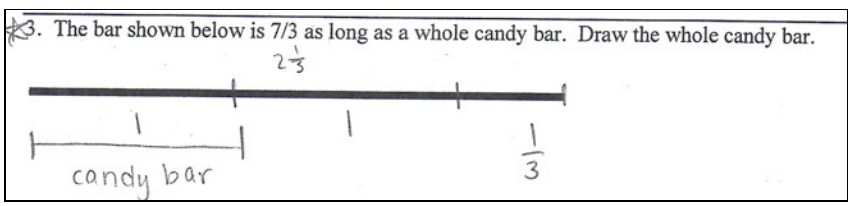

Figure 3. PST used a mixed number to answer an IFS task.

Also, almost half of the 13 PSTs examined demonstrated some level of confusion in their responses to IFS items. Even when they provided correct representations and answers, these PSTs exhibited a lack of confidence in their work. For example, one PST explained, "I'm having trouble understanding the amounts when the given amount is over 1." Another PST expressed confusion when the given diagram (which represented an improper fraction) was a half circle: "I got confused looking at the diagram because the picture doesn't look bigger than a whole." The confusion unveiled in responses like these raise further questions about these PSTs' construction of an IFS.

The analysis of the 13 PSTs' responses suggests that some PSTs may have actually interiorized 3UC, but their use of procedures and algorithms potentially mask the coordination of the three levels of units. In addition, some issues related to the IFS tasks, such as relying on mixed numbers, may have been overlooked and resulted in a false identification of that PST constructing IFS. As such, our results call for further investigation.

Discussion

We found that assessing the interiorization of Three Level Units Coordination (3UC) in PreK–8 preservice teachers is challenging. Primarily this challenge arose from the PSTs' automatized mathematical procedures that upper elementary and middle school students may still be in the process of learning. Instead of having to coordinate three levels of units on these tasks, PSTs may just be using their procedural knowledge for dividing fractions or finding equivalent fractions. These computational procedures mask evidence for or against the interiorization of 3UC. While PSTs may be able to use a procedure (see Figures 1 and 2) to find a correct answer, they often do not demonstrate evidence for or against their ability to view 3/4 of a whole pizza as three 1/4 pieces, where four such pieces would make up a whole pizza, and that each 1/4 piece contains two 1/8 pieces.

Likewise, we found assessing whether or not PreK–8 preservice teachers have constructed an iterative fraction scheme (IFS) to also be challenging. Two reasons for this challenge are the PSTs' use of mixed numbers and lack of confidence. Hackenberg (2007) claims that when students cannot consider an improper fraction as a number in its own right, and rather must change the number to a mixed number, then the student has not constructed an IFS. Some of the 13 students found to have constructed an IFS prior to interiorizing 3UC may have "fooled" us into believing they had constructed an IFS even though they relied on mixed numbers to reach (and represent) their solution. In addition, the PSTs were often confused about the problem statement as well as their solutions to IFS items. In retrospect, we wonder whether we should have inferred that a PST has constructed an IFS when they exhibit confusion and lack confidence in their solutions.

From part of the analysis of our data, it seems that some PreK–8 preservice teachers may have constructed IFS without the interiorization of 3UC. However, because of the problematic assessment of these items, as noted above, it is difficult to make a definitive conclusion. Because PSTs' difficulties with fraction concepts and fraction computations is an enduring challenge for researchers and teacher educators alike, this current research should be expanded. One starting point is to redesign the items used in our current assessment (which were originally designed for upper elementary and middle school students) so that PSTs cannot use procedures and algorithms to solve; instead, they must rely on their constructed schemes and interiorized operations. Another expansion of our research is to conduct clinical interviews with the PreK-8 preservice teachers to better understand whether or not they had truly interiorized 3UC or constructed an IFS.

In addition to the advancement of research, changes should also be considered in required mathematics courses for PreK–8 preservice teachers. For example, instead of using language that reinforces part-whole thinking (i.e., describing the fraction 4/5 as four equal-sized parts out of five), instructors can emphasize language that encourages a more iterative way of thinking (i.e., describing the fraction 4/5 as 4 equal-sized parts, each of which is 1/5 of the whole). In addition, instructors can incorporate more instructional tasks and activities that involve improper fractions, such as asking PSTs to model and describe improper fractions using representations or manipulative materials such as Pattern Blocks. These practices may help PSTs move towards interiorizing the coordination of three levels of units and constructing an IFS.

With the call for students as young as fourth and fifth grade to operate with higher-level schemes and operations (CCSSO, 2010), it is imperative that future PreK–8 teachers also be able to operate with higher-level schemes and operations. Both research and practice can help to accomplish this goal.

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SEEKING ATTENTION TO STUDENT THINKING: IN SUPPORT OF TEACHERS AS INTERVIEWERS

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In this paper, we analyze efforts to encourage teachers' attention to student thinking through a professional development (PD) program. We describe three groups of teachers within the same program who completed different types of assignments, either conducting interviews, planning classroom activities, or both. In both types of assignments, teachers were prompted to explicitly address student thinking. Teachers attended to specifics of student thinking when conducting and analyzing interviews, but struggled to do so when planning activities. While acknowledging the value of sustained attention to revision of lessons, reviewing classroom videos, and utilizing different forms of classroom discourse, we argue that conducting and analyzing interviews is an underused activity that can and should be an important part of teachers' professional development if we seek to encourage attention to student thinking.

Keywords: Teacher Education-Inservice (Professional Development); Classroom Discourse

Introduction and Theoretical Perspective

Student-centered teaching in mathematics classrooms has been at the heart of reform movements and curricula in recent years. We take the perspective that attention to student thinking is important, but that learning to notice and understand what students are doing is a difficult process. Recently, Schoenfeld (2011) has called us to consider how the practice of attending to student thinking may be developed. In the present paper, we respond to this call and argue that engaging teachers in conducting and analyzing mathematical interviews with students is extremely productive. To support this claim, we describe the work of teachers in an extended PD program that had the dual goals of enhancing teachers' mathematical content knowledge of functions and a functional approach to mathematics and enhancing teachers' understanding of students' mathematical thinking.

Attempts to enhance mathematics instruction in the United States (US) have embraced the ideas of student-centered or responsive teaching in varying ways. For example, Chapin, O'Connor, and Anderson (2009) describe types of classroom talk designed to elevate and utilize student contributions, such as restating student ideas in other terms and asking other students to address a proffered student idea. These strategies, and others designed to encourage rich student discussion in the mathematics classroom, are valuable, as is helping teachers to develop them. However, teachers cannot utilize these productively unless they are able to quickly understand what a student is saying or determine what questions to ask to clarify the student's thought. Thus, attention to and understanding of student thinking must underpin and accompany any strategies for encouraging classroom discussion.

Similarly, PD often includes activities in which teachers create, review, and revise lesson materials and implemented lessons. The Japanese "lesson study" model (e.g., Lewis, 2000) includes these kinds of activities, and they are the focus of the recent call from Hiebert and Morris (2012) to shift attention away from teachers and towards the artifacts of teaching, including constantly revised lesson plans. Hiebert and Morris state that a key feature in the plans would be that "students' likely responses to instructional tasks and questions are predicted to allow teachers to plan how to use students' thinking during the lesson" (p. 95). Here, they, too, lend their support to the importance of attending to student thinking. In both of these models, being able to understand the students is essential to the work that the teachers are being asked to do, and again must either precede or develop along with the focus activity of revising materials. There is also a logistical challenge to the idea of using lesson creation and revision as part of PD in the US. As seen both in lesson study and in Hiebert and Morris' vision, these are practices to be taken on and sustained by groups of teachers. This is at odds with most PD for in-service teachers. Teachers may be from different schools using different curricula; they may also have time allocated for PD only sporadically. As a result, any attempt to focus sustained attention on lesson plans is interrupted, as teachers are not able to repeatedly implement and revise these "artifacts." While some schools have recognized that teachers may benefit from shared planning time and collaboration with their peers, it remains to be seen how widely this will be sustained and whether or not researchers will be able to access these practices and determine whether and how they enhance teachers' attention to student thinking.

However, research has shown that teachers can change how they attend to student thinking after PD that encourages this attention directly. For example, the video clubs described by van Es and Sherin (2008) show that teachers engaged in discussing videos of their teaching shift what they attend to, or "notice." Past work from Carpenter, Fennema, Peterson, Chiang, and Loef (1989) showed the results of a PD course in which teachers learned about student thinking in mathematics. Their approach, Cognitively Guided Instruction (CGI), included examination of student explanations. The teachers who took the course had significantly higher scores in knowledge of student strategies for their students, as compared to a control group of teachers. In addition, the students of the teachers who had participated in the CGI PD spent significantly less classroom time on number fact problems, yet they did significantly better than the students of control group teachers on questions of this type on a standardized test.

This work has been continued by Jacobs, Lamb, Philipp, and Schappelle (2011), using resources from CGI to provide sustained PD that included examining written work and video from classrooms. They showed that teachers improved in the extent to which they could attend to student thinking after extended participation in the PD. They assessed this by evaluating each teacher's responses as the teacher watched a videotaped interview with a student. Since Jacobs et al. consider teachers' viewing of a video of an interview to be suitable ground to elicit and measure teachers' attention to student thinking, this implicitly supports our suggestion that teachers have much to gain from analyzing interviews. We suggest that this becomes an even more powerful activity when the teachers conduct the interviews themselves and when the interviews are used to develop, and not just assess, the practice of attending to student thinking. By carrying out the interviews themselves, teachers have the benefits of evaluating the resulting video, but also the experience of attending to and responding to the student thinking in the moment.

Interviewing has been a powerful tool for researchers to gain rich insight into student thinking and capabilities. However, we argue that we have not sufficiently tapped this resource for teachers. We take the position that interviewing, as outlined by Ginsburg (1997), is not so very different from the forms of classroom talk endorsed by Chapin et al. (2009). In order to use "productive talk moves" in the classroom, a teacher has to be in the position of attending to students' thinking. We argue that through interviewing, teachers can take advantage of concentrating on student's responses without the responsibilities of managing a classroom in order to (1) begin to build a knowledge base of student ideas around particular mathematical topics, and (2) develop skills in attending to student thinking that they can then extend into the classroom.

In this paper, we report on our work with middle school mathematics teachers from multiple districts and schools engaged in a single PD program. The program focused on deep mathematical content knowledge of functions and a functional approach to mathematics as well as attention to student thinking. To encourage teachers to attend to student thinking, assignments included watching and responding to classroom videos, examining pieces of student work and classroom transcripts, and responding to questions asking teachers to predict what they thought students would do with different mathematical scenarios. In addition, we provided teachers with assignments in which they could choose to plan a lesson or to conduct an interview with a student. By comparing cases of three groups who completed different kinds of activities, we illuminate some of the benefits of having teachers conduct interviews as a way to consider student thinking, as well as describe challenges inherent in using lesson planning with the same intent. We argue that conducting and analyzing interviews with students can allow teachers to examine student thought in depth and that this should be put to use in PD.

Method

The data for this paper come from a PD program that offers three graduate-level semester-long courses, conducted partially online. To date, the program has had three cohorts of approximately 60 teachers each, from nine school districts. Here, we analyze the work of some of the teachers in the third course of the first cohort.

In this course, we had four three-week units, each unit concluding with an assignment that was framed as "Engaging Students." For each of these four assignments, teachers worked in groups of two to four; each group could choose to conduct and analyze interviews with students, or they could create a learning activity. We did not require teachers to develop a full-class activity that would extend for an entire class period, nor did we require them to use any particular lesson plan format.

Table 1: Assignment details

First option: Interview	Second option: Activity plan
With your group, discuss and compare what you each found in your interview. Together, write a brief report on what you found in your set of interviews. Make sure you include the following in your group report: What did the set of your group's interviews show about students' ways of thinking about inequalities? What did students say or do that surprised you? Use evidence from the transcripts of the interviews to support your ideas. How might the students' ways of approaching this problem help or hinder their understanding of equations and inequalities in future mathematics? What more would you like to be able to ask your students in order to better understand their thinking?	What is one mathematical idea, preferably relating to this unit's content, that your students have difficulty with and that you hope to address with an activity? What understandings do you think your students already have that can form a foundation for improved understanding of that idea? What misunderstandings do you think your students have that may hinder their understanding of that idea? What would you like to know about your students' understanding of that idea? Focus on students' understanding of the mathematical idea, rather than (or in addition to) their performance of specific tasks or algorithms. Work together to design a single activity (not necessarily a complete lesson) appropriate to your grade levels that addresses the idea you identified, builds on their understandings, and/or

	addresses their misunderstandings. Design the activity so that you will learn something new about your students' reasoning.

The option to create an activity was intended as a way to allow teachers to apply their knowledge of student thinking to teaching practice. The text of the assignments is shown in Table 1.

In both cases, teachers were asked about student thinking and asked to describe their students' thinking about the content included in the interview or activity. PD facilitators gave feedback to assignments in writing via an online forum.

Table 2: Assignment choices

	Number of teachers	Grades taught	Unit 1	Unit 2	Unit 3	Unit 4
Group 1	4 teachers	5, 9	learning activity	learning activity	learning activity	learning activity
Group 2	2 teachers	7, 8	interview	interview	interview	learning activity
Group 3	3 teachers	6, 8	interview	interview	interview	interview

We chose three groups of teachers to analyze here. Group 1 chose to do learning activities for all four units. We selected this group because they were the only ones in the entire cohort who made this choice, not completing any interviews. Group 2 chose to do three interview assignments, followed by a learning activity for the fourth unit of the course. We selected this group, the only one to follow this specific pattern of choices, because we theorized that the interview assignments might better prepare them for the learning activity assignment. Group 3 chose to do four interview assignments. We selected this group because it represented the opposite end of the spectrum from Group 1. There were multiple groups (nine groups, out of nineteen in the cohort) selecting only interviews; from those, we chose Group 3 because they shared similar characteristics with Groups 1 and 2 as indicated through classroom observations carried out at the outset of the PD program using the Reformed Teaching Observation Protocol (RTOP) (Sawada et al., 2002). Together, these three groups illustrate a variety of paths followed by teachers.

For each group, one researcher reviewed the teachers' written analysis, the interview transcripts and/or activity plans, the online forum discussions, and the PD facilitators' feedback and produced a "thick description" (Geertz, 1973) of each group's work. Next, a second researcher reviewed each of the artifacts again and revised and added to the description.

The results we present below allow us to provide insights into the utility of these assignments and offer points of consideration for PD design.

Results and Implications

Analysis of the teachers' work showed that when teachers created learning activities, their attention to student thinking was not as detailed, not as specific, and did not form as substantial a portion of their work. They did address student thinking in some cases, but it was because the assignment required it. That is, they were responding to our questions in order to complete the assignment, not because attention sprang organically from planning an activity.

Group 1: Four learning activities – beginning to address generalities of understanding

In their learning activity for Unit 1, the group members planned an activity in which different types of candy represented positive and negative numbers and unknowns. However, it was unclear how the structure of the activity itself took into account their students' thinking. In response to the question, "*What understandings do you think your students already have that can form a foundation for improved understanding of that idea?*", they wrote only, "*Our students have the foundation of how solve an equation with the goal of isolating the variable.*" Here they made a general statement about what their students might be able to do, but did not provide evidence or specifics to support their claim. In response to the question, "*What misunderstandings do you think your students have that may hinder their understanding of that idea?*", they listed four related items: "*Vocabulary word - inverse // Inverse means the opposite in effect. The reverse of. // Some students lack the understanding of what an inverse operation is // Solve for an unknown within the equation.*" Here, they pinpointed the absence of a skill, but did not address what it is that their students *do* think or give any information other than the summary assessment. The Unit 2 work was similar.

By the Unit 3 learning activity, there is some evidence that Group 1 did try to respond to the instructors' requests that they provide more evidence and specifics about student understanding. They stated that, "*some of our students tend to be stronger graphing information than verbally interpreting it.*"In this statement, the group made some progress in attending to student thinking: they avoided listing mathematical competencies as "student understandings," as they did in Units 1 and 2, and instead addressed differences among students. The statement suggests that the group felt that their students had different competencies to build upon. However, note that the statement is still general and lacks supporting evidence. This continued in Unit 4, with general comments forecasting difficulties their students would have with the activity such as, "*Finding the connection between the word problem and the graph will be difficult for [the students].*"

Group 2: Three interviews and a learning activity – progress and regression

In conducting their interviews for Unit 1, Group 2 focused on correct and incorrect answers from students, following up more on incorrect responses and pushing for correct notation. While the correct/incorrect answers were also a focus of their written analysis, they did state that they understood why their student gave a particular incorrect answer: "*I feel that this was an overgeneralization of solving for the variable… As a side note, this makes sense to me. In the process of practicing the solving of inequalities, students made the observation that it 'was like solving an equation.'*" They also suggested a way to investigate the student's understanding: "*we would be interested in how this student would graph the inequality on a number line that has no values listed.*" Taken together, these two quotes demonstrate that Group 2was beginning to consider why students might give particular answers and how they might follow up on their perceptions of student thinking.

In Units 2 and 3, Group 2 continued to try to meet the assignment and instructors' requests for specific evidence and for details about student thinking. The group's primary focus was still on what students didn't understand in comparison to what they did; for example, they wrote, "*the 8th grade student didn't really to seem to understand the problem clearly.*" While they did not articulate details about students' thinking, they did identify specific issues and consider how to address them. In Unit 2, the group noted that a student seemed to have "*difficulty with the coordinate grid*," so their suggestion was to remove the grid and see how the student would approach the problem. In Unit 3, they directly addressed both their attempts to understand student thinking and their challenges: "*His misunderstanding of the line being diagonal confuses us. He seems to think the line on the apples graph is diagonal but the line on the oranges graph is not. (Refer to [line in interview transcript].) It was interesting watching him gesture with his hands to determine that both lines were indeed diagonal.*"

In Unit 4, with each teacher having previously completed three interviews, Group 2 decided to create a learning activity, asking students to find the length of the diagonal of a rectangle with an area of 90 cm^2 and length and width that were consecutive integers. While they included mention of having students explain their strategies, the background they gave on student thinking to justify the activity was quite general: "*Students are able to use formulas to solve problems as long as the problem is straightforward and there is only one step involved.*" The course assignment did not require Group 2 to implement and analyze the activity, but this group chose to do so. In the process, they tried to return to addressing specifics of student thinking: "*many of the students totally fell apart and started trying to do some different things instead of using the Pythagorean Theorem. One student in particular added 44 and 46 to get 90. A few other students tried to add some other numbers together. One student divided 90 by 2 to get 45 and then divided 45 by 2 again to get 22.5.*" They listed the student strategies, but did not try to determine why the students had used these, nor did they offer suggestions for what they might do when confronted with these student ideas, as they had done in Units 2 and 3.

Group 3: Four interviews – developing focus on individual students

As with Group 2, the Unit 1 interviews from Group 3 focused on students' correct and incorrect solutions. They had asked students to shade a number line to show a<12, and their analysis of the interviews included evaluative comments like, "*One student did not seem to understand what "less than" meant, he actually included 12 as a possible solution.*" While the students' accuracy was at the forefront, the group did demonstrate that they were considering specifics of what students were doing: "*Zero was a point of confusion for the inequalities. For example, when a student answered verbally he believed that all the solutions had to stop at zero, but when asked to place the "a's" on the number line he realized the solutions could include values less than zero.*" The Unit 2 work was similar.

In Unit 3, the group designed their own interview tasks. They stated that they designed this activity, which used drawings to depict a scenario that they then asked the students to graph, because the students struggled with graphing in the Unit 2 interviews and they wanted to think of ways to scaffold a graphing activity: "*we hypothesized that the images would aid the students in both activities if the student began with the image problem.*" They refer to specifics from their Unit 2 interviews as they're justifying their choice of topics here; however, when they shift to discussing the actual items, they revert to generalities. For example, to address the question, "*What understandings do you think your students already have that can form a foundation for improved understanding of that idea?*" they write, "*variables, linear relationships, coordinate plane*". The use of a list of topics, rather than what students understand about the topics, is similar to the work of Group 1, as described above. While the teachers in the group did go on to conduct the interviews, their analysis of the interviews was only a few sentences long, and mainly addressed whether or not the students thought the pictures were helpful.

In Unit 4, Group 3 again used the interview tasks supplied by the PD program. The group shifted away from their prior emphasis on correct answers, and mentioned wanting students to understand the full scenario in the problem. However, their statements often did not include specific supporting evidence; for example, "*Student 2 was able to independently arrive at two solutions and eventually was able to explain a deeper understanding of the scenario than student 1.*" After the departure in Unit 3, they again considered what individual students were doing, but they did not progress in providing evidence as much as Group 2 had.

Implications

As seen in the examples above, when teachers creating a learning activity did address student thinking, it was often generalized because they didn't have a case to examine. However, when teachers conducted interviews, we can see more instances where their analyses maintained a focus on the specifics of student thinking. For example, Group 2 wrote descriptions of the students' progression through the problems, pointing to specific moments in the transcript to justify times when they claimed the students were confused. In addition, while Group 2 exhibited attention to student thinking during the three interview assignments they completed, the activity plan that they devised for the fourth assignment, after the interviews, showed the same type of generalizations as seen in the work of Group 1. This shift away from attending to student thinking was also seen in the Unit 3 work of Group 3, in which they focused on designing interview tasks, rather than on conducting and analyzing their interviews. While Group 3 had not demonstrated the depth of attention to student thinking that we saw from Group 2, the Unit 3 work of designing the interviews seemed to detract from the attempts they had made in Unit 1 and Unit 2 to focus on specifics about what students were doing. While all groups show some evidence of shifts in their work, and all groups show room to continue to develop their attention to student thinking, these cases begin to illuminate the advantages of interviews.

In planning activities, we were asking teachers to recall student thinking from the past. Conversely, when the teachers conducted interviews, they had the video and written work, as well as recent memories. The specificity of what they had available to them allowed them to remain focused on students' thinking. While this specificity would also be a benefit of having teachers watch classroom videos, as in Van Es and Sherin (2008), the interviews afforded teachers the opportunity to focus on one student at a time without having to manage as many other tasks. This was the case both while conducting the interviews and while reflecting upon them. This advantage emerged in the example above from the Unit 4 activity plan by Group 2. When they reflected on their activity after implementation, they reverted to listing many student answers, rather than taking a deeper look at particular instances of student thinking.

Certainly, this analysis uses only a few cases of groups of teachers. Thus, it shows us what is possible when teachers conduct and analyze interviews, but of course it cannot conclusively say what the results would look like in a large-scale comparison of groups of teachers doing different series of assignments. It is also important to note that here we are analyzing teachers' work in groups, but each group has multiple teachers working together on joint final products. These joint analyses reflect the work of the group, but it's not possible to know how each individual contributed or what they might have done differently if they were working on the same task alone (see Bautista, Brizuela, Glennie, and Caddle (2014) for an examination of this issue). An additional complication is that when teachers chose to do interviews, the feedback from facilitators after each assignment was more consistently focused on the teachers' attention and interpretation of student thinking. When teachers planned activities, the facilitators tried to address student thinking, but also commented on the structure of the activity. This may be valuable feedback on an important task of teaching, but it means that facilitators devoted less time and space to supporting teachers' understanding of student thinking. As with the teachers, facilitators lacked a case to examine, and it adversely affected the support they were able to offer.

Researchers have much to learn about how to promote teachers' attention to student thinking, including how it may grow and evolve while teachers conduct and analyze interviews with students. While future work should include more rigorous examination of a larger number of groups of teachers, this analysis highlights some lessons for teacher educators and PD providers. Certainly, activity planning could be undertaken in a thorough and specific way, with constant revision, as well as videos of implementations, as suggested by Hiebert and Morris (2012). Equally, analyzing classroom video (e.g. Van Es & Sherin, 2008) can enrich teachers' understandings as well. Our analysis does not detract from these other approaches; rather, it shows the rich opportunities that arise when teachers conduct and analyze interviews with students. As mentioned, many PD opportunities involve teachers from different grades, different schools, or who are using different curricula. Interviews can be conducted and analyzed even within these constraints, and teachers' attention to student thinking can be enriched as a result.

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EXAMINING THE ROLE LESSON PLANS PLAY IN MATHEMATICS EDUCATION

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Formal lesson plans have long been touted as a best practice in mathematics teacher preparation. Experienced teachers frequently view formal lesson plans as nonessential to the planning, implementation, and evaluation of instruction. We discuss results from an online survey designed to make the perspectives of 60 prospective and practicing mathematics and special education teachers regarding lesson plans explicit. Practicing teachers identified their use of formal lesson plans as a reflective tool and for organization purposes, whereas for prospective teachers lesson plans served as a guide and for accountability reasons. Finally, we describe future mathematics teacher education engagements designed to promote productive yet practical perspectives of formal lesson plans.

Keywords: Teacher Beliefs; Teacher Education-Inservice; Teacher Education-Preservice

Introduction

Planning for a lesson has long been recognized as a primary factor impacting the efficacy of classroom instruction. According to Brahier (2013), "The effectiveness of a lesson depends significantly on the care with which the lesson is prepared" (p. 141). Focused lesson planning has been shown to support teachers' implementations of cognitively demanding tasks, help teachers anticipate students' cognitive challenges, and support the generation of questions teachers can ask that promote and elicit student thinking (Boaler & Staples, 2008; Franke & Kazemi, 2001; Henningsen & Stein, 1997). Smith, Bill, and Hughes (2008) assert, "One way to both control teaching with high-level tasks and promote success is through detailed planning prior to the lesson" (p. 133).

As introduced by Morine-Dershimer (1977) and described by Schoenfeld (1998), a teacher's lesson image is "the teacher's envisioning of the possibilities and contingencies related to a lesson" (p. 17). Furthermore, a teacher's lesson image includes the teachers' expectations for how students will engage with certain tasks or activities, what students might find straightforward or challenging, and potential student responses to the lesson's tasks and activities and how the teacher expects to deal with them (Schoenfeld, 2010, p. 233). As such, a teacher's lesson image incorporates everything related to how the teacher imagines the lesson will unfold (Schoenfeld, 1998, p. 18). Although the idea of a lesson image is more preponderant in literature with reference to experienced teachers, one need not have taught a lesson in order to have some image for how instruction might play out. Therefore, prospective and early career teachers should be motivated to imagine and anticipate how students might engage with instruction, envision the understandings and ways of thinking students might learn from alternative instructional approaches, and the ways in which discourse invites mathematical thinking and reasoning (Grouws & Shultz, 1996; Thompson, 2002).

One tool that encourages teachers to make their lesson images explicit, and potentially objects of thought and reflection, is the formal lesson plan. According to Brahier (2013), "A lesson plan…is a road map that can be used by the teacher to provide structure to the lesson" (p. 165). Furthermore, written lesson plans help motivate teachers to think deeply about their classroom tasks and activities and attempt to anticipate how students might interpret a task, the methods or strategies (correct and incorrect) students might use to make sense of the task and work toward a solution, and how those "strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn" (Smith & Stein, 2011, p. 8). As such, formal lesson plans permeate teacher preparation programs in general and mathematics methods classes in particular. As described by Kagan and Tippins (1992), "In virtually every teacher education program, considerable time is spent teaching novices how to write detailed, linear lesson plans" (p. 477).Although much less pervasive, research in professional development in mathematics education has included formal lesson plans as a data generating and analysis component (e.g., Burns & Lash, 1988; Morine-Dershimer, 1977; Smith, Bill, & Hughes, 2008). In addition, the creation and implementation of formal lesson plans and reflecting on how students engaged with instruction has been used with practicing teachers as a means to support teachers' development of instructional practices that promote "framing and solving problems, looking for patterns, making conjectures, examining constraints, making inferences from data, abstracting, inventing, explaining, justifying, challenging, and so on" (Stein, Grover, & Henningsen, 1996, p. 456).

Unfortunately, lesson plans have typically been viewed by teachers as a script or directions for executing a lesson that emphasizes procedures and structures, with "limited attention to how the lesson will help students develop understanding of key disciplinary ideas" (Smith & Stein, 2011, p. 76). According to Kagan and Tippins (1992), "Traditional university coursework may exaggerate the importance of daily lesson plans…[and] an emphasis on detailed written lesson plans may even be somewhat detrimental in that it masks the importance of improvisation" (p. 478). Moreover, research has consistently shown that "experienced teachers do not use written lesson plans…[and] at most…jot down an outline or list of topics to be covered during the lesson, using a cryptic shorthand" (Kagan & Tippins, 1992, pp. 477-478). Practicing teachers tend to regard formal lesson plans as useful only for student teachers or when they need to plan a new unit, perhaps with new standards, or as a required component of a formal administrative observation of their instruction (Kagan & Tippins, 1992).

Purpose of the Study

Disparity between prospective and practicing teachers, regarding the expectations of and value to developing, discussing, and revising formal lesson plans, highlights a need to better understand these distinctions from a situational perspective (Peressini et al., 2004), where teacher learning is "understood as a process of increasing participation in the practice of teaching, and through this participation, a process of becoming knowledgeable in and about teaching" (Adler, 2000, p. 37).In this article we present results from a study designed to better understand teachers' perspectives on the role formal lesson plans can and do play in the teaching and learning of mathematics. Specifically, the study was designed to address the following research questions:

- What are practicing (or in-service) teachers' perspectives on the role lesson plans play in their instructional practices?
- What are prospective (or pre-service) teachers' perspectives on the role lesson plans play in their instructional practices?
- How do practicing and prospective teachers' perspectives regarding lesson plans compare and contrast?

Methods

Study participants consisted of two samples: (a) 28 practicing teachers comprised of middle (grades 5-8) and secondary (grades 9-12) school mathematics teachers and intervention specialists (special education teachers); and, (b) 32 prospective teachers comprised of early childhood (grades K-3), middle childhood (grades 4-9), secondary (grades 7-12), and special education (grades K-12) license seeking teacher candidates. Potential participating teachers were emailed a link to an online survey designed to make explicit their perspectives on the role formal lesson plans play in their practice (see http://kentstate.az1.qualtrics.com/SE/?SID=SV_cCou1lOt7M930zj)

Survey Respondents (Study Participants)

The survey response rate for practicing and prospective teachers was 31.1% (28 of 90) and 20.1% (32 of 159), respectively. Practicing teachers ranged from first year math teachers to those with 30 years of experience. A comparison of the number of respondents by grade band, specialization, and teacher education program is displayed in Table1 below.

Table 1: Teacher Survey Participants by Grade Band, Specialization, or Program

Practicing Teachers	Number of Respondents	Prospective Teachers	Number of Respondents
Intervention Specialist	13	Special Education Licensure Program	21
Math Content, Grades K-2	0	Grades K-3 Licensure Program	3
Math Content, Grades 3-5	1	Math Content, Grades 4-9 Licensure Program	3
Math Content, Grades 6-8	5	Math Content, Grades 7-12 Licensure Program	5
Math Content, Grades 9-12	11		

Analysis

A situative perspective suggests that knowledge, beliefs, and practices are indissoluble from the situations in which they occur (Putnam & Borko, 2000). As such, learning to teach mathematics "occurs in many different situations—mathematics and teacher preparation courses, pre-service field experiences, and schools of employment" (Peressini et al., 2004, p. 67). The samples of teachers examined here represent individuals at different ends of a teacher-learning trajectory: teachers at the mathematics methods stage (prior to student teaching) and teachers at various levels of experience as practicing teachers.

In the following section, we examine and compare practicing and prospective teachers' perspectives on the role formal lesson plans play in their practices. We focus on two specific comparisons: (a) the role of lesson plans for prospective teachers, and (b) the role of lesson plans for practicing teachers. Analysis involved both qualitative and quantitative methods.

The role of lesson plans for prospective teachers. Survey respondents were asked their perspectives on the role lesson plans serve prospective teachers. Practicing teachers indicated a wide spectrum of perspectives. Sample responses included, "Prepares you to think about all of the things that can occur in a period…makes you start thinking about how to organize the time in class" and, "They help a pre-service teacher realize and get used to every aspect that is involved in teaching on a daily basis. It helps with time management and relating teaching to things that are meaningful in students' lives."

Individual members of the research team, which consisted of the course instructor and three graduate students enrolled in a graduate level course on mathematics education research, examined practicing teachers' responses and developed themes with which to categorize these responses (Strauss & Corbin, 1998). The entire research team then reviewed and discussed each category, category (theme) descriptors were made consistent, and teacher's responses were re-classified to support coding reliability. The final categories arrived at through examination and discussions align, to a degree, with Clark and Peterson (1986) "types" and "functions" of planning. In order to provide a better understanding of the categories (or themes) the research team settled on, it will be beneficial to exemplify what we considered a representation of each (Table 2).

Table 2: Sample Practicing Teachers' Responses in Regards to Corresponding Categories

Category (Theme)	Sample Practicing Teacher Response
Confidence	"The lesson plan can add to their confidence…"
Guide	"[Lesson plans give] you have a general idea of what you want to accomplish and how you are going to do it." "[Lesson plans] help them understand what they have to know."
Instructional Flow	"[Lesson plans] do help a new teacher understand the flow of the lesson..."
Keep on Track-Accountability	"[Lesson plans]…make sure standards are taught."
Keep on Track-Locally	"[Lesson plans] forced me to put things on paper, such as time spent on an activity…" "It helped me to have typed lesson plans early on because included everything I was to cover in the lesson."
Keep on Track-Globally	"[Lesson plans] help a new teacher understand the flow…of the week/month…[and] can help a new teacher…transition from one unit to the next."
Organization	"[Lesson plans] helped [me] to learn what goes where and how to find the resources." "[Lesson plans] help structure their day when feeling overwhelmed."
Reflective Tool	"[Lesson plans] prepare you to think about all of the things that can occur in a period." "Thinking about how to tie lessons to each other as well as the standards looking for connections."
No Productive Purpose	"None."

Prospective teachers also indicated a wide variety of perspectives regarding the role formal lesson plans serve a prospective teacher. Responses ranged from, "They make you thoughtfully decide what to teach and how to teach it so that it would be effective," to "They serve as a guide to the teacher so they can accomplish what they want, and do it in an organized fashion."

The process by which the research team categorized prospective teachers' responses followed the same stages as described above for practicing teachers to support coding reliability. After individually examining and categorizing teachers' responses, the entire research team reviewed and discussed each response and its categorization, re-classifying responses as needed. The percentage each category (i.e., theme) was indicated by prospective and practicing teachers is displayed in Figure 1.

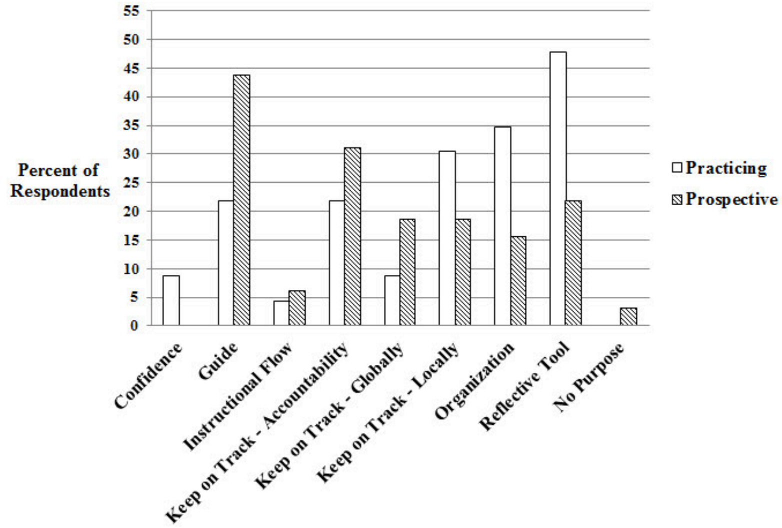

Figure 1: Practicing and Prospective Teachers' Responses to the Role that Formal Lesson Plans Serve Prospective Teachers

There were substantial differences between the percentages of practicing and prospective teachers' responses for "Reflective Tool," "Guide," and "Organization." Practicing teachers indicated formal lesson plans serve prospective teachers as a "Reflective Tool" 26% more and "Organization" 19.2% more than did prospective teachers. Taking into account that our sample of prospective teachers had been limited, in general, to observing K-12 instruction and tutoring individual students as part of their respective prior and current mathematics methods course field experiences, it is not surprising they would identify formal lesson plans as a "Reflective Tool" to a much smaller degree than practicing teachers. Regarding "Organization," practicing teachers used the term in the sense of helping a prospective teacher "prepare" or be "well planned" for a lesson. As observers or class tutors, our sample of prospective teachers would have limited understandings of how and what to prepare for pragmatically. Therefore, it seems reasonable that prospective teachers would indicate "Organization" to a much smaller degree than practicing teachers.

Prospective teachers identified "Guide" as a role formal lesson plans serve prospective teachers (i.e., themselves) 22% more than did practicing teachers. Prospective teachers used the term "Guide" in ways similar to how practicing teachers employed the term; that is, in very general ways. For example prospective teachers' responses included, "[It] will be a guide to help with my instruction"; whereas sample practicing teachers' responses, included, "[A] basic outline." Both groups of teachers used the term "Guide" in the sense described by Kagan and Tippins (1992), where a teacher simply "jot[s] down an outline or list of topics to be covered during the lesson, using a cryptic shorthand" (p. 478). Although our sample of practicing teachers may plan their lessons mentally, without committing much to paper as described by Kagan and Tippins (1992, p. 478) and suggested by their identification of a lessons plan as a "Reflective Tool," prospective teachers (especially at the mathematics methods stage) lack the experiences to think of lessons in terms of students developing understandings and skills, rather than in terms of topics.

The role of lesson plans for practicing teachers. Survey respondents were asked their perspectives on the role lesson plans serve practicing teachers. Practicing teachers indicated a wide range of perspectives regarding the role that formal lesson plans serve a practicing teacher (i.e., themselves). Sample responses included that lesson plans "help better organize the teacher and to keep track of what they taught or modified, and what is working and not working" to "I feel it is burdensome." Prospective teachers also indicated a wide array of perspectives regarding the role they envisioned formal lesson plans serving in their future as a practicing teacher. Responses ranged from, "They will help me improve my teaching by allowing me to look back at what I taught and fix my mistakes. It is a way to better my teaching," to "Formal lesson plans will be a requirement that I will do because it is required but not because it is valuable to me or my time."

Individual members of the research team examined and categorized teachers' responses using those categories (or themes), if possible, described earlier in Table 2. Next, the entire research team reviewed and discussed each response and its categorization, re-classifying responses as needed. These discussions again supported coding reliability. The percentage each category (i.e., theme) was indicated by prospective and practicing teachers is displayed in Figure 2.

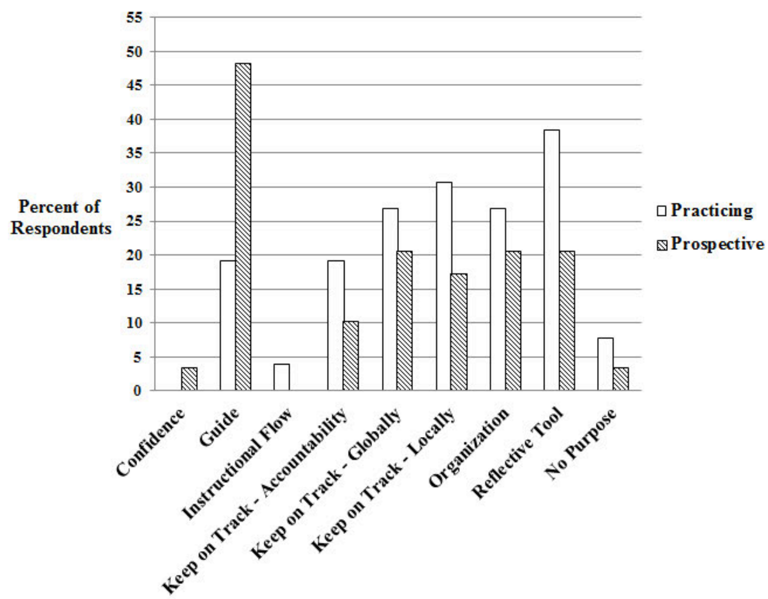

Figure 2: Practicing and Prospective Teachers' Responses to the Role that Formal Lesson Plans Serve Practicing Teachers

There were substantial differences between the percentages of practicing and prospective teachers' responses for "Guide" and "Reflective Tool." Prospective teachers indicated formal lesson plans serve practicing teachers as a "Guide" 29% more than did practicing teachers. As with teachers' responses to the role lesson plans serve prospective teachers discussed in the previous section, both groups of teachers used the term in very general ways, as an outline or list of topics to be covered during the lesson (Kagan & Tippins, 1992). Practicing teachers identified "Reflective Tool" as a role formal lesson plans serve practicing teachers (i.e., themselves) 17.8% more than did prospective teachers. As indicated in the previous section, such differences could be accounted for by prospective teachers' lack of experiences at designing and enacting instruction.

Discussion

In this report we described and compared prospective teachers' (at the mathematics methods stage of their respective licensure programs) perspectives of the role formal lesson plans can and do serve in mathematics teaching and learning with practicing teachers' perspectives. Analyses of teachers' responses to survey questions designed to make teachers' perspectives explicit indicated that our sample of prospective teachers had reasonable perceptions of district and school expectations they will encounter, regarding lesson plan requirements, as early career teachers—at least compared to our practicing teacher sample. In addition, we described how analyses suggest that lesson plan activities for prospective teachers at the mathematics methods stage should: (a) promote and reinforce a focus on student thinking and learning, rather than a focus on covering topics; (b) minimize the potential for interpretations that convey formal lesson plans as something done simply by mandate; and (c) model and engage teachers in authentic planning, enactment, and reflection sessions. Furthermore, analyses suggest that universities and licensure programs should seek consistency in their mathematics methods courses regarding: (a) resources faculty promote to their students (i.e., prospective teachers) and (b) the amount of time prospective teachers should anticipate spending developing and revising their lesson plans once they have entered the field.

Prospective teachers' inclination to view formal lesson plans as a "Guide" aligns with Kagan and Tippins (1992) suggestion that lesson plans be defined as a brief outline of instructional procedures to be used to supplement teachers' guides and other curricular materials and resources (p. 477, 488).Rather than pushing for lesson plans to be viewed as a "Reflective Tool" or a means to keep instruction "On Track," as identified by our sample of practicing teachers, mathematics methods instructors should allow for students (i.e., prospective teachers) to initially view lesson plans an outline or guideline. According to Kagan and Tippins (1992), once enacted, these lesson plans should be revised to reflect the "spontaneous modifications that occurred during class (p. 488)," thus becoming a record of interactions. Such a process has the potential to promote a more natural transformation of prospective teachers' perspectives of the utility of formal lesson plans toward student learning; thus, supporting prospective teachers' development of productive lesson images. It seems reasonable to expect prospective teachers' experiences at developing lesson plans, attempting to enact lesson plans, and reflecting on these attempts to vary somewhat across licensure programs and universities. The number and content of mathematics methods courses prospective teachers take, the amount and context of field experiences, and the faculty assigned to teach mathematics methods courses all have significant impact on these experiences. Results presented here do not address these distinctive experiences. Future research should explore how such potentially disparate experiences impact prospective teachers' expectations of the realities of mathematics teaching.

With a situative lens, a focus on teachers' perspectives regarding formal lesson plans supports the development of models of teachers' understandings and ways of thinking at two distinct points (i.e., contexts) along a teacher-learning trajectory: the mathematics methods stage, prior to student teaching, and the practicing teacher stage. Although each of these "stages" is idiosyncratic, with the practicing teacher stage itself encompassing a continuum of experiences and contexts, such a focus supports the development of productive learning-to-teach situations for prospective teachers. Such situations have the potential to be successfully re-contextualized in prospective teachers' future K-12 classrooms (Peressini et al., 2004, p. 70).

Finally, this study did not include one important set of data points, those of prospective teachers' perspectives during student teaching. As such, future research should explore teachers' perspectives on the role formal lesson plans serve at three distinct stages of a teacher-learning trajectory: prospective teachers enrolled in program-specific mathematics methods courses, prospective teachers during student teaching, and practicing teachers—including those teachers serving as cooperating or mentor teachers during student teaching.

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FINDING VOICE: TEACHER AGENCY AND MATHEMATICS LEADERSHIP DEVELOPMENT

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In the course of engaging with a Mathematics and Science Partnership (MSP) project, we planned a yearlong Leadership Academy that fit under the "train the trainers" model for professional development. Midway through, teacher leaders rejected a traditional conception of leadership based on expertise, individualism, and the transmission of knowledge. This proved to be a moment of critical mass wherein the project was democratically reinvented around a model of shared leadership where teacher leaders were positioned as ambassadors of a culture of inquiry. We were able to document three catalysts for this shift as well as the conditions that existed such that these catalysts could prove effective at producing change. We present here our findings on what happens when we allow teachers to take power and experience agency in a teacher leadership development program.

Keywords: Teacher Education-Inservice; Teacher Knowledge; Elementary School Education

Introduction

Answering the RFP for a state MSP grant, we planned a yearlong Leadership Academy that fit under the "train the trainers" model for professional development. We proposed to engage nine teacher leaders in an intensive study of their own practice and the Measurement and Data strand of the Common Core State Standards for Mathematics (CCSSO, 2010). At the conclusion of their study, the Teacher Leaders (TL) would plan and enact a two-week-long institute for their colleagues in the district. Our proposal adopted a leadership development stance akin to a game of telephone. As Mathematics Teacher Educators, we understood teacher leadership as an automatic byproduct of providing long-term, high-quality professional development to teachers and giving them a platform to share what they'd gained with others.

We were shocked mid-way through when teachers who had been happy to participate in our curricular activity balked at the thought of identifying as teacher leaders. One wrote, "I am not sure that I really want to become a teacher leader…just because you try something doesn't make you an expert and teachers may be afraid of being perceived as representing themselves in that way."

In this moment, we confronted a living contradiction (Whitehead, 1989) in our assumptions and stance. At all times, we believed that teachers deserved agency and voice and intended to empower them as instructional leaders. However, the game of telephone positioned teachers as message receivers and ourselves as the adjudicators of expertise (a title that the teachers rejected) and denied teachers the very things we had intended the program to develop.

We were able to recognize the contradiction in the midst of our project having previous experience with such work (Cox et al., 2014). Our story focuses on the impact of that dissonance and its role in our process of reconstructing our leadership development program midstream. We will present here our findings on what happens when we allow teachers to take power and agency in a Teacher Leadership Development (TLD) program.

Leadership Development

Current reform initiatives have led the mathematics education community to consider the professional development of teachers as leaders. There are multiple models of leadership development and relatively little research on the effectiveness of these models. As such, we have made decisions to situate our work at the intersection of best practices for professional development of teachers of mathematics and their development as teacher leaders. With current demands for professional development programs that will reach large numbers of teachers, the idea of teachers themselves participating in the professional development of their colleagues is appealing to the mathematics education community. Koellner, Jacobs, and Borko (2011) call for the purposeful preparation of a cadre of leaders who can implement effective high-quality PD. They identify three features that are critical for the effective preparation of these leaders, that coincide with the features of quality PD for teachers: "(1) fostering a professional learning community, (2) developing teachers' mathematical knowledge for teaching, and (3) adapting PD to support local goals and interests" (pg. 116).

Like so many professional development providers Koellner, Jacobs and Borko (2011) frame their PD around three design principles: "fostering active teacher participation in the learning process, using teachers' own classrooms as a powerful context for their learning, and enhancing teacher learning by creating a supportive professional community" (p.117). Engaging teachers in worthwhile mathematical tasks, creating opportunities for reflection on their learning, supporting teachers to critically analyze their practice, analyzing student work, considering a multiplicity of teacher moves as possible for a given teaching episode, are all examples of the types of engagements used by mathematics teacher educators to create a high quality PD experience for teachers that attend to these three principles.

Underlying a large number of leadership development projects we find a predominance of a design principle where teacher leaders experience, as learners, the PD that they will provide to others. In general, a collaborative and problem-based approach that is situated in problems of practice are used. These approaches are identified by Davis, Darling-Hammond, LaPointe, & Meyerson (2005) as features of high quality leadership development programs as they discuss the preparation of highly effective school principals.

This theory about the development of leaders produces leaders who exemplify expertise, who stand apart from other teachers whom they call colleagues, and who take on a separate role as "leader" that may include duties such as facilitating district professional development, coaching, classroom observation and peer review of teaching. In contrast, the shared leadership model suggests that teacher leadership is less about standing apart and more about standing beside. Schlechty (2001) describes shared leadership as, "less like an orchestra, where the conductor is always in charge, and more like a jazz band, where leadership is passed around among the players depending on what the music demands at the moment and who feels most moved by the spirit to express the music," (p. 178).

In our experiences, shared here, we describe the dissonance that is created when a leadership group favors a shared leadership model, but engages in a professional development experience aimed at establishing expertise and elevated status.

Project Background

Our MSP was designed to meet the recommendations of best practices in the development of teacher leaders as described by the federal MSP program. We began with the selection of a small cadre of teacher leaders who would remain classroom instructors throughout and after the project. From there, we developed a three-strand curriculum (content, pedagogy, leadership) to be delivered in a Leadership Academy conducted over the course of a school year. We adhered to three principles when designing the curriculum:

- Teachers should encounter a variety of engaging and interactive activities (c.f. Even, 1999; Nesbit et al., 2001) conducted in a constructivist environment (Khourey-Bowers et al., 2005).

- Learning should be individualized and grounded in professional inquiry and specific classroom practice (Khourey-Bowers et al., 2005).
- A learning relationship should be at the heart of the academy and focus should be placed on establishing and supporting partnerships between individual teachers and university partners.

Our Academy met six times in cycles, each cycle having the same basic structure. We'd start each cycle with a seminar where we met as a whole group including nine teacher leaders, six university partners, and one evaluator. In the weeks following a seminar, we'd spend time on individual reflection and meet in partnerships. The Academy would culminate in a 60-hour Summer Institute for math/science teachers in their district. At this institute, teacher leaders would engage their colleagues in conversations about their mathematical and pedagogical work in the Academy as well as in questions of future practice.

University partners were thoughtfully matched (prior to our first seminar) to teacher leaders based on pedagogical interests or expertise and the personal preferences of faculty members about school or geographical location. Thus, one-on-one relationships were established early in the project and remained consistent throughout, documented as a best practice (Howe & Stubbs, 2003). We designed a series of partnership activities. From mathematical problem solving to curriculum alignment activity, to conducting mini action research projects into student reasoning (Even, 1999), to the activity of facilitating public discussion (Harris and Townsend, 2007; Howe and Stubbs, 2003; and Nesbit et al., 2001), our teacher leaders would be immersed in inquiry and intellectual exploration.

Measurement and data and the eight mathematical practices within the CCSSM (CCSSO, 2010) were selected by district administrators as target areas for district improvement. Thus, those two areas formed the mathematical basis for our seminar curriculum. We had intended there to be a two-way flow of influence between the seminars and the partnership activity. The interplay of ideas between large and small groups, 1.would keep our curriculum and professional learning grounded in real practice; 2. help the group note contextual similarities and differences; and 3. scaffold learning for all participants. We had intended to establish a professional community of practice (Lave & Wenger, 1991) for the teachers. Teacher leaders would gain authority, capable of establishing a line of inquiry (mathematical or pedagogical), pursuing that inquiry alongside partners, and presenting their findings to a larger group. In this way, teacher leaders would be empowered to be reflective practitioners and to engage in a rigorous examination of their own practice and the mathematical culture of their buildings and district.

We intended the summer institute to be a replication of the activity of the Academy on a smaller scale. There would not be enough time for individual teachers to conduct their own inquiries into classroom practice, but they could benefit from the experiences and data collected by the teacher leaders in the Academy. Like the transitive property or a *game of telephone*, we expected the lessons of the Academy would translate to good district-wide professional development, an expectation supported by Wallace et al. (1999) and Miller et al. (1999).

Methodology

Presented here is one slice of a two-year project aimed at uncovering what happens when we allow teachers to take power and agency in a Teacher Leadership Development (TLD) program. Data relevant to this study were taken from transcripts of LA Seminars, partnership meetings, and also teacher blogs and journaling responses. Data spans the first year of our MSP project.

Data were transcribed and analyzed by the researchers. Each researcher conducted multiple readings and interpretations using a hermeneutic process (Kinsella, 2006). Interpretations were substantiated and validated as we contrasted our interpretations and sought common ground, thus enriching our understanding of our teacher and the development of their identity as leaders.

Findings

Data shared here come from three key episodes in our project. We refer to these episodes as *catalysts* because of their potential to leverage change. The three episodes, *Mentoring Peers?, Angela's Rejection of Leadership*, and *Leading Through Practice* stand in concert with other events, however stand apart in the sense that they had a profound impact on the planning and enactment of our Leadership Development Program.

Mentoring Peers?

In our third seminar, we asked TLs to consider leaders within their communities, district, and lives. We invited them to describe their mentors and asked, " To whom do you listen?" Each of us shared anecdotes about those in our lives who had been our teachers and when we felt we had strong relationships with mentors.

In the ensuing conversation, TLs shared their previous experiences with leadership models (such as state mentorship programs) that had indicated to them that assigned partnerships rarely produced chemistry and that the mere act of assignment destroyed a part of the intended relationship.

When TLs imagined leaders or described individuals who had an impact on their own practice, the relationships seemed more serendipitous and unplanned. Leadership happened in the small conversations outside classroom doors (Terry), in the examples witnessed by accident (Marissa), or in the lessons that were co-planned, co-constructed, and co-enacted by two eager colleagues who were not organized into a false or assigned hierarchy (Molly and Shannon).

Furthermore, teachers challenged the notion of models of leadership that implied a mentoring experience with peers. Angela described her experiences as a mentor when she was asked to help her team implement what she had learned in a professional development experience about Professional Learning Communities (PLC).

> "We have weekly PLCs and at my grade level they don't work. I attended the training in Saint Louis about 5 years ago but no one else in my grade level did. It didn't work well when we tried to come back and tell the other teachers what we had experienced. We basically pretend each week to collaborate in a meaningful way but teachers are not invested in the process." (Angela)

Angela's Rejection of Leadership

Reflecting on the conversations about mentoring in her blog post, Angela provided an abrupt catalyst for change. Angela called into question the model of leadership that had been here-to-fore assumed (emphasis hers).

> "I am not sure that I really want to become a teacher leader," wrote Angela, "I think that people choose the "mentors" that they have in their lives and I am very leery of being in a position of "leadership" when it has been forced upon ANY one."

She had begun to question her position within the district and as a "MSP Leader". Angela equated leadership with expertise. She was adamant that she did not want to be perceived as someone with expertise--or more specifically, as someone who *considered herself an expert.* Angela was worried that, by association with the MSP, others would assume that she was an expert, or even worse, assume that she thought that she was an expert.

> "As I watched the presentation given by two teachers today, I thought of all the teachers in the district who have done similar things for much longer than twice and somehow putting people and ideas in front of others is in some way suggesting that they are experts. Could this be why

no one wants to be watched teaching? Just because you try something doesn't make you an expert and teachers may be afraid of being perceived as representing themselves in that way. We are always trying new things and how many of our new ideas really stand the test of time."

In this same time period, in a small group meeting, Angela expressed a new vision of what leadership might entail. She later wrote, "I don't want to be a part of a group of leaders. I want to be a part of a leadership group." To unpack this statement as a unified group and eventually negotiate what leadership would mean to us would mean redefining leadership and constructing a new purpose for our MSP project.

Leading Through Practice

In seminar one, we were still playing the role of the PD providers, wanting to plan and account for every minute of the four hour block of time we had set aside. Our rationale -- teachers' time is valuable, we need to make sure they take away something worthwhile from a Saturday morning spent with us. We decided on an activity that seemed important for the roles of leaders. We decided to collaboratively create a timeline of children's experiences with measurement (the mathematical content focus of our funded PD project) when going through their school district. Following this activity, time was allotted for teacher leaders and university partners to begin to articulate inquiry projects as they considered the teaching of measurement.

The rich conversations that ensued, the difficult questions that were asked, the thoughtfulness with which the group considered the timeline of experiences were beyond our expectations. While staying within the predetermined activity, teachers used their voices to provoke shifts in the agenda. Advocating for herself, one teacher (Ellen) remarked that it made no sense for her to engage in a cycle of inquiry about student reasoning and thinking about measurement and data since she wasn't slated to teach the topics. Another group of teachers (Molly, Shannon, Christine, Anne) noticed within the timeline the activity of measuring one's foot in Kindergarten, 2nd grade, 3rd grade and again at fifth grade. The group, rather than jumping to conclusions, wondered whether that was needless repetition or if there was something useful to gain. They proposed it as a question for the larger group.

It was clear to us that the teachers knew their curriculum well, they had given much thought as to the placement of topics in their own grade level, but had never had the opportunity to hear from their colleagues and create a mental picture of the vertical alignment of a child's experience. We were fortunate to have the voices of a few participants who had experiences in different grade levels -- either because they had taught at different levels and recently been reassigned (Ellen, Molly, Shannon), or because they were intervention specialists (Marissa) and lived in multiple classrooms of different grade levels. This activity generated an opportunity to share their expertise across grade levels and across buildings.

In addition to advocating for an individualized approach to our planned cycles of inquiry, teachers used a collective voice to exert pressure on our planning of future agendas for the Academy. During seminar two, Marissa and Terry offered to share a brief "fifteen-minute" story about an episode that occurred in Terry's classroom in the time between our meetings. Their short story evolved into a two-hour discussion with audience participation. Their peers' interest in their work was a great surprise to them. The many questions and the request for ideas as to how they would recommend modifications for the task for other age groups generated a rich discussion and positioned Marissa and Terry as having pieces of practice that were worth sharing and discussing. What began with excitement about what had transpired in their classrooms turned into a curricular experience for others where we continued to interpret and analyze the event. The result was empowering, not only to them, but also to others who then followed their lead. The remaining seminars were filled with such presentations and examinations of practice.

Discussion

Allowing these catalysts to have an impact on the structure of our MSP required three conditions: 1. a culture of non-evaluative listening; 2. the willingness of all participants to unpack our expectations and experiences with leadership; and 3. a conscious choice on the part of university partners to equalize the power relationships within the partnership. We will establish each of these conditions below along with how they facilitated change.

First, we were predisposed to listen to participants and hear their comments without judgment. We made the conscious choice to step back from our planned curriculum and forward into a space that allowed for multiple funds of knowledge (Moll, 1992). In stepping back, we gave teachers agency to construct alternative paths based on professional motivations and interests.

Each participant in the Academy began to shape a vision of what it would mean to take part in the MSP and what we would be doing during the Academy. The original plans for the Academy quickly disintegrated giving way to a new concept and a new reason for being. We are not naive to think that all of us shared a common vision, but our visions were converging to at least a few common aspects, these being that the teachers would be presenting for a good part of the Academy seminars, and also that issues were arising in the smaller partnerships that were valuable to bring to the large group for their deliberation.

Second, we committed to the work of consciously unpacking our expectations and experiences with leadership. This required the participation of every person involved. It allowed for a shift from thinking about leadership as belonging to or living within the individual toward thinking about the power of the collective. As a group, we have more power than as individuals to change perception, to lift our voices and have our ideas heard by others in our community. This requires that we shift our perception of leaders as those with expertise toward where leaders are willing to seek a deeper understanding and share that inquiry process publically with others.

Marissa and Terry were moved by the spirit that Schlechty (2001) describes and we, the participants, were inspired by their music to ask our own questions, to examine our own lived classroom experiences, and to modulate their melody in our own style and timbre. Their experience, because it was shared, reverberated throughout our group.

Third, as members of this partnership who had the power to determine the curriculum and structure of the project, we, the university partners, consciously chose to equalize the power structure and to give teacher leaders agency in their own professional development. At the outset, we did not view teachers as agents of and for change, but as agents that needed to be changed (Roesken, 2011).

As a result, in the course of the Academy, participant (university partners and teacher leaders) roles in PD began to blur and shift. From recipients or attendees of PD, teacher leader roles would morph to planning and delivering PD themselves. Our roles as university partners morphed from the planners and deliverers of PD to a place where we were active participants reflecting on our personal practices as teachers of teachers as well as grounding our practice in the real lived experiences of local classrooms. In this way, we were able to define expertise to include multiple funds of knowledge and in blurring the lines between the providers and the receivers, the teacher leaders have autonomy in crafting, (re)shaping, and enhancing their practice (Diaz-Maggioli, 2004). In as much as the teacher leaders perpetuating this model in the summer institute, they passed that autonomy on to their colleagues within the district as well.

Conclusion

In the course of this project, teacher leaders found professional voice, not only defining for themselves what needed to be accomplished beyond classroom doors, but also envisioning their role in accomplishing it. This has direct implications for personalizing local and state reform policies and implementing the Common Core State Standards. Ultimately, what we have learned is that it is not a matter of trusting that teachers will be productive with neither assigned direction nor oversight. It is the act of not even questioning them that ensures both agency and voice.

As a group, we also reformulated what it means to lead. What began as a game of telephone morphed into a model of shared leadership. Teacher Leaders became ambassadors of our culture of inquiry, but also remained a part of the community within their buildings and district. This was an integral facet of the experience for the teachers like Angela who did not want to be perceived as held aloft as experts in their field, entirely distinct from their colleagues. By establishing an environment where teachers had agency, we had opportunities to come to know ourselves in relation to one another. We realized that we are not only individuals, but are also framed by our participation within the community. Leadership identities were shaped by our shared experiences, but also by coming to grips with our histories in the district, in our schools, and by coming to understand how we are viewed by others because of how we use and have used our professional voice(s).

Lastly, by challenging conceptions of expertise and including multiple funds of knowledge, Academy learning opportunities were based on the realities of the classroom and occurred naturally in the course of investigating classroom practice in small supportive partnerships. This learning was rigorous, supported discussion of content and pedagogy, and had an immediate effect on classroom activity.

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PRESERVICE TEACHERS' STRATEGIES TO SUPPORT ENGLISH LEARNERS

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Although English language learners (ELLs) are one of the fastest growing groups of students in the United States, many teacher preparation programs have yet to require preservice teachers (PSTs) to receive training in effective practices for teaching ELLs. We examined a four-week field experience pairing an elementary PST with an ELL. We examined the strategies PSTs used to support ELLs as they implemented cognitively demanding mathematics tasks. Through interviews, observations, and written reflections, we found that the PSTs tried to support students, with varying degrees of success, by allowing for multiple modes of communication, including visual supports, pressing for explaining, and checking for understanding. Implications for teacher preparation are discussed.

Keywords: Equity and Diversity; Elementary School Education; Teacher Education-Preservice

Recent curricular reforms have emphasized the importance of engaging all students in rich mathematical activity (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO), 2010). Rather than passively participating in mathematics classrooms, children are expected to actively engage in the mathematical practices. The shift toward engagement in the mathematical practices has come with an increase in the linguistic demands of the mathematics classroom. For example, students are expected to justify their solutions and to critique the reasoning of others. To engage in these practices requires extensive communication skills in not only everyday English, but also the academic language of mathematics.

In conjunction with curricular reforms, the demographics of U.S. public schools are undergoing significant changes. English language learners (ELLs) are one of the most rapidly increasing groups of students (Wolf, Herman, & Dietel, 2010). In states such as Texas, California, and Florida where ELLs have made up a sizeable portion of the school population for a number of years, teacher education programs are required to prepare preservice teachers (PSTs) to teach ELLs. Many states with historically low numbers of ELLs are now experiencing dramatic increases in their ELL population. This has driven the need for these states to consider ways to prepare teachers to work with the realities of today's classrooms.

Despite estimates that nearly every teacher in the U.S. now has at least one ELL student, teachers are still underprepared to teach ELLs. In 2008, Ballantyne and colleagues found that only about a third of teachers had received training in strategies to support ELLs. The need for greater attention to strategies for teaching ELLs is further evidenced by the persistent achievement gap among ELLs and their native English-speaking peers (Fry, 2008). There is widespread agreement that teacher education programs must prepare PSTs to work with ELLs. Increasingly, researchers in mathematics education have examined PSTs' work with diverse groups of students (e.g., Aguirre et al., 2012; Foote et al., 2013; Turner et al., 2012; Wager, 2012), though these studies did not focus on ELLs in particular, but on the broader group of students characterized as culturally and linguistically diverse. Because ELLs are simultaneously learning language and mathematics, meeting the needs of ELLs in the mathematics classroom may require teachers to learn skills and knowledge specific to supporting

ELLs. In this study we examined a four-week field experience for PSTs to engage in mathematics with ELLs. The following question guided our work, *What instructional supports do PSTs enact when implementing cognitively demanding mathematics tasks with ELLs*? The answer to this question was important in helping to provide a foundation from which to build further work in determining effective means of preparing PSTs to teach ELLs.

The present study will provide greater insight into PSTs' work with ELLs in particular; a topic few researchers have examined. Downey and Cobbs (2007) and Pappamihiel (2007) conducted empirical studies that offered fieldwork opportunities for PSTs to teach mathematics to ELLs. Downey and Cobbs' study was situated in a university-based teacher preparation program that required elementary PSTs to complete one-on-one tutoring fieldwork with a culturally diverse student. As a result, the PSTs deepened understanding of the relationship between cultural diversity and mathematical learning. In the Pappamihiel study, content-area PSTs who spent 10 hours with an ELL changed their views and recognized that more acceptance and adaptation are essential of multicultural perspectives. Fernandes (2012) observed how middle school mathematics PSTs noticed ELLs' understanding in mathematics through task-based interviews. He found that the PSTs started adopting ELL strategies from an intervention course and became aware of ELLs' needs and challenges. McLeman and colleagus (2012) found that field experiences with ELLs in conjunction with reading ELL literature were valuable in helping PSTs learn instructional strategies for ELLs and helping them understand linguistic complexity such as academic language. These studies suggest that field experiences with ELLs have promise, therefore we wanted to understand how one such experience might help PSTs develop strategies for supporting ELLs in enacting cognitively demanding tasks.

Perspectives

We frame this study in a situated-sociocultural perspective (Moschkovich, 2002) of learning. Because we are interested in the supports the PSTs provided, we focus this study on the learning and experiences of the PSTs. We view learning as discursive activity and that PSTs participate in a community of practice as they draw on a variety of resources in developing sociomathematical norms with their students (Moschkovich, 2002; Yackel & Cobb, 1996). That is to say, we focus on what the PSTs are capable of and the ways they draw on these capabilities to extend their own learning that will, in turn, support ELLs.

In considering the strategies the PSTs might employ, we draw on the work of Chval & Chavez (2011). Chval and Chavez described seven, research-based strategies they characterized as key to supporting ELLs' mathematical proficiency. These strategies included: (1) connecting mathematics with students' prior knowledge, (2) fostering a classroom environment that is rife with language and mathematics, (3) allowing for the use of multiple modes of communication, (4) including visual supports, (5) connecting mathematical representations to language, (6) recording key ideas and representations, and (7) discussing students' writing (Chval & Chavez, 2011). These strategies support students' use and development of academic language (Cummins, 1980) in conjunction with their development of mathematics and served as a framework to guide our examination of the PSTs' work with the ELLs.

Methods

The purpose of this study was to examine the supports elementary PSTs employed when enacting cognitively demanding tasks with ELLs. Four PSTs—Kimberly, Hannah, Morgan, and Fiona—participated in the study, all of whom were juniors in a four-year undergraduate elementary education program at a large research university. Each PST was white and spoke English as her native language. One PST, Kimberly, stated that while not fluent in Spanish she was able to communicate in that language. All four PSTs had limited prior experiences working and learning about ELLs and were eager to put their limited knowledge into practice.

The four ELLs were native Korean speakers. Kyong-Tae, Jin, and Ho-Min had each been in the United States for about six months, Hwa-Young had been in the United States for a little over a year and was also a fluent Japanese speaker. We purposefully (Patton, 2002) selected students enrolled in classes specifically for ELLs but were at or above grade level in mathematics. This allowed the PSTs to gain experience with students who were not yet fully fluent in English without having to also support students who struggled greatly in mathematics.

The field experience centered on the PSTs' weekly, one-on-one meetings with their assigned ELL. The field experience spanned four weeks and each meeting lasted approximately 30 minutes. Prior to each meeting, the PSTs were given a cognitively demanding task that they were to enact with the student. The PSTs were asked to complete a lesson plan detailing their plan for the meeting and given free reign to modify the task and make use of any resources they wished. During the meeting, the PSTs enacted their plan with their ELL.

Data Sources & Analysis

We used qualitative methods in order to gain rich descriptions of the PSTs' interactions with their ELLs (Patton, 2002). Each PST completed a survey before and after the field experience. These surveys contained both open-ended and Likert scale items and provided insights into the PSTs' teaching experiences, prior experiences with ELLs and thoughts about issues of equity and diversity in the classroom and their teacher preparation program. We collected the PSTs' written lesson plans that detailed the learning objectives, their procedures, planned modifications, and assessments. After each meeting, the PSTs crafted a written reflection.

For each weekly meeting, the PSTs would arrive half an hour before their ELL student to participate in a pre-meeting interview. In these video recorded interviews we asked the PSTs to discuss their lesson plans and planned supports in depth. We also observed and video recorded the meetings with the ELLs during which the observer took field notes on moments on which to follow up with the PST. Immediately following each meeting, we conducted a video recorded post interview with the PSTs. The post interview investigated the PSTs' immediate reactions to the meeting, things she would do differently if she could do the meeting again, and thoughts for the subsequent meeting. All video data was fully transcribed.

Using themes previously described from Chval and Chavez (2011) and additional themes that emerged from initial rounds of analysis, we generated a list of codes to use in our subsequent analysis of the interview and meeting data. To establish inter-rater reliability (Patton, 2002), two coders worked to code each data source. From this initial analysis, we further refined our codes and recoded the data. Then, we collapsed the codes into larger categories. For the present study, we examined the most commonly occurring categories that were present in all four of the PSTs' data sets to better understand what strategies the PSTs used and how they supported the ELLs in enacting the tasks.

Findings

Each of the PSTs employed a number of strategies to support their ELLs during the weekly meetings. These supports were both intentional and unintentional. The following sections describe those supports that were most frequently employed across all four PSTs. These supports included using multiple modes of communication, using visual supports, pressing for explanations and meanings, and checking for understanding.

Allowing Multiple Modes of Communication

All four PSTs encouraged their ELL to make use of multiple modes of communication during the weekly meetings. We defined multiple modes of communication as teacher actions that encouraged/allowed students to communicate meaning and thinking through the use of speaking, writing, gesturing, drawing, manipulating, and/or using a first language as they grow in language and mathematical proficiency (Chval & Chavez, 2007).

Across the data set, when PSTs referred to speaking as a mode of communication, they did so in one of two ways: as an alternative to writing (when unable to) or as a means to explain or communicate one's process (i.e. reaching a solution), thinking, or ideas. When Morgan employed this strategy during week two, she asked her student "And this, this time can you try, maybe, telling me what you're doing? Talking about it?" With this question, Morgan was asking her student to verbally explain his process and thinking while problem solving.

During the pre-interview sessions, all four PSTs described drawing as another way to explaining one's thinking in lieu of writing or speaking. The PSTs readily identified that their ELLs may have difficulty communicating in English and discussed the need for other methods of communication to relay their mathematical thinking. There was one occurrence when Hannah verbalized this to her student during a session. During week 2, she said, "Can you draw some over here? Let me help you out (began to draw 24 circles on the paper by 6 x 4)." In this instance, Hannah offered an alternative method of communication to her student; however, she also took over this communication by drawing the representation, removing some challenging aspects of the problem and reducing the cognitive demand.

When PSTs used this strategy with regards to writing, they did so in several ways. During pre-interviews, the PSTs identified that writing would be used to communicate explanations, equations, processes, and thinking and to improve language development. While working with students, three of the four PSTs made use of this strategy to provide an additional method of communicating, specifically for explanation. In all instances, with two exceptions, requesting students to write their explanations did not impact the cognitive demand of the task. In the two instances where the cognitive demand was impacted, the PSTs stated a pathway for solving the problem after stating. For Fiona, in week 2, she stated, "So, how can you write a sentence to explain how each stadium is compared to each other? So are they all the same number? Is one stadium bigger than the other stadium?" For Morgan, in week 2, she stated, "If you need, you can write it out. And you can start subtracting numbers." In both statements, in addition to suggesting the student write their thinking they also suggested a pathway for reaching a solution, which thereby reduced the overall cognitive demand of the task.

Providing Visual Supports

Throughout the four weeks, each of the PSTs included visual supports in her lesson planning and implementation. We defined visual supports as concrete objects, videos, illustrations, or added emphases (bolding, color-coding, etc.) on written tasks. These supports were used in various ways by the PSTs but generally involved the use of manipulatives or images.

Hannah, Fiona, and Kimberly used manipulatives in the meetings with their ELLs. These manipulatives included alien cutouts, connecting cubes, and small vehicles intended for the students to use in their exploration. In some cases the PSTs used the manipulatives to demonstrate a certain solution strategy as in the following excerpt from Fiona's second meeting.

Fiona: It's impossible? Let's work through this, ok, because it's possible. It's kind of confusing, I know. So, if I have [moves one of each 2, 3, and 4 eyed cutouts of creatures into center of table]—how many eyes do I have?

Jin: 9 eyes
Fiona: How many more do I need to get to 24 eyes?
Jin: 15
Fiona: 15. So is there a way I can use the rest of these [cutouts] to make 15 eyes?

Throughout the lesson, it was Fiona, and not her student, that used these manipulatives, leading to an imposed solution strategy. This lowered the cognitive demand on the student by removing the challenging aspect of developing a solution. In the majority of instances involving manipulatives, the PSTs elected to implement them in a similar manner to Fiona.

Finally, all four PSTs used imagery in their lessons. For example, Kimberly drew a picture of a group of coconuts to help define "pile." Although many images were used as linguistic supports, some were utilized as scaffolds for mathematical learning. In the fourth task, in response to Morgan's student's ineffective attempts to find the perimeter of a complicated shape, she drew a simpler example for him to examine. This figure was intended for him to explore his own strategy in an easier setting before returning to the larger figure, applying his exploration, and correcting his mistake. However, as the following excerpt shows, this did not occur.

Morgan: Ok. Alright, so you just added- so can you tell me what you did? What did you add together? All of the what? What did you add all of? You added all- all the sides?
Ho-Min: Yes,
Morgan: Sides, together?
Ho-Min: Yes.
Morgan: Yeah, all sides. Ok. So on [the first problem], when you did this for perimeter, you added all the sides, but you made- you made new sides. With these lines. See, you added these also. But with this shape, see here-

Morgan stopped requiring the student's reflection and instead used her drawing to show him his mistake and how to solve the problem. When the PSTs used such scaffolds, they often relied on them as a crutch themselves, prompting the use of strategies that lowered the cognitive demand.

Most visual supports were implemented to allow students to organize their mathematical thinking. However, there was often conflict between the planned intention and their actual use. When the PSTs relied on the visual supports too heavily, they tended to think for the students and provide a solution method or an idea instead of encouraging the student to use the supports in his or her own way. Thus, the visual supports more often led to lowered cognitive demand, even though the PSTs may not have been intended to do so.

Pressing for Explanation and Meaning.

PST actions that pressed a student to explain his/her solution or meaning of task statement were one of the prevalent strategies that all of the PSTs frequently used throughout the four weeks. The most popular format they used to press students to explain about their solution was questioning and they usually asked students to explain more details and rationale about their work. However, they sometimes took on command forms such as "Explain what you mean by that" or "Tell me about how you figured that one out."

The PSTs most frequently prompted students to justify their solution and/or provide more details about their solution strategies. Hence, the PSTs' press for student explanation usually came after students finished or stopped their solving process. They used various types of questions ranging from general inquiries such as "Why do you think so?" or "How did you find out?" to more specific ones such as "Why did you decide to add these and not something else?" or "Where does the 12 come from?" Other ways the PSTs employed this strategy included asking students about the meaning of the task statement, asking clarifying questions, asking for students' current thinking, asking for students' plans to solve the task, and pressing students to think further.

Each PST favored one particular approach in pressing for meaning. For example, Kimberly's use of this strategy involved asking her student for justification and more details. Hannah used instances of her student's work to ask questions such as "I saw you looked over your piece of paper over here. What were you just thinking?" This type of question evidenced her careful observation of her student's activity and was particularly effective in eliciting her student's explanation of her mathematical thinking.

When the PSTs used this strategy, in general it seemed to provide students an opportunity to speak with longer expressions and to think more deeply about their solutions. However, this was not always the case. Consider the following excerpt.

Kimberly: Ok. (he writes) Ok, how did you find that?
Kyong-Tae: Mm, (long pause) I don't know. I just found.
Kimberly: Yeah, well I saw you do it and I was thinking about how I would have done that in my head, so I saw what you did, first was you found the two- eyed creatures right? And you found five of those and that gives you how many eyes?

In the first line, Kimberly asked her student to explain about his solution pathway. However, the student could not come up with an explanation after a long pause. Kimberly did not provide him with support to find appropriate words or use guiding questions; instead, she provided her interpretation on his work. In this case Kimberly took on much of the thinking for the task.

Most of the instances in which the PSTs pressed for explanation sought to maintain the cognitive demands of the given task. By asking students to explain their thinking after arriving at a solution, the PSTs maintained the initial intent of the tasks. There were several instances, as in Kimberly's excerpt above, in which the PSTs pressed for explanation but did not support students or hold students accountable for responding to the questions. This led to the PSTs suggesting specific solutions, or moving to another task without further discussion.

Checking for Understanding

Each of the PSTs frequently checked for their ELL's understanding during task implementation. Checking for understanding usually occurred after the PSTs presented the student with the task and allowed them time to read it. For example, in week two Kimberly gave Kyong-Tae a task about space-creatures and said, "Ok, well this is the first one that I put together for you. And I want you to read it before anything else. [Very Short Pause] Okay, are there any words on there that you don't know?" Asking if there were any words the ELLs were unfamiliar with was a common strategy to check for understanding among the PSTs.

Once the student identified unfamiliar words, the PSTs similarly responded by defining the words for the child by attempting to connect to the child's prior knowledge. For example, Hwa-Young did not know what *creature* meant and Hannah tried to explain by connecting to other words Hwa-Young might know.

Hwa-Young: Creature.
Hannah: You know what that is.
Hwa-Young: Nope
Hannah: Okay. So, a creature is a broad term that covers multiple animals, so it could be, do you know what a monster is,
Hwa-Young: Yes.

Hannah: Do you know what an alien is
Hwa-Young: Alien?
Hannah: Like, UFO, like Ti-Yoong Ti-Yoong (make a sound of UFO).
Hwa-Young: Oh, oh, I know.
Hannah: Okay, so monsters, aliens, bugs, animals, they are all considered creatures,
Hwa-Young: Oh.
Hannah: So, today, it's going to be the space creatures, so it's going to be aliens. That would say.
Hwa-Young: Okay. Aliens.

In this excerpt, Hannah attempted to define the term *creature* for Hwa-Young by comparing it to other terms such as alien or monster. She also pantomimed a UFO in an attempt to support Hwa-Young in understanding the concept.

This support for language was common among the PSTs when finding the student was unfamiliar with a term after checking for understanding. Checking for understanding was most often used to check for *linguistic* understanding, not mathematical understanding. As such, there were several missed opportunities to support their students' mathematics misconceptions because they assumed the students' misunderstandings stemmed from language, not mathematics.

Discussion & Conclusion

During the four-week field experience, we found that the PSTs employed a number of strategies in attempting to support their ELLs. These strategies were not prompted by the research team, but rather by their own experiences working with the students. Each of the PSTs reported no prior experiences specifically focused on the mathematics education of ELLs in their teacher preparation program. As such, this study provides some insight into the strategies teachers might draw on without further formal preparation.

The PSTs drew on a limited number of strategies in supporting their ELLs. Over time the number of strategies did not increase but the ways in which they used the strategies changed, as did the frequency in which they employed the strategies. For example, Morgan, whose student was hesitant to speak, drew more on multiple modes of communication for her student throughout the weeks. In addition, she broadened her range of acceptable communication from spoken and written to include drawing and gestures throughout the experience. However, Morgan was unable to employ strategies that allowed her to elicit detailed descriptions of Ho-Min's mathematical thinking. His thinking was often shown by a calculation on a page or gesture to a solution, leaving Morgan to assume the process behind these artifacts. Morgan and her peers' experiences suggest that while the PSTs did learn from prior experiences with their ELLs, this experience was not sufficient to fully meet their students' needs.

We also found that the PSTs had difficulty determining whether students' struggles stemmed from linguistic or mathematical misunderstandings. As such, the PSTs typically supported the linguistic aspects of the task without considering possible mathematical misunderstandings. Further, they typically attempted to take on the mathematical thinking in an effort to support the students. This implies that PSTs should be provided with guided experiences that help them begin to support both the mathematics and linguistic needs of their ELLs while also maintaining the cognitive demand of mathematics tasks.

In analyzing the PSTs' final reflections and surveys, we found that completing the field experience left the PSTs with the view that ELLs are capable of any task. This is in contrast to earlier thinking present in their pre-surveys that particular tasks are more appropriate for ELLs. Further, the PSTs were also more aware of their lack of preparation to teach ELLs following the experience. This implies that the field experience allowed the PSTs to better understand the need for further learning in this area. Though further research is needed to understand how teacher preparation programs might better prepare PSTs for the increasing number of ELLs, this study suggests that PSTs' natural inclinations to support ELLs is not sufficient to support both the linguistic and mathematical needs of ELLs. Teacher educators should build on PSTs' natural inclinations and provide further support to help them learn to better accommodate ELLs. PSTs need explicit instruction and experiences enacting both mathematical and linguistic supports with ELLs to help build on ELLs' cultural, mathematical, and linguistic resources as they help them develop linguistic and mathematical proficiency.

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RESEARCH IN MATHEMATICS EDUCATIONAL TECHNOLOGY: TRENDS IN PROFESSIONAL DEVELOPMENT OVER 40 YEARS OF RESEARCH

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The effective use of digital technologies in school settings calls for appropriate professional development opportunities for inservice teachers. How has professional development shifted in support of mathematics teachers integrating these technologies in their teaching? This study explored the impact of digital technologies in mathematics inservice professional development over the past four decades and examined how various technologies, content strands, grade-level bands, teacher outcomes, and student outcomes were being used to design mathematics professional development on integrating technology. This study provides recommendations to mathematics teacher educators as they transform professional development to meet the challenges faced in integrating new and emerging technologies in their instruction.

Keywords: Teacher Education-Inservice; Teacher Knowledge; Technology

What it means to teach mathematics has changed over the past four decades. The development and availability of mathematics educational technology is a key factor in how the mathematics classroom looks different than 10, 20, 30, or 40 years ago. Professional development provides opportunities for inservice teachers to experience new methods of both teaching and learning mathematics with technology and to collaborate with colleagues about pedagogical strategies they use when implementing these technologies.

Guskey (2000) defined professional development as, "those processes and activities designed to enhance the professional knowledge, skills, and attitudes of educators so that they might, in turn, improve the learning of students" (p. 16), which is the definition used in this study. Professional development can be short-term or ongoing and may take many forms such as workshops, sessions during teacher inservice training, institutes and sessions in the summer, individual classroom interactions, lesson study, professional learning community models, online sessions, or video-series. When professional development is effective, teachers may take new ideas back to their classrooms and implement new strategies and use technology in different and/or new ways.

To address ongoing challenges in mathematics educational professional development, we sought to examine extant literature in the field, analyze trends therein, and facilitate present and future improvements to both teaching and learning. The research questions that guided the present study were: (1) What types of technology and content areas have been the focus of professional development research over time? (2) What types of technology and grade bands have been the focus of professional development research over time? (3) What types of outcomes are used to measure effectiveness of mathematics educational technology professional development; have they changed over time, and how do they vary across grade levels?

Conceptual Framework

Two frameworks were applied to the analysis of a database of mathematics educational technology studies identified by a systematic review: Technological Pedagogical Content Knowledge (TPACK) and Comprehensive Framework of Teacher Knowledge (CFTK). The TPACK framework (Mishra & Koehler, 2006; Niess, 2005) describes the unique set of knowledge needed to effectively integrate technology into the classroom in conjunction with appropriate pedagogical content knowledge (as in Ball, Thames, & Phelps, 2008; Shulman, 1986). TPACK extends beyond knowledge of how to use the technology proficiently and encompasses a deeper and transformed knowledge for understanding how subject matter, pedagogy, and technology are integrated to provide richer learning experiences. Subsets of the TPACK framework include Technological Knowledge (TK), Pedagogical Knowledge (PK), Content Knowledge (CK), Technological Pedagogical Knowledge (TPK), Technological Content Knowledge (TCK), and Pedagogical Content Knowledge (PCK). TCK was used to support research question one while TK was used to support research question two.

CFTK (Ronau & Rakes, 2011) describes an all-encompassing structure for studying and understanding the complex nature of knowledge required for teaching as a highly complex interaction of multiple aspects of teacher knowledge across three dimensions: Subject Matter and Pedagogy (Field dimension), Discernment and Orientation (Mode dimension), and Individual and Environment (Context dimension). These teacher knowledge outcomes and an extension of these outcomes to student outcomes were applied in the analysis of research question three.

Method

The present study is part of a larger, more comprehensive study that analyzed mathematics educational technology papers published between 1968 and 2009. For the literature search of the comprehensive study, we followed a systematic process based on the techniques outlined by Cooper, Hedges, and Valentine (2009) and Lipsey and Wilson (2001); for example, we defined constructs before coding, defined keywords before conducting the literature search, defined a coding process, trained coders, and cross-checked results. To obtain the overall sample, a wide array of databases were searched using terms to restrict the sample, based on two criteria for inclusion: (1) The paper must examine a technology-based intervention (e.g., technology, calculators, computers), and (2) The paper must be focused on the learning of a mathematics concept or procedure (e.g., mathematics, algebra, geometry, visualization, representation). We searched the following database platforms (and the databases within those platforms): EBSCOWeb (ERIC, Academic Search Premier, PsycINFO, Primary Search Plus, Middle Search Plus, Education Administration Abstracts), JSTOR (restricted to the disciplines of Education, Mathematics, Psychology, and Statistics), OVID, ProQuest (Research Library, Dissertations & Theses, Career & Technical Education), and H. W. Wilson Web (Education Full Text). We also examined the bibliographies of the papers we identified through this search in order to identify potentially relevant papers that were missed in our searches. For further details about this process, see Ronau, Rakes, Bush, Driskell, Niess, & Pugalee (2014). Altogether, our literature search of the comprehensive study resulted in 1,210 papers.

In order to code the 1,210 papers, we created a Microsoft Access database with over 200 variables, with each coder paired with each of the other coders (i.e., six coders = 15 coding teams) so that each paper was both coded and cross-checked. The new coding format created a counter-balanced design with all six coders, providing a way to maximize construct validity and inter-rater reliability of the coding. Our overall inter-rater agreement was 91.5% (Number of Agreements out of the Total Number of Possible Agreements), from which we concluded that the inter-rater reliability for the comprehensive study was high. Upon completion of coding for the comprehensive study, a filter was applied to the database to extract the papers that were coded as teacher development. Next, each paper was read to make certain it aligned with Guskey's (2000) definition of professional development and to verify that the professional development was non-credit bearing (not part of a degree program) and took place after initial teacher certification (preservice education was not included). Using this criterion, 21 of the 1,210 papers were retained for the present study.

Results

None of the papers in our subsample of 21 professional development papers were found in the 1960's or 1970's. For each subsequent decade, the ratio of professional development papers per total number of technology papers in mathematics education was 2/48 in the 1980's (4.14%), 3/320 in the 1990's (.94%), and 16/818 in the 2000's (1.96%). To answer research question one, the number of times each technology type was used as compared to content strand per decade was analyzed (see Table 1).

Table 1: Technology Type Compared to Content Strand by Decade

Decade and Content Strand	Calculator	Computer Software	Interactive Whiteboards	Internet	Probeware & Motion Detectors	Other Technology
Total for 1980	**2**	**0**	**0**	**0**	**0**	**1**
Number	1	-	-	-	-	-
Not Specified	1	-	-	-	-	1
Total for 1990	**0**	**3**	**0**	**1**	**1**	**0**
Algebra	-	3	-	1	1	-
Total for 2000	**7**	**13**	**1**	**3**	**5**	**2**
Algebra	2	3	-	-	2	-
Algebra, Geometry, Data Analysis	1	-	-	-	1	-
Algebra, Probability & Statistics	1	2	-	1	-	-
Algebra, Geometry, Calculus	-	1	-	-	-	-
Number	-	-	-	-	-	1
Probability & Statistics	-	2	-	-	-	-
Not Specified	3	5	1	2	2	1
Technology Type Total	**9**	**16**	**1**	**4**	**6**	**3**

Note. $N = 21$ papers. The number of papers per decade is not always the sum of the row because some papers addressed more than one technology type and/or content strand area. The Other Technology consisted of computer programming, personal digital assistants, and video clips.

The content strands addressed within the professional development papers in the 1980's were limited to number content or was not specified, with calculators used in both studies. Algebra content was the only content strand addressed in the 1990's, and the technology for all three papers was computer software, specifically graphing software in two papers and presentation software in the third paper. In the 2000s, there were six different reported content strands. The algebra content strand was addressed most often (six of 13 papers). In the three papers that solely included algebra content, one paper discussed using graphing software and two papers discussed using spreadsheet software. One of the two papers coded as algebra and probability and statistics content discussed using geometry software while the other paper used spreadsheet software. The paper coded as algebra, geometry, and calculus content discussed using geometry software. The computer software used in the two probability and statistics content papers was statistics software. Professional development papers written in the 2000's displayed the greatest variety in technology use, with a growing use of the Internet. Many papers (n = 6; 37.5%) however, did not specify the content addressed in the professional development.

To answer research question two, the technology used and teacher participants' grade-level band reported in the papers was analyzed (see Table 2). Across all three decades, two papers included K-5 teachers; one included a combination of K-5, 6-9, and 10-12 teachers; seven included 6-8 teachers only; five included both 6-8 and 9-12 teachers; five included 9-12 teachers only; and one did not specify the grade-level band.

Table 2: Technology Type Compared to Grade Band by Decade

Decade and Grade Band	Calculator	Computer Software	Interactive Whiteboards	Internet	Probeware & Motion Detectors	Other Technology
Total for 1980	**2**	**0**	**0**	**0**	**0**	**1**
K-5	1	-	-	-	-	-
K-5, 6-8, 9-12	1	-	-	-	-	1
Total for 1990	**0**	**3**	**0**	**1**	**1**	**0**
Unspecified Grade	-	1	-	-	-	-
6-8, 9-12	-	1	-	1	1	-
9-12	-	1	-	-	-	-
Total for 2000	**7**	**13**	**1**	**3**	**5**	**2**
K-5	-	-	1	-	-	-
6-8	2	7	-	2	1	2
6-8, 9-12	2	4	-	-	2	-
9-12	3	2	-	1	2	-
Technology Type Total	**9**	**16**	**1**	**4**	**6**	**3**

Note. N = 21 papers. The number of papers per decade is not always the sum of the row because some papers addressed more than one technology type and/or grade band. The Other Technology consisted of computer programming, personal digital assistants, and video clips.

The most common technology used was computer software, followed by graphing calculators, probeware, Internet, other technology, and interactive whiteboards. Technology for grades 6-8 and 9-12 teachers varied more widely compared to grades K-5. Professional development for grades K-5 in the papers was limited to either calculators or interactive whiteboards. Professional development for grades 6-8 or 9-12 teachers, on the other hand, included calculators, computer software, the Internet, probeware and motion detectors, and other technology, including computer programming, personal digital assistants, and video clips.

To answer research question three, we analyzed outcomes (student and teacher) being addressed in the professional development papers, organized by grade band and decade (see Table 3). While the total number of professional development studies was21, the total count of outcomes as shown in Table 3 was51, as many studies had more than one outcome. Teacher outcomes were measured more often than student outcomes. Teacher orientation was the most common outcome measured, which was measured in 15 studies (2, 3, and 10 from the 1980's, 1990's, and 2000's respectively), followed by 10 studies (0, 2, and 8 across the three decades) which measured teacher knowledge of pedagogy, and 9 studies (1, 2, and 6 across the three decades) that measured teacher knowledge of subject matter.

Table 3: Outcomes Compared to Grade Band by Decade

Decade and Grade Band	Student Achievement	Student Orientation	Student Behavior	Teacher Knowledge Subject Matter	Teacher Knowledge Pedagogy	Teacher Knowledge Discernment	Teacher Knowledge Orientation	Teacher Knowledge Individual	Teacher Knowledge Environment	Teacher Orientation
Total for 1980	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**2**
K-5	1	-	-	-	-	-	-	-	-	1
K-5, 6-8, 9-12	-	-	-	1	-	-	-	-	-	1
Total for 1990	**1**	**1**	**0**	**2**	**2**	**0**	**1**	**0**	**2**	**3**
Unspecified Grade	-	-	-	1	1	-	1	-	1	1
6-8, 9-12	-	-	-	1	1	-	-	-	1	1
9-12	1	1	-	-	-	-	-	-	-	1
Total for 2000	**1**	**2**	**1**	**6**	**8**	**3**	**1**	**2**	**1**	**10**
K-5	1	-	-	1	1	1	-	-	1	1
6-8	-	-	-	2	3	1	-	-	-	4
6-8, 9-12	-	1	-	-	1	1	-	1	-	2
9-12	-	1	1	3	3	-	1	1	-	3
Outcome Total	**3**	**3**	**1**	**9**	**10**	**3**	**2**	**2**	**3**	**15**

Note. N = 21 papers. The number of papers per decade is not always the sum of the row because some papers addressed more than one outcome and/or grade band.

Teacher knowledge of Discernment or Individual Context in relation to technology was not addressed in professional development research until the 2000's.Both of these knowledge constructs tend to be student-centered and are not as easily observed or measured. Since only three studies included K-5 teachers, there were no trends to describe how the outcomes varied across grade level. With regard to student-related outcomes, student achievement (n = 3 out of 21 papers, 14%) and student orientation (n = 3 out of 21 papers, 14%) were studied most often. As the numbers for student outcomes were so small, there were not clear patterns or trends through the decades.

Discussion

This study found very little published research on professional development focused on technology in teaching and learning mathematics, which was surprising given the long-standing calls from professional organizations such as the National Council of Teachers of Mathematics (1989, 2000) for mathematics teachers to incorporate new technologies in the classroom. Only 21 studies of 1,210 total studies in our sample of mathematics educational technology papers addressed professional development. Limited published professional development research impairs the ability to advance the field of mathematics educational technology professional development. Shavelson and Towne (2002) stated, "Scientific studies do not contribute to a larger body of knowledge until they are widely disseminated and subjected to professional scrutiny by peers" (p. 22). Therefore, a reasonable direction for researchers who conduct mathematics educational technology professional development is to measure the outcomes of their efforts and publish the results of their work. Also, researchers might focus on conducting and publishing further research on professional development with K-5 teachers as only three of the 21 studies included these teachers. Since many students are currently required or will be required to take a computer-based state standardized assessment, some K-5 classrooms now have some sort of 1:1 structure in place (e.g., laptop carts, chromebooks, iPads). Research is needed on how to use these resources to enhance mathematics learning effectively, and professional development research is needed to help providers improve teachers' ability to integrate technology in their classrooms effectively. Furthermore, how technology can enhance learning is content-specific, and too few of the published studies reported the content area that was addressed during the professional development. Future studies need to explicitly identify the content area being studied so that teachers and researchers can build on the work appropriately. Also, the effectiveness of the professional development with any explicit measure related to student knowledge, orientation, or behavior was often omitted. These results align with Sztajn's (2011) concerns, who argued that norms and standards for reporting on professional development studies are needed. Still, consistent with Supovitz and Turner (2000), such standards must include the specific content area(s), grade band(s), and technology type addressed in the professional development. The constructs of the study must be clearly and explicitly stated, threats to validity discussed, and research methodologies clearly articulated as purported in the scientific principles 3 and 4 (Shavelson & Towne, 2002).

Future Directions

The historical data we analyzed in mathematics educational technology literature denotes a clear need for future research measuring the effectiveness of professional development focused on technology in teaching and learning mathematics. Professional development efficacy can be measured by using observational tools, teacher knowledge assessments, and teacher surveys to assess teachers' change in practice. These changes should evaluate instructional activities and practices, classroom discourse, the fidelity in which the curriculum is implemented, teacher knowledge, and teachers' beliefs. Another reasonable direction is to measure changes in student learning/achievement, although we recognize the challenge in collecting such data and the necessity of an extensive length of professional development to gather the data to evaluate this change.

Research analyzing technology-focused professional development in mathematics education could make great strides in mitigating the field's traditional challenges by pointing to specific research-supported methods for improving future professional development.

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SOCIAL CAPITOL, SOCIAL NETWORKS, AND LESSON STUDY: SUSTAINING MATHEMATICS LESSON STUDY PRACTICES

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I investigate the sustainability of lesson study as mathematics teachers who participated in a 3-year professional development partnership engage in a district scale-up lesson study professional experience. This study answers three questions: (1) what are K-12 teachers' conceptions of sustaining mathematics lesson study, (2) what practices of lesson study continued after the grant as reported by participants, and (3) what are supportive and constraining factors in continuing lesson study after external funding ends when there is both reported desire from teachers as well as some district support? Survey and interview data are analyzed using grounded theory and social network analysis for patterns in and structure to activities. Findings suggest rich conceptions of lesson study, the continuation of particular lesson study practices, and the importance of integration and linkage as factors that supported or hindered lesson study.

Keywords: Policy Matters; Teacher Education-Inservice (Professional Development); Instructional Activities and Practices

Introduction

Though lesson study shows promise as a vehicle for professional development (Stigler & Hiebert, 1999; Yoshida, 2012) and has been described as a sustainable form of professional development, little research exists on what would help to support teachers in continuing lesson study past the life of a grant. Why do some mathematics teachers of lesson study continue while others do not, particularly when teachers express interest and have some district support? Therefore, to develop a better understanding of sustainability with respect to lesson study, the field is in need of a deeper understanding of how to support the continuation of lesson study.

The purpose of this report is to examine teachers' conceptions and practices of lesson study, and factors that supported and constrained teachers' ability to continue to engage in lesson study. The setting for this study is unique in that participants surveyed and interviewed come from one network of U.S. mathematics teachers of grades 3 through Algebra 1 (students 8–14 years old) who participated in a three-year professional development partnership that used lesson study and who expressed interest in and had some district support for continuing lesson study after grant funding ended. In this report, I answer the following research questions:(1) what are K-12 teachers' conceptions of sustaining mathematics lesson study, (2) what practices of lesson study continued after the grant as reported by participants, and (3) what are supportive and constraining factors in continuing lesson study after external funding ends when there is both reported desire from teachers as well as some district support?

Background

Lesson study has been described as a vehicle for developing and sustaining professional learning communities whose goal is to improve instruction (Yoshida, 2012). Research on mathematics lesson demonstrates the potential to enhance teachers' knowledge about mathematics content (Alston, Pedrick, Morris, & Basu, 2011; Fernandez, 2005; Lewis, Perry, & Hurd, 2009; Meyer & Wilkerson, 2011; Robinson & Leikin, 2012; Yoshida 2012), change teaching practice (Hart & Carriere, 2011; Murata, Bofferding, Pothen, Taylor & Wischnia, 2012; Olson, White & Sparrow, 2011), nurture professional communities of teachers (Lieberman, 2009; Lewis, Perry & Hurd, 2009; Saito, Khong,

& Tsukui, 2012), and help teachers understand how to teach mathematics aligned to reform efforts (Lee & Ling, 2013; Lewis & Takahashi, 2013; Takahashi, Lewis & Perry, 2013). These foci – enhancing mathematical content knowledge, changing teachers' practice, nurturing professional communities, and helping teachers teach in ways aligned to reform efforts – are ways in which lesson study has contributed to the improvement of learning and teaching mathematics, benefiting both teachers and students. Yet researchers call for more research in lesson study (Fernandez, 2005; Lewis, Perry & Murata, 2006). Future research pathways would be impossible without teachers continuing to implement and engage in lesson study.

Although many educators involved in lesson study research and work describe it as a sustainable form of professional development, little research exists that seeks to understand aspects of engaging in lesson study that ensure its continued success (cf. Gero, 2015; Lewis & Perry, 2014; Saito, Khong & Tsukui, 2012). Factors that hinder lesson study include engaging in collaboration, observating a lesson, the potential critique of a teacher's lesson and teaching, and the collision with the existing culture in districts with the tenants of lesson study (Gero, 2015). Saito, Khong & Tsukui (2012) found that faith in meetings, support and enthusiasm from principals and other senior teachers, and the desire to retain respect from external parties supported teachers in continuing to organize PLCs with lesson study. This study furthers research on these factors to add to a deeper research base on continuing lesson study.

Theoretical Perspective

The theoretical model of lesson study used in this study is based on Japanese Lesson Study (Fernandez, 2005), which consists of teachers collaboratively (a) *investigating* content and setting goals for the research lesson, both content-focused and broader site based goals; (b) *planning* a research lesson that seeks to inquire into how students learn a particular topic or sets of topics; (c) *teaching* and *observing* a live research lesson while gathering student data; and (d) finally, *debriefing* on specifics of what was learned from the lesson as well as more generally about teaching and learning mathematics (Lewis, Perry, & Hurd, 2009). Optionally, teachers may modify their research lesson and opt to teach it a second time, collecting data on student thinking and debriefing again.

Yet to understand how mathematics teachers engage in professional activities like lesson study requires understanding how they are situated within their site and district. Consequently, this study is shaped by the perspective that teaching is embedded within institutional settings like classrooms, school sites, and districts with teachers members of communities (Cobb, McClain, Lamberg, & Dean, 2003). Additionally, understanding how mathematics teachers engage in professional activities requires understanding the nature of collaborative activities that the teachers engage in both in informally arranged groups and formally arranged groups by the school or district.

Supporting the work of teachers in complex institutional settings also requires attention to different types of resources supporting teacher work (Gamoran et. al, 2003). These include *material resources* (physical objects or information like curriculum or activities), *human resources* (qualities of people that can be changed like training someone to be a math coach), and *social resources* (attributes of relationships, roles or modes of communication like connections to math coaches and other people). I examine social resources for this study, which is one way to understand conditions for sustainability (Gamoran et al., 2003).

Sustainability is defined as maintaining generative practice, or to keep growing and learning (Franke et al., 2001; Gamoran et al., 2003). I use Gamoran and colleagues' (2003) framework for conditions for sustainability to inform data collection and analyses, which was derived from an economic growth model (Woolcock, 1998). To understand how social capitol is embedded in groups among complex institutional settings, Gamoran and colleagues describe the four conditions for sustainability as integration, linkage, organizational integrity, and synergy. *Integration* refers to shared values, mutual expectations, levels of trust, and norms. *Linkage* refers to the social relations that attract resources. *Organizational integrity* refers to the effectiveness of the organization in distributing human and material resources. Finally, *synergy* refers to whether the efforts of the teacher community is aligned with the efforts of the school and district. For this study, I restrict my analyses to integration and linkage.

To conceptualize and document social resources like integration and linkage, I use the perspective of social network wherein the goal is to understand how individual actors are embedded in social structures by examining relationships among actors in addition to attributes of individuals (Carolan, 2014; Daly, 2010).

Methods

Participants

A subgroup of six primary teachers, one principal, and one district administrator is selected from a larger data set to examine in detail due to the high concentration of former grant teachers at one site and reported support by their principal.

Context

The study began with a survey administered on the last day of the former partnership where approximately 75% of 80 teachers described an interest in continuing lesson study. Thus, all participants in the current study recently participated in this three-year university partnership that sought to improve teachers' instruction on algebraic thinking. The three-year partnership was structured to include a 40-hour week long summer institute for teachers focused on mathematics content, student thinking, and pedagogy; four rounds of lesson study during each school year that utilized Japanese Lesson Study; and mathematics coaching. For the lesson study component, eighteen groups of 3-6 teachers each engaged in two lesson study cycles per year and observed two lesson study cycles per year. Groups were arranged to consist of cross-site and cross-district participants but are now reconstituted groups as relations among teachers shifted with the conclusion of the grant.

Data Collection and Analysis

Data were collected after external support from the university ended, or the first school year, 2013-2014, following the conclusion of the grant. A survey and interview instrument were constructed and administered. The survey results were collected in October 2013 and asked for lesson study cycles and components completed, resources needed for lesson study, support given by principal and fellow teachers, and any additional comments. From these surveys, I gathered participants to engage in individual, semi-structured interviews and asked about who teachers worked with, the nature of their activities, their work with lesson study, resources that support their work, resources that support lesson study, and changes they would make if they were to do another cycle of lesson study. I used a snowball technique for collecting interviews, which involved interviewing those participants named by interviewees (Carolan, 2014; Cobb, McClain, Lamberg & Dean, 2003; Cobb, Zhao & Dean, 2009). I then interviewed principals of participants to learn more about the work of teachers at their school site.

Data were analyzed using grounded theory methods (Corbin & Strauss, 1990), with the methods of open coding and constant comparison methods used to derive themes in the data. Analyses were also informed by Woolcock's (1998) conditions of sustainability on integration and linkage and analyzed using social network analysis (Carolan, 2014). Specifically, egocentric networks were inferred from interview data and analyzed for qualities such as like density and structure.

Results

I first characterize participants' conceptions of what it means to engage in lesson study using data from interviews. I then report on which aspects of lesson study have continued as reported by participants on surveys and during interviews. Finally, I describe factors that support and hinder teachers' potential to continue engaging in lesson study using analyses on survey and interview data.

Teacher' Conceptions of Lesson Study

To address the first research question on teachers' conceptions of lesson study, I analyzed teacher responses to characterizations of lesson study. Their responses targeted three general areas to varying degrees – (a) the structure or protocol associated to engaging in lesson study, (2) the nature of the activities that comprise lesson study, and (3) the focus or purpose of the described structure or nature of activities.

Structure. Most all participants described the structure of lesson study to include planning, teaching, observing the research lesson, collecting data during the research lesson, and then debriefing on the research lesson. Five participants included the goal setting stage in addition to the planning. There was a strong emphasis on student thinking for all participants, and often for all components. Ways in which the conceptions of structure varied included whether or not the participant described a second enactment of teaching, observing, and debriefing on the research lesson.

Nature of Activities. As mentioned before, most all teachers described activities as including goal setting, planning the research lesson with attention to questioning techniques and student misconceptions and responses, observing the lesson while one teacher taught the lesson and others collected student data, and finally debriefing the lesson where the teacher of the lesson would comment first on what went well and changes they would make to improve the lesson based on their goals.

Focus. Two teachers described the focus or purpose of lesson study as a way to unpack the teaching practices (e.g. understanding assessment, standards, or student thinking). Primary teacher Gillian[1] reported that, "It's more of a philosophy of how to approach what you're doing professionally in the classroom. It's the philosophy of teaching, if you will." Four participants described it as an activity to better understand student thinking. Jimmy described lesson study as useful for observing student thinking and "to be the one standing back and listening. And, you know, asking the kids to explain themselves." One participant described the focus of lesson study as improving mathematics content knowledge as well as pedagogy. One participant described lesson study as a way to create polished lessons, in addition to a way of understanding student thinking.

In summary, teachers conceptions of lesson study aligned with how experts in the lesson study literature describe lesson study. This finding is significant with the strong presence and focus of student thinking in conversations.

Practices of Lesson Study That Have Continued

Most teachers (N=4) reported engaging in one cycle of lesson study in the beginning of their school year (See Table 1). During this cycle of lesson study, teachers reported to engage in goal setting, planning the research lesson, observing the research lesson and collecting student data while one teacher taught, debriefing on the research lesson with final reflections. This effort was initiated and supported by the district. It differed from former grant efforts in that there was less time for planning (three hours on average versus six hours), shorter time between the planning and enactment of the lesson, and only one cycle of lesson study planned for the year. Those study participants that participated in this effort served as facilitators of lesson study for teachers who had never participated in lesson study.

Though not all participants completed a round of lesson study, participants reported continuing practices of lesson study. Most significantly, all participants reported engaging in the practice of analyzing student thinking. For example, one primary school teacher participant named Bertha

Table 1: Practices of Lesson Study Reported to Continue

Table 1

Practices of Lesson Study Reported To Continue

	Teachers						Administrators	
Practice	Bertha	Carmen	Jimmy	Ben	Mia	Gillian	Ron	Kono
Complete Cycle of Lesson Study		X	X	X	X			X
Unofficial Cycle			X		X	X		
Goal Setting	X	X	X	X	X	X		X
Planning	X	X	X		X	X		
Observing							X	X
Analyze Student Thinking	X	X	X	X	X	X	X	X
Debriefing	X	X	X		X	X	X	

highlighted one of her conversations with her colleague by recalling her notes from a lesson study:

> The kids were just kind of brain storming what is multiplication. And one of the notes that the kids came up with that Carmen and I had on our notes was, multiplication is, you know, a bunch of things. But one of the things that stood out was when you multiply, the value always increases. I think that's what it was. And then somebody was having a conversation, where they said, 'Yeah.' And I always though that, too. But then they're talking about, 'Yeah, but what about when it's multiplied by 1 or 0, it does not increase.'

Mia, a primary school teacher participant, described an example of the role that student thinking played in her work with colleagues. She gave the example on how to modify a multiplication task with fractions to a multiplication task with whole numbers to help scaffold a problem for a student. "If they're not understanding that it is 3/4 of one half, and it is getting smaller. Why? Because they're not understanding that it's groups of. Oh, so that's multiplication. So we have to come back with, if you have 3 times 5, bring it to an array." These two examples highlight the role that student thinking played in the reported activities of participants.

None of the teacher participants reported observing their colleagues' lessons, though administrators like the principal and Teacher on Special Assignment (TOSA) reported observing teachers. Most teacher participants reported planning and debriefing together, oftentimes during formally arranged time like Professional Learning Community (PLC) time and during informally arranged times like lunch or sporadically throughout the day. Gillian reported meeting at a coffee shop after school to debrief with fellow teachers on what students did during the lesson.

> And it was not, umm, full scale lesson study, I would say. It was more, why don't we try this lesson. And then after the lesson we collaborated. We met at Starbucks to talk about, you know, how everything went. And, and share information on what the kids were doing. So I would, I would call it kind of like a mini lesson study. Cause we didn't, umm, we didn't do the observing of each person doing the lesson. So it was more that we, we had common planning and debriefing. Which I think is a good option when you can't get release time.

Factors that Supported and Hindered Lesson Study

Two factors that have the potential to support the continuing of lesson study as a form of professional development include integration and linkage. Integration levels, referring to the shared values, mutual expectations, levels of trust, and norms, were reported low for those teachers who described it as unlikely to engage in lesson study with their colleagues. For instance, both Bertha and Carmen reported shared values of what constitutes effective teaching and mutual expectations in terms of wanting professional collaboration time to focus on students' mathematical thinking and designing lessons that elicit their thinking. Yet their two colleagues in their formally arranged grade-level PLC did not share similar goals (See Figure 1). Bertha exemplifies this theme in the following data:

> And then on my team, not everybody values math the same way, not that I love math. I really don't really like math, but it intrigues me because I don't know about it and I want to know more about it. But I think that. I'll say some teachers on our team don't really see the value behind the lesson study because they haven't been through it. They don't know what it is, and they just know that it's, oh it sounds like a lot of work. It sounds like a lot of time. It sounds like a lot of planning. I don't have time for that. I'm just going to do the lesson that I've always done.

Thus, Bertha reported a difference in the way her PLC members valued mathematics. Carmen, her fellow PLC member also interviewed for this study, explains that one reason for the difference in values on mathematics could be explained by the engagement in union policy by the other two members.

> There's a clause in there that says in PLC that it has to be teacher driven, and teacher, like, decided upon. So, if two of the four people on the grade level want to do lesson study, but two other people, or one other person who doesn't want to do lesson study, we can't make those people do lesson study.

These data exemplify instances of low levels of integration and signify a challenge of continuing lesson study with these colleagues.

To exemplify linkage, I present egocentric network data in Figure 1. In this figure, vertices indicate participants and undirected edges between a pair of vertices indicate a reported significant professional relationship in the form of work related activities between the two participants. For instance, the single edge *BC* between vertices *B* and *C* represent activities such as engaging in planning conversations, focusing on students' mathematical thinking, and also conversations about mathematics content.

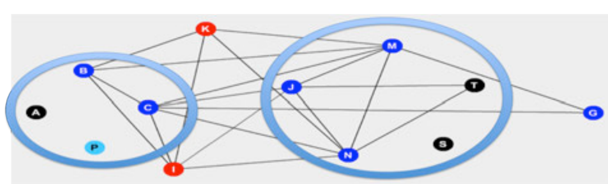

Figure 1. Network of participants derived from interview data. Formally arranged grade level groups are indicated by a circle. Red vertices represent administrators, blue vertices represent participants interviewed for the current study, light blue vertices represent former grant teachers not interviewed for the study, and black vertices represent teachers not interviewed for the study and not a former grant teacher. The circles each represent a formally arranged grade-level PLC group.

The *size* of each participant's neighborhood, or the other vertices that each vertex is connected to, range from two to six. In other words, teachers reported to exchange information while engaging in activities ranging with two to six specific colleagues, with an average number of connections to others out of participants for this study being 4.75 (vertices A, P, T, and S were not participants for this study and consequently are not calculated for size). The *density*, or the extent to which a participant's connections are connected to one another, is found by dividing the number of ties for one participant by the total number of potential ties to other participants. For instance, the total number of ties in this network can be calculated as 8!/2!(8-2)! This number, which counts the total number of ways 8 participants could be connected to one another exactly once, is 28, making the density of this network a total of 19/28, or approximately 67%. The *distance*, or the mean of the shortest path lengths among all connected pairs of participants, for each participant ranges from 1.14 (for instance, vertex C is connected to all but one by 1 path, and connected to K by a path of length 2, making the mean of 1+1+1+1+1+1+2 equal to 1.14) to 1.5 (for instance, vertex G is connected to all but two by a path of two, and connected to M and C by a path of length one, making the mean of 1+1+2+2+2+2+2 equal to 1.5).

These three measures – size, density, and distance – suggest a way to quantify the measure of linkage. These moderate levels of linkage suggest some potential in continuing lesson study.

Conclusion

Little research exists on issues surrounding the sustainability of lesson study for mathematics teachers. This research examined practices of teacher communities that get reorganized when relationships among teachers shift; in particular, when relationships and funding between teachers and university faculty who engaged in mathematics lesson studies end.

Findings from this study highlight teachers' conceptions of what it means to engage in lesson study and reported practices of lesson study that continued past the end of the grant. Findings also highlight the need to attend to social relations among teachers and administrators in one district to better understand issues of sustainability. It was evident that many teachers from the former grant wanted to continue to engage in lesson study. Additionally, districts attempted to put in place supports for these teachers to continue to engage in lesson study. Integration and linkage were shown to be important factors in continuing lesson study; low levels of integration or linkage suggested low potential for continuing while high levels of integration or linkage suggested high potential for continuing.

Endnote

[1]All names used in this report are gender preserving pseudonyms.

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BUILDING PEDAGOGICAL CAPACITY THROUGH TASK DESIGN AND IMPLEMENTATION

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We traced the impact of a sequence of five research-based professional development sessions on a cohort of mathematics teachers' Mathematical Knowledge for Teaching. The sessions focused on task design and implementation as a means of building teachers' pedagogical capacity. Findings revealed that teachers' pedagogical knowledge pertaining to student thinking, if not their practice, was influenced by the activities they experienced.

Keywords: Teacher Education-Inservice; Mathematical Knowledge for Teaching; Instructional Activities and Practices

Introduction

Improving the quality of Mathematical Knowledge for Teaching (MKT) among teachers has been at the forefront of mathematics education reform agenda for quite some time. Within the last three decades, advances have taken place in defining accurately and precisely what mathematical knowledge for teaching might mean and the various dimensions that are embedded in the construct (Ball, Thames, & Phelps, 2008), developing theoretical models that inform how teacher learning of this body of knowledge might be best grounded (Borko, 2004), and identifying features of effective professional development programs that facilitate such learning for teachers (Garet, Porter, Desimone, Birman, & Yoon, 2001).

Less clear however, are the specific domains of MKT of teachers that are enhanced by their participation in research-based professional development programs or empirical data that support changes as the result of their newly acquired knowledge (Sztajn, 2011). In this paper, we will describe the findings of an exploratory research project in which we traced the impact of a series of professional development sessions focused on task design and implementation on teachers' pedagogical content knowledge.

Background

Building on two of Shulman's categories of teacher knowledge (content knowledge and pedagogical knowledge), Ball, Thames, and Phelps (2008) defined six domains of Mathematical Knowledge for Teaching (MKT), which they defined as the "mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). While Ball and colleagues distinguished between content knowledge (Subject Matter Knowledge) and pedagogical knowledge (Pedagogical Content Knowledge) as Shulman did, they more specifically designated three domains within each. Within Subject Matter Knowledge, they defined Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge (HCK). The CCK domain includes knowledge of mathematics that students must learn, while the SCK domain includes knowledge of mathematics that is specific to the classroom environment (e.g. analyze student errors). HCK is described as "a view of the larger mathematical landscape that teaching requires" (Hill & Ball, 2009, p. 70).

Within Pedagogical Content Knowledge, Ball and colleagues (2008) distinguished between Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). In the KCS domain, teachers must understand how students may come to understand a concept, while in the KCT domain teachers must make instructional decisions that best facilitate student learning. KCC requires that teachers know the standards and curriculum not only for a specific mathematics course, but also across grade levels, courses, and subject areas.

Guskey (2003) argued that the primary goal of professional development (PD) programs is to "bring about change in the classroom practices of teachers" (Guskey, 2002, p. 381). In order to achieve this goal, researchers have identified characteristics of teacher PD programs that tend to advance teacher learning and shifts in practice. Among them include opportunities for sustained interactions over time, collective participation by participants, and activities that are content-focused and grounded in the teachers' everyday practice (Garet et al., 2001).

The use of mathematics tasks in PD sessions has been found to influence teachers' teaching knowledge and their instructional practices. Through activities with mathematics tasks, teachers can increase their capacity to implement the curriculum with coherence across multiple grade levels (Ferrini-Mundy, Burrill, & Schmidt, 2007), improve their problem solving skills (Guberman & Leikin, 2013), and increase their content knowledge of mathematics (Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007).Boston (2013) found that teachers' capacity to identify cognitive demands of tasks, as well as identify the opportunities particular tasks provided to elicit student thinking, was increased by participation in a PD program that focused on the cognitive demand of tasks. What remains to be learned about the use of tasks in PD programs is how the modification of traditional tasks and their implementation in the classroom builds teachers' pedagogical capacity.

Drawing from this body of scholarly ideas, a series of 5 PD sessions for mathematics teachers who were expected to serve as instructional leaders in their respective schools was designed. Prominently, relying on Cognitively Guided Instruction (Carpenter, Fennema, & Franke, 1996), we intended to engage teachers in building their own knowledge of mathematics through investigating children's understanding of mathematics content and using that knowledge to guide instruction. In implementing the sessions, we capitalized on five practices recognized to be pivotal to orchestrate productive mathematics discussions (Smith & Stein, 2011). The primary focus of our work was facilitating knowledge development of the teachers through the creation and implementation of rich mathematical tasks, and then considering different solution strategies (appropriate or inappropriate) students might use on the tasks as a way of anticipating what may need to be addressed in instruction.

The Five Practices for Orchestrating Productive Mathematics Discussions includes the planning and selecting of appropriate tasks for classroom activity (Smith & Stein, 2011) since this venue has been recognized to have an impact on the mathematics students engage with (or not) in the classroom (Hiebert & Wearne, 1993). Coupled with the understanding that textbooks and teacher resources are often limited in their offering of tasks that require reasoning of children (Thompson, Senk, & Johnson, 2012), two of the PD sessions were focused on task selection and design in motivating mathematical thinking among school learners. Brown and Walter's (1983) problem posing framework served as the primary guide for organizing teachers' activities during these sessions. The teachers were first presented with a task, and asked to list its attributes. They then removed constraints from the task and rephrased it in a way that could substantially extend learners' thinking. The goal was for the modified tasks to reflect characteristics of rich mathematical questions: tasks that provide multiple entry points, multiple solution strategies, and opportunities for students and teachers to develop deeper mathematical connections (Stein, Grover, & Henningsen, 1996). Our research goal was to determine the knowledge teachers may have gained from such experiences based on their classroom implementation.

Methodology

Setting & Participants

The participants in this study consisted of 12 mathematics teachers expected to serve as instructional leaders in their own respective schools using a coaching model. The participants' prior mathematics teaching experience ranged from 1-31 years, with a mean of 13.9 years. All teachers engaged in a year-long PD program that was grounded in principles of Cognitively Guided Instruction (Carpenter et al., 1996). This approach was used to help teachers develop capacity towards shifting classrooms from teacher-centered orientation to a more student-centered environment. The PD sessions that served as the basis for our exploratory inquiry consisted of series of five sessions that lasted approximately 3 hours each (a total of 15 hours). The fourth session in this series used the problem posing framework of Brown and Walter (1983) to assist teachers in learning about how to modify mathematics tasks to advance students' mathematical thinking. The teachers were asked to revise a task that already existed in their practice or curriculum materials, implement it the classroom, and then reflect on what they learned from doing so.

Data Collection

Upon the creation of the new tasks to be implemented in the classrooms, the teachers were asked to identify mathematical and pedagogical goals they intended to meet using the new tasks. They were also asked to collect samples of student work (correct or incorrect) and to comment on what they may have gained from the experience. The participants submitted their reflections/assessments electronically. For this particular study, the teachers' original and revised tasks, as well as their responses to four questions served as the primary sources for data analysis: (1) How is your revised task different from the original task? Describe the process you used to adapt the task from the original task.; (2) What additional questions did (or would) you or your teacher ask to elicit, scaffold, or extend student thinking?; (3) What insights into your students' mathematical thinking and understanding did the revised task provide that the original task may not have been able to provide?; and (4) How would you adapt or implement this task differently in the future and why?

Data Analyses

The original and revised tasks submitted by the teachers were coded using Stein, Smith, Henningsen, and Silver's (2000) categories of level and kind of cognitive demands and/or thinking processes in mathematics classrooms: high-doing mathematics; high-use of procedures with connections to concepts, meaning, and/or understanding; low-use of procedures without connections to concepts, meaning, and/or understanding; and low-memorization. An example of our coding procedure is show below, illustrating the original and the revised tasks submitted by one teacher:

> *Original*: There are twenty-one shells. The shells are evenly divided among three students. How many shells will each student get? A. 6 B. 7 C. 8 D. 9

> *Revised*: Susie has 24 gumballs. She wants to share them with some friends. She wants to make sure each friend gets the same amount of gumballs. How many different ways can you come up with for Susie to share her gumballs with friends? Show all the ways below.

The original task was coded as low-use of procedures without connections to concepts, meaning, and/or understanding, and the revised task was coded as high-use of procedures with connections to concepts, meaning, and/or understanding. Once the codes were generated, further analysis determined strategies the teachers used to revise the original task.

The teachers' responses to the four reflection questions were coded first using the Mathematical Knowledge for Teaching framework (Ball et al., 2008). The unit of analysis was one sentence. If two or more sentences referred to the same idea, the two sentences were coded once, and if a sentence contained two or more phrases that fell into different codes or ideas, the phrases were coded separately. Three independent researchers coded the teachers' responses. Coding results were compared among the researchers; if disagreement occurred, discussions were held to reach agreement on the coding for each response. Disagreement on the coding occurred and was remedied five times.

The following excerpt from one of the teachers' responses to question (1) serves as an example of our coding procedure:

> The original task was simple math. There was not a lot of higher level thinking involved. The students had worked on this type of equivalent fractions for some time. It came pretty easy to them. We needed to create a task that would challenge the students and take the math they learned and apply it to a real life situation.

The first two sentences were coded once as Specialized Content Knowledge, the second two sentences were coded once as Knowledge of Content and Students. The last sentence was coded twice as Knowledge of Content and Teaching (one code for "challenge the students" and one code for "take the math they learned and apply it to a real life situation").

The codes that were identified for the responses to each question were tallied corresponding to each participant. Since question (2) primarily elicited responses in the MKT domain of Knowledge of Content and Teaching, the question-type framework as described by Boaler and Brodie (1994) served as a second level of analysis for the teachers' responses for question (2). The responses to question (4) also overwhelmingly fell into the KCT domain, so they were coded on a second level as either related to content or pedagogy.

Results

Original and Revised Tasks

The coding results for cognitive demand level and type of the original and revised tasks are summarized in Table 1.

Table 1: Cognitive demand level and type for original and revised tasks (n=12)

	High-doing mathematics	High-procedures with connections	Low-procedures without connections	Low-memorization
Original task	0	0	10	1
Revised task	0	7	5	0

Note that all of the original tasks were low level cognitive demand (one of the tasks did not provide enough information to be coded), and that 7 of the 12 revised tasks increased the cognitive demand level to high. The teachers who increased the cognitive demand from low to high did so by removing constraints in the task to allow for multiple solutions, asking students to provide explanations and visual representations, introducing new mathematics content, and requiring students to give non-examples.

Five teachers submitted original and revised tasks that were both categorized as low-procedures without connections. While they did not increase the cognitive demand of the task, these teachers added a context or changed the language of the original problem to revise their task. For example, one teacher's original task, a worksheet of 3-digit subtraction problems, was revised to include three 3-digit subtraction problems placed in a context, one of which included "There are 300 dogwood trees currently in the park. Park workers cut down 115 dogwood trees today. How many dogwood trees are left in the park?".

Reflection on the Task and its Implementation

Six teachers responded to all four questions, and six teachers responded to questions (1) and (2) only. Table 2 is a summary of the MKT codes, specifically in the domains of SCK, KCT, KCS, and KCC that were generated from the teachers' responses to each question, illustrating the knowledge domains that may have been influenced as a result of the activity.

Table 2: Domains of teacher knowledge elicited by four reflection questions

	Specialized Content Knowledge (SCK)	Knowledge of Content and Teaching (KCT)	Knowledge of Content and Students (KCS)	Knowledge of Content and Curriculum (KCC)	Total
Question (1)	11	32	7	4	54
Question (2)	0	57	0	0	57
Question (3)	11	14	5	0	30
Question (4)	0	13	0	0	13
Total	22	116	12	4	154

For question (1), in which the teachers were asked to compare their original and revised tasks, 11 of the 12 teachers made statements that compared the tasks in terms of the different levels or types of thinking the tasks may (or may not) elicit from children. For example, teachers stated that the revised task was more open-ended than the original and would thus require students to use their own thinking to solve the problem, or that the new task would require a higher level of thinking than the original task. Six teachers explicitly stated that they removed a constraint or piece of information in the original task to make the revised task more open-ended. Five teachers responded that the revised task would require different mathematics content than the original task.

For question (2), the teachers stated questions they asked or would ask to elicit, scaffold, or extend student thinking. In total, the teachers stated 57 questions, all coded as KCT. 66.7% of the questions asked students to go deeper than the revised task, categorized in the Boaler and Brodie (2004) question-type framework as extending thinking (n=18); probing, getting students to explain their thinking (n=14); exploring mathematical meanings (n=4); and linking and applying (n=2). Questions posed by the teachers in these categories included "How might your ideas change if Susie includes herself in the equal sharing?" (extending thinking) or "Do you think those represent the same solution or different solutions?" (exploring mathematical meanings). The remaining 33.3% of the questions geared students toward finding the answer to the posed task. These questions fell into the categories of orienting and focusing (n=12), gathering information (n=5), inserting terminology (n=1) and establishing context (n=1). Questions posed by the teachers in these categories included

“What is this problem asking you to do?” (orienting and focusing) and “How did you know this was a subtraction problem?” (gathering information).

For question (3), the 6 teachers who implemented the task reported what they learned about student thinking as a result of implementing the revised task. All six of the teachers claimed to gain insight into their students’ thinking, their misconceptions, and where they may struggle with the content. For example, one teacher stated “it was apparent which students recognized that there was a missing addend and that an efficient way to find the missing addend is to subtract.” Four of the six teachers reported improving their SCK as a result of implementing the task, stating that the task provided opportunities for students to use multiple representations and solution strategies, and for the teacher to interpret student work and student thinking on their particular topic.

For question (4), the 6 teachers who implemented the task reflected on how they would implement the task differently in the future. The teachers’ responses to this question generated 13 KCT codes, 5 related to content, and 8 related to pedagogy. For an example of a pedagogy modification, one teacher stated that she might “have different students solving the same problem but with different numbers of gumballs. This may allow for more connectedness about numbers being divided up in different ways.” Pedagogically, a different teacher thought he would “give students more time to work through the task, provide manipulatives, encourage collaboration and communication, and strategically have students share ideas.”

Summary

All of the teachers revised a traditional task to be implemented in the classroom with the intent to offer more open-ended venues for student explorations. Of the 6 teachers who revised a low cognitive demand task to a high cognitive demand task, 3 implemented the revised task in their respective classrooms. The reflections of these 3 teachers revealed that their revised task gave them more insight into student thinking, the reflection comments offered by these individuals indicated that they felt the new questions allowed their students to learn more mathematics and make deeper connections as a result. These 3 individuals also reported that they felt more efficacious in guiding the classroom discussions in a manner that deviated from additional tell, show, and correct pattern.

In their reflections, all 12 teachers reported ways in which their Mathematical Knowledge for Teaching may have been influenced by their participation in the PD sessions and related activities. Overwhelmingly, the knowledge the teachers claimed to gain could be described as pedagogical, as 75.3% (n=116) of the teachers’ responses described ways their Knowledge of Content and Teaching was impacted as a result of the activity. The teachers claimed to have gained knowledge of strategies they could utilize to modify a traditional task to create a task that is more open-ended, allows for multiple solution strategies, and requires students to use their own thinking. The 6 teachers who implemented the task cited ways the new task enabled them to better facilitate mathematical discussions that focused on student thinking and making mathematical connections.

Six teachers who implemented their revised task in the classroom cited ways that this process influenced their Knowledge of Content and Students. One articulated her heightened awareness of student thinking as she stated “it was clear that the students did understand the relationship between multiplication and division. This is something that the original task would not have brought out about student understanding.” Two of the teachers noted that students had difficulty finding new strategies that were different than the strategies their teacher had taught them. This leads us to conclude that the enactment of the revised task in the classroom, not just its revision as an isolated activity, is critical in impacting teachers’ pedagogical knowledge.

Conclusion

The main goal of our exploratory research was to determine what aspects of teachers' Mathematical Knowledge for Teaching could be influenced by PD sessions guided by the principles of CGI and with a focus on designing rich, open-ended tasks. The reports of these participants indicated that their pedagogical knowledge pertaining to their understanding of student thinking, if not their practice, was influenced by what they had learned as a result of implementing the revised tasks in their respective classrooms. Additionally, the teachers who revised the traditional task from low to high cognitive demand claimed their students learned different mathematics content and made richer connections as a result of engaging with the revised task.

The National Council of Teachers of Mathematics (2000) and the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010) require that children have the opportunity to engage with mathematics in ways that foster understanding, sense making, and reasoning. No longer is the traditional drill and practice classroom environment acceptable to meet these standards. But teachers must be given learning opportunities that may foster the pedagogical and content knowledge necessary to facilitate such a classroom environment. The PD program and sessions that informed this research gave teachers valuable tools and increased pedagogical capacity necessary to take their current classroom materials and adapt and implement them in ways that may help children reason more deeply about mathematics to meet these new standards.

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RISKY BUSINESS: MATHEMATICS TEACHERS USING CREATIVE INSUBORDINATION

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In an era of high stakes education and the persistence of racism, classism, and the politics of language, there is evidence that teachers may benefit from learning creative insubordination, the bending of rules in order to advocate for all students to learn mathematics. Even so, we know little about how or why teachers decide to take risks when stakes are high. This study examines the experiences of secondary mathematics teachers moving from pre-service to full-time teaching and their choices of whether or not to use creative insubordination in their working contexts. It highlights three rationales that justify taking risks: 1) Changing the minds/practices of others, 2) Projecting an identity one can be proud of, and 3) Modeling advocacy behavior for bystanders. Implications for future research and teacher education are offered.

Keywords: Equity and Diversity; Teacher Knowledge; Teacher Beliefs, Teacher Education-Preservice

In an era of high stakes education and the persistence of racism, classism, and the politics of language in society, professional development for mathematics teachers has started to expand beyond developing forms of pedagogical content knowledge that draws upon deep understanding of mathematics (Ma, 1999; Hill, et al., 2005) and students' funds of knowledge (Civil, 2002; Turner et al., 2012; Aguire & Zavala, 2013) to include an understanding of privilege, oppression, and political knowledge (Willey & Drake, 2013; Bartell, 2011; Gutiérrez, 2012, 2013a, b, in press). The importance of political knowledge and creative insubordination is underscored by the derailing of successful mathematics departments such as Railside High and Union High (Boaler & Staples, 2008; Nasir et al., 2014; Gutiérrez, 2013a; Gutiérrez & Morales, 2002). That is, these mathematics departments had long histories of success with low income Latin@ students who normally do not have positive experiences or reach advanced levels of mathematics while in high school. Yet, the politics of their districts kept teachers from either maintaining the practices they had developed or eventually pushed teachers out who were beaten down by a climate of alienation and deprofessionalism. The stories of these departments indicate that professional development around issues of pedagogical content knowledge and commitment of teachers to all learners may not be enough to sustain success in student learning when larger political debates about public schooling and testing arise. As such, some teacher education programs have adopted a broader lens of equity, including actively providing opportunities for teachers to not only deconstruct the deficit narratives that circulate in schools, but also speak back to those narratives through direct actions. One of those forms of speaking back through actions is creative insubordination, the bending of rules in order to advocate for one's students.

Creative Insubordination in Mathematics Teaching

Our project began using the term "creative insubordination," having heard it first in activist circles in the late 1970s and early 1980s and growing up. Later, we learned that in their ethnographic work conducted in Chicago Public Schools, Crowson & Morris (1985) found "widespread rules and directives violations among site-level administrators" that they labeled partly as "creative insubordination" because these violations were benign and counter-bureaucratic and substituted the

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principal's values for those implicit in organizational policies directed from above. Summarizing various studies, Roche noted:

> Creative insubordination has two main purposes: to ensure that the system directives do not impinge unfairly or inappropriately on teachers and students and to avoid the possible backlash that outright defiance might incur. Crowson (1989) and Haynes and Licata (1995) argue that when principals use creative insubordination, the counterbureacratic behaviors they adopt often contain a moral element designed to balance antieducational consequences." (Roche, 1999, 257-8)

Our work (Gutiérrez et al., 2013; Gutiérrez & Gregson, 2013) builds upon and extends the early research on creative insubordination by connecting it with teachers and showing its usefulness within the context of secondary mathematics. With respect to mathematics teaching, creative insubordination includes the following acts: creating a counter-narrative to the achievement gap; questioning the forms of mathematics presented in school; highlighting the humanity and uncertainty of mathematics; positioning students as authors of mathematics; challenging deficit narratives of students of color; renaming a course to reflect the fact that it only covers Western, Euclidian geometry, not all geometries that are practiced in the world; refusing to go along with procedures at a workshop that asked teachers to publicly endorse the Common Core State Standards in mathematics; and convincing a co-teacher that the mathematics being taught needed to reflect a more rigorous curriculum so that students understood why procedures worked.

Elsewhere, I have described the concept of *political conocimiento for teaching mathematics* (Gutiérrez, 2012; 2013a) that connects mathematical content knowledge, pedagogical knowledge, knowledge with communities, and political knowledge within a community of like-minded individuals. This concept of political conocimiento for teaching mathematics takes into consideration the history of mathematics teaching and learning in a global society. In addition, I have articulated both a model of teacher education that supports the development of political knowledge, as well as taxonomy of strategies for creative insubordination and language practices that keep teachers from being dismissed in political situations (Gutiérrez, 2014). This study seeks to extend that knowledge by asking: What compels mathematics teachers to take risks in their working contexts to advocate for historically marginalized students and their learning when the stakes are high and the benefits of taking risks are not always clear?

Risk taking is normally viewed as a process whereby an individual weighs the costs and benefits of an action and finds the benefits outweigh the costs. The majority of the research conducted on individuals taking risks falls within the area of risky sexual behaviors, risky business ventures, or risk taking in health. Within education, there is some evidence that both support from leadership (Blase, 2000) and strong internal feelings of power and a sense of responsibility (Anderson & Galinsky, 2006) can encourage individuals to take more risks. Moreover, some researchers have suggested that risk taking may be a sign of great teachers (Brazeau, 2005). However, outside of the context of innovating their pedagogy to align with reform-based mathematics, we know very little about the risk taking behaviors of mathematics teachers in a context of political situations.

All of the participants in this study acknowledged the utility of creative insubordination (which requires taking risks) in their student teaching and current working contexts as well as expressed a desire to use it. However, not all of them did so. As such, this study sought to understand the phenomenon of risk taking for mathematics teachers, how they interpreted risks and how they decided whether or not to take a stand in a political situation that involved power dynamics.

Methodology

This study is part of an ongoing, NSF-funded, longitudinal investigation of secondary mathematics teachers who have been provided with an alternative teacher education program that foregrounded issues of equity, social justice, creative and rigorous mathematics, and political knowledge. Over a period of 2 years each, four cohorts of teachers (n=19) participated in a 3-hour bi-weekly seminar, a partnership with a Chicago public high school teacher, several professional development sessions including an annual summer boot camp and conference attendances, a weekly after-school mathematics club, and biweekly individual mentoring sessions. These structural aspects of the teacher education model supported several conceptual goals, including broadening and challenging knowledge, noticing multiple interpretations, developing an advocacy stance, and rehearsing creative insubordination (Gutiérrez, 2013a). Rather than offering a set of "effective practices" to follow (Bartolomé, 1994), one guiding principle of this teacher education model was "The Mirror Test:" the ability to look oneself in the mirror everyday and say, "I'm doing what I said I was going to do when I entered the profession of mathematics teaching." This notion of a mirror test underscored the idea that being a great teacher for students who have historically been marginalized (e.g., students who are black, Latin@, low income, English learners, and/or immigrants) means carrying out one's practice in a way that is consistent with one's philosophy and ethical stance.

The data for this study draw from transcripts of 78 seminars and selected mentoring sessions between 2009 and 2014. Transcripts were coded (Lincoln & Guba, 1985) to identify major themes around the interpretation and use of creative insubordination, including types of political situations in which teachers find themselves (e.g., battles over curriculum, use of technology, high stakes testing), types of power dynamics involved (e.g., teacher-student, teacher-parent, student-student, teacher-administrator), underlying issues (stereotypes of who is good at math, deficit views of students of color, watering down of curriculum, lack of low income students in advanced courses or higher tracks), types of creative insubordination strategies teachers used under different contexts (e.g., seek allies, challenge with evidence, turn a rational issue into a moral one). Because I was interested in the phenomenon of risk taking as it related to creative insubordination, I began analyzing data by first sorting participants into categories of high and low use of creative insubordination. I identified five teachers who consistently reported using creative insubordination in their student teaching and full-time teaching, three who never or very rarely used creative insubordination in either context, and the rest who sometimes used creative insubordination. Focusing on the high users of creative insubordination, I turned to their rationales for why they did so in any given situation as well as the questions they raised to others who chose not to use creative insubordination. I was most interested in how these high users justified taking a stand on an issue when others in the group might not have done so if faced with the same scenario. From there, I looked at the infrequent users of creative insubordination to understand better what challenges they saw that seemed insurmountable or that did not seem worth risking their status or relationship with colleagues, students, or administration. I used member checking (Lincoln & Guba, 1985) to corroborate findings. I report on the trends here.

Findings

A review of the transcripts indicated that teachers are often willing to take risks when they immediately benefit from doing so. For example, one teacher in our work group didn't pause before defending herself when a student of hers was leaving her classroom and made a disparaging remark about whites. However, teachers are more cautious when they are risking their status or credibility with colleagues to defend students or a more rigorous form of mathematics.

Changing the Minds of Others

Teachers in this study tended to be most likely to take risks in their teaching practices with students and their interactions with others when they were optimistic that their actions would have some positive effect on the decision-making process of others. In particular, when faced with political situations, ones involving power dynamics, they tended to gauge whether they could diplomatically disagree with someone and get an alternate view onto the table in a way that would not be dismissed. They considered the language they would use and the contexts in which they found themselves so that they were the most effective.

One high user of creative insubordination was faced with a cooperating teacher who had low expectations for students in an Algebra II course populated by older students. He laid out an argument to the cooperating teacher about the internal consistency of mathematics presented in class (how students needed to relate polynomials with integers) because he knew that an argument based solely on what students were capable of was not going to sway him.

> He showed me the curriculum, topics that I needed to cover. There's like synthetic division, but there's nothing else like…And, synthetic division is basically just an algorithm. And, I'm gonna have to just say, "it works." …I don't want to argue against him because I don't want to step on his toes. I need to be wise in what I say so I don't sever that relationship with my coop. I just think about it in my head, then I try to think about how he would take it. From there, I try to compromise my own words. I was going to say this, but I try to come up with a better wording. I try not to say things right away. I know that the issue is very fresh and he may think I am acting on impulse. I try to make sure there is no room for my statement to be misconstrued as I'm just challenging him. Trying to be wise in my timing on things. I know he feels pretty strongly about things and I try to my best of ability, weighing the risk and reward. Is this going to be something worth fighting over? A lot of times, I'll say, "Let's wait a bit and see what happens."

This teacher, a high user, talks about his process for weighing the risks, and sometimes choosing not to fight a particular battle. In doing so, he suggests that he considers who he is talking with and in what context to help him choose his words wisely so that he is not simply dismissed. In this sense, his focus is partly on changing the mind of the person he faces.

Projecting an Identity that One Can be Proud Of

Another rationale that arose for the high users that did not arise for the infrequent or rare users of creative insubordination was the idea that risk taking is worthwhile even if one is not sure of the outcome for others because it reflects who you are. When faced with a school meeting where the school administrator suggested that the achievement gap was due to black student culture, one teacher clarifies that she was not sure how the administrator would respond to her correcting his statement publicly, but she was also projecting a particular identity that she could be proud of. She says,

> Okay, this is also the boss and I've only taught one year. I don't know how I'm going to say what I want to say without sounding like, "I straight out disagree with you" in a respectful way…[I was] Just feeling, a lot of fear…But, I have to let myself be known to people. This is the kind of person I am, this is what I believe in, and if you wanted to talk about something that is important with respect to race, I'm the kind of person to talk to.

Her rationale highlights the fact that she was willing to risk her status with a superior not just because she thought she might change his mind about black students and the achievement gap, but because her actions were signaling something about herself to others. She counsels others in the teacher group who are less inclined to take risks when they feel uncomfortable to consider how they'll feel about themselves in the end.

> It's about changing our behaviors not our ideals. It's about not saying you stand for problem solving and then not doing any problem solving in your class.

This same teacher faced a black student who when she passed him over to help another student when he raised his hand, publicly called her out as racist. She felt the need to stand up to him in a similar way, less because she thought he was serious about thinking she was racist or because she was trying to change his mind, but because she wanted other students in the class to know what kind of a teacher she was, someone who was comfortable talking about race.

Returning to the case of the high user who faced a cooperating teacher with low expectations who did not want to increase the rigor of mathematics in his class, the high user explains that he wanted to change his mind, but that was not his only goal.

> That's just not how I do things. I didn't know if I was going to change his mind. But, I just needed him to let me do what I wanted. I couldn't be him.

This high user of creative insubordination expressed a similar rationale for standing up to a student who made a stereotypical comment about certain people being good at mathematics. He witnessed a student who was struggling with a problem turn to another and exclaim, "Put on your Asian hat and help me do this problem." He explained,

> I don't know if he really thinks that or they're just joking, but I couldn't just stand there and, I don't want to signal that that's okay in my class. Maybe that's okay in other classes, but not for me.

Again, we hear the rationale less about weighing the benefits and risks for impacting the student who made the stereotypical comment and more about what it signals to his class about the kind of teacher he is and what is allowable in his class.

Modeling Advocacy Behavior for Bystanders.

Mentioned less often than changing the minds of others or projecting a particular identity was the idea that even if one's actions didn't convince others to change policy or their beliefs about which students were capable of advanced mathematics, using creative insubordination had the possibility of influencing others who were bystanders to stand up in similar ways to such comments. Again, this rationale was more present among the comments made by high users of creative insubordination than those who were infrequent or rare users.

One teacher (a less frequent user) was faced with a superior in the mailroom and wasn't sure how to respond to his stereotypic comment about Asians being good in math.

> He said it in an offhand way. There was a student who was Asian and was not good in math. He said, "She's Asian and you wouldn't expect that." He said it so quickly and kept moving on with the story… I work there, so I can't have him mad at me. I don't want him to say something to my boss… being the younger person talking to an older person, is it my place to be saying something? What are the consequences of speaking up? I know him well enough to know that had I said something, he wouldn't have any respect for what I said. So, is it even worth mentioning?

Three fellow teachers, high users, counsel her,

> T1: I think so,

T2: I think so too. One thing, it's easy to make assumptions about people that protect us from putting ourselves out there. I'm not saying that you did…It's really easy to assume something about someone to keep me from getting out of my comfort zone. You could say how he would have responded, but to say something, even if you are casual about it, he is probably going to reflect on his comments regardless of how you think of what he is thinking. I think it's important for us to get out of our comfort zones because it is how we are going to grow and how we are going to make a difference…

T1:… the first step is by putting yourself out there; you are making a change within yourself. Then, you can help others make the same change. How can I help others in my classroom take that risk to see others differently if I cannot do that? How can I start to live more like what I believe in? The little things are what matter.

T3: I really like what was just said. I think it is really good that a lot of us are feeling this discomfort. I don't know if I would have acted. I think if I hadn't, I think I would have looked back and said, "Shoot, why didn't I? It's good that we are even feeling that drive and that tension. It's there because you are taking a real risk when you are doing that. Especially as a white person, that risk doesn't benefit us. We can, it benefits us by not taking the risk. Maybe the person you are talking to might not get the message. But, you may become a role model for the observer. That other person will see you taking that risk. Especially as a white person. That is really strong and a way to start making change too.

Another high user gives a similar explanation for standing up for students even when it doesn't benefit her as a white person. A rare user of creative insubordination asked her how she gets the guts to do what she does. She responds that when faced with racist comments about her students of color, she does not naturally know what to say or how to say it, but she takes the risk because of what it says to other white people.

> I'm white and it is really hard to get your privileges pointed out to you, and you know you didn't earn them and you… One privilege is to be able to walk around this discomfort and to be able to walk away from it…I feel really nervous right now, that fear, feeling uncomfortable. Because we are in the majority, we can just choose to leave the conversation. It's extra important for us to get into these uncomfortable situations. We need to commit ourselves to not taking it and not walking away from it because we can signal to others what is possible.

She recognizes that standing up to racist comments by individuals one faces is not always for the benefit of that person changing, but for other bystanders to witness what it looks like for a white person to stand up for the rights of others.

Conclusions

Teachers in this study were provided with professional development that supported their understanding of and inclination to use creative insubordination, to not simply go along with the status quo but to stand up to racist comments, deficit-based perspectives on students, mathematical practices that emphasized procedures and memory over conceptualization, and/or stereotypes of who is good at mathematics. However, some teachers were more likely to stand up for their students, even if it meant risking their status or relationships with others.

There was a greater propensity for high users of creative insubordination to rely on a rationale that extended beyond the likelihood that their actions would be met by a change in belief or action by the person with whom they were facing in a political situation. These individuals tended to consider what kind of identity they were projecting to others, as well as whether their actions might provide an incentive for others to also speak up or advocate for historically marginalized youth and their rights to learn rigorous mathematics. That is, they were often willing to take a stand even if they knew their arguments or suggestions for alternate policies or actions were likely to fall on deaf ears. They did so knowing that their choice not to go along with the status quo was a means by which they could look themselves in the mirror each day.

Understanding the rationales of high users of creative insubordination can help teacher educators and professional developers to emphasize the importance of looking beyond the immediate gain of winning an argument with a colleague or changing a policy. The choice to focus on the projection of one's identity and the possibility of influencing bystanders suggests a longer term approach being used by teachers who seem more willing to take risks to advocate for all students to learn meaningful mathematics.

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CONNECTING MULTIPLE MATHEMATICAL KNOWLEDGE BASES: PROSPECTIVE TEACHERS' CONCEPT MAPS OF ASSESSING CHILDREN'S UNDERSTANDING OF FRACTIONS

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This study investigates how 20 prospective elementary teachers make connections among children's multiple mathematical knowledge bases in their thinking about assessing children's understanding of fractions. The researcher facilitated concept-mapping tasks to examine the ways the prospective teachers linked concepts related to children's lives and experiences and children's mathematical thinking. This paper focuses on high-level tasks as a potential entry point to build stronger connections between assessing children's understanding of mathematics and children's multiple mathematical knowledge bases in teacher education. Additionally, I discuss implications for teacher educators and considerations for further research.

Keywords: Teacher Education-Preservice; Teacher Knowledge; Equity and Diversity

A role of teacher education programs is to provide support for prospective teachers to develop professional skills that are specific to and required for teaching. Mathematics teacher educators may provide opportunities for prospective teachers to recognize and validate children's many ways of knowing mathematics, which is especially powerful for addressing the needs of students who are traditionally marginalized in the mathematics classroom. Mathematics education researchers emphasize the importance of eliciting and building on children's mathematical thinking in teaching mathematics (e.g. Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Jacobs, Lamb, & Philipp, 2010). Supplementing this focus on children's thinking, more recent research calls for incorporating children's home and community-based mathematical *funds of knowledge* in mathematics teaching to support student learning of mathematics (e.g. Aguirre et al., 2012; Turner et al., 2012). When teachers consider children's funds of knowledge, they ultimately have more resources to draw upon and inform their teaching practice to design meaningful classroom experiences that incorporate their students' knowledge and experiences outside the mathematics classroom. There is limited research, however, on how prospective teachers connect the role of assessing children's understanding of mathematics to incorporating children's funds of knowledge in their mathematics teaching. The purpose of my study is to better understand in what ways prospective teachers make this connection in the context of an elementary mathematics methods course. In this paper, I address the following research question: In what ways do prospective elementary teachers link concepts related to children's multiple mathematical knowledge bases to assessing children's understanding of fractions?

Theoretical Framework

Aguirre and colleagues (2012) defined children's *multiple mathematical knowledge bases* as children's mathematical thinking and children's community, linguistic, and cultural funds of knowledge. By building stronger connections among children's multiple mathematical knowledge bases, teachers may have opportunities to learn from their own practice while making intentional instructional decisions that support their students' learning in meaningful ways by connecting students' funds of knowledge to their mathematics classroom experiences. In this study, I am interested in the connections that prospective teachers make between children's multiple mathematical knowledge bases and their thinking about assessing children's understandings as part of their professional practice.

Given the intricacies of teaching practice, I draw on concept mapping as a tool that provides prospective teachers with a learning opportunity to create a representation of their thinking about relationships between concepts related to teaching practice. The research literature on concept mapping in teacher education provides evidence that its use as a research tool is valid and robust. Scholars have used concept map artifacts to examine teacher knowledge about mathematics content (e.g. Williams, 1998; Hough, O'Rode, Terman, & Weissglass, 2007) and knowledge about teaching skills (e.g. Beyerbach & Smith, 1990; Koc, 2012).Other researchers studied the validity of using concept maps as a research tool in education. For example, Miller et al. (2009) examined the capability of concept maps as a research tool by studying pre- and post- concept maps by 251 prospective and practicing teachers. The authors used a concept map scoring method (Novak & Gowin, 1984) and found that participants' concept map scores distinguished expert to novice levels in conceptual understanding and growth over time. My research question focuses on how prospective teachers make connections among concepts related to children's multiple mathematical knowledge bases, and I am using concept maps to gather evidence of these connections by examining how concepts in their maps are linked together. I based my research design on work by Hough and colleagues (2007) who partially used qualitative content analysis to compare teachers' pre-maps and post-maps from beginning to end of a professional development program.

Method

I conducted this study within the context of a larger research project that produced modules designed to teach prospective elementary teachers to make stronger connections in mathematics lessons between children's mathematical thinking and children's lives and experiences. In my research, I collected data from20 prospective teachers enrolled in a15-week elementary mathematics methods course using activities from these modules. This paper reports data from and analysis of individual concept maps constructed in the last week of the course and individual reflections on the connections within the concept maps. All prospective teachers created these artifacts during their regular class meeting time and location; however, the methods course instructor was not present during these activities. All names in this paper are pseudonyms.

Data Collection

During the last class meeting of the semester, I explained concept maps and the concept map activity to the whole group. In this activity, I prompted prospective teachers to create a concept map to represent their knowledge about *assessing children's understanding of fractions*. I selected "fractions" as a specific mathematics topic to consider because it was a major topic in the mathematics methods course. I explicitly instructed prospective teachers to think about this concept map activity with a network structure, rather than a hierarchical structure with downward flow, to encourage making connections among any and all related concepts on their maps. I emphasized connections among concepts because I wanted prospective teachers to focus on indicating links between concepts to answer my research question.

First, prospective teachers individually created a concept map on a legal-sized sheet of paper with the option to draw the map entirely by hand or to write concepts on provided sticky notes to place on the sheet of paper. I did not provide prospective teachers with any initial concepts to consider, but I did provide black pens for this part of the activity. Next, I asked prospective teachers to have a focused discussion about evidence of connections among children's multiple mathematical knowledge bases in their concept maps. These discussions occurred in groups of four people. After a few minutes, I interrupted small group discussions to direct attention to specific concepts related to children's multiple mathematical knowledge bases that I provided on a sheet of paper: Children's mathematical thinking; Problem solving strategies; Making sense of students' mathematical ideas; Students' personal experiences; Students' interests and activities; Students' home and community knowledge bases (e.g. regular routines, places in community); and Funds of knowledge (e.g. cultural, community, and linguistic resources).

I asked each group to review the list and determine if any of these concepts connected to their concept maps. I also asked each group to discuss how they would change their concept maps to include any of the provided concepts but explicitly instructed them not to change their maps during the discussion. While prospective teachers had discussions in their groups, I collected all black pens and distributed red pens. I designed this activity in two parts with different colored pens to distinguish and collect data for both parts (before and after focused group discussion). After discussion, prospective teachers individually revisited their concept map to make any changes with red pens. Finally, prospective teachers individually wrote a brief reflection about evidence of children's multiple mathematical knowledge bases in their map.

Data Analysis

I used content analysis techniques to examine links and concepts within prospective teachers' concept maps by specifically looking for evidence of children's multiple mathematical knowledge bases in their maps. I first coded all concepts in the maps using three initial categories: Mathematical Concepts, Assessment, and Children's Multiple Mathematical Knowledge Bases. I created these a priori categories based on my research question and concept map prompt. I started analysis by coding five concept maps at a time with the initial coding scheme and identifying any concepts that did not seem to fit well in any of the categories. Through multiple rounds of this iterative process, I refined my coding scheme to include emergent categories based on patterns from my analysis. Table 1 contains my final coding scheme, definitions, and examples from concept maps.

Table 1: Final coding scheme with examples

Category	Definition	Notes	Examples
Mathematical Concepts	Includes all concepts related to fractions (types and parts of fractions, definitions) or number & operations more broadly	Same as a priori category	Numerator; Reciprocal; Common denominator
Representations and Tools	Includes representations such as number lines, manipulatives, and examples of fractions	Emerged from iterative coding process	Number lines; Pie charts; ½
Teaching Practices	Includes examples of and concepts related to assessment (design and types), activities, tasks, and instructional planning	Broader category of a priori *Assessment* category	High level tasks; Formal assessment; Differentiation
Children's Mathematical Thinking	Includes examples of and concepts related to students' prior knowledge, solution strategies, specific common understandings and misconceptions	*Children's Multiple Mathematical Knowledge Bases*	Problem solving strategies; Misconceptions; Seeing students' thinking
Children's Lives and Experiences	Includes examples and concepts related to children's funds of knowledge: linguistic, community, home, and cultural knowledge		Funds of knowledge; Relate to students' interests; Culturally relevant

Mathematical Concepts remained a category, Representations and Tools emerged as a new category, and I broadened the a priori Assessment category to include all concepts related to Teaching Practices. I decided to split the Children's Multiple Mathematical Knowledge Bases category to make a distinction between *Children's Mathematical Thinking* and *Children's Lives and*

Experiences because I wanted to compare how prospective teachers represented both concepts in their maps. Finally, I used the prospective teachers' brief reflections as another source of data to examine how prospective teachers saw evidence of children's multiple mathematical knowledge bases in their concept maps.

Results

Across the group, nearly half (48.89%) of all links in the maps connected concepts related to children's multiple mathematical knowledge bases to concepts I coded as Teaching Practices. In particular, seven prospective teachers made a direct link between children's multiple mathematical knowledge bases and *high-level tasks*. A key finding from my analysis suggests that high-level tasks may be a possible entry point to strengthen connections between children's multiple mathematical knowledge bases and assessing children's understanding of mathematics. In this section, I will highlight three examples of how prospective teachers made these connections with particular attention to high-level tasks in their concept maps and reflections.

As shown in Figure 1, Avery has *high-level task* as a concept directly connected to *assessing children's understanding of fractions* with a cluster of concepts also connected to it. After editing the map, Avery added *problem-solving strategies* to this cluster of concepts, which I coded as a concept related to children's mathematical thinking. Avery also added *connect to students' experiences*, *use students' names*, and *connect appropriate funds of knowledge* to this cluster, which I coded as concepts related to children's lives and experiences.

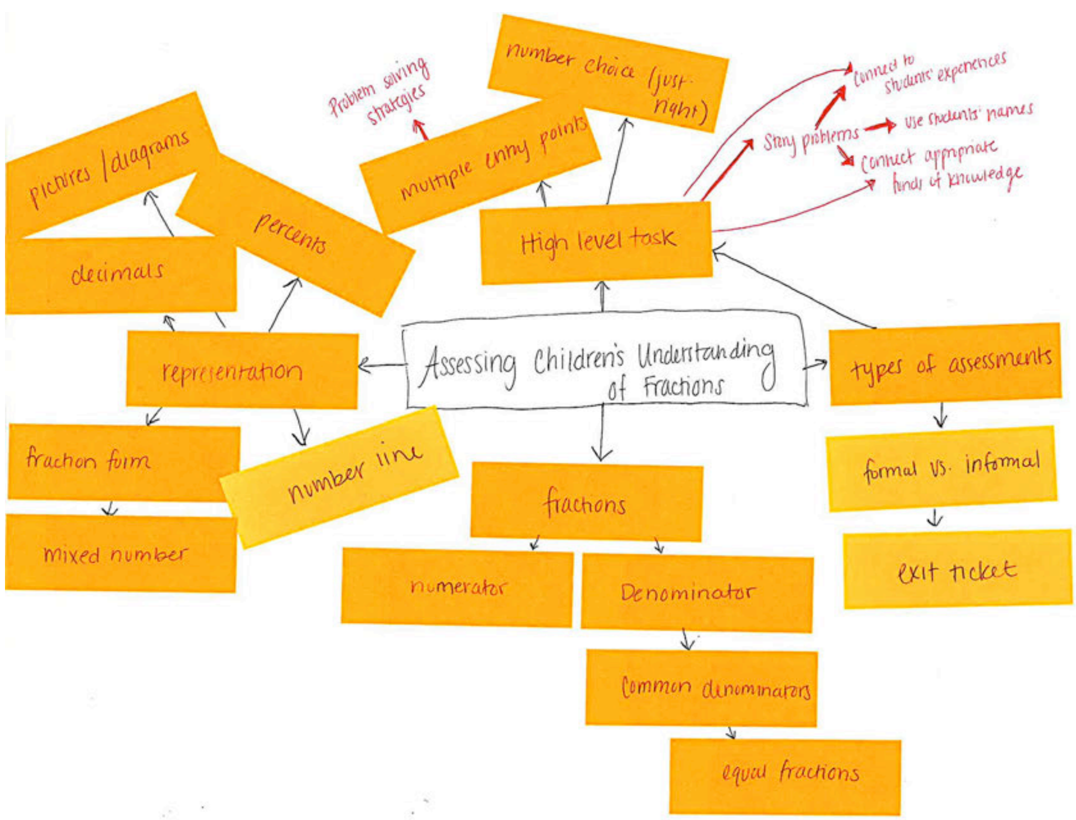

Figure 1. Avery's end of semester concept map.

In the end of semester reflection, Avery explicitly wrote about seeing evidence of connecting children's multiple mathematical knowledge bases to assessing children's mathematical understanding by using high-level tasks:

> *Avery*: I made the biggest connection between developing *high-level tasks*[emphasis added]with relating students' experiences and funds of knowledge. When assessing students and how they think we need to make sure that all students relate to the problem and can understand the context or background of a problem. The material needs to be relevant to every child so that they can one day use their knowledge in the real world. Even something as small as changing the names in a story problem will increase student interest and motivation.

Avery points to a connection between creating high-level tasks to assess students' mathematical understanding and using relevant information about students' funds of knowledge. Avery also notes that it is important to acknowledge students' relationship to mathematical problems, including the problem context and background. Part of this relationship may be related to student motivation, but Avery focuses on the potential utility of mathematics in students' lives outside of the mathematics classroom.

Similarly, other prospective teachers indicated evidence of children's multiple mathematical knowledge bases by connecting these concepts to high-level tasks. Figure 2 shows Morgan's end of semester concept map with these connections. In this map, *utilize students' funds of knowledge* is the lead concept of an added cluster directly connected to *assessing children's understanding of fractions, fraction vocab*, *number talks*, *exit tickets*, *high-level tasks*, and *allow multiple representations*. Additionally, Morgan added a direct link between *high-level tasks* and *allow multiple representations* in the edited map.

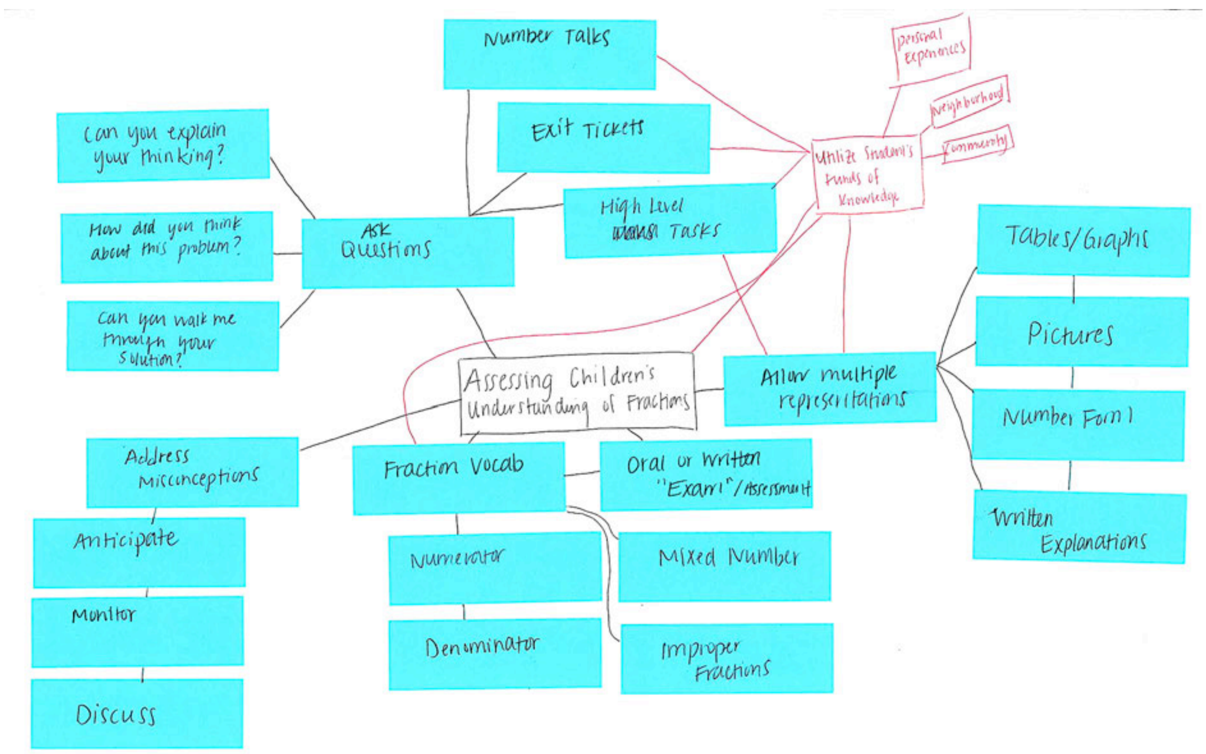

Figure 2. Morgan's end of semester concept map.

In the brief reflection, Morgan explained that *high-level tasks* and *multiple representations* connected to concepts related to children's multiple mathematical knowledge bases:

> *Morgan*: Creating *high-level tasks* [emphasis added] that provide students with multiple entry points into problems allowing them to think and use strategies that makes sense to them will tap into students' prior knowledge of what they already know and what strategies they are comfortable using. Allowing students to use *multiple representations* [emphasis added] also connects to their experiences of what types of strategies they have used in school before and what representations they prefer to use.

Morgan's brief reflection provides evidence of connections between high-level tasks and students' mathematical knowledge and strategies from in-school mathematics experiences. It is not clear, however, in what ways Morgan is making connections between high-level tasks and students' funds of knowledge. Similarly, it is not clear in what ways Morgan is making connections between multiple representations and students' funds of knowledge, even though there are links between these pairs of concepts on Morgan's map.

One prospective teacher, Harper, noted that there was evidence of concepts related to children's mathematical thinking before editing the map, such as *number talks*, *seeing students' thinking*, and *high-level tasks*. After editing the map, Harper added more concepts related to children's lives and experiences:

> *Harper*: At first, the only discussion I had about mathematical thinking was in describing how we can use number talks to see students thinking, and allowing them to explore different strategies with *high-level tasks* [emphasis added]. Once I edited the map, I added things about students' home life, community, personal experiences, etc.

Harper initially highlighted eliciting students' mathematical thinking through number talks, which is an activity in which students participate in 15-minute conversations about computation problems to communicate their mathematical thinking. Harper also describes how high-level tasks provide opportunities for students to explore different problem solving strategies, which is also closely connected to students' mathematical thinking. Harper does mention a shift to focusing on students' funds of knowledge only after the discussion and editing process.

An Interesting Case to Explore

Out of all the prospective teachers' brief reflections, Parker was the only participant who claimed to not see evidence of children's multiple mathematical knowledge bases in the end of semester concept map (see Figure 3). Parker explained that there was no evidence of concepts related to children's multiple mathematical knowledge bases in the map because of Parker's ways of thinking about assessing and about using funds of knowledge in mathematics teaching:

> *Parker*: I think there is no evidence because I thought of assessing in the pedantic sense. I thought that experiences of the children would go more along with the *actual teaching* [emphasis added] of concepts… I could add funds of knowledge to the concepts that I stated as being a part of a formal assessment.

I highlight Parker's reflection as an interesting case because I heard multiple prospective teachers voice similar thoughts during small group discussions about funds of knowledge being more directly related to the process of teaching mathematics rather than assessment, which could follow the act of teaching. From the methods course materials, I have evidence that prospective teachers adapted existing tasks and curriculum materials, but I have little evidence that prospective teachers have designed assessments at this point in their preparation program. One prospective teacher told me during the whole group discussion that they have experience adapting problems to align with students' needs although they have not created assessments in the course. Consequently, this evidence made me wonder about how prospective teachers' made sense of the phrase "*assessing children's understanding*" in the root of this concept map activity, and ultimately, how their understandings influenced the construction of their concept maps.

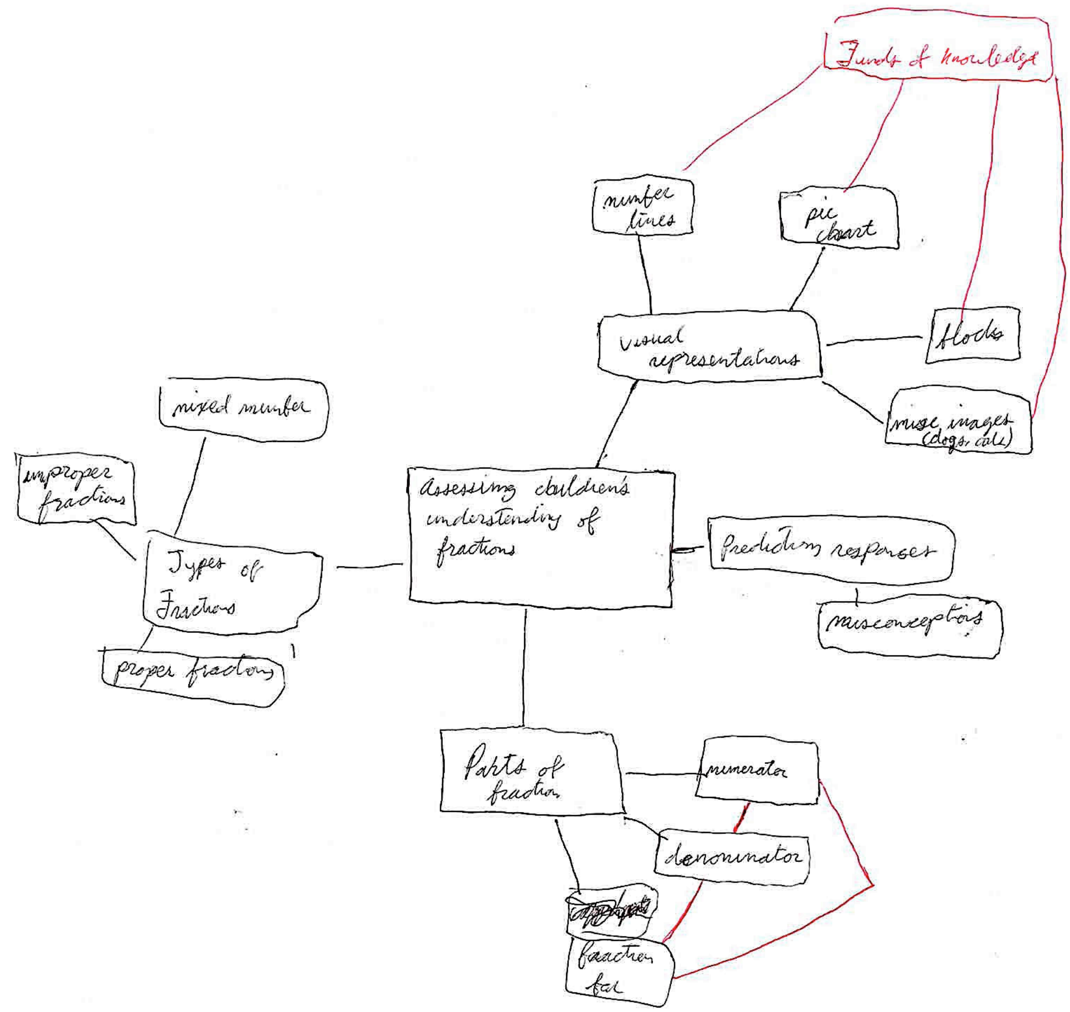

Figure 3. Parker's end of semester concept map.

Discussion

This study investigates in what ways prospective elementary teachers make connections among *children's multiple mathematical knowledge bases* in their thinking about assessing children's understanding of fractions. In the following, I describe implications for teacher educators and considerations for further research.

Teacher educators may support prospective elementary teachers to make explicit and stronger connections among concepts related to children's mathematical thinking, children's lives and experiences, and assessing children's understanding of mathematics. One entry point to better support these connections involves emphasizing *high-level tasks* as a concept related to teaching practices that connects to *both* children's mathematical thinking and children's lives and experiences. From my analysis of concept maps and brief reflections, I found strong evidence that the prospective teachers in my study made connections between high-level tasks and children's mathematical thinking. Based on an interesting case I found in the data, another implication for teacher educators is to be cognizant of a possible perception that funds of knowledge, including children's lives and experiences, are not used or useful in assessing children's understanding of mathematics. I would recommend a stronger focus on assessment as one of many teaching practices and the role of assessment for equitable teaching practices that serve the needs of all students, especially those who are traditionally marginalized in the mathematics classroom.

For further research, I would be interested in gathering more information about how prospective teachers make sense of the concepts related to *children's multiple mathematical knowledge bases*. Although prospective teachers made connections among these concepts, I noted that the concepts used more general language and did not include specific examples of how children's multiple mathematical knowledge bases connected to *children's understanding of fractions*. More specifically, I would like to examine *whose* multiple mathematical knowledge bases are represented in these concept maps. Are particular racial or ethnic groups of students in mind when we use the phrases *funds of knowledge* or *multiple mathematical knowledge bases*? Where do understandings of multiple mathematical knowledge bases come from (e.g. content in course readings/activities, prior experiences with children, etc.)?

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IDENTIFYING SPACES FOR DIVERSE LEARNERS' MULTIPLE MATHEMATICAL KNOWLEDGE BASES IN EXISTING CURRICULUM

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This project examines how prospective elementary teachers (PSTs) framed the idea of drawing on multiple mathematical knowledge bases (MMKB)—children's mathematical thinking and funds of knowledge— for diverse learners, in the context of adapting curriculum. We analyzed 47 written reflections of PSTs' analyses of an existing mathematics curriculum. Using inductive analysis, we identified four themes related to how PSTs evaluated the curriculum and identified possible spaces for small adaptations. Findings describe how these four themes related to incorporating MMKB. We discuss implications for mathematics teacher education.

Keywords: Equity and Diversity; Teacher Education-Preservice; Elementary School Education

The field of mathematics teacher education has begun to address how the cultural, linguistic, and socioeconomic positionality of students impacts their learning opportunities (Zevenbergen, 2001). Unfortunately, the focus on children's sociocultural identity in the classroom often appears as a separate subset of study from the more traditional focus on the psychology of mathematics education (Aguirre et al., 2012). Typically, children's home- and community-based knowledge receives much less attention in teacher education than children's mathematical thinking (e.g., problem types, solution strategies, etc.; Carpenter, Fennema, Peterson, Chiang & Loef, 1989) (Aguirre et al., 2012). In mathematics teacher education, the emphases on children's mathematical thinking and on children's home- and community-based knowledge and experiences remain largely disjointed, leaving prospective teachers (PSTs) ill-equipped to meaningfully integrate both important sources of mathematical knowledge and learning.

Research continues to reveal how a majority white, female, middle class teaching force struggles to effectively teach a diverse student population (Sleeter & Milner, 2011). This enduring challenge in education, more broadly, has serious implications for mathematics teacher education at all levels. The purpose of this paper is to examine how elementary PSTs make sense of addressing the needs of historically underrepresented populations in school mathematics during their mathematics methods course. In particular, we examine PSTs' work on a curriculum analysis assignment to better understand how PSTs frame meeting the needs of historically underrepresented populations through mathematics curriculum adaptation.

Theoretical Perspectives

Theoretical perspectives from two strands of research on elementary mathematics teaching and learning guide this project. First, the extensive body of research on mathematics instruction that centers on children's mathematical thinking (e.g., *Cognitively Guided Instruction*, Carpenter, et al., 1989) provides a basis for developing PSTs' knowledge of children's mathematical thinking in ways that change beliefs and shape classroom practices. Second, research that documents the benefits of drawing upon the cultural, linguistic, and community-based knowledge of historically underrepresented groups (Ladson-Billings, 2009; Turner, Celedón-Pattichis & Marshall, 2008) guides PSTs' development of leveraging home- and community-based knowledge in mathematics instruction. In particular, this project draws on the theory of *funds of knowledge*(FoK) for teaching. FoK refer to the "historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being" (Moll, Amanti, Neff, and

Gonzalez, 1992, p. 133). Using students' FoK for mathematics teaching means that classroom instruction utilizes the cultural, linguistic and cognitive resources from home or community settings to promote students' learning of the standard mathematics curriculum in school settings (Moll et al., 1992).

Although both strands of research are well developed, they remain disconnected in mathematics teacher education, as mentioned above. As a result, the field of mathematics education lacks a deep understanding of how teachers might learn to integrate the focus on children's mathematical thinking with the emphasis on home- and community-based knowledge. This project aims to bridge these two bodies of research by guiding K-8 PSTs to use children's *multiple mathematical knowledge bases* (MMKB) to support student learning. In this paper, we refer to MMKB as the integration of children's mathematical thinking and children's FoK (Aguirre et al., 2012). More specifically, the research question for this project is: How do K-5 PSTs frame the idea of drawing on MMKB, specifically for historically underrepresented student groups, in the context of adapting curriculum?

Methods

The research presented in this paper is part of a larger project, TEACH Math. In this section, we briefly discuss the goals and methods of the larger project and provide details about the specific data collection and analysis that produced the findings presented here.

Research Overview

The TEACH Math project aims to transform elementary mathematics teacher preparation so that new generations of teachers will be equipped with powerful tools and strategies to increase student learning and achievement in mathematics in our nation's increasingly diverse public schools. The project involves iterative refinement of three instructional modules for elementary mathematics methods courses designed to explicitly develop teacher competencies related to mathematics, children's mathematical thinking, and community and cultural FoK. Across these three modules, PSTs develop specific knowledge, beliefs, and dispositions related to MMKB.

Research has occurred at six university sites across the United States, with data on PSTs' work in all three modules collected from elementary mathematics methods courses at each of these sites. We analyzed data collected at one university site, a large university in the Midwest located near a small city with an increasingly diverse population. For this analysis, we used data collected from an activity in two K-5 mathematics methods courses at this university, each with a different co-principal investigator (PI) as course instructor.

In the activity selected for this analysis, Analyzing Curriculum Spaces, PSTs analyzed an existing elementary mathematics curriculum to identify opportunities for accessing, building on, and integrating children's mathematical thinking and children's home and community-based mathematical FoK (i.e. MMKB). We refer to places in the curriculum where teachers can make these types of small adjustments as curriculum *spaces*(Drake et al., 2015). This activity is one of four in the Classroom Practices Module in which PSTs learn to analyze classroom practices through four lenses: teaching, learning, task, and power and participation. In this specific activity, PSTs use a tool, Curriculum Spaces Table, designed by the co-PIs, to guide PSTs' identification and adaptations of curriculum materials that create spaces for eliciting and building on children's MMKB.

The Curriculum Spaces Table has three sections. In the first section, PSTs identified the central mathematical goal or ideas of the lesson. In the second section, they answered questions about the different phases of the lesson (e.g. launch, explore, summarize). The questions included: (1) What makes the task(s) in each phase good and/or problematic?; (2) What are opportunities for activating or connecting to family/cultural/community knowledge in each phase of the lesson?; (3) How does each phase of the lesson open spaces for making real-world connections?; (4) What are opportunities for students to make sense of the mathematics and develop/use their own solution strategies and approaches?; (5) What kinds of spaces exist for children to share and discuss their mathematical thinking with the teacher and the class?; (6) Where does the mathematical authority reside in the lesson? In the final section, PSTs proposed possible adaptations for the lesson phases or the overall lesson.

Data Collection and Analysis

Data for this paper included PSTs' written reflections on their use of the Curriculum Spaces Table to analyze the *Stickers: A Base-Ten Model* lesson from Grade 3 *Investigations in Number, Data, and Space* (TERC, 2008, p. 26-33). We analyzed a total of 47 written reflections. Participating

Primary Theme	Definition	Subthemes & Definitions (if applicable)
Learning Supports	Aspects of the lesson/teaching provide supports to facilitate student learning of mathematical content, including ***scaffolding*** (i.e. gradually decreasing the need for learning aids as students' comfort with language, concepts, etc. increases); ***teacher questioning***(i.e. various forms of questioning recognized to support learning (Boaler & Brodie, 2004)); or ***differentiation*** (i.e. individualized adaptations of lessons/tasks).	**General Learning Supports:** Considerations that will arise in essentially every classroom setting; no reference to FoK.
		Learning Supports for Diverse Learners: *give diverse learners access to the mathematical content*; particularly important when the class included culturally and linguistically diverse students or special education students; may or may not reference FoK.
Prior Knowledge	Aspects of the lesson/teaching are ***relevant to the students' prior knowledge***. The teacher or students can provide this connection to prior knowledge.	**School-Based Knowledge:** Prior knowledge that arose in a school-based setting.
		Funds of Knowledge: Prior knowledge that arose from community/family/cultural knowledge or experiences.
		Not-Specified: Source of prior knowledge is not specified
Motivation	Aspects of the lesson/teaching are ***relevant or familiar to students*** for the purpose of ***engaging or motivating students***.	
Children's Mathematical Thinking	Aspect of the lesson/teaching accessed, built on, or integrated children's mathematical thinking, including references to: (1) students' mathematical ***explanations and justifications***; (2) orchestration of mathematical ***discussion*** (Smith & Stein, 2011); and (3) specific mathematical ***features of the task*** (e.g. manipulatives, multiple representations, multiple strategies).	

Figure 1: Coding Themes & Definitions for Content of Written Reflections

PSTs were reflective of national demographics (i.e. mostly white, middle-class females). In the written reflections, PSTs discussed strengths and limitations of the lesson, spaces they identified for eliciting and building on children's MMKB, and the ways in which using the Curriculum Spaces Table aided in their analysis.

We analyzed the written reflections through an iterative coding process. The first two authors of this paper began the coding process separately, analyzing two of the written reflections and noting themes. We compared our initial impressions and used the themes to develop an initial codebook. We continued coding separately, comparing analyses, and revising the codebook until we produced a final (seventh) version of the codebook (Figure 1). Throughout our development of the codebook, we continually looked for confirming and disconfirming evidence of the identified themes (Erikson, 1986).

Using the final version, we coded the remaining written reflections together, discussing discrepancies and reaching consensus on coding. Written reflections were coded at the paragraph level (as denoted by the participant or roughly 10-15 lines) because surrounding sentences (or turns) provided important context for identifying themes. The codebook provided exhaustive codes (i.e. every paragraph received at least one code), but primary codes were not mutually exclusive.

We used the codebook described above to identify major themes related to the *content* of PSTs' written reflections, and we created a second coding stream to identify themes related to PSTs' *evaluation* of the curriculum. This secondary coding stream represented three major themes in PSTs' analyses: (1) Strength; (2) Weakness; and (3) Curriculum Space (Figure 2). Coding in these two streams allowed us to examine both the specific aspects of the lesson/teaching that PSTs identified in their curriculum analysis and whether PSTs framed those aspects of the lesson as strong/weak or spaces for adaptation. We linked codes in this second stream directly to codes in the first stream. For example, sometimes PSTs discussed ways in which the lesson provided learning supports, and they clearly identified those learning supports as strengths of the lesson. Such a paragraph would receive a "learning supports-strength" code. In some cases, PSTs did not clearly evaluate lesson aspects as a strength/weakness or a space for adaptation, and we did not use the second coding stream. Codes in the second stream were not mutually exclusive because some paragraphs included a discussion of both strengths and weaknesses of the same aspect of the lesson and because curriculum spaces were generally identified alongside weaknesses.

Primary Theme	Definition	Subthemes & Definitions (if applicable)
Strength	PSTs evaluated some aspect of the lesson/teaching as strong.	
Weakness	PSTs evaluated some aspect of the lesson/teaching as weak or limited.	**Too little** – Weakness or limitation resulted from too little of an aspect (e.g., too little support for diverse learners)
		Too much – Weakness or limitation resulted from too much of an aspect (e.g., too much explaining/telling by the teacher)
Curriculum Space	PSTs identified an aspect of the lesson/teaching where the curriculum could be adapted. This code was used both when the PST made a specific adaptation suggestion (e.g., I would let the students try the problem on their own first.) or when the PST talked more generally about spaces in the curriculum (e.g., The lesson needs to be adapted for English Language Leaners (ELL).)	

Figure 2: Coding Themes & Definitions for Evaluation by PSTs

Findings

In this section, we share the major themes that emerged across the written reflections. We found that attending to children’s mathematical thinking represented the most dominant theme (Table 1), and we present these findings first. Attention to the needs of historically underrepresented populations of students, specifically, emerged under several different themes - learning supports, prior knowledge, and motivation. We present findings related to each of these themes separately. Overall, FoK and diverse learners received lesser attention in PSTs’ curriculum analysis.

Table 1: Percent of total codes (n=1086) by specific code/theme

Children’s math thinking	Learning supports		Prior knowledge		Motivation	Other
35.6%	17.7%		11.8%		7.8%	27.1%
	General	15.2%	School	4.2%		
	Diverse	2.5%	FoK	4.7%		
			Unspecified	2.9%		

Attention to Children’s Mathematical Thinking

PSTs paid considerable attention to aspects of the lesson that accessed, built on, or integrated children’s mathematical thinking (Table 1). PSTs identified both strengths and limitations in the ways the lesson attended to children’s mathematical thinking, and they recognized spaces for adapting lessons to integrate, build on, or elicit children’s mathematical thinking beyond opportunities already offered in the lesson. Consider the following excerpt:

> The launch of the lesson is very important in making the lesson effective in promoting students’ learning…Students might only use one method of solving the task if they were not taught multiple ways to see the numbers. The lesson provides opportunities for students to explore but the lesson could summarize more in a group discussion format. In doing a discussion students could share their solutions and allow other students to ask questions, compare, or justify their own thinking…

In this reflection, the PST identified aspects of the launch as strong but also recognized limitations in and suggested adaptations for eliciting student thinking, particularly in the summary discussion.

Learning Supports

Among all relevant themes for our research question, learning supports, both general and specific to diverse learners, represented the second most common theme in written reflections (Table 1). Learning supports for diverse learners represented 2.5% of all codes. This means that fewer than 15% of all reflections on learning supports focused on diverse learners, specifically. In other words, overwhelmingly, PSTs seemed to focus on general learning supports in their analysis of the lesson. The following PST identified general learning support as a strength.

> This lesson…gives [students] a way to visualize the patterns and fully understand what it means to add by ten as opposed to just giving them a problem and hoping they figure out the answer. Giving students ways to remember patterns is more beneficial in my opinion.

When PSTs did discuss learning supports for diverse learners, a greater proportion of the codes occurred alongside codes for weaknesses than alongside codes for strengths in the lesson. The following excerpt is from a rare instance where a PST identified strengths of the lesson in regards to supporting the needs of diverse learners:

> This lesson states, "As frequently as possible, refer to strips as *strips of 10* to reinforce the groupings of 10s and 1s". When teaching I think it is important to pay attention to the vocabulary you are using and try to keep it the same throughout. Especially when thinking about ELL students…using the same words to describe something throughout a lesson can help decrease confusion of word meanings and phrases.

More commonly, PSTs expressed that the lesson inadequately addressed the needs of diverse learners: "This lesson also limits ELLs. At no point in the lesson do I see any talk of ELLs so that's something that could limit their learning." Despite being able to identify learning supports for diverse learners as a weakness in the lesson analysis, however, only approximately 11% of all suggested adaptations related to learning supports focused on the needs of diverse learners, specifically.

Prior Knowledge

Prior Knowledge represented one of the most identified categories relevant to how K-5 PSTs framed the idea of drawing on MMKB, specifically for historically underrepresented student groups, in the context of adapting curriculum (Table 1). PSTs discussed home- and community-based FoK with roughly the same frequency as school-based prior knowledge. Figure 3 provides example excerpts of school-based prior knowledge, funds of knowledge, and prior knowledge with an unspecified source.

Examples of Prior Knowledge Themes	
School Based	"The lesson connects back to Sticker Station done in second grade so it utilizes [students'] background knowledge."
FoK	"Aspects of the lesson plan that stand out as especially important for making the lesson effective in promoting students' learning are…that the lesson opens spaces for making connections to their family/cultural/community knowledge."
Non-Specified	"I do believe that it is possible to be responsive to students' thinking and background knowledge while also using the curriculum materials."

Figure 3: Examples for Each Category of Prior Knowledge

Only a small percentage of PSTs identified prior knowledge (combined) as a limitation in their lesson analysis; however, when PSTs did identify prior knowledge as a weakness, they overwhelmingly focused on FoK, specifically. Consider the following excerpt:

> I thought the lesson was limited at making a connection to cultural, community, and/or family knowledge. It was great that the lesson used stickers which students were familiar with from the sticker station store in 2nd grade. This would hook student's interest and motivate them for the lesson.

As the above quote illustrates, school-based prior knowledge was rarely identified as a weakness in the analysis of this lesson. PSTs who identified FoK as a weakness in this lesson also generally discussed adaptations related to FoK. The focus on adapting the lesson to attend to FoK, however, represented less than 10% of all identified spaces for adaption. For example:

> The teacher could have had students create their own problems using something from their culture, home, or of interest that could be bundled in a set of 10 or used individually; for example, my case study student was very interested in basketball and played it at home with family often. He could create a problem having one team of basketball players represent the 10's place and an individual basketball player representing the 1's place. They could then share these

> problems in partners or with the whole class. I feel if students could connect with the problem better, it would better promote students learning.

Motivation

Attention to engagement and motivation in the lesson plan was common throughout the reflections, receiving about 8% of the codes. PSTs identified aspects of the lesson that attended to motivation in both strong and limited ways. Nonetheless, about 10% of codes related to curriculum adaptations identified motivation as a space for improving upon the lesson, suggesting an overall desire to improve engagement and motivation in the lesson.

> If students are forced to complete tedious, uninteresting tasks, their motivation for learning the material will decrease…However, if a teacher is allowed to use the requirements of the curriculum materials as a guide to the objectives students are expected to learn and revise the material to the interests of the students, both motivation and success in learning the material will increase.

Additionally, as a specific focus of the study, students commonly referred to engagement and motivation when discussing the purpose of integrating students' FoK. The quote in the prior knowledge section above offers one example of this connection, but the following excerpt also illustrates how PSTs linked FoK to motivation.

> [This analysis] helps me to think how I should engage real-life objects with my lesson plans and how it would help [students] to have clearer understandings…[students] not only easily learn how to add and subtract multi-digit numbers, but also learn how to engage their class lessons in their real life.

We interpreted this PST's use of "engage" as evidence of attention to students' motivation or interests.

Discussion and Conclusion

Even though the mathematics methods courses in the TEACH Math project aimed to integrate a focus on children's mathematics thinking and children's FoK, the emphasis given by the PSTs still seems unbalanced. Of all the themes discussed by the PSTs in their curriculum analysis, less than 10% focused on children's FoK specifically (i.e. prior knowledge: FoK or learning supports for diverse learners). PSTs overwhelmingly attended to aspects of the lesson focused on children's mathematical thinking (over 30%). These findings suggest that the two emphases, children's mathematical thinking and children's FoK, remain disjointed. Other teacher educators/researchers have observed similar trends when PSTs use lesson analysis tools that attend to both children's mathematical thinking and FoK (Aguirre, Zavala, & Katanyoutanant, 2012). Even though FoK received less attention than we might have expected given the methods course goals, the fact that some PSTs emphasized FoK and suggested lesson adaptations, while still attending to children's mathematical thinking, offers hope of balancing these two important emphases in mathematics teaching.

The equal attention to school-based prior knowledge and FoK offers some hope of guiding PSTs to consider MMKB in their lesson analysis, but prior knowledge, overall, received only limited attention in PSTs' analyses. These findings suggest a need to understand how PSTs frame FoK. In the rare instances when PSTs attended to children's FoK, they commonly discussed the practice in terms of motivating or engaging students. In science education, teachers' rationale for attending to FoK ranges from motivating students to increasing access to the content to changing the content itself (Barton & Tan, 2009). A deeper understanding of the reasons PSTs focused on FoK in the context of mathematics curriculum analysis could shed light on more effective ways of balancing the emphasis on children's mathematical thinking with FoK. The range in motivations for attending to FoK raises questions about which framings promote consideration of MMKB and challenge deficit views of historically underrepresented students.

Findings from this study also have implications for research on how teachers learn to adapt lessons to attend to the needs of historically underrepresented students, once they have identified the need for such adaptions. Those PSTs who attended to FoK in their reflections overwhelmingly identified a need for incorporating more focus on children's home- and community-based knowledge into the analyzed lesson. The ability of some PSTs to identify spaces for and to recognize the importance of incorporating MMKB suggests that this type of analysis holds promise. Nonetheless, PSTs did not necessarily suggest specific lesson adaptations or specific spaces in the lesson where adaptations might occur. This finding suggests that PSTs might need more support in thinking about how to adapt mathematics lessons to integrate, meaningfully, children's mathematical thinking with children's FoK. Further research is necessary to support PSTs in analyzing lessons and adapting them in order to support the needs of diverse students.

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EXAMINING EFFECTS OF IMPLEMENTING AN EDTPA TASK IN AN ELEMENTARY MATHEMATICS METHODS COURSE

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This mixed-methods study examines effects of implementing a mock edTPA task on prospective elementary teachers' perceptions of teaching effectiveness. Results from the Mathematics Teaching Efficacy Beliefs Instrument document a significant change in participants' beliefs that they can effectively teach mathematics. Qualitative results illuminate participants' growing confidence in their understanding of elementary mathematics, their ability to recognize and attend to children's thinking, and their use of pedagogical tools and resources to support children's learning.

Keywords: Teacher Education-Preservice; Teacher Beliefs; Teacher Knowledge

Background and Purpose

Teacher preparation and initial certification are undergoing significant changes as a result of new policy. Teacher performance assessments, including edTPA, are at the center of these policy changes and play a consequential role in some states for prospective teachers' eligibility for initial certification. One teacher preparation program recently implemented edTPA, and data for this study were collected during the first semester that edTPA had consequences for program graduates. This study occurs in the context of a mathematic methods course preparing prospective teachers for the math task in the elementary education edTPA. Current research is suggesting that it is important for mathematics content courses to introduce the academic language needed to be successful with teacher performance assessments (Lim, Moseley, Son, & Seelke, 2014). However, a national survey conducted by Masingila, Olanoff, and Kwaka (2012) shows the majority of elementary mathematics content courses are taught by instructors in mathematics departments. Therefore, by default, teacher preparation programs must take the lead in preparing prospective elementary teachers (PSTs) for teacher performance assessments such as the edTPA.

This study examines how one elementary mathematics methods instructor prepared PSTs for edTPA through a focus on Cognitively Guided Instruction (CGI) as an approach to instruction. In the mathematics methods course, the final course assessment was the implementation of a mock edTPA task. It is hoped the findings of this study will provide insights into ways to prepare PSTs for successfully completing edTPA, while maintaining a focus on effective pedagogy in elementary mathematics. This study seeks to answer the following two research questions:

- What changes occur in elementary prospective teachers' mathematics teaching efficacy beliefs during a mathematics methods course implementing a mock edTPA task?
- What are the perceptions of PSTs about their mathematics teaching effectiveness upon completion of a mathematics methods course implementing a mock edTPA task?

Theoretical Perspective and Related Literature

edTPA is a standardized high-stakes assessment modeled after the Performance Assessment for California Teachers (PACT). PACT was developed by researchers and teacher educators at Stanford

University. The intent of edTPA is to move the measure of PSTs' effectiveness from an individual university responsibility to a state or national level (Sato, 2014). Several prominent organizations, including AACTE, CAEP, NCATE, and CCSSO, have supported the need for teacher performance assessment to predict PSTs' effectiveness. edTPA is composed of four teaching tasks designed to focus on planning, implementing, and assessing instruction based on a central focus selected by the PST. The fourth task concentrates on mathematics instruction, and PSTs are required to consider student's mathematical thinking to plan, implement, and assess instruction. PSTs are then evaluated on their ability to analyze their effectiveness as a mathematics teacher implementing instruction focused on student's mathematical thinking and learning.

In the field of mathematics education, numerous studies have linked teachers' pedagogical beliefs and pedagogical content knowledge to students' learning of mathematics (Peterson, Fennema, Carpenter, & Loef, 1989; Campbell, Nishio, Smith, Clark, Conant, Rust, DePiper, Frank, Griffin, & Choi,2014). PSTs' affect (e.g., emotions, attitudes, and beliefs) and knowledge undoubtedly also play an important role in learning, including successfully completing the edTPA Task 4. Teacher affect has been conceptualized as a continuum (Philippou & Christou, 2002). Feelings and emotions have been found to be short lived, highly charged, and unstable. Feelings and emotions are on one end of the continuum, with beliefs on the other end. Beliefs have been found to be more cognitive in nature and also more stable. One belief construct important to PSTs' learning in mathematics methods and eventual classroom teaching is teaching efficacy beliefs. Mathematics teaching efficacy beliefs have been considered as two-dimensional: involving personal mathematics teaching efficacy and mathematics teaching outcome expectancy. Personal mathematics teaching efficacy is the beliefs a teacher holds about his or her skills and abilities to teach mathematics effectively. Mathematics teaching outcome expectancy is a teacher's belief that effective teaching will yield positive student outcomes regardless of external factors. These beliefs impact PSTs' perspectives and understandings of subject matter, therefore this can impact how PSTs perceive and understand elementary mathematics content and pedagogy (Fives and Buehl, 2014; Philipp, 2007;Swars, 2005; Richardson, 1996; Pajares, 1992).

Cognitively Guided instruction (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999) is an instructional model focused on children's mathematical thinking. It is designed to support teachers' instructional decisions in ways that allow them to connect the informal knowledge of students' mathematical thinking with the formal mathematics. A majority of the instructional time in a CGI classroom should be dedicated to discourse (i.e., dialogic discussion about mathematics). During discourse, teachers make real-time decisions about children's knowledge and orchestrate the mathematical discussion through intentional questioning. Professional development focused on CGI has proven to enhance teacher's pedagogical content knowledge and shift their beliefs (Carpenter et al., 1989; Carpenter et al., 1993; Fennema et al., 1996; Swars, Smith, Smith, & Hart, 2009). Students in classrooms aligned with the pedagogy of CGI have also been shown to perform better on number sense tasks than peers not receiving similar instruction (Higgins & Parsons, 2010).

Methods

Participants and Context

The participants in this study were 33 PSTs enrolled in two sections of a 3 credit-hour mathematics methods course. Their ages ranged from 19-48 (30 females and 3 males). They were in the second semester of their initial teacher certification program at a large, urban university in the southeastern United States. The duration of the teacher preparation program was two years and used a cohort model. This teacher preparation program includes campus-based courses and a field component for the first three semesters. The PSTs attended courses two days a week and were in their field placement for two days per week. The field component followed a developmental model, progressing from prekindergarten through fifth grade during three semesters. The fourth and final semester consists of a full-time student teaching experience.

Course Design

As mentioned, the PSTs were enrolled in a mathematics methods course, which consisted of a 14-week semester and met one day a week for 2 and ½ hours. The instructor taught the course from a social constructivist perspective. The primary goal was to develop beliefs and practices that aligned with the practices described within the NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014). The purpose of these practices is to integrate the content outlined in the Common Core State Standards (CCSS-M) with NCTM's curriculum standards and principles (NCTM, 2000). Therefore, the methods course used the practices in *Principles to Actions (PtA)*to frame course assignments and instructor feedback on those course assignments, classroom discourse, and learning activities during course sessions.

Learning activities in the course were focused on experiences that supported the PtA. For example, one principle states to "elicit and use evidence of student thinking" (NCTM, 2014). Therefore, PSTs were asked to conduct interviews with students in one-on-one settings during their field experiences. Once they completed the interview they would submit their final analysis for feedback and an assignment grade. As they continued to grow and have experiences with eliciting students' thinking, they were introduced to materials that would assist them in facilitating whole group lesson implementation. The students were then asked to select a central focus and anticipate students' potential strategies with a story problem they designed.

The final assessment component of this course was a teacher performance assessment (edTPA) mock task. This task focuses on how PSTs are able to plan, implement, and analyze elementary students' mathematical thinking. A central focus must be selected that allows students opportunities to develop conceptual understanding, procedural fluency, and problem solving knowledge. The PST must speak to these types of knowledge when they are analyzing the elementary students' mathematical thinking. The purpose of the mock assessment task was not only to familiarize PSTs with the evaluation process, but to introduce the academic language of the edTPA assessment, alongside the academic language of the mathematics content standards and principles for teaching.

The primary course goal was for PSTs to align their pedagogical beliefs and practices with the PtA teaching practices, and a secondary goal was for the PSTs to have a successful experience with the mock edTPA task. In order to meet these goals it is important to have a pedagogical model that integrates these theoretical ideas, so the primary text for the course was *Children's Mathematics: Cognitively Guided Instruction* (Carpenter et al. 1999). Cognitively Guided Instruction (CGI) was the approach to instruction selected to allow PSTs to construct ideas about student's mathematical thinking and problem solving capabilities, strengthen their own mathematical content knowledge, and to build their confidence teaching mathematics through implementation of CGI in their field placements.

As shown in Figure 1, this course began with introducing the Common Core State Standards for Mathematics (CCSS-M) to situate the relevance of CGI within PST's current field placements and future classrooms. Then, language and ideas in course learning activities and assignments immersed students in CGI. Students began with early number concepts and progressed through number and operations leading into algebraic thinking. As students became more familiar with the CGI framework and implementation materials, the course transitioned into the introduction of the mock edTPA task. The language and focus then joined the ideas of CGI with the expectation of the edTPA task. This task was broken into individual components to allow PSTs an opportunity to continue working with CGI as they merged these new understandings with a teacher performance assessment task. Therefore the course began with the language of the standards, immersed PSTs into the pedagogical model (CGI), and concluded with a teacher performance task (edTPA).

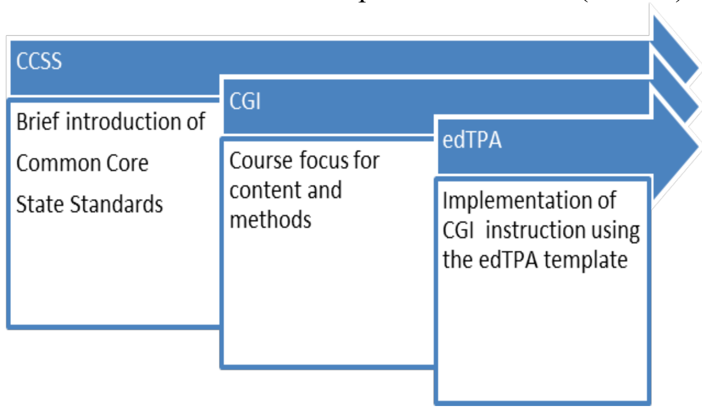

Figure 2. Course Design Overview

Data Collection

This study includes both qualitative and quantitative methods of data collection. Data collection occurred during one semester. The instructor of the course collected any materials related to coursework (i.e. course assignments, daily writings, etc.). Another researcher administered the survey data and conducted six interviews, without the instructor present at either of those events.

Quantitative data presented here were collected during the first and last day of the mathematics methods course using the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI). The MTEBI survey consists of 13 items on the Personal Mathematics Teaching Efficacy (PMTE) subscale and 8 items on the Mathematics Teaching Outcome Expectancy (MTOE) subscale (Enochs, Smith, & Huinker, 2000). The subscales address the two-dimensional aspect of teacher efficacy. The PMTE subscale examines the PSTs' beliefs about their abilities to be an effective mathematics teacher. The MTOE subscale examines the PSTs' beliefs about their abilities to increase student learning through effective mathematics teaching regardless of external factors. This instrument employs a five-item Likert scale, with a higher score correlating with teacher effectiveness. Possible scores on the PMTE subscale range from 13 to 65; MTOE subscale scores range from 8 to 40. These subscales have high reliability (Chronbach's alpha = .88 for PMTE and .81 for MTOE) and represent independent constructs based on confirmatory analysis.

The qualitative data consists of end of the semester semi-structured interviews that were conducted with six PSTs selected randomly from the 33 PSTs in the methods course. The participants were recruited through an email that was sent from the researcher conducting the interviews. The purpose was to elicit conversation around PSTs beliefs, course experiences, and the mock edTPA task. The interview protocol consisted of 11 multi-part questions.

Results

Research question 1: What changes occur in elementary prospective teachers' mathematics teaching efficacy beliefs during a mathematics methods course implementing a mock edTPA task?

Table 1 presents the Likert scale value means and standard deviations as well as those for the PMTE and MTOE subscales. Table 2 shows the analysis of dependent sample t-test, using an alpha level of 0.05. The Likert scale values on the overall MTEBI and the PMTE subscale show a significant increase in teacher effectiveness beliefs. PSTs became more confident in their beliefs about their ability to implement effective mathematics instruction. Therefore, mathematics teaching efficacy beliefs did positively change during the mathematics methods course. The change in scores on the MTOE subscale was not statistically significant, so the slight decline in mean scores should be interpreted as unchanged expectations for student outcomes and the statistically significant change for the overall MTEBI should be attributed entirely to the change in the means for the PMTE subscale.

Table 1:MTEBI Likert Scale Value Means, Standard Deviations, and Mean Differences

	Pre		**Post**		**Change**
Instrument	**Mean Likert Value**	**SD**	**Mean Likert Value**	**SD**	**Mean Difference**
MTEBI	3.54	0.48	3.75	0.40	0.21
PMTE subscale	3.55	0.64	4.31	0.46	0.76
MTOE subscale	3.53	0.61	3.32	0.68	–0.21

Table 2: MTEBI Dependent T-Test Results

Instrument	**T-value**	**p-value**
Overall MTEBI	3.12	.004*
PMTE subscale	7.93	.000*
MTOE subscale	1.91	.065

* Statistically significant difference in the mean for Pre vs. Post.

Research Question 2: What are the perceptions of PSTs about their mathematics teaching effectiveness upon completion of a mathematics methods course implementing a mock edTPA task?

The qualitative interview data indicate that PSTs recognize the importance to teaching effectiveness of assessing and understanding students' mathematical knowledge prior to and during instruction. "I think they should learn based off of what they know" (Participant # 2, interview, December 9, 2014). Another recurring theme was that effective teaching allows for multiple solution strategies to be considered, which was different from the PSTs' own mathematical learning experiences, "I use to think that there was just one way to learn math…. I don't really think that there is one way I saw kids [describes a variety of solution strategies]" (Participant # 3, interview, December 9, 2014). Also, some participants indicated they were able to be effective because they were introduced to resources that they could rely on in their own classrooms such as, "CGI interviews"(Participant #6, interview, December 9, 2014) and "formative assessment" (Participant #4, December 9,2014). Finally, others made general statements about their growing confidence, such as, "I have seen, from the beginning of this course to the end of the course, I have seen a progression in my own math…so I do feel more prepared coming out of it" (Participant #3, December 9, 2014) and "I was terrified to teach math…but now I feel a little more confident" (Participant # 2, December 9, 2014).

Discussion

With the limited research on how edTPA is shaping and changing teacher education it makes sense to begin with considering PSTs' teaching effectiveness beliefs during a time when they are preparing for an experience that will assess how well they are able to analyze their teaching effectiveness during a sequence of mathematics lessons and a re-engagement lesson based on their analysis of student assessment data. As teacher educators consider how to merge teacher performance assessments into their teacher education programs, this study is one example that if a pedagogical model such as CGI has been previously shown to shift pedagogical and efficacy beliefs, then edTPA can become a useful part of the overall methods course experience without unduly dominating such a course. The PSTs perceptions of their effectiveness proved to be strong because of their ability to implement CGI in their field experiences and come back to the university classroom to engage with creating the edTPA Task 4 documents. The field experiences and the shift in beliefs seemed to ease the potential tension about edTPA and how it should be completed during their future student teaching experience.

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INVESTIGATION OF MATH TEACHERS' CIRCLE THROUGH A ZONE THEORY LENS

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Math Teachers' Circles (MTCs) are an innovative, problem-solving focused approach to professional development. This model provides teachers opportunities to develop their problem-solving skills as well as help them to communicate with others on classroom implementation of problem-solving activities. As with any professional development with teachers, it is important to explore the impact of this model in terms of teachers' learning and development. In this report we provide our implementation of a zone theory lens provided by Goos as a way to investigate the MTC model. Initial analysis implementing this particular theoretical lens helps us gain insights in ways to improve this new model of professional development activities for future participants.

Keywords: Teacher Education-Inservice; Problem Solving; Mathematical Knowledge for Teaching; Assessment and Evaluation

Introduction

Problem solving is one of the critical topics in mathematics education that has been the focus of research and curriculum reform internationally (e.g., Common Core State Standards Initiatives 2010; NCTM 2000; OECD 2004 & 2010). As its importance in students' learning of mathematics is palpable, in-service teachers also need support in developing and improving their problem solving abilities as well as its classroom implementation (Anderson 2005; Hiebert et al. 1996). One innovative professional development approach to address these areas is the Math Teachers' Circle (MTC) model, which emphasizes developing and improving teachers' problem solving skills in the context of significant mathematical content. It is designed after Math Circles for secondary students, which originated in eastern Europe, migrating to the US in the mid-1980s. Secondary students in these circles engage in solving challenging mathematics problems with guidance from the mathematicians who facilitate these sessions. In this spirit, participating teachers in an MTC engage in solving math problems geared towards the level of teachers, rather than the level of their students, designed and facilitated by mathematicians. During each session teachers solve problems, discuss problem-solving strategies and different solutions, as well as possible implementations of problem-solving activities in their classrooms. Generally, each MTC provides a weeklong summer workshop, followed by monthly evening sessions during the academic year. This model has been gaining momentum in the US, with 71 active MTCs in 36 states. White, Donaldson, Hodge and Ruff (2013) and White (2015) provide additional information about the MTC model.

Being a relatively new model (since 2006), the evaluation of MTCs requires an investigation of effectiveness from various lenses-both theoretical and practice based ones. The purpose of this report is to describe our initial attempt at implementing a theoretical model described by Goos (2009) to explore possible contributions of the MTC models to teachers' development in the area of problem solving.

Theoretical Framing

The evaluation of professional development activities is a complicated task. Many researchers provide practice-based perspectives to design, implement and evaluate professional developments and outline effective practices. For example, Mewborn's (2003) review of research identifies three key elements of successful practices. In general professional developments are stated to be effective if they (1) provide opportunities in which teachers engage with mathematical concepts and also focus on their students' learning of such concepts; (2) are situated in school-context in which teachers could implement and authenticate ideas; and (3) provide opportunities in which teachers discuss issues related to their and their students' learning within a supportive group of participants and network.

These key elements are part of a more theoretical model proposed by Goos (2009), which provides additional aspects contributing to the outlined key elements of effective professional developments. This particular perspective is an extension of Valsiner's (1997) model for understanding learner's development that stems from Vygotsky's Zone of Proximal Development construct (1978).

In their model, Goos et al. (2007) describe three zones, each of which focuses on different aspects of teacher learning and development: the Zone of Proximal Development (ZPD), the Zone of Free Movement (ZFM), and the Zone of Promoted Action (ZPA). The ZPD in this model refers to teachers' *knowledge* and *beliefs* on content and pedagogy. In other words, teachers' development of content knowledge, pedagogical content knowledge, their beliefs about mathematics and about teaching and learning of mathematics are considered within this zone. The ZFM, on the other hand, focuses on "constraints and affordance within the professional context" (p.26). In particular, teachers' *perceptions* related to their profession such as insights on their students' ability, views on curriculum, standards, and assessments they implement, and connections to other teachers and districts, and how such perceptions develop or change through participation in professional development activities are considered in this zone. The third zone, the ZPA, focuses on the *professional development strategies* that are being introduced either formally through structured workshops or informally through communication with other colleagues.

Goos et al. (2007) observed that these zones complement each other in the effort to describe teacher's learning and development. In particular, Goos et al. state, "For teacher learning to occur, professional development strategies [ZPA]must engage with teachers' knowledge and beliefs [ZPD]and promote teaching approaches that the individual believes to be feasible with their professional context [ZFM]" (p. 26).

In this exploratory study, we investigated ways in which the MTC model contributes to participants' learning and development by identifying aspects related to each zone. Though this particular theoretical lens was only implemented in the data analysis phase of the evaluation, this exploration provided insights on improving the planning stages of the professional development model, as discussed in our conclusion.

Methods

Participants and Data

The participants of this study were 129 in-service teachers who attended one of the six MTCs summer workshop hosted during 2011 summer. The number of participants and their years of teaching varied by site, as summarized in Table 1. There were more MTC sites during this particular

Table 1: MTC sites and participants

Site	Number of participants	Years Teaching Mean (SD)	Range
A	23	7.26 (5.2)	1-18
B	20	9.4 (8.8)	0-31
C	16	12.3 (10.3)	0-35
D	19	11.7 (8.7)	0-28
E	20	8.2 (5.3)	1-19

F	27	14.6 (6.9)	2-30
All	129	10.8 (8.0)	0-35

summer; however only six of them both qualified and agreed to participate in this exploratory study. To qualify, a site had to offer at least a workshop of four full days in length and with primarily middle-school level teacher participants, though many also allowed secondary teachers to participate. These sites were geographically diverse. All sites focused on problem-solving activities by engaging participants to solve challenging problems. Some sites also included activities in which participants explicitly discussed pedagogical approaches and implementation of problem solving in their classrooms.

Analysis

In this study we focus on data collected by two instruments: the Learning Mathematics for Teaching (LMT) instrument developed by Hill, Schilling and Ball (2004) and an exit survey developed by one of the authors. All sites administered two subscales of the LMT instrument at the beginning and the end of the workshop. These subscales were used to measure mathematical knowledge for teaching (MKT) of Number Concept and Operations, and Proportional Reasoning. These particular items were used to get an insight on the participants' development of content and pedagogical content as described in the ZPD.

In addition to the LMT instrument, we investigated the ideas relating to the ZPD by qualitatively analyzing the end of workshop exit survey. These questions focused on participants' overall experiences, asking them to comment on their thoughts about the workshop, their learning, other participants' impact on their learning, their anticipation of changing their teaching practices as a result of this workshop, useful aspects of the workshop as well as their comparison of this workshop to other professional developments. Participants' responds were transcribed and analyzed using open and axial coding procedures as described by Strauss and Corbin (1990). Each researcher developed codes that describe participants' experience as a learner and a teacher. These codes were refined by the research team and themes were developed. At the final stage of the analysis, themes relating to the ZPD, ZFM and ZPA from each site were analyzed across workshops to get an insight of the overall experience of all participants.

Results

The LMT instrument was used to explore the participants' development of mathematical knowledge for teaching, addressing aspects outlined in the ZPD. As a condition of using the instrument, we must report standardized scores only by converting raw LMT scores to standardized (z) scores using the scoring tables provided at the instrument-training workshop. An example of the

Table 2: The LMT: Standardized Scores for Number Concepts

Site	Number and Operations		
	Pre	Post	Difference
A	-.41 (1.0)	.06 (1.0)	.48 (.53)*
B	.23 (1.0)	.58 (.74)	.35 (.86)*
C	.12 (.85)	.30 (.91)	.18 (.75)
D	-.50 (.66)	-.21 (.73)*	.28 (.63)*
E	.65 (1.1)	.91 (1.1)	.26 (.72)*
F	.37 (.98)	.56 (.96)	.18(.47)
All	.05(1.0)	.33(97)	.29(.67)*

*Notes.(1)*Scores are standardized and are presented as *M (SD)*. *Pre* = pretest score; *Post* = posttest score; *Difference = Post – Pre*.

(2) *= Planned comparisons showed a significant difference between pre- and posttest scores ($p < .05$).

standardized pre-, post and differences is provided in Table 2 for the Number Concept and Operations.

The analysis of the LMT indicates that the planned comparison t-tests show an increase in the Number Concept and Operations scores that was significant with all sites combined, *M(SD)=.29(.67)*with *p <.00001*.However, we have not observed a similar increase in Proportional Reasoning content that was also administered in the LMT instrument.

In addition, we conducted the repeated measure ANOVA, which revealed a significant main effect of Test Form, $F(1, 112) = 76.31$, $p < .001$. Overall, Proportional Reasoning scores were significantly higher than the Number Concept and Operations scores (M(SD) = .29(.67) and -.10(.67), respectively), and this pattern was consistent across all six sites. The interaction of Test Form x Workshop Site was not significant, $F(5, 112) = 1.46$, $p = .21$.

Table 3: Repeated Measures ANOVA: LMT Standardized Scores

Source	SS	df	MS	F	p
Between subjects					
Site	68.27	5	13.65	5.48**	.00
Error	279.19	112	2.50		
Within subjects					
Test	21.03	1	21.03	76.32**	.00
Test * Site	2.02	5	0.40	1.44	.21
Test (Error)	30.86	112	0.28		
Time	0.96	1	0.96	4.28	.04
Time * Site	0.476	5	0.10	0.42	.83
Time (Error)	25.13	112	0.22		
Test * Time	4.45	1	4.45	18.82**	.00
Test * Time * Site	0.40	5	0.08	0.34	.89
Test * Time (Error)	26.49	112	0.237		

Note. Site = Workshop Site 1, 2, 3, 4, 5 or 6. Test = Number Concepts and Operations or Proportional Reasoning. Time = Pretest or Posttest. ***p* < .01.

There was also a significant main effect of Time Administered, $F(1, 112) = 4.28$, $p = .04$. Post-test scores were higher on average than pre-test scores (M(SD) = .45(.92) and .36(1.05), respectfully). The interaction of Time Administered x Workshop Site was not significant, $F(5,112) = .424$, $p = .83$, indicating that all six sites shared a pattern of differences between pre- and post-test administration and supporting the combination of data across sites for the planned comparisons tests.

There was significant interaction effect of Time Administered x Test Form, $F(1,112)=4.452$, $p<.001$. Overall, there were gains from the pre- and post-test scores for Number Concept and Operations scores, whereas there were losses for the Proportional Reasoning scores (M(SD) = .29(.67) and -.10(.67), respectively),and this pattern was consistent across 5 of the 6 sites. However, the interaction of Test Form x Time Administered x Workshop Site was also not significant, $F(2, 112) = .35$, $p = .889$. This indicates that the pattern of pre- and post-test scores for each form was not significantly different across sites and further supports combining the data across sites for analysis in the planned comparison tests.

The exit survey results were analyzed qualitatively to capture general themes within all zones: ZPD, ZFM, and ZPA. The main themes reflecting ideas that relate to the ZPD were different types of learning such as learning of math content, problem-solving strategies, teaching strategies and overall comments on learning. The themes that captured the ideas referring to the ZFM were teachers' perspectives on student learning, plans for classroom teaching and problem-solving activities and teaching, and the Common Core State Standards perspectives. The main themes observed reflecting the ideas for the ZPA were the challenges that participants experienced during the workshop, collaboration and engagement experience, and general comments on the structure of the workshop.

Responses to the question, "please tell us your thoughts about the workshop," were varied, demonstrating that the culture within the individual sites was unique. However, the dominant themes were reported by the participants mostly related to the ZFM and ZPA, such as collaboration with other teachers; engagement in the workshop activities and perspectives on teaching strategies in the classroom. Even though themes related to the ZPD were also observed, these were not observed in the cross-case analysis. For example, Site A participants reported that learning problem-solving strategies was impactful.

Most participants (64.12%) commented that collaboration with others was the most important support they received during the workshop. This particular theme directly relates to the structure of the MTC model, referring to the ZPA. This theme was observed in response to the question, "please comment how the support you received from others impacted your learning." A participant from Site E commented that the support from others was most impactful in a way that it was, "positive, reflective, [they] bounced ideas off each other, check[ed] mechanisms, asked/answered why—had seen/shared several different approaches to the same question."

Another theme indicating a possible development in the ZPD but also ideas referring to the ZFM was observed in the responses to the question, "Do you anticipate changing anything about how you teach mathematics as a result of the workshop. If so, in what ways?" The majority of participants (63.84%) stated that they learned teaching strategies in the workshop that they planned on transferring to their classroom. An exemplary quote from a participant referring to the ZFM was about this teacher's perspective on student learning is, "I realize that it's ok to give 'hard' problems for students to solve. The process to solve takes patience and, as a teacher, I need to encourage students to be creative in their thinking to build good reasoning."

Participant engagement was the most common theme that highlights the unique structure of the MTC model in area of the ZPA. This theme emerged in response to the questions, "Please comment on any differences or similarities that struck you about this workshop compared with other professsional development workshops you have attended in the past (if applicable)" and "Please comment on what you considered to be the most useful aspects of this week." Participants from workshop Sites A, C, D, E, and F all commented that the MTC workshops were more engaging than other professsional development workshops they had attended. A participant from site F commented that,

> most workshops are how to get children excited about learning. This workshop goes beyond with a chain reaction. The instructor was excited, making the teacher excited, which in turn will make the students excited. Some workshops I have attended in the past, I felt like the teacher link was the smaller part. Each link needs to be equal to gain strength.

The most common theme coming from participants of workshops at sites B, C, and D was in regards to the usefulness of collaborating with other teachers: this was also the most common theme from the combined responses.

Overall, this initial analysis to investigate the MTC professional development model through a zone theory perspective highlights the possible impact that the MTC could provide to participating teachers in the ZFM and ZPA. In addition to the possibility of increasing participants' development and learning of mathematics and mathematical knowledge for teaching the exploration of other zones provides a framework to structure the future MTC workshops.

Conclusion

In this report we shared a theoretical approach to explore the MTC professional development model. The MTC model is an innovative professional development in which participants develop their abilities and beliefs on problem solving which are the main ideas of the ZPD. However, as previous research on effective professional development address the development in the ZPD is not enough to make change in teachers' professions and special focus and investigation are needed on the areas that are highlighted in the ZFM and ZPA. We use this zone theory approach was to gain insights on possible impacts of the MTC model. This particular approach provided a mean to understand aspects such as importance of engagement with other teachers and impacting teachers' personal motivation during their participation. This initial attempt needs further exploration by collecting data from facilitators of each MTC on the structure of the MTCs, observing each MTC during workshops and conducting follow-up interviews and surveys with the participants.

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USING WRITING TO ENCOURAGE PSMTS' REFLECTIONS ON AMBIGUITY IN MATHEMATICAL LANGUAGE

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Literature suggests that the mathematical language of teachers impacts a student's understanding of math concepts. When teachers unintentionally use ambiguous language, students' understanding of a subject can be negatively affected. We share background on specific instances in which teachers can create confusion with the language they use, and we investigate both pre-service teachers' and college algebra students' concepts of three common terms in mathematics: Solve, Evaluate, and Simplify by asking both groups to unpack their understanding of these terms through a writing prompt. We compare the language used by both groups in their definitions. Preservice teachers' reflections on their experience with the writing prompt are also examined to identify ways that such a task can help them identify gaps in their own understanding and in their thinking about student learning.

Keywords: Teacher Education-Preservice, Classroom Discourse, Mathematical Knowledge for Teaching

Introduction

By the time students have entered a college mathematics class, we might assume that they have developed a clear understanding of some basic terminology. Similarly, as teachers we might expect that we also have a clear definition in our minds for the words we use in our academic language everyday. However, when pressed to really examine our understanding of certain terms, it is possible that several gaps in understanding may come to light.

Teaching requires a sensitivity to the need for precision in mathematics (Ball & Bass, 2003). Mathematical terms are perhaps more precisely defined than those in many other disciplines (Barwell, Leung, Morgan, & Street, 2005), and ambiguities can only be accepted when there are shared experiences and assumptions across a community of learners (Jamison, 2007). However, we know that simply reading or hearing a precise definition of a mathematical term does not guarantee that a learner will attribute the same given meaning to the term. Meaning-making is dependent on an individual's construction experiences surrounding such mathematical expressions (Brown, 1997). It is important for teachers to consider ways in which they may unintentionally influence students' learning through their own ambiguous use of terminology and academic language in mathematics. Even simple vocabulary can have a large impact on students' understanding (Boulet, 2007; Gay, 2008).

In this study, we discuss the use of a tool for reflection called *writing to learn mathematics* (WTLM) to help preservice teachers not only examine their own understanding of certain vocabulary terms but also investigate how college algebra students understand them. Our study is housed within a unique college teaching seminar that supports the development of preservice secondary mathematics teachers' (PSMTs) knowledge about both mathematics and pedagogy as they work as the instructor of a college algebra course. We are interested in how both PSMTs and college algebra learners define words that they have been using in mathematics for many years. Specifically, we examine the following three questions in this study: (a) How do college students (specifically college algebra students and PSMTs) define the common terms, *solve, simplify, and evaluate* in mathematics? (b) What are PSMTs' reactions when required to interrogate their own understanding of these common mathematical terms? (c) What can PSMTs learn from reading students definitions of the terms?

Background

Ambiguity in Mathematical Language

It is a common misconception in and out of the classroom that mathematics is a subject composed of "arcane rules for manipulating bizarre symbols something far removed from speech and writing" (Jamison, 2007, p. 45). This misconception places mathematics in a negative light. The meaning of mathematics almost literally gets lost in translation as vocabulary and terminology seem to take a backseat to the repetition of procedure and blindly following steps to solve a problem (Boulet, 2007). Jamison (2007) provides one suggestion for why students may struggle in mathematics classes:

> Ordinary speech is full of ambiguities, innuendoes, hidden agendas, and unspoken cultural assumptions. Paradoxically, the very clarity and lack of ambiguity in mathematics is actually a stumbling block for the neophyte. Being conditioned to resolving ambiguities in ordinary speech, many students are constantly searching for the hidden assumptions in mathematical assertions. But there are none, so inevitably they end up changing the stated meaning–and creating a misunderstanding. (p. 47)

The preciseness of math terminology juxtaposed with the implicit, sometimes vague definitions contained in natural language, may prevent students from developing adequate meanings; they are constantly switching between math and every day speaking. Problems may arise especially when there are shared meanings with everyday words, or when vocabulary used in natural language has a very different meaning in the mathematical context (Rubenstein, 2007). Table 1 shows some examples of potential vocabulary problems that teachers may overlook when trying to understand students' conceptual difficulties.

Table 1: Vocabulary Issues (adapted from Thompson & Rubenstein, 2000, p. 569)

Category of Potential Pitfall	Examples
Some words are shared by mathematics and everyday English, but they have distinct meanings.	*number*: prime, power, factor *algebra:* origin, function, domain, radical, imaginary *geometry*: volume, leg, right *statistics/probability*: mode, event, combination
Some mathematical words are related, but students confuse their distinct meanings.	*number:* factor and multiple, hundreds and hundredths *algebra:* equation and expression, solve and simplify *geometry:* theorem and theory *statistics/probability:* dependent and independent events

Students and teachers need to know the meaning of math vocabulary words and terminology in order to communicate in the classroom (Gay, 2008, NCTM, 2000, Thompson & Rubenstein, 2000). More specifically, it is important for teachers and their students to have the *same* (or at least similar) understanding of the words being used to convey ideas, objects, and actions. There is much literature devoted to the development of mathematical vocabulary, particularly in the context of students studying a field of mathematics (e.g., geometry, algebra, statistics) for the first time, or for students learning mathematics in a second language. However, by the time students are in college, we expect them already to have learned the basic vocabulary associated with a topic such as college algebra. In this study, the focus is not on teaching vocabulary, but on probing existing understanding of common vocabulary terms in mathematics.

It is challenging for beginning teachers with little classroom experience to skillfully incorporate academic language for productive discourse. Gay (2008) shares examples of witnessing such challenges when preservice teachers ask students to do such things as “Graph this expression” and “Evaluate 6^3, 12^4, and n^4 if n=3” (p. 218). She suggests the importance of helping pre-service teachers be aware of the impact of their use of vocabulary on their students’ learning and understanding and shares strategies she has used with preservice teachers to help them build their understanding of mathematics terms, such as graphic organizers, concept circles, and the use of analogies. Below, we discuss the use of writing in mathematics as an effective tool for unpacking one’s understanding of the language used in mathematics.

Addressing the Problem

Nathan and Petrosino (2003) posit that, “discursive and reflective methods that are already commonplace in professional development and teacher education programs can serve as the basis for interventions aimed at aligning teachers’ views with accurate models of student reasoning and development” (p. 924). *Writing to Learn Mathematics* (WTLM) is one such method that incorporates writing prompts into content and methods courses to support the understanding and teaching of concepts and procedures. Through writing, learners (including PSMTs) can engage with mathematical content in new ways (Author and colleagues). In alignment with WTLM, careful reflection on written work can influence perspectives on teaching and learning in mathematics. Researchers suggest that giving learners opportunities to write in the content domain can play a major role in helping them to develop their voice in that domain (Kaplan, Fisher, Rogness, 2009; Thompson & Rubenstein, 2000). And can play a significant role in advancing and assessing learning (Inoue & Buczynski, 2011; Miller, 1992). Incorporating writing in the mathematics classroom offers multiple benefits to PSMTs committed to understanding the diverse ways in which students learn, discover, and create.

Research on WTLM focuses heavily on benefits it provides to the students doing the writing (Inoue & Buczynski, 2011). Investigations have also focused on the benefits that teachers can gain from reading their students’ writings in mathematics (Adu-Gyamfi, Bosse, Faulconer, 2010; Miller, 1992, Quinn & Wilson, 1997). In this study, we take this further to look at what teachers can learn from engaging in the writing prompts themselves before giving them to students.

Theoretical Perspectives

We believe that well-developed subject- matter knowledge is critical for effective teacher preparation. However, as PSMTs develop more expertise in their field, they can easily forget what novice students find easy and difficult to learn in mathematics (Nathan & Petrosino, 2003). One reason to explicitly teach WTLM stems from the existence of such *expert blind spots* (Nathan & Petrosino, 2003), where “teachers’ subject-matter expertise often overshadows their pedagogical knowledge about how their novice students learn and develop intellectually in the domain of interest” (p. 906). PSMTs need opportunities to engage with mathematics in ways that interrogate and reframe their current understandings and to perturb their basic idea of “knowing” mathematics. WTLM provides a tool for both expanding content knowledge and assessing students’ understandings of mathematics (Miller, 1992).

Reflective practice is one means of supporting WTLM’s incorporation in the classroom (Quinn & Wilson, 1997). It allows teachers to consider the implementation of novel practices in the classroom (Foss, 2010). When enacted as “a deliberate way of thinking leading to change in action,” (Shoffner, 2008) reflection allows PSMTs to develop and refine the knowledge needed to guide their teaching (Spalding & Wilson, 2002). Thus, reflection can support PSMTs’ understanding of teaching mathematics through consideration of prior understandings, past experiences, and current beliefs (Stockero, 2008).

Working within a constructivist framework where learning is viewed as a process of transformation or modification of existing ideas, we see WTLM and reflection as powerful tools for perturbing PSMTs' existing ways of thinking in mathematics. They provide a way from them to interrogate understanding, enhance their thinking, and reflect on ways in which they and their students understand mathematics.

Methods

The participants in the study were senior PSMTs enrolled in a unique mathematics seminar at a university in the United States where they receive course credit to teach College Algebra. As course instructors, they were fully responsible for teaching the class to 20-30 students three days a week in 50-minute class sessions. They also had the added requirement of attending a seminar (taught by the first author) each class day to discuss pedagogical issues from the day and the mathematics that they would teach next. The College Algebra course was overseen by a course coordinator who designed the syllabus, pacing guide, and common exam and online homework sets. The added responsibilities for the undergraduate teachers were writing lesson plans, creating and grading their own quizzes twice a week, and proctoring exams.

Over the course of a semester, PSMTs were given WTLM prompts related to college algebra topics. In this study, we focus on the first prompt given, which dealt with mathematical vocabulary. We chose to address understanding of the common terms solve, simplify, and evaluate. The prompt given to the PSMTs was stated: **Explain the difference between the directions "Solve, Evaluate, and Simplify" in math problems. Then write an example using each with the expression 3(x+2)-x.** This question comes from a set of writing questions at the end of a chapter in Sullivan and Sullivan's (2004) algebra book. We chose it because of the target courses' focus on algebra and because confusion amongst these terms is mentioned in the literature (Thompson & Rubenstein, 2000).

After writing a response to the prompt in class, PSMTs engaged in collaborative reflection through asynchronous web discussions on a course wiki (c.f., Shoffner, 2008). They were asked to post their prompt response and post a reflection on the experience of answering the prompt, and were required to read and respond to other posts from their fellow PSMTs. They next created a quiz for their college algebra students (referenced below as "students") that included the same WTLM prompt, and posted another reflection on the WIKI to discuss that they had learned from reading their students' responses.

We collected prompt answers and reflections from 31 PSMTs over four semesters. These serve as our primary data source. In Fall 2014, we also collected prompt responses from 185 college algebra students. All data have been analyzed using content analysis to identify patterns in responses (Creswell, 2007) to address the research questions above. Our analysis included an examination of the following: (a) common language used to define the terms; (b) common misinterpretations of the terms; and (c) common themes in the PSMTs' reflections.

Findings

Definitions of Solve, Simplify and Evaluate

We examined the glossaries of several algebra textbooks to see examples of some formal definitions of the terms solve, simplify and evaluate. One McGraw Hill online algebra text gave the following three definitions: *evaluate* means to find the value of an expression; *solve* is the process of finding all values of the variable that make an equation a true statement; and *simplify* means to write an expression in simplest form. In comparison, we found that both the PSMTs and college algebra students used similar language in some of their definitions, but provided many more specific ideas from their experiences with these terms. Additionally, many of the college algebra students (and some PSMTs) struggled significantly with defining evaluate.

Table 2 shows a count of the number of acceptable vs. unacceptable/incorrect definitions.

Table 2: Response Results

	Unacceptable Definition			Acceptable Definition			No Definition Given		
	Solve	Eval	Simp	Solve	Eval	Simp	Solve	Eval	Simp
Students (N=185)	16	133	15	169	30	168	0	22	3
PSMTs (N=31)	1	5	0	30	25	31	0	1	0

Solve was not a difficult term for the PSMTs or students to define. Over 90% of both groups provided an acceptable definition – some formal, some very informal. Table 3 shows the most common language used across both the PSMTs and students. For example, a majority of both groups mentioned that solve involves either finding a value of the given variable (similar to the textbook definition above) or finding an answer/solution to the problem (without specific mention of variables). A small percentage (3.9%) of the students felt that solve and simplify were actually the same thing, and another 7.8% defined solve as producing an "exact" answer – neither of these ideas was found with the PSMTs. Both groups made use of the language of "isolating" the variable or "getting the variable on one side" in their definitions.

Table 3: Results for Solve

SOLVE	Same as Simplify	Find value of variable	Find an answer/solution	Isolate variable	Exact
Students (N=185)	7 (3.9%)	65 (35.7%)	67 (36.8%)	9 (5%)	14 (7.8%)
PSMTs (N=31)	0	12 (38.7%)	8 (25.8%)	7 (22.6%)	0

Simplify was also relatively easy for the both groups to define, with 91% and 100% providing an acceptable response. Interestingly, in their reflections, several PSMTs mentioned this term as the one that gave them the most difficulty. This was due to the fact that they did not know how to define it without using the word "simplify" in their definition. As seen in Table 4, we did find that the most common language used by the college algebra students was to get something into the simplest terms or simplest form, which matches the textbook definition above. However, we also found that 14.6% of the students and 9.7% of PSMTs referred to getting an "equation" in simplest form, rather than an "expression." It is possible that this could be partially explained by how the responders may have been thinking about the purpose the simplification - some members of both groups of learners made reference to a need to simplify first before solving an equation. Thus, their reference to an equation makes sense for them in this context

The most common language in the PSMTs definitions for simplify involved "combining like terms." They were also more likely to mention specific strategies for simplifying such as "factor" or "cancel," while a larger percentage of students used the language of "breaking it down" to describe what was happening. Both groups had members who referred to the idea of "reducing" in their definitions in some way.

Table 4: Results for Simplify

SIMPLIFY	Combine like terms	Reduce	Break it down	Simplest terms/for m	Involves an equation	Factor	Cancel

Students (N=182)	13 (7%)	28 (15.1%)	28 (15.1%)	85 (46%)	27 (14.6%)	20 (10.8%)	1 (0.5%)
PSMTs (N=31)	18 (58.1%)	8 (25.8%)	1 (3.2%)	13 (41.9%)	3 (9.7%)	12 (38.7%)	9 (29%)

Evaluate was the most difficult for both the PSMTs and the college algebra students. Only 16.2% of students provided an acceptable definition of this term, and while 83% of the PSMTs were able to come up with a suitable definition, their reflections after answering indicated that most of them struggled to do so in the task. When members of both groups did have an acceptable definition, it was common to find the language of "plugging in" in their answer. The most common incorrect responses among the students was that evaluate meant the same thing as solve (28.8%) or simplify (10.4%) (see Table 5). Only a small percent (3.7%) used the language similar to the textbook definition above, though all of these students said that evaluate meant to "find a value of the equation" (while eight out of the nine PSMTS who used this language said "find a value of the expression). What we found most interesting where the definitions that attributed a non-mathematical definition of evaluate to this term. Some examples include: "To evaluate an equation is to determine how you are going to solve the given equation;" "evaluate means tell what kind of problem it is"; and "evaluate is to analyze a given expression in order to find further information" (the latter was from the one PSMT who answered with a non-math definition).

Table 5: Results for Evaluate

EVALUATE	Same as Solve	Same as Simplify	Find the value	Non-math definition	Plug in
Students (N=163)	47 (28.8%)	17 (10.4%)	6 (3.7%)	16 (9.8%)	12 (7.4%)
PSMTs (N=30)	2 (6.7%)	0	9 (30%)	1 (3.2%)	10

PSMTs' Reflections After Answering the Prompt and Reading Student's Responses

We next analyzed the PSMTs wiki reflection posts to identify common themes in the ideas that they posted and discussed asynchronously with each other. The left column in Table 6 includes general forms of the most common themes found across the PSMTs' reflections after they answered the writing prompt or after they read their students' answers. These themes are paraphrased from PSMTs' writings, while the quotes in the right column are one example of a direct quote from a teacher that fits the theme. Each one here appeared in some form in more than five different PSMTs' reflections and was therefore identified in our coding as an area of interest.

Table 6: Common Remarks from PSMTs' Reflections

	Theme	**Example Quote**
Reflections on mathematical	I had a hard time differentiating among the three words	"At first, I really had no idea what the differences were between the terms."
	I had a hard time putting my answer into written words	"Once I started writing my ideas down, I felt stupid. I couldn't find the words to describe what the process was for each one."

understanding	I take things like this for granted	I think we are so conditioned to know what to look for we don't look at the directions unless we are confused... If [it says] 3(x+2)-x=0, we'd solve."
Reflections on pedagogy	This activity caused me to think about what I do or my own experiences	"I never realized how much informal language I use, and how I often find myself neglecting the more formal language of mathematics."
	I need to change what I'm doing in class	"I am going to be more strategic with the words I use in class to be sure that my students fully understand and that I am truly meaning what I am saying."
Reflections on student thinking	I can see why students struggle in mathematics	"Grading my quizzes today, I realized that a lot of students tried to factor when the problem was to expand or expanded when they were told to factor. I think if I had not done this prompt before grading, I might have been a little more judgmental. I probably would have thought: Can't these kids read directions?"
	If we change what we're doing, it will be easier for students to learn	"I think that by using the term evaluate, we can get our students to think about a problem more deeply."
	I learned a lot about the different ways that students think about mathematics	"When reading what the students write, I get to see an alternative way of thinking about a problem or a word, which helps me become a better educator."

As seen in the table, the PSMTs' reflections can be grouped into three main categories: reflections on their own mathematical understanding surrounding mathematical language/vocabulary; reflections on their own pedagogy; and reflections on their students' thinking and learning. The most common statement in the reflections (16 out of 31 PSMTs) was that struggling to define these terms made them feel like they were being placed in their students' shoes and that they could empathize with students' struggles. The second most common reflection focused on PSMTs' awareness that they have become too comfortable with terms in mathematics and the realization that they do not tend to pay attention to them in problem solving– instead they felt they usually let the mathematical symbols "tell" them what to do.

Conclusions

Teachers impact students in some obvious ways, but the focus for us is how they unintentionally influence students through ambiguous terminology and practices. Given the emphasis for a focus on developing student conceptual understanding and sense making in mathematics (NCTM, 2000), there is a critical need to understand and support effective practices that PSMTs can engage in to overcome their blind spots and enhance their content and pedagogical content knowledge in secondary mathematics. Our findings suggest that the use of writing prompts that unpack and encourage reflection on existing understandings of common mathematical terms can serve as one such effective practice to use in teacher preparation.

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PRESERVICE TEACHERS' FRACTIONAL CONCEPTS IN SOLVING ADVANCED FRACTION PROBLEMS

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This study investigated how pre-service teachers' fractional concepts are related to solving problems involving advanced fractional knowledge. 96 Participants took a written test including three fractions questions about fraction concept, fraction comparison, and multiplicative relationship involving fraction quantities and composite units. The data were analyzed using an inductive content analysis approach. Findings suggest that many of our PSTs developed limited understanding of fraction sub-constructs and thereby were not able to solve the three problems. Another finding is that when PSTs relied on only one sub-construct such as part-whole, they tend to provide incorrect answers. This finding implies that PSTs' understanding of fractions as measure and operator may be a foundation of solving problems involving advanced fractional knowledge.

Keywords: Teacher Education-Preservice; Teacher Knowledge

Objectives or Purpose of the Study

The purpose of this study is to characterize profiles of the mathematical competence of pre-service teachers in the topic of fractions. Especially this study investigates how pre-service teachers' fractional concepts are related to solving problems involving advanced fractional knowledge. Since Shulman (1986) coined the notions of content knowledge and pedagogical content knowledge, many researchers have investigated what teachers know and how they know about mathematics and teaching mathematics over the past three decades. They reported teachers' insufficient knowledge about teaching mathematics, in particular, elementary in-service and pre-service teachers' lack of understanding about whole numbers, fractions, and fraction operations (e.g., Behr, Harel, Post & Lesh, 1993; Ma, 1999; Mack, 2001; Steffe, 2003).

This line of research studies suggests that teachers should develop a profound understanding of fundamental mathematics (Ma, 1999) and mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), which encompasses knowledge that teachers use in teaching practice such as selecting and using effective representations, deftly assessing students' work, and providing appropriate remediation. In particular, teachers' capability to solve problems differently and to use different representations of mathematical ideas is considered to be an important aspect of mathematics education. Empson (2002) highlighted that "the key in fraction instruction is to pose tasks that elicit a variety of strategies and representations" (p. 39) and, that representational models used by teachers (e.g., pizzas, number lines, and fraction bars) engaged and facilitated students' learning of initial fraction knowledge.

However, despite a large number of research studies focusing on teachers' problem-solving, representational knowledge, and computational skills (e.g. Eisenhart, Borko, Brown, Underhill, Jones, & Agard, 1993), several questions still remain unanswered regarding preservice teachers' ability to solve problems, their ability to translate from one mode of knowledge to another, and possible difficulties preservice teachers have in connecting different modes of knowledge. Lester and Kehle (2003) highlighted that "far too little is known about problem solving" and in particular, about teachers' problem solving strategies and their abilities to transfer in problem solving (p. 510). This issue should be addressed in the teacher education of preservice teachers of mathematics at all levels.

As a way to characterize profiles of the mathematical competence of pre-service teachers in the topic of fractions, this study investigates how pre-service teachers' fractional concepts are related to solving problems involving advanced fractional knowledge. The research questions that guided the study were: (1) How do PSTs solve fraction problems and what strategies do they use? (2) How does PSTs' conception of fractions relate to problem solving ability involving advanced fractional knowledge?

Theoretical Framework

Studies on Fractions (Five sub-constructs)

According to research, understanding fractions clearly indicates understanding five possible constructs that fractions can represent (Clarke, Roche, & Mitchell, 2008; Sibert & Caskin, 2006). Although the meaning of part-whole is dominantly used to represent fractions in mathematics textbooks, many mathematics researchers believe that students would understand fractions better when they are exposed to the other meanings of fractions (Clarke et al., 2008). The first construct is *part-whole*, which goes well with an example of shading parts out of a whole. The second construct is *measurement*. Measurement includes identifying a unit length and then iterating the unit length to determine the length of an object. The third construct is *division*. When considering a sharing context, people can connect division to fractions. For example, in the context of finding a person's share to fairly share 3 candy bars with 4 people, each person will receive ¾ of a candy bar. The fourth construct is *operator*. That is, fractions can be used to indicate an operation. For example, when John has $21 and Mary has 2/3 of John's money, Mary's money will be 14 dollars, which indicates 2/3 of 21 dollars. Students who represent "two-thirds the amount of 21 dollars as "2/3 x 21" may have developed a concept of a fraction as an operator. The last construct is *ratio*. That is, fractions can be used to represent part-part ratio or part-whole ratio. For example, when there are four red marbles and seven blue marbles, the ratio 4/7 could be used to indicate the ratio between red marbles (part) and blue marbles (part). Also, the ratio 4/11 could represent the ratio between red marbles (part) and total marbles (whole). In this study, we provided pre-service teachers with three fractions questions, which particularly used part-whole construct and operator construct. According to Usiskin (2007), operator construct is not stressed enough in school curricula although just knowing how to represent fractions using part-whole construct doesn't guarantee knowing how to operate with fractions in other areas of curriculum where fractions occur (Johanning, 2008).

Studies on Teacher Knowledge (MKT and CK)

Shulman (1986) put forward the idea that teachers have specialized knowledge of teaching, which is what differentiates a teacher from a subject matter specialist. This notion is referred to in his article as the distinction between *content knowledge, pedagogical content knowledge,* and *curricular knowledge. Content knowledge* is subject matter knowledge of mathematics. *Pedagogical content knowledge* is topic specific pedagogical knowledge needed to teach mathematics. *Curricular knowledge* refers to educational programs made for the teaching of specific subjects and topics at a given grade level, including instructional materials available in relevant programs, and affordances and constraints in the use of such specific instructional materials (Shulman, 1986). More recently, Ball and her colleagues (1990; 2008) proposed a new term, *Mathematical Knowledge for Teaching* (MKT), which is defined as "a practice-based content knowledge for teaching built on Shulman's (1986) notion of pedagogical content knowledge" (Ball, Thames, & Phelps, 2008, p. 389). Mathematical Knowledge for Teaching focuses on identifying students' mathematical thinking and on the understanding required to teach specific topics in mathematics. MKT distinguishes mathematical content knowledge into three strands: *common content knowledge*, *knowledge at the mathematical horizon*, and *specialized content knowledge*. C*ommon content knowledge* is described as the mathematical knowledge held in common with others who know and use mathematics in various professions or occupations, which is what Shulman meant by his original *content knowledge*. *Specialized content knowledge* is described as mathematical knowledge about the ways that mathematics is taught to students by a teacher. *Knowledge* *at the mathematical horizon* refers to an awareness of how mathematical topics are related over the span of mathematics included in the curriculum.

This study examines pre-service teachers' content knowledge such as specialized content knowledge and knowledge at the mathematical horizon. That is, our study focuses on pre-service teachers' specialized content knowledge in fraction concept (part-whole construct) and fraction comparison, and how this specialized content knowledge of different topics are horizontally related to advanced fractional knowledge, which deals with multiplicative and reversible reasoning involving fractional quantities and composite units.

Method

Participants and Contexts

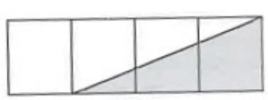
Data from this study came from 96 PSTs from two different university sites. Participants were either in their sophomore, junior or internship year of elementary teacher preparation programs—one from at a large northeastern university and the other from at a large southwestern in the United States. Each PST was enrolled in an elementary mathematics methods course. Mathematics methods courses were designed to support PSTs' knowledge development for teaching elementary mathematics.

Data Sources

A written task was used for the study, which asks three important fractional knowledge: (1) Fraction comparison (recognizing the difference between the actual numbers and fraction of the number and comparing the size of fractions); (2) fraction concept (the concept of referent units to represent a partial portion of a whole in given area models); (3) Multiplicative relationships involving fraction quantities and composite units (reversible reasoning including composite units of fractions). The provided questions are as follows (see Table 1).

Table 1: Main Task of this Study

1. At both Rivers High School and Mountainview High School, ninth graders either walk or ride the bus to school. 6/7 of the 9th grade students in Rivers High School ride the bus, while 7/8 of the 9th grade students in Mountainview High School ride the bus. If there are 40 9th grade students who walk at Rivers and 25 9th grade students who walk at Mountainview, in which school do more students ride the bus? In which school do a greater fraction of the students ride the bus? Explain your strategies or solutions as much as in detail.
2. For each picture shown below, (i) write a fraction to show what part is shaded. For each picture, (ii) describe in pictures or words how you found that fraction, and why you believe it is the answer.

   (1)  (2) 
3. Merlyn spends $60 of her paycheck on clothes and then spends 1/3 of her remaining money on food. If she $90 left after she buys the food, what was the amount of her paycheck? Explain your solution method as much as in detail. You may use representations (e.g., diagrams, rectangles, number line etc.).

Data Collection and Data Analysis

Three questions about fractions were administered to the entire class in four mathematics methods course sections towards the end of the spring semester in 2014. Qualitative and quantitative analyses were conducted. In particular, for the written response, we used an inductive content analysis approach (Grbich, 2007). We initially organized raw data into an Excel spreadsheet, read all of the responses, and created codes based on the raw data. More specifically, data analysis involved five processes: (a) an initial reading of each PST's response, (b) identifying correctness of the responses, (c) exploring the subcategories under each analytical aspect according to the number of correct responses and their problem solving strategies that PST demonstrated, (d) coding the categories and subcategories, and (e) interpreting the data quantitatively and qualitatively (Creswell, 1998).

Summary of Findings

In this section, we present overall findings from the written task and detailed analysis depending on PSTs' cognitive levels in fractional knowledge. Because we divided the cognitive levels according to the number of correct answers among three questions, it would be helpful to be aware of what fractional knowledge is involved in each three question. The first question is focused on comparing the size of fractions and recognizing the difference of comparing the size of the actual numbers and fractions. To find in which school a greater fraction of the students rides the bus, students need to compare the sizes of the given fractions 6/7 and 7/8 using procedural knowledge or conceptual knowledge. However, to find in which school more students ride the bus, students may use measurement construct by focusing on unit fraction and iterations (multiplicative relationship). That is, considering the given fractions for bus riders at two schools were 6/7 and 7/8, students would be able to know that 1/7 and 1/8 indicates fractions for walkers, which corresponds to the number of students given in the questions, 40 students at Rivers and 25 students at Mountain view. Thus, students can find the number of bus riders by multiplying 40 by 6 or 25 by 7 because 6/7 or 7/8 can be created by iterating a unit fraction, 1/7 or 1/8 six times or seven times, respectively.

The second question is focused on concept of fractions, more specifically, understanding of referent units to represent a partial portion of a whole in the given area models. To find a fraction of shaded parts in the diagram, students need to pay attention to what is considered as a referent unit and how to represent the different sizes of continuously or discontinuously shaded portions using the same referent unit (a whole). Depending on students' cognitive levels, some students could solve this problem by finding the total area or portion of the shaded part or by finding fractions of each shaded portion and adding them together, or by finding a fraction of the smallest portion, setting it as a unit and counting the total number of the unit through iterations.

The third question is focused on understanding of multiplicative relationships and reversible reasoning when composite units of fractions are included. To find the amount of original paycheck, students need to know 2/3 of the remaining money equals \$90 because 1/3 of the remaining money was spent for food. Also, students need to understand that 2/3 is created by iterating a unit fraction (1/3) twice and thus the amount of money corresponding to 1/3 can be found by dividing \$90 by 2. From here, students will be able to easily figure out the amount of money corresponding to 3/3 and so the original paycheck can be found by multiplying 45 by 3 and adding it to 60 via reversible reasoning.

Among the three questions, we think the third problem is most difficult followed by the first and second questions. The third question requires understanding fractions as operators and requires multi-steps to solve the problem. The first question may be solved by either using measurement sub-construct or ratio sub-construct. The second question, which may be easiest, can be solved by relying on part-whole sub-construct. Because each question that we asked was created by focusing on specific fractional knowledge, we first divided PSTs' cognitive levels depending on the number of correct answers and further examined their strategies used to solve the questions. More specifically, we divided PSTs into three categories depending on the number of correct answers. PSTs who got all three correct answers were sorted into level 3, and PSTs who got two questions were assigned to level 2. PSTs with all incorrect answers to the three questions were considered at level 0. 19 PSTs were assigned to level 3 and 36 PSTs were assigned to level 2. Also, there were 28 PSTs corresponding to level 1 and 16 PSTs corresponding to level 0. In the next section, we present each level in more detail.

PSTs at Level 3

PSTs at level 3 demonstrated following features in terms of problem solving strategies and fractional knowledge across the three questions (see Table 1). PSTs at level 3 demonstrated a good understanding of fractions as part-whole, measure, and operator in solving problems. In particular, they tended to often use multiplicative reasoning, which is related to iterations in the measure construct to solve word problems involving fraction quantities. Also, the PSTs showed that they had a clear concept of unknown and were able to create equations to find unknowns based on the concepts of unknown and the understanding of fractions as measure and operator.

Table 2: Features of PSTs at Level 3 in problem solving strategies and fractional knowledge

Level 3	Features
Fractional knowledge	• Understand the part-whole construct of fraction • Understand the measurement construct of fraction by using iteration (multiplicative reasoning) to solve the problems • Understand the operator construct of fraction by considering fractions as a multiplicative operator on knowns or unknowns • Have a concept of unknown and create equations to find the unknowns
Problem solving strategies	• Understand problem clearly • Often use drawing to make better sense of the problem situations • Tend to use multiple ways of solving the problems

Regarding fractional knowledge used to solve each question, in the first question, PSTs at level 3 compared the size of two fractions by using a conceptual strategy (i.e. 1/7 is bigger than 1/8 and thus 7/8 should be greater than 6/7 because 7/8 took out less fraction than 6/7), converting fractions into decimals (i.e. 7/8 [= 0.87] is greater than 6/7 [= 0.85]), or finding common denominators procedurally (i.e. 7/8 [=49/56] is greater than 6/7 [=48/56]). However, to find in which school has more bus riders, most PSTs tended to identify a unit fraction for walker from the given information and use the multiplicative relationship to figure out the number of bus riders at two schools (Figure 1).

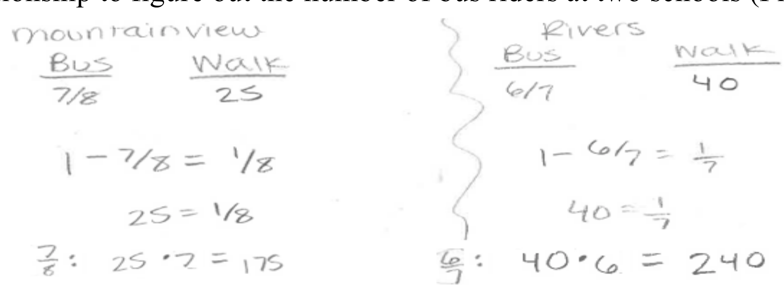


Fig. 1 Solution to find in which school has more bus riders in question 1.

Also, in solving the second question, most PSTs used a strategy to divide all parts into small equal sizes (1/16 or 1/32), set the equal size as a unit, and count the number of iterations of the unit (See Figure 2). This strategy shows that PSTs at level 3 understand measurement construct of fractions.

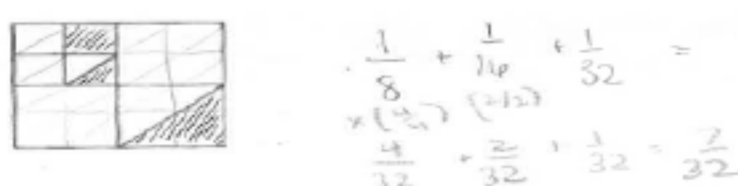


Fig. 2 Solution to find a fraction of shaded parts.

Finally, in solving the third question, two-thirds of PSTs found the amount of original paycheck by using multiplicative relationship from measurement construct of fraction. For example, PSTs first found the amount of money corresponding to 1/3 of remaining money by using the multiplicative relationship that 2/3 can be created by iterating 1/3 twice and the given information that \$90 is 2/3 of the remaining money. Then PSTs added the money spent on clothes (\$60) to the entire remaining money, which consists of 2/3 (\$90) and 1/3 (\$45). Figure 4 presents PSTs' solution method where they use their understanding of unknown and operator construct of fractions. PSTs set variable x as the original paycheck that she wanted to find and then represented the amount of money spent on food as (x-60) × 1/3 by considering 1/3 of remaining money was spent on food after spending \$60 on clothes.

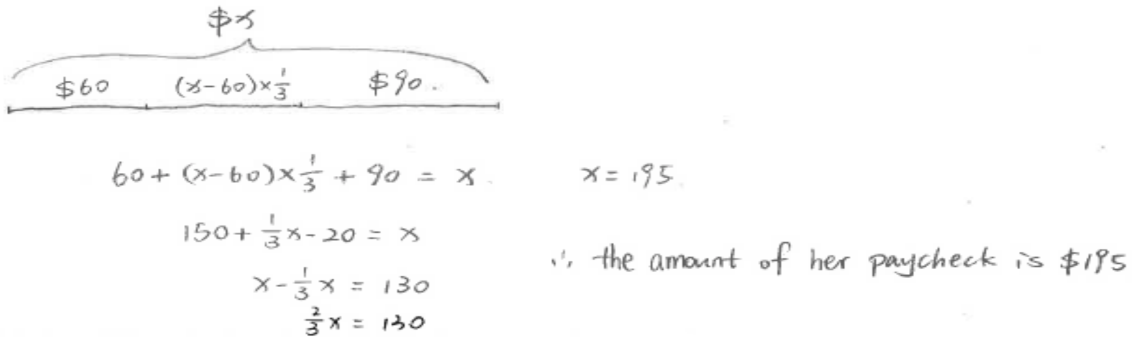


Fig. 3 Solution to use unknown and operator construct of fractions in question 3.

PSTs at Level 2

PSTs at Level 2 got two questions out of the three questions. We divided three sub-categories within level 2 depending on which questions PSTs got. For example, level 2-1 designates PSTs who got correct answers in both questions 3 and 2 (level 2-2: PSTs who got a correct answers in question 3 and 1 / level 2-3: correct answer in question 1 and 2). Although there are three sub-categories in level 2, we consider level 2-1 is highest, followed by level 2-2 and 2-1 due to the problem difficulty, which will be explained in detailed. We observed many similarities between PSTs at level 3 and those at level 2-1. 15 PSTs at level 2-1 demonstrate a good understanding of fractions a measure and an operator. However, these PSTs did not understand the problem correctly to question 1 and provide only one of the two sub-questions. With respect to problem solving strategies, they tended to represent the problem using drawing and solve each problem in multiple ways.

However, PSTs at level 2-2 seem not fully understand fractions as an operator. Although four PSTs provided a correct answer to the question 3, which requires multi-step multiplicative thinking, they often ignored denominators and solved it like whole number problems. 5 PSTs were categorized at level 2-1 where they were not able to recognize known and unknowns in correctly. In solving question 3, they showed difficulty in using measurement concept (e.g., 2/3 is twice of 1/3). In solving question #1, they relied on guess and check by either using given fractions.

PSTs at Level 1

PSTs at Level 1provide only one correct answer and consequently three subcategories exist in this level. For example, level 1-1 indicates PSTs who got a correct answer in question 1; level 1-2 includes PSTs who got a correct answer in question 2, and level 1-3 with PSTs who got a correct answer in question 3). Because each problem requires different level of problem difficulty, we consider PSTs at 1-3 is the highest level followed by those at level 1-1 and at 1-2. 13 PSTs were at 1-1; eight were at level 1-2; and six were at level 1-3. Table 3 shows the features of problem solving strategies and fraction knowledge at this level.

Table 3: Fractional knowledge and problem solving strategies by PSTs at Level 1

Level/Description	Features
1-1: Correct answer to question 1 (Middle)	*Fractional knowledge* • Be able to apply fraction as a ratio and measurement. • Yet, show lack of understanding of fraction as operator/ multiplicative thinking. *Problem solving strategies* • Although some used logical reasoning, rely on guess and check.
1-2: Correct answer to question 2 (Lowest)	*Fractional knowledge* • Develop an understanding of fraction as part whole. • Understand the measurement construct of fraction by using iteration (multiplicative reasoning) to solve the problems. • Lack understanding of fraction as measurement, operator, or ratio. *Problem solving strategies* • In solving # 2, relying on visualization by rearranging all parts together to form one simple fraction or viewing half of 3 squares. • Often do not understand the problem correctly. • Make computational errors/ Misrepresent the multiplicative relationship.
1-3: Correct answer to question 3 (Highest)	*Fractional knowledge* • Have some understanding of multiplicative thinking involving fractions. • However, have limited understanding of fraction as ratio or measurement. *Problem solving strategies* • Tend to use guess or backward/ Often do not recognize the problem correctly. • Misrepresent the second denominators.

PSTs at Level 0

15 PSTs were categorized at Level 0 where none of the answers are correct in the three fraction problems. In Question 1, these PSTs showed limited understanding of fraction as a ratio or measurement by either focusing on only one part of the two. In solving the question 2, they tended to miss the denominator and consider the problem as involving whole numbers. In solving Question 3, they showed lack of understanding of unknown and were not able to construct or solve algebraic equation. In particular, most of these pre-service teachers failed to identify the multiplicative relationship by recognizing the referent unit incorrectly.

Discussion and Concluding Remarks

In this study, we investigated how pre-service teachers solved three fraction problems that require different sub-constructs. By tracing their correctness and problem solving strategies in the three problems, we also explored how PSTs' use of fractional concepts is related to solving problems involving advanced fractional knowledge. First, we found that many of our PSTs developed limited understanding of fraction sub-constructs and thereby were not able to solve the three problems. Another finding is that when PSTs relied on only one sub-construct such as part-whole, they tend to provide incorrect answers. Based on our findings, we hypothesize that PSTs' understanding of fractions as measure and operator may be a foundation of solving problems involving advanced fractional knowledge.

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MINDSET IN PROFESSIONAL DEVELOPMENT: EXPLORING EVIDENCE OF DIFFERENT MINDSETS

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This exploratory case study investigated the role of mindset (i.e., fixed mindset vs. growth mindset) as elementary teachers participated in a professional development focusing on mathematics. Data were collected on two participants with opposing mindsets. Attention was given to their interactions in collaborative group settings as well as their views regarding working with successful and struggling mathematics students. Results indicated that although the two participants engaged in the same professional development activities, their engagement led to different interactions within those activities.

Keywords: Teacher Beliefs; Teacher Education-Inservice

Introduction

To support all students in developing deep mathematical understanding, "students [must] have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential" (National Council of Teachers of Mathematics [NCTM], 2014, p. 59). Although the literature abounds with descriptions of effective teaching and learning (e.g., Franke, Kazemi, & Battey, 2007;NCTM, 2000, 2014; National Research Council, 2001), teacher educators recognize that for most mathematics teachers the classroom practices described in the literature represent a reconceptualization of mathematics teaching (Sowder, 2007). Professional development (PD) is a key mechanism for supporting teachers' development of effective teaching practices (Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010;Sowder, 2007) and equity in mathematics.

PD experiences that are sustained, focused on worthwhile tasks, and provide immersion experiences in productive teaching practices can challenge teachers to transform their practice (Elmore, 2002; Hawley & Valli, 1999; Loucks-Horsley et al., 2010). Despite these known characteristics of effective PD, supporting change in mathematics classrooms continues to be daunting (Franke et al., 2007). Cooney (1999) posited that teachers' belief in teaching as an act of giving information to students represents an obstacle to change in the mathematics classroom. Sowder (2007) stated, "Many of teachers' core beliefs need to be challenged before change can occur" (p. 160). Recognizing that a teacher's beliefs influence his or her perceptions of effective instruction (NCTM, 2014;Pajares, 1992), much research on PD has sought to address changes in teachers' beliefs and changes in instructional practice, with varying results (see Philipp (2007)).

Recently, *mindset* has emerged as a term used to describe the belief that either mathematics ability can be cultivated in all students or mathematics ability cannot be changed (Dweck, 2006). NCTM (2014) reported research showing that "believing in, and acting on, growth mindsets versus fixed mindsets can make an enormous difference in what students accomplish" (p. 64). Responding to these different mindsets may be an important notion for progressing the field forward. Just as teachers' differing beliefs influence their PD experiences, we wondered how teachers' differing mindsets influence their engagement in PD experiences. The purpose of this study was to examine how teachers with opposing mindsets interacted with the content of a PD. Specifically, the research question that guided our work was: How do elementary teachers of different mindsets interact in a mathematics-focused PD?

Theoretical Framework

Dweck and Leggett(1988)described a social-cognitive model of motivation and personality that framed a theory of implicit conceptions of the nature of ability based on work in goal orientation and behavioral patterns. This implicit theory and its research base continues to support work in fields including educational psychology and mathematics education (e.g., Aronson, Fried, & Good, 2002; Blackwell, Trzesniewski, & Dweck, 2007; Dupeyrat & Mariné, 2005). The evidence supporting its generalization to other domains (Dweck, Chiu, & Hong, 1995; Dweck & Leggett, 1988) and the instrument used to measure its constructs (Dweck et al., 1995) provided the theoretical framework for this study.

Dweck and Leggett (1988) posited that an individual's implicit assumptions about the nature of an ability lead directly to the type of goals he pursues regarding that ability and the behaviors he exhibits when faced with challenges (Dweck et al., 1995; Dweck & Leggett, 1988). They described the *entity* and *incremental theories*. Individuals assuming an entity theory tended to view attributes as fixed, uncontrollable entities and adopted performance-oriented goals to gain or avoid judgment regarding the ability. In contrast, individuals espousing an incremental theory tended to view attributes as malleable and subscribed to learning goals focused on improvement of the ability (Dweck, 1986; Dweck & Leggett, 1988). These mindsets and their associated goals created "a framework for interpreting and responding to events" (Dweck & Leggett, 1988, p. 260) that promoted observable behavioral patterns when the ability under consideration is challenged. Maladaptive, *helpless* responses characterized by lowered performance and the avoidance of challenges were associated with entity theories. Their adaptive counterparts, *mastery-oriented* responses, were associated with incremental theories and characterized by the pursuit of challenges and persistence when faced with failure (Diener & Dweck, 1980; Dweck, 1975; Dweck & Leggett, 1988).

Although the tenets of implicit theories were initially established through the characterization of an individual's own intelligence, Dweck and Leggett(1988) proposed a framework through which its generalization to other attributes and domains occurred, culminating in the validation of an instrument used to assess individuals' implicit theories for multiple attributes (Dweck et al., 1995). The authors predicted that for any attribute of personal significance, "viewing it as a fixed trait will lead to a desire to document the adequacy of that trait, whereas viewing it as a malleable quality will foster a desire to develop that quality" (Dweck & Leggett, 1988, p. 266). Additional evidence supported that the model holds for generalization to traits beyond the self, such as the character and attributes of other people and the world (Dweck & Leggett, 1988; Erdley & Dweck, 1993). This application of the model suggested further observable characteristics of an individual's interactions based on their implicit theories. Those with fixed mindsets should be seen to reject change in themselves and others and draw simplified conclusions from brief experiences. In contrast, those with growth mindsets should be seen to encourage growth in other individuals and organizations and experience a sense of control relative to their environment (Dweck & Leggett, 1988).

Methodology

To examine the role of mindset in PD, we used case study methodology, specifically a holistic, multiple-case design (Yin, 2014). We viewed this as an exploratory case study, with the purpose "to identify research questions or procedures to be used in a subsequent research study, which might or might not be a case study" (Yin, 2014, p. 238).

This study occurred within a PD project serving 82 kindergarten through sixth grade mathematics teachers. The project was in its second year of external funding and represented a partnership between a university and four rural school districts in a southeastern state. Although the project included academic year meetings and demonstration lessons, the project component of focus in this report was the ten-day summer institute occurring during the second year of the project in which teachers met in grade-level groups (i.e., K–2, 3–4, 5–6) and engaged in immersion and practice-based experiences (Loucks-Horsley et al., 2010), with a mathematical focus on fractions and their operations.

The twelve-item Likert style mindset survey included Dweck et al.'s (1995) nine items related to mindset in relation to intelligence, morality, and world. (For evidence of reliability and validity see Dweck et al.) In addition, we created three survey items pertaining to a teacher's point of view of their students' mathematical abilities. The addition of these items is supported by the work of Dweck and Leggett (1988).

We developed four semi-structured interview protocols along with an observation protocol for data collection. Interview questions related to participants' strong academic students, struggling students, and their role as mathematics teachers working with these students. The observation protocol, employed during observation of participants in PD activities, consisted of six categories: evaluation of situation, dealing with setbacks, challenges, effort, criticism, and success of others. In each category, observable behaviors for each mindset were recorded. Participants were video-recorded during institute activities.

On the first day of the summer institute all teachers completed the mindset survey, which was scored following Dweck and colleagues' (1995) protocol. Four participants representing each mindset were interviewed and then two teachers, Ms. Fitzgerald (fixed mindset) and Ms. Gorman (growth mindset) were selected. Both Ms. Fitzgerald and Ms. Gorman (pseudonyms) participated in the grades 5–6 PD group, which had as its primary focus the modeling of multiplication and division with fractions. At the time of the study, Ms. Fitzgerald, a Caucasian female, had completed eight years as a classroom teacher and Ms. Gorman, an African American female, had completed nine years of classroom teaching.

To analyze the data, we drew on the organization and analysis procedures described by Yin (2014). We developed a case study database for each participant, organizing and compiling all data chronologically. We began by employing an inductive strategy for analyzing participant interviews. This strategy led to the development and refinement of a set of codes representing relevant concepts. In addition, we relied on our theoretical framework to guide the analysis of interviews and observation data, identifying evidence of both fixed and growth mindsets. The validity and reliability of our findings are supported by the use of multiple sources of evidence (construct validity), replication logic (external validity), and a case study protocol (reliability).

Results

In this section, we present the results from our analysis. Each case will be presented separately, organized according to the three themes that emerged during analysis: intelligence, goals, and challenges.

Ms. Fitzgerald (Fixed Mindset)

Intelligence. When asked if intelligence was something that could change, Ms. Fitzgerald stated, "I don't think so. I think it's something that you are born with and what you learn through school is what you learn." During her interviews, Ms. Fitzgerald distinguished between students who were naturally talented at mathematics and those who were talented at literacy:

> I think that you are either literature based or you're mathematically based. . . . The literature-based [students] are slower learners. . . . The mathematically based [students], they can do it through

> computations. They can do it with manipulatives. They can usually draw diagrams.

She also spoke of instructional practices related to students of different ability levels.

> I would give [students in the high ability group] a higher level task and give my inclusion group a lower level task. But all that was aiming towards the same goal. . . . For my inclusion group, I might pull a fourth grade task on perimeter versus my high group, I might pull a sixth grade task. And then my middle group, keep them on target with a fifth grade task. But in the end, we are all working towards the same goal.

Ms. Fitzgerald indicated that students have a tendency to disengage when faced with challenging material, thus leading to behavioral issues. In addition, as she reflected on her own frustration when faced with a challenge, she said: "Now I see how my kids get frustrated. . . . I think that I would back off a little bit if a task seemed to be too difficult for them. I don't want them to get this frustrated."

Goals. Ms. Fitzgerald indicated that she was guided by performance-oriented goals for herself and her students, which was evidenced in three ways. First, Ms. Fitzgerald focused on solutions to problems. She said:

> I would let [the struggling students] do their own models first but we might go to an algorithm a little bit quicker with them and try to work backwards if they can't get the models on their own. Solve the problem and then try to create a model that matches and then hopefully later on they could go to the model first.

Ms. Fitzgerald indicated that if a student cannot draw the model right away, then she has the student find the solution using an algorithm, thus emphasizing the model as another procedure to be learned. Then, the student can use the solution as a means for drawing the correct model.

Ms. Fitzgerald also placed a great emphasis on testing and algorithmic proficiency. She stated:

> I go straight to the algorithm and I would like for my kids to have different ways to do it because when it comes time for [the state's constructed response assessments], they can't just do the algorithm. They have to be able to draw some type of model or picture.

She saw her instructional role in working with the students as guiding them to the answer.

> I have to step away and let them do their thing, not tell them the right answer, step back, let them struggle a little bit, and guide themselves to the correct answer. And if they get it wrong, it's ok. . . . I might give them an easier task the next day, something that's still within . . . the same unit, but something that may be a little easier, not as confusing for them. After that, lead into some instruction because they always have to have some instruction.

Challenges. During the PD, Ms. Fitzgerald faced the challenge of developing models (i.e., either pictorial or concrete representations) for fraction problems involving multiplication and division. Ms. Fitzgerald consistently demonstrated helpless responses. She made statements such as, "I couldn't tell you because I don't know," and "Yeah, but that's probably not right," which indicated a negative self-cognition. In addition, Ms. Fitzgerald offered statements that seemed to be intended to divert away from the discussion such as, "I'm more of an algorithm person," and, "I need to go to the K-2 class and then maybe I can do something in there." When selected to respond during the institute, she would offer statements such as "We're nowhere yet" and "I'm thinking I don't know."

Ms. Fitzgerald expressed continued frustration when faced with the challenge of drawing models. "I can't read the models. I can't draw the models. So I was very frustrated. That made it hard to participate in the tasks and the problem sets." Similarly, Ms. Fitzgerald stated, "Because I've tried – like our problem sets, you have to draw the model, no algorithm and I just look at it and go – oh, I'm defeated again today, I can't do my homework." When asked how she might overcome the challenge, Ms.

Fitzgerald said, “Um, just forge ahead and really try to wrap my brain around it.” She also indicated that more time should be spent “breaking [the model] down and simplifying it to where everybody can understand it.”

Ms. Gorman

Intelligence. When asked if intelligence was something that could change, Ms. Gorman stated, “I look at this as saying people can make mistakes, and you can learn from your mistakes, and you can change.” She explained that although some students understand procedures more quickly, *all* students can gain a stronger understanding of mathematics through struggling with difficult material and exploring and explaining their thinking.

> I think that in the long run they would have a better understanding of math, they would have the critical thinking skills to think through it, whereas some of the others . . . they could do the procedures but couldn't explain why. . . . I had some [high ability] kids who, at the beginning of the year, 'The answer is 24', 'How do you know that?', 'I just know.' They knew the procedures of it, they could tell you the answer, but couldn't explain. It didn't mean anything. . . . I was almost excited when they would get something incorrect because that gave me the chance to help them think through it.

Ms. Gorman’s goal of supporting all students in understanding mathematics led her to describe her instructional practices in working with low achieving students.

> Definitely take some steps back. . . . Let them struggle. . . . to a degree. . . . you don't want them to struggle so much that they're like "I'm done." But to let them struggle to see if they can figure out – a way of persevering through it. And then offer my help and then walk away.

Ms. Gorman described her own struggles in working with high-ability students:

> This past year I had five [high ability] kids, which is a lot to have in one room. They were very procedural and what I found was hard for them was, they were like 'I just know,' but they couldn't visually show me or tell me why. . . . So, with them, having more advanced type tasks, questions, or even be able to have questions to be able to ask them, I think that's one of my struggles. Even though I know I'm supposed to ask advancing and assessing questions, I think that even when I'm looking at it I have a hard time being able to pinpoint what question to ask at that moment quickly and walk away.

Goals. Ms. Gorman’s statements suggested that she was guided by learning-oriented goals, as indicated by her tendency to focus on developing mathematical understanding. She described the role of student exploration and explanation in developing this understanding and linked these to changes in her instructional practice.

> Being able to connect with how they are thinking, I would say it's a lot more engaging, just because I am pulling it out and trying to find manipulatives, and getting the kids to explain their answers, and I've been trying to share with the kids multiple ways that they can get to it, and I usually don't share until they open the door first. So, I kind of build off of their ideas.

She indicated her role in working with students was to ask good questions and serve as a guide.

> So as far as my teaching strategies go, it's just taking some steps back, making sure I'm asking the right types of questions, to get the kids to go in a certain path, so even if they're steering off the wrong way my questioning needs to be correct in that I'm pointing them back in the right direction. If they have no clue at all, you know, just helping them to find that starting point without giving too much. And without them always using my strategy.

Finally, she acknowledged the role of drawing models in supporting the development of mathematical understanding. The models served as a tool for making sense of the mathematics and making connections.

> A lot of times I think . . . the lower kids tend to not have number sense or not really know what the numbers represent. And so with pictures, and things like that--with the labeling--I think that would help them to progress to a better understanding of what it is they're doing.

Challenges. When discussing the challenges associated with understanding the models for multiplication and division of fractions, Ms. Gorman's responses most often could be categorized as mastery-oriented responses. Ms. Gorman attributed her struggles with understanding the models to the fact that she had not been taught these models as an elementary student. She said, "I feel like I'm trying to figure some of these [visual models] out because I was not taught that way. . . . I thought I knew math. I do not know math the way I thought I knew math."

She felt it was important, however, to understand the models so that she could utilize them in her classroom.

> I don't [understand] the pattern blocks. . . . I would not use those in my classroom because I would feel like if I could not explain it to the kids with a clear understanding – because that to me would become one of the tools that they could use to figure out the problem. But they can't use something if I can't explain it efficiently.

During summer institute sessions, she frequently continued to work with one particular model until she felt sure, even as others around her moved on to a different representation.

In reference to the variety of models displayed by her peers, Ms. Gorman described herself as being overwhelmed as her lack of understanding had been exposed and she aimed to understand all of the different models.

> I'm very overwhelmed. Just in the sense of – obviously I know how to divide and multiply fractions – but having to show pictorially how to multiply and divide fractions is totally different. Because I never have really had to do that. . . . And then another thing that was overwhelming is that you are passing these posters around and you are looking and somebody like me whose mind is just constantly going and I see a different way to work out something then I'm like – ooh! You know, I want to work it out this way. And then there's another way. And then another.

She believed that to overcome the challenge she needed more practice with the models. She said, "Before I took it to the classroom I would need some more practice . . . just so I could feel comfortable in what I'm saying, so that they could try it and understand what they were doing."

Discussion and Conclusion

Supporting all students in learning meaningful mathematics is a daunting task, and one that requires a shift in how teachers view effective mathematics teaching (Sowder, 2007). PD represents the primary means towards supporting teachers in developing this view (Loucks-Horsley et al., 2010) and much is known about key components of effective PD (Elmore, 2002; Hawley & Valli, 1999; Loucks-Horsley et al., 2010). The impact of PD is often hindered, though, as a result of teachers' beliefs about instruction (Sowder, 2007). Although much research has examined the role of beliefs in teachers' PD (Philipp, 2007), the role of mindset has only recently emerged and has yet to be examined in this way.

As a result, the purpose of our study was to examine how teachers' mindsets influenced their participation in a PD setting. Throughout the PD, the teachers worked with representing the multiplication and division of fractions with models, a task to which many teachers had not been previously exposed. In this sense, Ms. Fitzgerald and Ms. Gorman were no exception. They each described the uncomfortable feelings associated with not knowing; yet how they responded to this challenge was quite different. Ms. Fitzgerald spoke of frustration, stating continuously that she did not understand the models and that she preferred to utilize algorithms. Ms. Gorman spoke of being overwhelmed, providing statements that demonstrated her persistence in trying to understand the variety of models available.

Additional differences between the two participants emerged when considering the purpose of understanding the models. Ms. Fitzgerald seemed to see the development of a model as an additional process for students to learn; a process that was needed in order to be successful on state assessments. Ms. Gorman indicated that the development of a model supported the development of students' understanding. Interestingly, Ms. Fitzgerald believed that students who understood mathematics made connections among the models and the algorithms. Unlike Ms. Gorman, however, she did not see this as an indication that students struggling with mathematics might benefit from tasks or activities that supported them in making those same connections. Rather, she felt these students did not hold the potential for understanding mathematics and instead needed more practice with the procedures.

These contrasting views influenced the instructional practices described by the two participants. While both mentioned the role of struggle in learning, Ms. Fitzgerald aimed to support her students in avoiding struggle, for example through the use of easier tasks. This practice was likely influenced by her belief in the inability to influence students' mathematical abilities. Rather, each student's achievement level should be identified and then appropriate tasks assigned. Ms. Gorman described the importance of productive struggle in supporting the learning process. Thus, students' mathematical abilities could be shaped by their learning experiences.

Across the three areas in our analysis (i.e., intelligence, goals, and challenges), the differences between views expressed by the two participants can be attributed to their opposing mindsets. Ms. Fitzgerald's propensity to see intelligence as unchangeable is clearly linked to her fixed mindset, along with her performance-oriented goals and helpless responses. Similarly, Ms. Gorman's view that intelligence is changeable combined with her learning-oriented goals and mastery oriented responses is clearly aligned with her growth mindset. As a result, the opposing mindsets resulted in different PD experiences.

The findings for this exploratory case study lead us to ask more questions. First, we believe that Ms. Gorman and Ms. Fitzgerald left the PD with greatly different experiences. How will the differences we observed in Ms. Gorman's and Ms. Fitzgerald's interactions during PD carry over into their classroom practice? In *Principles to Actions*(NCTM, 2014) explicit connections are drawn between growth mindsets and mathematical success for all students. Further, Blackwell, Trzesniewski, and Dweck (2007) reported being able to manipulate mindset. What interventions can we include in PD so that we orient teachers toward a growth mindset? How might findings from these questions impact future PD and even work with pre-service teachers? Finally, how can we help teachers build ways to work productively with students of different mindsets? Knowing that "believing in, and acting on, growth mindsets versus fixed mindsets can make an enormous difference in what students accomplish" (NCTM, 2014, p. 64), we must also continue to follow this investigation into the differences that can be made in what teachers accomplish when attention is given to mindsets.

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AN EXPLORATORY ANALYSIS OF A VIRTUAL NETWORK OF MATHEMATICS EDUCATORS

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This study is part of a larger research design project that aims to cultivate an online community of mathematics educators. The purpose of this smaller study is to suggest interventions that will further support community cultivation efforts. Social network analytical methods are used to study project participants' interactions in virtual spaces. Investigation focused upon the connectivity of network structure, the most prominent members of the network, the presence of a core-periphery structure and the relationship between participant types and prominence. Analysis suggests that supporting interactions between newly involved and more long-standing participants will enhance community cultivation efforts. And, four participants emerged as most influential in the network, therefore being most effective in controlling and spreading novel information. These participants are suggested to receive additional professional development.

Keywords: Teacher Education-Inservice

Educational research and policy call for mathematics instruction to be student-centered, with a focus on argumentation and negotiation. While these calls were made over two decades ago, classroom instruction still does not reflection this “reform” oriented vision of classroom instruction (Stigler & Hiebert, 1999). One initiative that has shown promise in supporting teachers in shifting instructional orientation from teacher-centered to student-centered is engagement in community. Involvement in community provides opportunities for teachers to critically examine daily instructional issues, analyze student work, and plan mathematics tasks to support learning. While community engagement has been linked with improved instructional strategies (Vescio, Ross, & Adams, 2008), there is a lack in understanding of how to support emergence of teacher communities, particularly those that take place in online mediums.

This study is part of a larger research project that implements and innovative professional development (PD) model designed to cultivate an online community of mathematics educators. Due to extensive interactional data available from the larger study, and a particular focus on structural features of the network, we used social network analysis to investigate the patterns of interactions within the network. Social network analysis is a quantitative analytical method derived from graph theory used to study the structure of social networks (Scott & Carrington, 2011). Taken together, social network analysis can provide data about relationships in social networks that can be informative for future interventions that would not have been evident by other investigatory approaches. From an educational researcher's perspective, structural information about teacher networks is crucial for the implementation of interventions designed to foster community cultivation. For example, if teachers are identified as being sparsely connected, an intervention can be designed to support interaction between these teachers in order to improve the flow and access of information in the network as a whole. Interventions that serve the purpose of the one just mentioned may be critical in the process of cultivating a community of educators. Part of the purpose of this research is to make such suggestions for interventions.

The analytical methods briefly introduced here (and further elaborated in the methods section) serve as the backdrop against which research questions are formulated to focus investigation in this study. Consequently, this study seeks to answer the following research questions:

- What are some of the structural properties of the social network?
- Who are the most central members of the participant network?
- To what extent is a core-peripheral structure present in the network?
- What is the relationship between participant type and centrality?

Literature Review and Conceptual Framework

This study is informed by literature on sociocultural learning theories (i.e. communities of practice and situated learning) and social network theory. These theories provide the conceptual lens to understand the structural relationships in the community of mathematics educators, and provide insights for interventions aimed at improving dissemination of instructional strategies and learning in the MathCom network.

Communities of practice

Communities of practice is a social perspective of learning that hinges learning upon individuals' enculturation into a community through increased participation within a social group. Three characteristics distinguish a group of individuals from a community of practice: mutual engagement, joint enterprise and shared repertoire (Wenger, 1998). Mutual engagement refers to the notion that communities develop and are maintained around engagement in shared practices. Engagement in meaningful activities of a community requires particular competencies that are valued by the community. Understanding of such competencies can be measured by the extent to which one interacts with community members. Therefore, identifying participation patterns both provides a sense of individuals' level of understanding of communal practices and the strength of community structure. Joint enterprise and shared repertoire refer to common goals, beliefs and a shared set of tools, respectively. While evidencing these aspects of community relies on more qualitative research, social network analysis is useful for investigation of mutual engagement to identify patterns of interactions (Wenger, Trayner, & de Laat, 2011).

Legitimate peripheral participation (LPP)

Communities of practice have a particular structure that is conducive to sustaining and maintaining engagement, which can be conceptualize through the lens of LPP. Communities consist of a core group of full participants, or "old-timers" who are experts in the community's practices (Lave & Wenger, 1991). However, if only experts are present, engagement typically becomes stagnant and interactions do not persist. Peripheral participants, or "newcomers", provide new perspectives and fresh outlooks to be considered by old-timers, which facilitates sustained engagement within a community (Lave & Wenger, 1991). Overtime, newcomers follow a trajectory of increased participation in the community, and eventually replace old-timers. The constant interchange of newcomers and old-timers is a core feature of a community of practice. Therefore, identifying places to support interaction between newcomers and old-timers may enhance community cultivation efforts.

Social network analysis and communities of practice

The discussion above underscores a perspective of learning that takes participation in a community as the unit of analysis. Researchers argue that communities of practice are comprised of social networks (Schenkel, Teigland, & Borgatti, 2001). Social network theory focuses upon the relational patterns embedded within a community and groups. In order for group norms to emerge and common beliefs to be present amongst the community, it is important for the social network structure to afford information flow. Recent studies have sought to identify central members—prominent members that have high levels of interaction within the group—in teacher communities to leverage their position in social networks to spread novel information (Daly, 2010). For example, through investigation of the importance of leadership for community success, Tsugawa, Ohsaki, and Imase (2010) conclude that in online communities, centrality can predict an individuals' ability to assume leadership responsibilities. In a similar study that sought to understand the relationship between individuals' centrality and successful implementation of reform, Atteberry and Bryk (2010) provide teachers professional development to act as "coaches" to spread information around school-based initiatives. In doing so, Atteberry and Bryk (2010) used social network analysis to study the relationship between teachers' centrality prior to receiving professional development and the success of the reform effort. They conclude that teachers with higher levels of centrality prior to implementation were more successful in dispersing novel information. In the current study, centrality measures are used to inform which participants, given additional developmental opportunities, may have the most impact on community cultivation.

Methodology

Social network analytical methods are used to investigate the "MathCom" network structure. In the following section these analytical methods are explained along with descriptions of the research setting, participants, data collection and processing, and data analysis plan.

Study Site and Participants

Data for this study is drawn from the MathCom project, which began in 2012. This project is designed to bring together and cultivate a virtual community of mathematics educators across the nation. A primary goal of the MathCom project is to foster mathematics educators' collective engagement in learning of mathematics and pedagogy. Accordingly, participants have been recruited in groups throughout the project as incremental efforts are being made to cultivate community. Over the past two years, multiple mediums of communication have been used as leverage points to support teacher interactions (i.e. face-to-face workshops, online classes and workshops, twitter discussion threads and email list serves) around different content, such as mathematics, instruction and assessment. Therefore, the research setting is in virtual spaces that foster online communication, such as twitter, online courses/workshops, and email listservs.

There are 82 participants (mathematics educators) involved in this study. Taking into consideration whether (and how) they participated in the two face-to-face summer institutes (at the end of Year 1 and Year 2 of the project) that focused on community and instructional development, we can distinguish between 6 types of participants:

- *2013 Fellows (n=12)* were part of the Year 1 institute
- *2014 Fellows* (*n=5*) began in Year 2 *and* were involved in the second summer institute.
- *2013 and 2014Fellows* (*n=12*) participated in both summer institutes.
- *Online participants (n=15)* were those who participated in the Year 2 institute solely via virtual video chat software.
- *Other(n=26)* participants participated strictly via online mediums
- *Staff(n=12)* are teacher educators responsible for virtual interactions with participants and developing project activities

However, given our interest in participants' interactions in the virtual spaces, we focused on the 67 who participated in those spaces.

Data Collection and Processing:

Social network analysis is based on the premise that social life consists primarily of relations and the patterns formed by these relations. Social network analysts study these patterns of interactions between individuals in a network (Scott, 2013). In order to conduct the social network analysis, we examined the records of interactions between participants in each virtual space (twitter, online classes and workshops, email listservs) between May 2013 and September 2014. Participant (*i*) is said to have "talked to" Participant (*j*) if the latter initiated a communication tie to the former. In Twitter, a tie exists if Participant (*i*) included (*j's*) hashtag in the message (Ediger et al., 2010). On email and Blackboard learning systems, a tie is defined if Participant (j) responded to an initial or any other posts/emails from another participant (*i*). From this information, we created one communication matrix that records the interaction between study participants. Each cell (*Xij)* of the matrix takes on a value indicating how frequently (number of times) participant (*i*) directs or initiates a tie to participant (*j*). The resulting 67 x 67 binary asymmetric matrix became the basis/input for all of the structural analysis presented in this paper. The social network analysis software, *UCINET* version 6 (Borgatti, Everett, & Freeman, 2002) was used in all data preparation and analysis. Analysis results are reported as normalized indices. A normalized measurement is one in which the numerical value that results from analysis is standardized so one could compare the particular analytical result to other networks with different sizes.

Data Analysis Plan

This study uses four social network analysis measures (i.e. density, centralization, centrality and core-periphery) (Borgatti & Everett, 2013; Kadushin, 2011) to address the research questions and describe the communication structure of the MathCom network. First, density and network centralization measures allow us to address the first research question: *what are some of the structural properties of the MathCom Network? Network density* describes the general level of cohesion in a graph (Scott, 2000). The centralization score describes the extent to which the flow of information is organized around particular individuals or groups of individuals. Second, centrality analysis is employed to answer the second research question: *who are the most central members of the MathCom network?* Centrality is used in social network analysis to identify important nodes or those that occupy influential positions in a network. Three variants of centrality measures are used in this study: in-degree, out-degree and betweenness centrality (Freeman, 1979). *In-degree* centrality measures the ties an actor receives from others, while *out-degree* centrality measures the ties an actor directs to others. Betweenness centrality is a measure that indexes the number of times a participant falls on the shortest path between two participants in the network. Third, we employ a core-periphery model to examine the extent to which core-peripheral structures are present in the MathCom network (Question 3). This method bifurcates the network into a discrete model consisting of two classes. Fourth, distinguishing levels of centrality for participant types is also of interest (Question 4). We used specialized Analysis of Variance (ANOVA) tests for social network data to examine the differences in centrality by participant type (e.g. whether 2013 fellows are more central then 13 & 14 fellows).

Findings

Research Question 1: What are some of the structural properties of the MathCom network? To address this question, density and centralization measurements were used to give information about network connectivity. The MathCom network resulted in a density of 8.3%. Out of the 5,852 possible directed edges in the network, 488 of these edges are present. The network centralization analysis resulted in a measure of 0.222. This indicates that the MathCom network is more representative of a network that has centrality evenly distributed throughout the network than one that is controlled by a few central members. Density and centralization give an overall representation of the structure of the network. And in this case illustrate that over the 15 months in which this data was collected, close to 500 connections were made and certain central members are potentially emerging as a centralization score of .222 indicates there is some heterogeneity among centrality in the network.

Research Question 2: Who are the most central members of the MathCom network? As will be recalled, in-degree and out-degree centrality scores index the level of activity of participants in the communication network. On average, the top 10 participants have normalized out-degree scores of about 1.082, which is about 10 times larger than the average out-degree scores for the middle ten participants (0.111), and ninety times larger than the scores for the bottom ten participants (0.0117) respectively. On an individual level, participants 11018, 11006 and 11003 are the most active participants: actors 11006 and 11018 for instance have an outdegree score that is about 35% higher than the next highest out degree in the network. The indegree analysis yield similar results to those of outdegree: actors 11006, 11018 and 11003 have the highest normalized indegree centrality scores. Betweenness centrality analysis also yields actors 11018, 11006 and 11003 falling in the top ten, however actor 11016 also yielded a high betweenness centrality score. While actor 11016 did not have as high a valued in/out-degree score, her dichotomized indices where among the highest. Given the importance of betweenness centrality for indicating potential for leadership qualities and the control of information, actor 11016 is considered as one of the most central members.

The above analysis suggests that actors 11018, 11016, 11006 and 11003 are the most central members in the network. Each of the identified actors are 2013 and 2014 fellows, meaning they have been a part of the project since it began in 2013, a year longer than other members that began participation in 2014. Over a year of the project, the identified actors will have had many more chances to participate than those that began in 2014, therefore being a possible reason for their increased centrality. However, based on centrality alone, and drawing from previous research, this analysis suggests these are the most central members and should receive additional professional development opportunities.

In addition, in this analysis staff members are omitted in identification of the most central actors. For example, staff members 1 and 7 are present in the top ten of each centrality measure, however they are left out of this analysis. Part of the purpose of this research is to provide additional professional development opportunities for participants that are identified through social network analysis as most influential in the network. Therefore, identifying staff members that are most central does not support research goals; although, staff members are included in analysis as part of the 67 individuals because many interactions occur between staff (who act as role models) and participants, thereby disseminating best practices in mathematics pedagogy.

Research Question 3: To what extent is a core-peripheral structure present in the network? To address this question, we fitted a categorical core-periphery model to the MathCom data. The core-periphery analysis provided sub-group densities, which can be used to examine the connectivity, between/within, the core and periphery groups. The density profiles also help evaluate the extent to which the derived groups approach an ideal core-periphery structure where 100% of ties are expected between core members, 0% between peripheral members, and most of the connections expected between the core and periphery members. The core-to-core density of the MathCom network is 59%, the core to periphery is 16.7%, the periphery to core is 9.6% and the periphery to periphery is 3.2%. Although this network does not completely approach an ideal core-periphery structure (and few are expected to be), the analysis indicates the presence of a core periphery structure in the network (see Figure 1).

Of the 15 participants in the core of the network, 9 are 2013 & 2014 fellows, 2 are 2013 fellows, 3 are staff, and 1 is an "other" participant that was not part of either institute but was rather active in online classes. The high concentration of 13 & 14 fellows in the core of the network is expected, due to increased opportunities to participate throughout the life of the project. Thus, participation in the summer institute seems to be related to communication activity in the network. The staff members in the core of the network are also reasonable results due to their frequent activity in twitter and email. However, a rather interesting result is participant 13010 falling into the core, as she was not part of either institute. Further research will explore the position of participants such as 13010.

Research Question 4: What is the relationship between participant type and centrality? The analysis above suggests that there are variations in centrality of participants in MathCom. One determinant of this communication activity is participant type (whether they are 2013 Fellows, 2013-2014 Fellows, etc.). Table 1 displays the results of an ANOVA permutation model examining the differences in mean (betweenness, outdegree and indegree) centrality of participants by participant type. The ANOVA analysis shows that 2013 & 2014 fellows have the highest centrality means in each category. Their average betweenness, outdegree and indegree centralities are about 3 times larger than the next largest mean. For example, the mean out degree centrality for 2013 fellows is 0.2452, while that of 2013 & 2014 fellows is 0.7148, which is 2.9 times larger then the 2013 fellows. Second, 2014 fellows have substantially lower in/out-degree (valued and dichotomized) than 13 & 14 fellows. For example, the in/out degree of the 2014 fellows is 14.8% and 10.6% of the 13 & 14 fellows, respectively. These findings illustrates a clear divide in the amount of participation between the two groups: with the 2013 & 2014 fellows much more active than 2014 fellows.

Discussion

Network structure

The above analysis clearly shows the presence of a core and periphery in this network. While the core-periphery analysis will “force” members into the two categories regardless of the nature of the connections in the network, the density measures suggest a rather densely connected core (59% of the potential connections are present) and a rather sparsely connected periphery (3.2% of the potential connections are present). In the ideal core-periphery structure the core-to-core density is 100% and the periphery-to-periphery density is 0% (Borgatti & Everett, 2000). While the results shown here are not quite representative of the ideal core-periphery network, the divide suggested by Borgatti and Everett is a theoretical model rarely (if ever) observed in practice.

Lave and Wenger (1991) suggest a core-periphery structure is critical to sustaining productive interactions within a community while maintaining the structure of a community. In particular, they note that constant interaction between core and peripheral members is critical for knowledge creation and distribution in a community. In consideration of this guiding theory, results of this study are used to identify connections that should be fostered between core and peripheral members to ensure the suggested interactional patterns. Through intervention, members that have been most recently introduced to the MathCom network (e.g. 2014 fellows) and are in the periphery, will be connected with members in the core that have been part of the project for a longer period of time (e.g. 2013 and 2014 fellows). Furthermore, as the periphery begins to decrease in size due to peripheral members beginning to evolve into more centrally located members in the community; new participants should be gathered in order to maintain the core-periphery structure.

Centrality

Recent literature suggests that higher levels of centrality in networks increase individuals’ likelihood of spreading novel information throughout the network (Atteberry & Bryk, 2010). In addition, Tsugawa, Ohsaki, and Imase (2010)mention that high levels of betweenness centrality are indicative of leadership qualities. Therefore, providing additional professional development opportunities to the most central actors may be an effective way of spreading novel information throughout the network in this study. The results of this study suggest that actors 11003, 11006, 11016 and 11018 should be given additional developmental opportunities in order to become "coaches" within the network that engage in practices with others to promote development of student-centered instructional strategies.

Conclusion

This study illustrates a quantitatively driven approach to enhancing professional development efforts. Social network analytical methods were used to identify opportunities for intervention that may have otherwise gone unnoticed. In particular, this investigation has evidenced the presence of a core-periphery structure and has identified 4 individuals that are most central in the MathCom network. These results provide justification for (1) providing professional development to specific individuals in the network, and (2) fostering relationships between specific groups of individuals in the network.

Table 1: Mean centrality measures and ANOVA tests of mean differences for each participant type

	2013 Fellows	*13 & 14 Fellows*	*2014 Fellows*	*Online*	*Staff*	*Other*	**One-Way ANOVA**		
Mean centrality							**F-Statistic**	**P-Level**	**R-Squared**
Betweenness	1.1381	0.2452	0.19	0.1125	0.0801	0.2357	3.966	0.003	0.108
Valued Outdegree	3.4667	0.7148	0.4265	0.2193	0.159	0.1331	4.44	0.015	0.127
Valued Indegree	0.5694	0.0762	0.0632	0.05	0.0498	0.0753	1.183	0.265	0.003
Dichotomized outdegree	0.679	0.211	0.079	0.066	0.066	0.0469	2.931	0.030	0.280
Dichotomized indegree	1.6592	0.2664	0.1535	0.0844	0.0712	0.0408	4.52	0.002	0.033
N	12	12	5	15	12	26	--	--	--

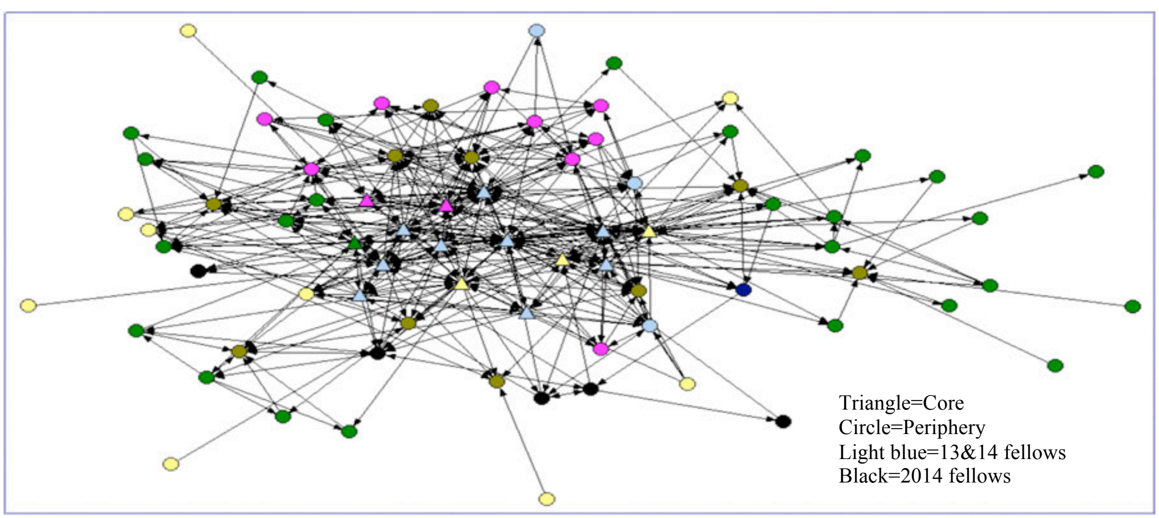

Figure 1: Communication interactions by participant type

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TEACHER NOTICING OF JUSTIFICATION: ATTENDING TO THE COMPLEXITY OF MATHEMATICAL CONTENT AND PRACTICE

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In this report, we will consider in-service elementary school teachers' noticing of the mathematical practice: justification. This study is part of a larger project evaluating the efficacy of a three year professional development built around attending to student thinking and promoting mathematical habits of justifying, generalizing and making sense. Noticing justification is a complex task requiring attention to both the (1) mathematical content and strategies and (2) the nature of the argument provided by a student. We have found that teachers' struggle to attend to both aspects simultaneously and offer a framework for considering teacher noticing of mathematical practices.

Keywords: Classroom Discourse; Teacher Education-Inservice; Reasoning and Proof; Standards

Reform curriculum and standards frequently treat mathematics as a dichotomous subject consisting of both content and practices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000). Teachers are expected to foster classrooms not just based on mathematical content goals, but that also promote practices such as justification and generalization. However, practice goals often remain more mysterious. For example, many teachers lack an understanding of just what justification is and how it would look if a student was engaged in justification (Knuth, 2002; Simon & Blume, 1996). *Professional noticing* is a lens for making sense of what teachers attend to in their classrooms, how they interpret student strategies and build upon them. We consider teacher noticing of justification in the context of a professional development (PD) designed to transcend particular mathematical content and focus teachers on the mathematical practices that support and sustain students' development and learning of mathematics. We attend to teachers' noticing of justification, considering the interplay between noticing of justification practice and content-specific mathematical strategies.

Noticing

Noticing in a professional setting is both a lens for making sense of what teachers see in the complex setting of a classroom and a skill to be developed in PD for current and pre-service teachers. Much of the work on noticing stems from Mason's (2002) *intentional noticing*. Intentional or professional noticing differs from everyday noticing as to what is attended to and how it is interpreted is influenced and focused by the professional experience and knowledge of the individual. Jacobs, Lamb, and Philipp (2010) developed a framework for teachers' professional noticing of children's mathematical thinking consisting of: *attending to children's strategies, interpreting children's understandings*, and *deciding how to respond on the basis of children's understandings.*

We build on Jacobs, Lamb, and Philipp's work to consider not just children's mathematical thinking related to content and strategies, but also as it relates to general mathematical practices. We are considering practices to be mathematical activity that is not dependent on particular mathematical content, bur rather is embedded in all areas of mathematics such as the practices of justifying or generalizing. For this report, we will focus on the practice of justification.

Justification

Justification is an essential practice in mathematics classrooms listed in both the Common Core State Standards for Mathematics (CCSS) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and the National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000). Our usage of mathematical practice will reflect the usage found in the CCSS where mathematical practices describe ways in which students engage in the discipline of mathematics. Justification is of particular importance as it provides "a means by which students enhance their understanding of mathematics and their proficiency at doing mathematics..." (Staples, Bartlo, Thanheiser, p. 447). Justification provides a means to *both* deepen understanding of various mathematical content and develop mathematical practices.

Despite its importance, defining justification is a perilous task. Our definition will most closely follow Stylianides (2007) definition of a proof:

- It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
- It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
- It is communicated with forms of expression (modes of argument representation) that are appropriate or known to, or within the conceptual reach of the classroom community. (p. 291).

Within the PD, we define *justifying* as:

Reasons with meaning of ideas, definitions, math properties, established generalizations to:

- *show why an idea/solution is true*
- *refute the validity of an idea*
- *give mathematical defense of an idea that was challenged*

As in Stylianides' definition of proof, justification derives its meaning from building on established facts in order to present a mathematical argument. However, in the PD, the emphasis switches from the product (a proof or justification) to the act of justifying. In this way, a student may be engaged in justifying even if his or her reasoning is incomplete or incorrect.

Previous work has shown that teachers and pre-service teachers may struggle to differentiate between justifications and non-justifications both when evaluating and creating their own justifications. Knuth (2002) found secondary teachers often evaluated non-proofs including empirical arguments as valid justifications for general cases. Pre-service teachers also often consider purely empirical arguments justifications for the general case (Stylianides & Stylianides, 2009; Martin & Harel, 1989). Simon & Blume (1996) illustrated that pre-service teachers provided alternate information when pressed to justify such as citing a rule for its efficiency or relying on procedures in place of a mathematical rationale. In light of these results, the noticing of student justification practices in classrooms is a challenging task. Lo and McCrory (2010), identified four factors necessary for teachers to promote justifying activity: (1) Knowing what counts as a valid justification for a given answer; (2) Familiarizing oneself with the struggles elementary school students may have; (3) Understanding how mathematical topics connect across operations and number systems; and (4) Knowing how to teach in a way that supports mathematical reasoning. (p. 150). These factors highlight some of the complexity involved for teachers to promote justifying activity. While the PD aims to address all four factors, we will be focusing on the first factor. Knowing what counts as a justification (and recognizing justifying activity) serve as an important aim when promoting justification.

Attending to justification practices provide an extra layer of complexity to the already complex situation of making sense of students' mathematical thinking. Noticing justification requires both attention to content and practice. That is, there must be mathematical content to be justified. However, mathematical content alone is insufficient; justification also requires an argument to provide a *why* for mathematical claims and decisions. For example, consider the task in Figure 1.

> [School Name Redacted]'s PTA provided 12 celery sticks, 17 carrot sticks, and 7 apple slices. If each student took 3 snack pieces, how many students would have a snack?

Figure 1: Task from Cycle 4

A non-justification for the solution of twelve students might be:

> "I added 12, 17, and 7. 12+17 is 29. 29+7 is 36. Then 36 divided by 9 is 12. So 12 students would get snacks.

While the student is explaining *how* they arrived at the answer, they are not providing any rationale for *why* this procedure produces the correct answer. There is mathematical content in this example, but no evidence of the practice of justification.

In contrast, a justification for the solution of nine students might be:

> "There are 12 of one snack, 17 of one snack and 7 of another. I could add all of the snacks together to find the total number of snacks since the types of snacks do not matter, which would be 36. Because division tells me how many groups of three are in 36, I could divide the 36 by 3. This would tell me how many groups of three snacks fit into 36 snacks. So 12students could each have three snacks."

In this example, the student used their knowledge of addition and justified why totaling the different snacks would be appropriate. They then justified their division decision by using the known definition and connecting to the context. By connecting to a known meaning, they not only used an operation, but provided a justification for it. The mathematical content and argumentation were both aspects of the excerpt.

The Setting and PD

The PD takes place at an elementary school in a mid-sized urban district that is engaged in a three year PD for third to fifth grade teachers. The PD uses a *Studio Model* where one teacher (the studio teacher) works with a consultant to plan math lessons and then opens their classroom to the other teachers while teaching this lesson. The remainder of the third to fifth grade teachers (resident teachers) help plan the lesson, observe the enactment, and then debrief the lesson. At this elementary school, the third to fifth grade teachers engage in a yearly summer course (3 days) and five studio cycles throughout the year. These cycles include two days of PD sessions. Day one consists of leadership coaching with the principal and planning with the studio teacher. Day two involves all third to fifth teachers and consists of working together to do mathematics related to the lesson, planning and enacting the studio lesson and then debriefing the lesson. For this study, we are going to focus on the first year of the PD with attention to the lesson debriefs.

The central focus of the PD is to get all students to habitually justify and generalize in order to make sense of mathematical problems and ideas. To this end a set of *Habits of Mind* (such as working through stuck-points), *Habits of Interaction* (such as critique and debate), and *Mathematically Productive Teaching Routines* are the focus of the PD throughout the first year (Foreman, 2013).

Resident teachers' attention is focused on student discourse, in particular on discourse related to justifying and generalizing during the studio lesson. The teachers are provided with observation tools to help focus their attention (see Figure 2). The teachers are encouraged to write down discourse they observe during the lesson within the categories of: *procedures/facts, justifying,* and *generalizing.* After the lesson is enacted, the teachers engage in discussion around the various discourse they noticed and characterize them with respect to the three categories provided. In the discussion they are asked to justify their categorizations. Using this tool helps teachers focus/attend to children's discourse rather than other aspects of the classroom.

Teacher ______________________________ Grade/Class________________ Date______________ Page_____

STUDENT DISCOURSE OBSERVATION TOOL

PF USING PROCEDURES/FACTS	**J** JUSTIFYING	**G** GENERALIZING
No evidence of reasoning • Short answer to a direct question • Restating facts/statements/rules • Showing or asking for procedures *Uses meanings, definitions, properties, known math ideas to describe reasoning when:* • Explaining ideas & methods • Questioning to clarify • Noticing relationships/connections But ***doesn't show why*** the ideas/methods work	*Reasons with meanings of ideas, definitions, math properties, established generalizations to:* • Show why an idea/solution is true • Refute the validity of an idea • Give mathematical defense for an idea that was challenged	*Reasons with math properties, definitions, meanings of ideas, established generalizations, and mathematical relationships as the basis for:* • Making conjectures about what might happen in the general or special cases or • Justifying a conjecture about what will happen in the general or special cases

Discourse Type	Student Mathematical Discourse * Indicates student thinking that I am especially curious about	Co-Inquiry Questions

Figure 2: Student Discourse Observation Tool. Reprinted from *Best Practices in Teaching Mathematics: How Math Teaching Matters* (p. 41), by L.C. Foreman, 213, West Linn, OR: Teachers Development Group. Copyright 2013 by Teachers Development Group. Reprinted with permission.

Methods

A member of the research team was present at each of the PD sessions and took detailed field notes. In addition, all PD sessions were video recorded. The field notes from year 1, provided the starting ground for identifying instances when teachers discussed student discourse and how it pertained to justifying, generalizing and procedures/facts. These episodes were then transcribed for analysis. Transcripts were analyzed for any instance of noticing during discussions of characterizing discourse. Each instance (measured as a turn in conversation) was then considered in light of whether the teacher noticed (1) the mathematical content, (2) the character of the discourse (justifying, generalizing, or procedures/facts), and (3) what evidence was provided for their interpretations and descriptions. The analysis focused on describing and interpreting aspects of noticing (rather than responding to student thinking), as the PD focused first on these aspects during year 1.

Results and Discussion

Each of the following excerpts comes from a teacher response when asked to share a piece of discourse from their discourse observation tool after the lesson. Teachers were prompted to characterize the discourse in terms of justifying, generalizing and procedure/facts. We begin each section by sharing the teacher-selected piece of discourse and then analyzing it.

Teacher Noticing 1 (Cycle 1)

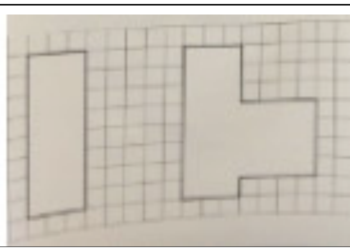
> Teacher: They [a pair of students] were very much engaged with each other and happily exchanging their different ways of looking at the problem.

No mathematical content or practices. This quote exemplifies noticing that is general and does not provide evidence of student discourse or any interpretation of mathematical content or practices. This teacher was commenting on level of engagement without evidence. She did note that they had different ways of looking at the problem, but without evidence, this statement does not reflect noticing of the mathematics in terms of content or practice. Because of the PD's structure (and the observation tool's focus), teachers were nearly always focused on students rather than teachers. However, this focus often swayed to their noticing of student affect rather than mathematical discourse.

Teacher Noticing 2 (Cycle 1)

> Teacher: I still think he's doing that justifying, defending the idea about that whole corner thing. I heard such awesome things, like, "How?," "I know because…," and then he's explaining, and then "I disagree, how did you get this number?" "Well, because you said this, this, this…did you know it was going to be this number" "Look. You said…" and he showed him on the paper!"

Noticing mathematical practice without content. The above noticing was of two students engaged in discussion as to whether the corners of the perimeters in Figure 3 should count once or twice in the total. The student used toothpicks to illustrate that a corner contributes two sides to the perimeter total. In this case, the teacher has noticed the practice of justification, but the evidence provided does not connect to the mathematical content. This noticing might be considered *keyword* justification noticing. While the teacher appeared to notice that a mathematical argument was being made, she did not report any student discourse that was mathematically focused. Noticing justification in this manner might reflect any number of justification conceptions and does not directly connect back to the definition of justifying being utilized in the PD. With the content of what follows, "I know because," an interpretation of justification lacks warrant. The "because" could be followed by a justification or it could be followed by a procedural explanation. Prior to attending to student mathematically thinking, it is unlikely a teacher could recognize justification beyond what might be a superficial interpretation.



What 2 observations are you going to share with your partner and why does it make sense?

Figure 3: Task from Cycle 1

Teacher Noticing 3 (Cycle 2)

> Teacher: But, then [student 1]– I really thought it was great when [student 1] on her own –cause we were just flipping it [a triangle] [motions upside down] you know what the kids would think

as right side up because we gave it to them “upside down” [air quotes] so he [student 2] thought it if you flip it, it’s right side up so when she [student 1] turned it lengthwise, “well does that look like it to you?”

Noticing mathematical content without practice. In this excerpt, a teacher is noticing the mathematical content. Students were asked to identify if various shapes were triangles. Many students felt that whether or not a shape was a triangle related to orientation (see Figure 4). In this exchange, one student recognized and rotated a triangle to provide an argument that the top was in fact a triangle. The teacher noticed this exchange including interpreting that the students were focused on orientation. In this excerpt, some *evaluative* comments were made such as, “I really thought this was great.” Despite the prompt to characterize this discourse, this teacher did not consider whether there was justifying, generalizing or using facts/procedures. This was a fairly typical response type within a subset of teachers. It is unsurprising that a teacher’s focus might be solely on making sense of the students’ mathematical strategies, especially if attending to students’ mathematical thinking is a shift from their typical teaching. Interpreting the nature of the discourse and making sense of the mathematical understandings requires attending to two different (though interrelated) facets of a complex situation.

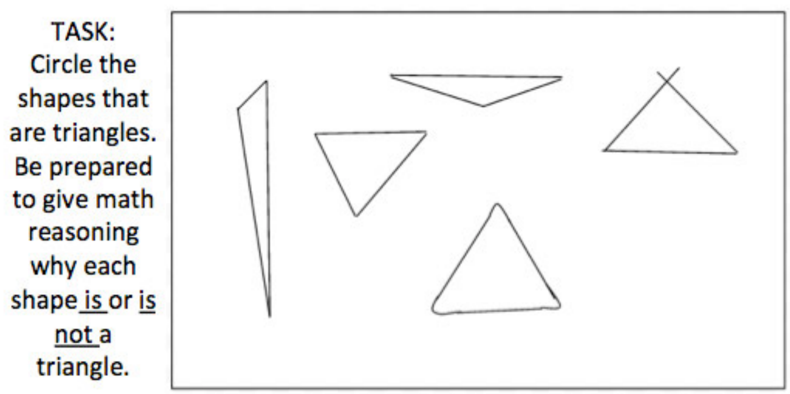


Figure 4: Task from Cycle 2

Teacher Noticing 4 (Cycle 4)

Teacher: [Student 1] was arguing with girl next to him [student 2] that she was wrong because she had the two and one remainders and the numbers added up to 11 even though [student 2] had 12 up there. They were focused on the remainders. He had 12 and knew he was right. When he talked to the whole group, it was when he said, “2 remainders plus 1 remainders equals 3” and he goes, “three divided by three equals one more group.” As soon as he said it out loud, you could see the light bulb flash and he was smiling and he told the girl next to her she’s right. The other girls said [inaudible]. He was having the whole conversation justifying it to himself.

Noticing mathematical content and practice. In this excerpt, the teacher is describing a debate between students based on the remainder from the prompt in Figure 1[1].The teacher in this case provided specific evidence of the mathematical discourse that included both content and interpreting the character of the discourse. While the connection between the mathematical content and characterizing of practice was tenuous, this excerpt represents one of the only cases of a teacher identifying a justification with concrete evidence. This might reflect both the complexity involved in attending to both aspects, but also the fact that this data is from the first year of the PD implementation. At this point, the teachers are in the beginning phases of attending to student reasoning in a meaningful way.

A Framework for Noticing Mathematical Content and Practices

We summarize the four different ways teachers notice justification in Table 1.

Table 1: Framework for Noticing Mathematical Content and Practices

	Does not notice mathematical practices	**Notices mathematical practices**
Does not notice mathematical content and strategies	Statements are general in nature and contain no evidence of students' mathematics.	Statements may provide evidence of students engaged in a practice but no mathematics content is included.
Notices mathematical content and strategies	Statements include evidence of students' mathematical strategies and content-specific interpretations.	Statements include evidence and interpretation of both content and practices.

Attending to and promoting student justification may require a high level skill-set including both a knowledge of the nature of mathematical justification paired with being able to notice and make sense of students' mathematical thinking and strategies around content. The characterizing student discourse tool provides both a tool for writing down discourse (which focuses teachers on students), and provides a stable definition for the discourse types of: *generalizing, justifying* and *using procedures/facts.* As teachers learn to notice their justifications, their conceptions of justifications should continue to develop. At the same time, characterizing discourse (with evidence) necessarily requires understanding of students' mathematical thinking around given content. In this way, characterizing discourse is a way to promote attending to students' mathematical thinking and making sense of their reasoning.

Our analysis of Year 1 data, provided insight into the current state of teacher noticing of practices such as justification and generalizing. Noticing justification is an incredibly complex skill requiring both a deep understanding of elementary mathematics content and understanding of justification as a mathematical practice. The teachers in our study frequently attended to one or the other, but rarely created robust interpretations of student discourse that addressed both the character of the discourse and the mathematical content understandings. In fact, there were no examples of attending to both during the thirty minute debrief discussions in the first three cycles.

Based on this analysis, we were able to illustrate examples of what teachers notice when told to focus on characterizing discourse in terms of justifying, generalizing, and procedures/facts. Prior to extensive professional development, teachers were not able to characterize discourse using evidence. Further, we argue that noticing practices is a difficult and complex task due to the requirement to notice not just mathematical practices cues, but also notice mathematical content. Finally, we have introduced a framework that has helped to organize and make sense of the way teachers were noticing to student discourse. We conjecture that teachers will continue to shift from noticing only certain facets of student discourse to an integrated view of mathematical content and practice through Year 2 and Year 3 of the professional development.

Endnote

[1]The division problem could be solved in one of two ways. The snacks could be first summed to arrive at the total number of snacks, and then divided by three. Alternately, each snack type could be divided by three first, leaving remainders of two carrot sticks and one apple slice.

Acknowledgement

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THE UNINTENDED CONSEQUENCES OF A LEARNING TRAJECTORIES APPROACH

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This case study, which is a part of a larger design study, examines teachers' uses of learning trajectories (LTs) in diverse classrooms. Specifically data were collected and analyzed for evidence of equitable use of LTs and learning trajectory-based instruction (LTBI). Qualitative analysis revealed that the use of LTBI did not guarantee equitable instructional practices. Moreover, a number of deficit-oriented themes emerged in one case.

Keywords: Learning Trajectories; Equity and Diversity; Teacher Beliefs; Teacher Education-Inservice

Introduction

Research has demonstrated that professional development (PD) that focuses on students' mathematical thinking can lead to changes in teachers' practice (Kazemi & Franke, 2004; Sowder, 2007). Learning trajectories (LTs) are gaining prominence as a representation of students' mathematical thinking, and are thus being used in professional development projects nationwide. While initial results of LT-focused professional development projects are positive, these results have not yet been teased apart to investigate if these benefits are present for traditionally underserved students.

In this paper, I present findings from a case study of teachers that participated in a yearlong professional development project focused on LTs. I begin this report with a brief discussion about learning trajectories. I focus this discussion on how research on LTs has developed over time, specifically in relation to their use in classroom instruction. I follow with a discussion of the theoretical framework that guided this study. I describe the context of the study (location and professional development) and present findings from one case that was particularly interesting and problematic. I conclude this report by discussing implications of the findings.

Background

Over the past two decades, a number of scholars have written about and argued for the use of LTs in mathematics and science education (Battista, 2004; Brown, Clements & Sarama, 2007; Clements & Sarama, 2008; Clements, Wilson & Sarama, 2004; Clements & Sarama, 2008; Confrey et al, 2009; Duncan, Rogat & Yarden, 2009). While the terminology differs (e.g., learning progressions vs. learning trajectories), many of these scholars agree that LTs have the potential to transform the teaching-learning process. LTs have shown promise because they: simultaneously attend to specific skills as well as broader concepts; they attend to how student thinking develops over time (NRC, 2007); they are empirically developed from work with students; and they attend to probable pathways students may take as well as common misconceptions in their mathematical development (Confrey, 2006).

Research on LTs has occurred along three primary fronts: a) developing and validating LTs in different mathematical strands (Battista, 2006; Clements & Sarama, 2007; Maloney & Confrey, 2010); b) using LTs to design curriculum and assessment (NRC, 2007); and c) exploring the ways teachers use LTs in instruction (Bardsley, 2006; Edgington, 2012; Mojica, 2010; Wilson, 2009). Regarding the latter, previous research on LTs has shown promise. Using knowledge of student thinking as represented by LTs, teachers have: a) set goals based on students' developmental level (Clements, 2007); b) described student work with greater detail (Wilson, 2009); c) assessed students more effectively (McKool, 2009); and d) anticipated students' strategies as well as misconceptions (Edgington, 2012). What remains unexamined is whether or not *all* students benefit from the promise of LTs as well as how teachers use their knowledge of LTs with traditionally underserved students. Specifically, are LTs used equitably?

Theoretical Framework

To examine whether or not LTs were used equitably in the classroom, I framed this study using Gutierrez (2007) equity framework. In this and other work, she articulates four dimensions of equity: access, achievement, identity, and power. Because Gutiérrez' initial framework was developed from her work with high school math departments and school districts, my first step in this work was to re-conceptualize this framework at the classroom level. Additionally, I conjectured ways in which teachers *might* use LTs to promote access, identity, achievement, and power. Table 1 represents a conjecture of the ways in which teachers might use LTs in to promote equitable instruction (Myers, Sztajn, Wilson & Edgington, in review).

Table 1. Conceptual Framework for LTBI and Equity (E-LTBI Framework)

	Teachers use their knowledge of LTs and LTBI to:
Access	Design instruction and instructional tasks such that they are accessible for all students.
	Identify and use up-to-date research based materials and technology
	Be accessible to and attend to all students in the class.
	Foster classroom discussions such that all students can participate and engage.
	Provide all students with opportunities to engage in rigorous mathematics.
Achievement	Set high, yet appropriate, academic standards for all students.
	Unpack and build upon their students' prior mathematical knowledge and use it as a basis for understanding more meaningful and complex mathematics.
	Select and use a variety of forms of assessment (e.g., formative, summative, projects, class discussions) to gauge student achievement.
Identity	Support the development of a robust mathematical identity
	Listen to and consider students out-of-school experiences and design instructional activities that incorporate elements from their homes and communities.
	Validate the use of students' own algorithms and strategies to solve problems.
	Assist students to build connections between the mathematics they learn and the broader world/society.
	Encourage students to engage in mathematical tasks according to their preferences and participate in mathematical discourse in ways that are comfortable for them.
Power	Ensure that students have voice in the classroom.
	Position students as experts in the classroom (this includes things they know in school and things they know from outside of school).
	Allow students to solve problems that are relevant to them (these problems can exist inside or outside of school).
	Encourage all students to present, justify, and defend their mathematical ideas/arguments.
	Help students to see themselves as sources of mathematical knowledge.

Methods

The Learning Trajectories Based Instruction (LTBI) project is a multi-year design project that examined teacher learning of LTs and how LTs were used in instruction (Sztajn et al., 2012). The participants in this case study represented a subset of teachers in a larger design study. The research question under investigation for the case study was: In what ways do teachers use LTBI to promote access, achievement, identity, and power in their instruction? In this report however, I address a smaller research question: *How do deficit perspectives manifest in the context of an LT-focused professional development project?*

Context

Learning trajectory-based instruction is defined as instruction that situates LTs at the center of the teaching practice (Sztajn et al., 2012). The goals of the larger LTBI project were to (1) explore the impact of LTs on elementary teachers mathematics instruction; (2) build a conceptual model of instruction that is centered on LTs; and (3) confront teachers' stereotypes of students (van Langenhove & Harré, 1999) by focusing on what students can do (as seen in LTs) and building upon it through supportive classroom practices (instead of focusing on what they cannot do).

The professional development began with a 30-hour summer institute in which participants learned about Clements and Sarama's (2009) LTs for early number and counting, addition and subtraction problem types, Smith and Stein's (2008) five practices for mathematical discourse, as well as formative assessment. Because the content of the LT was dense, the LTBI research team organized multiple LTs from Clements and Sarama's work into what we called "Learner Profiles"(Myers, Sztajn, Wilson & Edgington, in review). These profiles—named perceptual child, direct modeling child, counting on child, place value child and multi digit child—provided a way for teachers to "chunk" information about students and created a broader perspective. Teachers participated in an additional 30 hours of PD throughout the school year where they engaged in professional learning tasks related to both the content of the LT and its use in instruction. These tasks included watching clinical interviews, analyzing samples of students' work, curriculum evaluation, and task design. Participants also conducted a number of activities with their own students, which served as the basis for project discussions.

While there were a number of initial conjectures from the project related to teacher learning, discourse, and positioning/stereotypes, three of these conjectures were particularly relevant to this case study. First, the LTBI team conjectured that the learner profiles would be a more manageable grain-size for teachers and offer them a productive way to talk about "where" their students were mathematically. Second, the LTBI team conjectured that LTBI could support teachers' focus on individual students and help teachers design instruction that meets their needs based on their current conceptions. Finally, the LTBI team conjectured that teachers would reduce their use of the words "low" and "high" to describe their students and begin describing their mathematical work using language from the LT.

The partner school for this study was a small elementary school in a suburban district in the southeastern United States. The enrollment was 370 students and the five-acre campus was situated in the heart of a historic district. Demographic data indicated that 36.2% of students are considered economically disadvantaged and 27.1% of students had limited English proficiency.

Participants

Seven teachers participated in the larger design study described above. Because the content of the learning trajectory focused on early number knowledge (including counting), four K-1 case teachers were selected for additional research. Case teachers participated in four interviews and three classroom observation cycles during a four-month period. At the start of this case study, the teachers had completed 40 hours of the professional development. The remaining 20 hours of PD occurred concurrently with the case study.

For the purpose of this paper, I focus on one case “Elizabeth.” At the time of the study, Elizabeth had 15 years of teaching experience. She was in her fourth year teaching kindergarten and had been teaching at the partner school for nine years. Elizabeth was selected as the focal case for this paper because she demonstrated a deficit orientation throughout the PD project.

Data Sources and Analysis

Data sources for this case study consisted of student mathematical portraits, interviews, classroom observations, and field notes. The mathematical portraits consisted of questions related to how teachers use LTs to: solicit evidence of students’ mathematical understandings, set goals for students, and create mathematical opportunities for students. These portraits were created so that the researcher could examine the ways in which teachers talked about individual students as well as groups of students. The mathematical portraits were revisited at each interview during the course of the study. Interview protocols addressed each dimension of the E-LTBI framework and additional questions were added as themes emerged during the study. An observation protocol was also developed to address the E-LTBI framework as well as other aspects of equitable instruction (Author, 2014).

Data analysis occurred in two phases: ongoing and retrospective. The goal of on-going analysis was to understand emergent themes as they related to the E-LTBI framework. All interviews and field notes were reviewed immediately and notes were made regarding preliminary themes. Both pre-determined and open coding was used during ongoing analysis. The pre-determined codes consisted of the four dimensions of the E-LTBI framework (access, achievement, identity, and power) as well as the LTBI instructional practices (task/learning goal, anticipate, monitor, select/sequence, and connect). While pre-determined codes resulted from the research questions and the LTBI instructional model, open coding allowed for the emergence of new themes. The following codes were developed during open coding: curriculum, high vs. low students, comparing, literacy, usefulness of LTs, using “high” students as exemplars, deficit orientations, and motivation. During retrospective analysis both within-case and cross-case analysis were conducted (Merriam, 1998).

Results

While a number of themes emerged during the data analysis, one particular manifested throughout Elizabeth’s case. At different periods during the study, Elizabeth displayed a deficit perspective towards certain students in her classroom. In some cases, she actually used language from the trajectory to justify her beliefs. Here, I present a number of examples that highlight Elizabeth’s deficit orientations as well as how she failed to use LTBI equitably.

Tracking

In my opening interview with Elizabeth, I asked her to talk to me about where each of her students was currently working in the learning trajectory. Elizabeth chose to group her students into four groups. She labeled these groups, “…lower group, next up from low, not quite counting on, and higher group.” Later in the study, I again asked Elizabeth to talk to me about her students and where they were working in the trajectory. Although the LTBI team conjectured that teachers would describe groups of students using mathematical evidence from the LT, this was not the case with Elizabeth. Throughout the study, she continued to use language of “low” and “high” and only described her “high” students with LT language.

Not only were these labels of “low” and “high” problematic, but also the demographics of the students that fell into these two groups were alarming. Of Elizabeth’s 17 students, eight of them were in the two “lower” groups and the remaining nine students were in the “higher” groups. Of the eight students in the lower groups, six were Hispanic, one African-American, and the other was White. Of the nine students in the “higher” groups, seven were White, one African American and one Hispanic. Throughout the course of the study, the only student that moved out of the lower track (according to Elizabeth’s assessment) was the one White student. As you will see in the coming examples, Elizabeth used LTBI to provide opportunities for her “higher” students while her “lower” students were denied the benefits of LT-based instruction.

This particular finding is problematic because one conjecture from the project was that teachers could use knowledge of LTBI to focus on individual students and design instruction to move them forward. At the conclusion of the study, Elizabeth indicated that her students in the higher group continued to achieve, while those in the lower group did not.

Access

One aspect of the LTBI PD was a focus on creating high demand tasks with multiple entry points. Teachers had opportunities to examine a variety of tasks and modify them such that students at multiple levels of the LT could access the task. When I asked Elizabeth to explain how she ensures that her tasks are accessible by students at various LT levels, she indicated that the students at the lowest level needed step-by-step instructions. Specifically, she stated:

> You take it step-by-step and for each step you plan for them not understanding it possibly. And for the ones it’s not accessible because they just don’t comprehend it…[make] sure through the activity progressively each step is accomplished and comprehended and then move on to the next step. Check for comprehension at the lowest level and once that’s clear, move to the next step.

In this example, Elizabeth indicated that the learning trajectory could be used as a rigid checklist to monitor students’ progress. In contrast to our discussions about using LTs develop open tasks that were accessible for students working at different profile levels; Elizabeth indicated that students at the lowest level needed more procedural tasks.

Another important component of access was that teachers would use LTs to design rigorous tasks for students at all LT levels. When I asked Elizabeth about how she used the LT to design these opportunities, she stated:

> I remember. . .. we do set appropriate academic standards, and I think since our last meeting when I mentioned that we thought that the idea of them counting to 100 and counting to 100 by tens would be pretty much asking too much but we did it every day and now a good majority of them can. Not successful for I’d say 50% of the class but we work it into. . . I work things into my daily routines so they got daily practice, and those that could achieve it at least have the opportunity to try and the opportunity to practice.

In this example, Elizabeth indicated that rigorous goals were appropriate for those that “could achieve.” Throughout the study, Elizabeth offered some students opportunities to be challenged mathematically, while those that were “lower on the trajectory” or “floundering” were given rote tasks.

Achievement

A focus of the achievement dimension is that teachers could use LTBI to set short- and long-term goals for individual students, assess their progress, and design instruction that builds upon students’ individual conceptions as identified in LTs. During a pre-lesson observation, I asked Elizabeth to think about the skills her students had demonstrated in class and how she would build upon students’ conceptions. Elizabeth responded:

> They have learned to count to ten, which is great because they didn't do that before. I'm trying to think. They don't have a number sense and…I guess it's hard to say what they do have, honestly, I guess because I worry so much about what they don't have. But they have gained since the very beginning of the year and just being able to count to ten, and even to twenty. Not completely accurately but they know if they put one number after the other, they're seeing things in order. I guess I should be more positive but really they're . . .. I don't know how to express it in a positive way. I worry that they don't grasp just the whole idea of numbers and amount, and why it's important.

This example highlights tension Elizabeth felt. While she was able to identify some progress that her students were making, her primary focus was what students did not know. She focused more on skills that the students did not demonstrate and she did not use the LT did not provide her with the agency to develop appropriate instructional plans.

Identity

In the identity dimension, teachers use their knowledge of LTs to solicit and validate various approaches to tasks. Another element of this dimension is that teachers help students build connections between in-school mathematics and out-of-school mathematics. When I asked Elizabeth how she could facilitate this in her classroom, she stated:

> I don't think they have the ability to really grasp say, say I think maybe fourth, fifth grade demography and geography. That might be something a little bit more of a mathematical connection but for their level and I'll introduce grander ideas or bigger numbers and they kind of get it but I think my kids who are challenged with second language and also just the beginning of number sense, they can't get it.

In this statement, Elizabeth indicates that English language learners or students developing number sense could not "get it." This statement also highlights that Elizabeth was unwilling to think of out-of-school mathematics in relation to her students lived experiences. Rather, her example of out-of-school mathematics was demography or geography.

Power

A key component of this dimension is that teachers see the mathematical potential of all of their students, allow them to take ownership of their ideas, and position them as experts in the classroom. As we were discussing the concepts of positioning students as experts, I asked Elizabeth to recall and example she previously shared with me of a student who counted and tapped her chin as she counted to keep track of the number. I suggested to Elizabeth that this could be an example where she could allow this student to share her counting method with the class. Elizabeth responded:

> Usually they can't express it because they don't have . . . They don't know what they're doing. They've just seen adults do it and they think that's counting…I assume some type of counting method that is used in that country, but I usually would ask them about it and they could continue doing that if it helped them . . .. But I wouldn't encourage the rest of the class to start doing it too. I just said this is the way we're going to do it here and if that's the way you do it, you can do that, that's fine.

I asked Elizabeth if she could think of an example when she would position students as experts and allow them to take ownership of ideas in the classroom. She provided the following example:

> We just had a center – a subtraction center – that was manned by Paul, Natalie and Jared. Natalie had her first grade workbooks and she asked me if she could have a center during math time that

if people were done they would be able to visit the math center and practice subtraction because she knew how to do it and she wanted to teach them how to do it.

Later in this conversation I asked Elizabeth to think of other students that she could position as experts or different types of expertise that existed in her classroom. She shared another example about Paul. These examples demonstrate Elizabeth's belief that only some students can be seen as experts in the classroom. The student referenced in the first example was one of Elizabeth's "lower" students. Paul, Natalie, and Jared on the other hand were her top three students. Throughout the study, I asked Elizabeth to share examples of students taking ownership of ideas in the classroom or demonstrating expertise. For Elizabeth, only her "high" students were referenced. These examples indicate that the trajectory was not enough to help Elizabeth see all students as possessors or knowledge therefore allowing all students to be experts.

Discussion

My goal for this paper was to present an important "unintended consequence" that emerged as a result of a professional development project. While other teachers in the case study exhibited growth/progress/demonstrated change, Elizabeth did not. Using the LT as a tool to focus on the students' individual thinking did not disrupt Elizabeth's beliefs about who could and could not do mathematics. While Elizabeth did demonstrate findings from other studies of LTs (e.g., using LTs to set goals for students), these findings did not cut across all subgroups of students in her class.

Previous work has shown promise that when content-focused PD and issues of culture, privilege and power are fully integrated; teachers can begin to acknowledge the contributions of all students and deficit orientations can be dispelled (Battey & Chan, 2010). Elizabeth's case presents a critical challenge for the field. When teachers have strong deficit perspectives, those beliefs will carry over to their professional development experiences. Therefore, although teachers may engage in new and innovative pedagogies, not all students will benefit from this new professional knowledge. This case highlights the fact that although Elizabeth was able to use elements of the LTBI in her instruction, they were not implemented equitably. Therefore, to avoid unintended consequences of content-only focused PD, teacher educators must consider simultaneously addressing issues of culture, power, and privilege.

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TEACHERS' UNDERSTANDING OF RATIOS AND THEIR CONNECTIONS TO FRACTIONS

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In this study, we considered how middle school teachers understood the relationship between fractions and ratios. We used two instruments to collect data from 11 teachers and relied on Knowledge in Pieces as a lens for considering understandings teachers have and how coherent those understandings are. From our analysis, we developed three main findings: participants did not have a single definition for ratios; they used specific vocabulary when discussing ratios; and their language evoked additive strategies rather than multiplicative relationships. Further, we concluded that they each had a number of knowledge resources, but that those resources may not yet be well-connected to each other. This has implications for professional development.

Keywords: Teacher Knowledge; Middle School Education; Rational Numbers

Purpose & Background

In middle school mathematics, teachers are asked to teach an array of concepts for which they may have only limited understanding. One such area, proportional reasoning, has increased in prominence and emphasis by being considered its own content domain in the Common Core State Standards for Mathematics (National Governors Association & Council of Chief State School Officers, 2010). Despite the importance and richness of the proportional reasoning domain there has been a disproportionate focus on it in research (Lamon, 2007). The limited research available on teacher knowledge of proportions indicates that like students, teachers struggle with proportional reasoning (e.g., Akar, 2010; Harel & Behr, 1995; Orrill & Brown, 2012; Orrill & Kittleson, in press; Orrill, Izsák, Cohen, Templin & Lobato 2010; Post, Harel, Behr, & Lesh, 1988; Riley, 2010). Strikingly, in one study, Post, Harel, Behr, & Lesh (1988) found that their sample of teachers in grades 4-6 were unable to correctly respond to ratio and proportion items developed for students in those grades. In fact, on the ratio items, the 167 respondents answered less than 50% of the items correctly.

Lamon (2007) explained that proportional reasoning is one of "the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites" (p. 629). Despite this there is little research on teachers' understandings of proportional reasoning (Ben-Chaim, Keret, & Ilany, 2007; Lamon, 2007; Lobato, Orrill, Druken, & Jacobsom, 2011). Existing research suggests that proportional reasoning is conceptually difficult for teachers. This is, in part, because it is possible to rely on rote algorithms such as cross multiplication to get correct answers while overlooking the multiplicative nature of the relationship (Berk, Taber, Gorowara & Poetzl, 2009; Lobato et al., 2011; Modestou & Gagatsis, 2010; Orrill & Burke, 2013). Researchers have also suggested that teachers hold naïve conceptions about proportions (Canada, Gilbert, & Adolphson, 2008; Lobato et al., 2011). For instance, Canada, Gilbert, and Adolphson (2008) found that in a sample of 75 pre-service teachers only 28 were able to reasonably interpret a unit rate (e.g., amount per dollar) as useful for determining which package was a better buy when comparing two different size packages of ice cream.

Past research indicates an important link between the amount of knowledge a teacher demonstrates and its organization (Bédard & Chi, 1992; Ma, 1999; Orrill & Shaffer, 2012). For instance, Orrill and Shaffer (2012) found that the least expert teacher in their study demonstrated many ideas about ratios and fractions that were not interconnected while the most expert teacher introduced many ideas that co-occurred more frequently, suggesting stronger connections between them. We hypothesize that these stronger connections are an indicator of greater coherence. This finding is consistent with research in cognitive psychology that suggests expertise requires both an accumulation of knowledge and organization of that knowledge (Bédard & Chi, 1992). It is also consistent with seminal work in mathematics education such as Ma's study (1999) that showed that teachers with more connections between their mathematical knowledge resources were better able to interpret a variety of mathematical situations. As highlighted by Thompson, Carlson, and Silverman (2007), teachers with incoherent understanding can only teach disconnected facts. In contrast, a teacher with coherent understanding has the potential to support students in developing coherent understandings. Thus, coherence is a salient aspect of teacher knowledge (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kaasila, Pehkonen, & Hellinen, 2010; Ma, 1999).

Two concepts that are important to proportional reasoning are fractions and ratios. Past research has found that the relationship between these two important concepts is not always clear (Clark, Berenson & Cavey, 2003; Sowder, Philipp, Armstrong, & Schappelle, 1998). This may, in part, be because of the organization of textbooks that frequently provide limited guidance on the definition of ratios and fractions and that deal with multiplicative structures in discrete unconnected ways, such that topics like the relationship between ratios and fractions are not shown or discussed (Clark et al., 2003; Sowder et al., 1998). These issues suggest that it is entirely plausible that teachers hold multiple knowledge resources about the relationship between fractions and ratios that are not coherently organized.

This study contributes to the growing knowledge base focused on teacher understanding by considering one aspect of proportional reasoning: relationships between fractions and ratios (e.g., Lobato & Ellis, 2010). Specifically, we considered the following questions: (1) how do 11 middle school teachers understand the relationship between ratios and fractions; and (2) how coherent are their understandings?

Theoretical Framework

Coherent & Robust Understandings

A coherent and robust understanding of ratios for middle school teachers must go beyond that of their students (Clark et al., 2003; Lobato & Ellis, 2010). Teachers need to understand that a ratio is a comparison of two quantities, where quantity is defined as "a measurable quality of an object-whether that quality is actually quantified or not" (Lamon, 2007, p. 630). A teachers' understanding of ratio should go beyond ways to express it, to include the understanding that a ratio is a multiplicative comparison and not an additive comparison (Lamon, 2007; Lobato & Ellis, 2010; Sowder et al., 1998). This is a critical understanding as the concept of ratio is considered crucial for the transition from additive to multiplicative reasoning (Sowder et al., 1998). Teachers need to be able to discern whether students are using additive or multiplicative reasoning (Sowder et al., 1998).

It is also important for teachers to understand the relationship between ratios and fractions. A fraction is more than simply a part-whole relationship. Fractions can be interpreted as a part-whole comparison, measure, operator, quotient, and ratio (Lamon, 2007). A common notion that students have is that all ratios are fractions – which is a limited conception given that ratios can be part-part relationships and given that a ratio is a comparison of two quantities, thus not a value that can be placed on a number line (Clark et al., 2003; Lamon, 2007; Lobato & Ellis, 2010). Teachers should both understand the relationship between fractions and ratios and have the ability to identify students' limited understandings to justify or refute them (Lobato& Ellis, 2010). And, teachers need to know that in many cases ratios can be meaningfully reinterpreted as fractions (Lobato & Ellis, 2010). For instance, in a salad dressing that is 2 parts of vinegar and 5 parts of oil, the ratio 2:5 expresses not only the part-part comparison, but also the multiplicative relationship—that there is 2/5 as much vinegar as oil.

Knowledge in Pieces

We rely on the Knowledge in Pieces theory (KiP; diSessa, 2006) for this study. KiP asserts that individuals hold understandings of various grain sizes that are used as knowledge resources in a given situation (Orrill & Burke; 2013). For novices, these knowledge resources are not well-connected to each other. As expertise develops, interconnections allow more knowledge resources to be invoked in appropriate situations. KiP offers a unique lens for exploring the development of expertise, which is dependent on the extent of the *coherency of knowledge* (Orrill & Burke; 2013). By coherency of knowledge we refer to multiple knowledge resources that are connected in robust ways allowing for *in situ* access to the resources. Coherence, combined with a robust set of knowledge resources, allows teachers to deal with complex situations in more efficient ways. This is consistent with previous research on expertise (e.g., Bédard & Chi, 1992), and Ma's (1999) concept of profound understandings of mathematics. We hypothesize that as a teacher develops coherence among knowledge resources, the teacher will be more fluent at teaching and doing mathematics.

KiP represents a departure from the deficiency model traditionally used in the study of teachers' knowledge. Much prior research has focused on what knowledge teachers do not "have" and the misconceptions that they do display. In contrast, KiP assumes that teachers have a wide variety of knowledge resources available to them, but that those resources may not be well connected. KiP also allows for identification of additional resources that could be important for a teacher to develop.

Methods & Data Sources

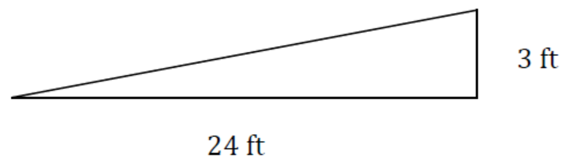
The participants were 11 middle school teachers (6 females) ranging from 1 to 18 years of experience from multiple schools within a single state. Data were collected from two interviews. One, the LiveScribe interview, was a paper-based protocol with think-aloud prompts that included 23 items. We mailed the interview protocol to the participants along with a LiveScribe pen, which recorded their spoken words as well as marks they made on their paper. The second source of data was from a 90-minute videotaped clinical interview including 18 items conducted with each participant. All recordings were transcribed verbatim to capture the knowledge resources evoked by the participants. We analyzed the data by focusing on the knowledge resources the teachers demonstrated on these tasks (not those resources they did not use). We used open coding (Corbin & Strauss, 2007) to identify codes for knowledge resources.

Interview Tasks

For the current study, we considered participants' responses to four tasks that focused specifically on the relationship of fractions and ratios. The Triangle task (Table 2), from the LiveScribe interview, explored teachers' understandings of the multiplicative relationship between the two sides of the triangle. All the other tasks were drawn from the clinical interview. Tasks 2 and 3 focused on situation related to salad dressing shown in Table 2. In Task 2, participants are asked to respond to one teacher's approach to making sense of the situation using an algorithm to find equivalent fractions. We then asked the teachers, "What does 2:5 mean as two-fifths? What is there

2/5 of in this situation?" Task 3 asked the participants to react to other teachers' responses to Task 2 as shown in Table 2. We considered only the second bullet point, "fractions and ratios are the same

Table 2: Interview Tasks

Task 1 Triangle Task
Some students in Mr. Warren's class have noticed that the ratio of 3 feet to 24 feet simplifies to 1 to 8. They also know that this ratio can be written as 1/8. However, they get confused about what the fraction 1/8 means in this situation.  a) What does the 1/8 mean in this situation? b) How would you explain that to your students?
Task 2? Oil & Vinegar Situation
Alexi made a batch of salad dressing using 2 tablespoons of vinegar and 5 tablespoons of oil. She would like to make a much larger batch that preserves the ratio of vinegar to oil. If she uses 15 tablespoons of oil, how much vinegar should she use?
Task 3 Teachers' responses to the oil & vinegar situation
• "I know that 6 is $\frac{2}{5}$ of 15. So I guess there's a two-fifth there." • "Fractions and ratios are the same thing." • "$\frac{2}{5}$ is a ratio here not a fraction. A fraction is a part-whole relationship like 2 T vinegar to 7 T of salad dressing, which is $\frac{2}{7}$ not $\frac{2}{5}$." • "I wonder if it has something to do with finding how much vinegar I would need for 1 part oil or how much oil for 1 part vinegar?"

thing" for this analysis. Finally, Task 4 asked each participant whether they believe fractions and ratios are the same. (Note: there are 4 participants who did not respond to Tasks 2 and 3 due to time constraints in the clinical interview).

Results

In our analysis, we found three main results related to our questions of how 11 middle school teachers understand the relationship between ratios and fractions; and how coherent those understandings are. First, the participants did not share a unified definition of ratios. Second, these participants used specific vocabulary to discuss ratios that differed from their fraction vocabulary. Finally, these participants relied on language that evoked a build-up strategy rather than language that suggests multiplicative relationships.

Multiple Definitions

Consistent with previous research (e.g., Clark et al., 2003), these participants seemed to draw from multiple knowledge resources in defining the relationship between ratios and fractions. The knowledge resources we saw among the 11 participants were comparison of two concepts, part-part and part-whole relationships, context as differentiating, and equivalence. For example, five participants focused on the similarity between the representations of fractions and ratios in talking about the relationship of the two concepts. For instance, Greg said in response to Task 3,

> Fractions and ratios are the same thing…I mean, a ratio can be written as a fraction, but again, you could write this as 2/7ths. … when you're thinking about it as a ratio it's important to define what you're comparing what the numerator and denominator are.

Mike and Alan also mentioned the idea of "comparing two things to each other" (Mike).

Nine participants relied on discussions of part-part and part-whole relationships. Bridgett and Alan both relied on the idea that ratios are part-part whereas fractions are part-whole, without clarifying whether ratios could be part-whole. For example, Bridgett explained, "When we first introduce ratios we say it's a part over a part and then we say for fractions it's a part over a whole." Allison, Ella, Mike, Larissa, and Greg added that ratios can be part-whole relationships. For instance, Allison explained that ratio is part-part but "sometimes it can be a part to whole relationship" as in the relationship of "the vinegar to the whole recipe" in Task 2.

A third set of knowledge resources considered context as differentiating ratios and fractions. David and Greg both discussed the need for using units (labels) with ratios. David asserted that fractions and ratios are the same except, "… with a fraction you don't need a unit. A ratio you should have some type of unit… you don't just put numbers." Larissa considered the need for context through word problems as a differentiating characteristic. She stated, "When I'm dealing with fractions I don't necessarily see it as a ratio unless it's in a word problem form."

Equivalence was the final knowledge resource on which participants drew. Greg, David and Ella explained that the 1/8 in Task 1 is a ratio rather than a fraction by referring to equivalent ratios. For instance, Greg explained that 1/8 is "the ratio between the two legs" and that we also could have "1-to-8, 2-to-16, 30-to-240, and those would still have the same ratio." Larissa, Ella, Greg and Allison relied on the use of similar triangles to demonstrate that all triangles similar to the given one have the same ratio as the original triangle.

Ratio Language

For these 11 participants, ratios evoked certain phrases. Most common among these was the phrase "for every", which was used by nine of the participants. For example, in describing the relationship in the Task 1 David said, "For every one foot on the short side of this triangle you have eight feet on the long side of this triangle." Similar language was used by six participants and two others used this language in Task 2.

We also noted that many participants used "*a* to *b*" when describing ratios versus "*a* out of *b*" or "*a*-*b*ths" language when describing fractions. For example, Ella justified her assertion that there is not a two-fifths in the oil and vinegar situation saying, "the two-fifths is not like two to five… like it is just fundamentally part to part. Pretty much all ratios are." Care in using differentiating language use was not consistent for all the participants.

Build-up Language

Our third main finding focused on the language selected by the participants. There was pervasive use of language that suggested additive reasoning. In particular, we found that10 of the 11 teachers used some variation of "for every" in their response. For example, when responding to part B of Task 1, Bridgette stated, "so every time you go up one you should go out eight points." This suggested a build-up strategy in that every time you add one to the short side, you add 8 to the long side. Another suggestion of additive reasoning came in statements of uncertainty about multiplication versus addition. For example, Allison noted, "for every one unit on one side the other side has eight times that unit. I almost said eight plus, but then that wouldn't work if it was eight plus so it has to be eight times that unit." This suggested a tie to addition for this participant. Only Autumn avoided use of this language in her responses.

Conclusions

We examined 11 teachers' understandings of the relationship between fractions and ratios and how coherent those understandings were. The lack of a dominant focus for ratios and fractions suggests that students may be hearing a number of different definitions from their teacher. This is consistent with previous research and could partially be attributed to a wide array of definitions presented in textbooks (Clark et al., 2003) as well as to a lack of a single definition of these concepts in the field (Lamon, 2007; Lobato & Ellis, 2010; Vergnaud, 1988). For our research, the use of these resources raise questions about the coherence of teachers' knowledge. Holding many definitions that do not seem well-connected could suggest knowledge structures that are not robust enough to support an array of student thinking. For instance, if these teachers tell students that ratios are part-part relationships whereas fractions are part-whole relationships students may infer that ratio cannot be part-whole. Teachers with coherent and accurate resources for the relationship of ratios and fractions may be better able to support students in developing coherent understanding of these concepts.

We saw that language and context both seem to be important in considering knowledge resources for ratios and fractions. Many of the participants relied on certain phrases when discussing ratios. The participants were not consistent with these phrases and some used them interchangeably, which obscures the coherence or lack thereof of the concepts. Thus, language and context seem critical for the development of coherency of knowledge in this domain.

We found that several important aspects of a coherent and robust understanding of ratios and fractions were not evoked by the teachers. For example, not all described ratios as a comparison of two quantities. Also only five participants were able to reason about the relative value of one quantity to the other and six participants were unable to reinterpret a ratio as a fraction in Task 1. These teachers seemed to have access to knowledge resources for fractions and ratios, but relied only on ratio understanding in some cases.

This study considers areas in which participants may lack coherence in their understandings. For example, part-whole discussions only happened in the context of the oil and vinegar situation. In contrast, build-up strategies, which are more elementary (Lamon, 2007), were found across the tasks. For a coherent understanding, we would expect to see strong connections between consistently used knowledge resources comprising a robust understanding of ratios.

Scholarly Significance

Teachers need robust understandings of mathematics to support students' learning (e.g., Baumert et al., 2010). However, little research has been done on teachers' understandings of proportional reasoning to uncover how they conceptualize the relationship between fractions and ratios. Knowing how teachers understand the mathematics they teach has practical implications for guiding the development of effective support opportunities for teachers.

The teachers in our study have access to a variety of knowledge resources for fractions and ratios, but they have not necessarily developed coherent connections between those resources. Returning to the idea that expertise refers to having more structured knowledge (Bédard & Chi, 1992), this work unveils some possible connections between knowledge resources that teachers rely on when differentiating between ratios and fractions. More research needs to be done to highlight the kind of knowledge and the organization of the knowledge needed for teaching ratios and fractions.

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SECONDARY MATHEMATICS METHODS COURSES: WHAT DO WE VALUE?

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Mathematics teacher education has been criticized, both internally and externally, for failing to identify shared practices and goals within teacher preparation programs. Work has begun to address this criticism at the elementary level but less exists at the secondary level. This paper reports on a national survey with responses from 116 secondary mathematics methods course instructors from colleges and universities. The purpose of the survey was to identify those topics, or "touchstones," in secondary methods courses that are widely valued. The survey asked participants to rank 41 potential "touchstones" of secondary mathematics methods courses on a scale from one to five according to those touchstones they value most in their methods courses. The results were quantitatively and qualitatively analyzed looking for important characteristics that would spur discussion about shared goals in secondary teacher preparation.

Keywords: Teacher Education-Preservice; Mathematical Knowledge for Teaching

Teacher preparation is striving for continual improvement, motivated both internally and by critiques from external entities. Within the teacher preparation community, scholars and practitioners consistently press for self-improvement through adherence to guiding principles (Grossman, Hammerness, & McDonald, 2009), attention to the needs of schools and communities (Darling-Hammond, 2006), and emphasis on evidence-based practices as shared through venues such as the *Mathematics Teacher Educator* journal. External groups, such as some economists (e.g., Harris & Sass, 2011) or the National Council on Teacher Quality, have also called for reform, citing the difficult-to-detect effects of teacher preparation programs on beginning teacher performance. There are counterarguments, however, to the external critiques (e.g., Heller, Segall, & Drake, 2013). For example, it is unwise to condemn university-based teacher preparation programs in general when they vary widely in their specific features.

Some of these varied features are the quantity or quality of field components, course requirements, the content of required courses, and the alignment and integration of various aspects of a preparation program. In mathematics teacher preparation, specifically, some scholars (Ebby, 2000; Youngs & Qiang, 2013) have focused on the importance of prospective teachers' field experiences and the alignment between those experiences and the kinds of mathematics instruction advocated in methods courses. Others have called for more content courses for prospective mathematics teachers (Conference Board of the Mathematical Sciences, 2012) or for thoughtful integration of mathematics subject matter in pedagogical methods courses (Burton, Daane, & Giesen, 2008; Steele & Hillen, 2012). These subject-specific teaching methods courses—in which prospective teachers develop skills and pedagogical content knowledge essential to them developing effective, ambitious, and manageable classroom practices—are commonly a central component of teacher preparation programs (Sowder, 2007) but the number, format, and foci of these courses vary widely from institution to institution (Kidd, 2008).

This study focuses specifically on the topics addressed in mathematics teaching methods courses because these courses are largely under the control of mathematics teacher educators and so can be a focused area of improvement to complement larger-scale programmatic efforts. We address secondary methods courses because of our experiences at this level and because it is less studied than methods courses at the elementary level. This study involved a survey of 116 methods instructors from across the United States for the purpose of determining what topics they value for inclusion in secondary mathematics methods courses. We sought to determine the extent to which secondary mathematics methods instructors agree in their valuations and to identify topics that are broadly valued by instructors that may potentially serve as shared foci, or what we refer to as *touchstones*, for these courses. By analyzing and discussing responses from a large number of mathematics teacher educators, we can, as a field, work to clarify and bring needed coherence to the curricula of these methods courses.

Background

With regard to research on mathematics methods course, much of the past work focused on courses for prospective elementary teachers. Mewborn (1999) and Ebby (2000), for example, analyzed prospective elementary teachers' reflections on and connections between these courses and their concurrent field experiences. Swars and colleagues (2009) traced changes in prospective elementary teachers' beliefs and specialized content knowledge as they progressed through a two-course methods sequence that had associated field experiences, finding that shifts in the prospective teachers' beliefs about mathematics teaching and learning and their sense of efficacy with mathematical content could be traced to specific features of the methods courses. Ball and colleagues (2009) have also worked on features of elementary methods courses, developing and studying curricula for such courses that include high-leverage teaching practices and aim to develop mathematical knowledge for teaching.

At the secondary level, some scholars have developed textbooks for methods courses (e.g., Posamentier & Smith, 2009; Rock & Brumbaugh, 2013) and there is a wide range of studies that examine specific topics within the context of methods courses (e.g., Stump, 2001) but not a great deal of research focused on the overall content of secondary methods courses explicitly. Two exceptions to this lack of research on methods courses overall are the work of Markovits and Smith (2008) and Steele and Hillen (2012), both of which deal with content-focused methods courses that integrate pedagogical development with the development of mathematical knowledge for teaching through "discernible mathematical and pedagogical storylines that are tightly connected" (Steele & Hillen, 2012, p. 54). For the present study, it is important to note that one principle for designing content-focused methods courses is to choose a narrow focus on a specific mathematical topic or pedagogical process. This choice, then, becomes centrally important to the methods course and the advocates of content-focused methods courses do not explicitly specify what those focal topics or processes should be. Our study is complementary because it concerns the specific topics that one might choose to include in a secondary methods course but it does not specify how one might design a coherent course around the chosen topics.

Currently, there are many different topics addressed in secondary mathematics methods courses (Arbaugh & Taylor, 2008; Kidd, 2008). In an article about elementary programs that applies equally well to secondary programs, Ball and colleagues (2009) argued that the lack of a shared professional curriculum for teacher preparation means that "[s]tudent teachers' learning opportunities reflect the orientations and expertise of their instructors and cooperating teacher" rather than "common agreements about the preparation required for initial practice" (p. 459). Although the lack of common agreements is certainly a concern, the diversity that currently exists provides a rich set of resources to draw upon as we work to establish common agreement.

To guide the field in drawing upon those resources and achieving systematic improvement, Arbaugh and Taylor (2008) laid out a framework adapted from Borko (2004). Their framework identifies three phases of research phases. The first phase involves studying a single course or single teacher preparation program. The second phase involves studying a single course or a single program feature that is enacted in multiple teacher preparation programs. The final phase compares multiple programs with varying features across multiple sites. Arbaugh and Taylor (2008) pointed out that "the vast majority of work in mathematics teacher education fails to surpass Phase 1" (p. 5). The research in Phase 1 provides a valuable literature base for the field, but the present study moves into Phase 2 by focusing on a specific program feature—secondary mathematics methods courses—across the United States and Canada. This is not to say that our study is the first endeavor into Phase 2 with respect to methods courses. Indeed, Taylor and Ronau (2006) analyzed 58 methods course syllabi from members of the Association of Mathematics Teacher Educators (AMTE) and found considerable variation in the types of assignments included on the syllabus and the stated goals and objectives for the courses. The present study complements Taylor and Ronau (2006) by focusing on topics within secondary mathematics methods courses rather than assignments and broad learning goals and by relying on instructor survey responses which can capture more than what is encoded in a syllabus.

We use the term "touchstone" to refer to potentially agreed-upon topics for inclusion in secondary mathematics methods courses. This term has historical roots in the notion of a physical, public stone to which community members could bring their precious metal to verify its authenticity. In our usage, we imagine a set of touchstones as a community-developed, public resource to which instructors could refer as they design and develop their own courses. We chose to use "touchstone" rather than the term "standard" because "standard" conveys an official or authoritative quality that we do not intend. Rather, if this initial work of identifying potential touchstones for secondary mathematics methods courses leads to a well-defined set, we intend for the set to form a resource that instructors have the option but not the obligation to adopt.

Toward that end, this study addresses the following questions: Which potential touchstones do instructors of secondary mathematics methods courses value the most highly? Which potential touchstones are valued to significantly different extents by different instructors?

Method

Survey

Drawing on seminal research in the field related to mathematics teaching and teacher education (e.g., Arbaugh& Taylor, 2008; Schoenfeld & Kilpatrick, 2008; Swars et al., 2009) as well as our own experiences with secondary mathematics methods courses, we compiled a list of potential touchstones to be used in a survey for methods instructors. Examples of these touchstones are "Enacting mathematical tasks," "Formative assessment," and "Digital tools and technologies (e.g., calculators)." We piloted this list with approximately 20 instructors and asked whether items could be removed or whether items we had omitted should be added. Revision then yielded a list of 41 potential touchstones (see Appendix) that we used for this study. Our goal for the list was to balance comprehensiveness and specificity by covering a full spectrum of topics without overwhelming survey respondents with an inordinate number of options or with options that were too closely related to allow for meaningful distinctions.

We chose to supply a predetermined list of touchstones rather than ask open-ended questions because an open-ended approach would have likely led to a wide variety of phrasing and terminology in the responses and possible idiosyncrasies of meaning, as described by Kidd (2008), that we would then have to interpret and categorize with possible concerns for the internal validity of the analysis. We recognize that, with a predetermined list, respondents also have to engage in interpretation of what we mean with various phrasing of the touchstones, but we felt there would be less variability in the respondents' interpretations as readers of touchstones than there would be in their responses as writers of touchstones. Furthermore, it is possible that, when responding to an open-ended question, a respondent may inadvertently omit a topic that is actually quite valuable to them only because they did not happen to bring it to mind in the few minutes they were responding to the survey. With the predetermined list, we were able to go through a multi-step process to assure that there were no serious omissions and we also included an open-ended item at the end of the survey asking respondents to list any touchstones that they highly value but were not on the list. Finally, the predetermined list lessened the time demands on the respondents and thus likely increased the response rate.

The survey was administered electronically with the following prompt: “Please tell us how important you feel it is for each of the following content items to be valued and addressed by secondary mathematics methods courses for preservice teachers.” The 41 touchstones were then listed with a five-point Likert scale ranging from “Not important” to “Very important.” We chose to ask about the instructors’ values rather than about their actual practice because the latter may elicit what instructors feel obligated to teach or what they are able to address in a limited timeframe rather than what they value in an ideal sense. In addition to the open-ended item about missing touchstones, there was also an open-ended item for general comments. The survey then gathered demographic information including professional title, secondary methods teaching experience, and academic home (e.g., college of education, department of mathematics).

Participants

The survey was sent to the approximately 940 members of AMTE, with the invitation email explicitly asking for responses from those involved in secondary mathematics methods courses. An item on the survey was used to verify that respondents were secondary methods instructors as opposed to other members of AMTE. Members of AMTE were chosen because the association is professional peers within the field who would most like participate and find value in the results of this study. It should be noted that AMTE’s membership is not necessarily representative of all mathematics teacher educators in the United States but rather those who are active with regard to professional organizations of teacher educators. Thus our results should not be construed as the representative values of secondary mathematics methods instructors in general. The results, however, can be interpreted as representing the values of many of the leaders in mathematics teacher education and those likely to be involved in shaping future directions in the field.

We received 129 responses and included 116 responses in the analysis. Of these, 70 were from individuals in colleges of education and 36 were from individuals in mathematics departments. The remaining 10 individuals had either joint appointments or another situation.

Data and Analysis

The data were analyzed using quantitative and qualitative methods as the data included numerical and free responses. First, the data were compiled on the 41 touchstones to determine basic descriptive statistics (mean, standard deviation) to understand which touchstones participants valued the most and the least. Independent t-tests and analyses of variants (ANOVA) were used to determine if certain groups separated by department or professional title varied significantly in how they valued any of the 41 touchstones. Second, the free responses were qualitatively analyzed to determine what touchstones participants perceived as missing and to identify themes in any of the additional comments offered.

Results

Due to space limitations, only the overall valuations of the touchstones and comparisons according to academic home will be reported in this paper.

Descriptive Statistics

Table 1 lists the 41 touchstones ordered by mean values. Note that every touchstone had a rating above 2.5 out of 5, which lends internal validity to the set of potential touchstones. Nineteen of the 41 touchstones had means within one standard deviation of the highest-rated touchstone and 0

Table 1: Touchstones Ordered by Mean Value

Touch stone	Description	Mean	St. Dev.
TS4	understanding of practice/process standards (e.g., CCSS, NCTM, NRC)	4.71	0.56
TS16	multiple representations of mathematical ideas	4.68	0.57
TS28	attending to student thinking and using student ideas to push understanding forward	4.68	0.58
TS35	mathematical knowledge for teaching	4.68	0.64
TS8	adapting, choosing, and generating mathematical tasks	4.61	0.59
TS20	productive classroom discourse	4.59	0.61
TS9	enacting mathematical tasks	4.55	0.68
TS6	lesson and unit planning	4.53	0.67
TS7	cognitive features of mathematical tasks	4.48	0.67
TS3	understanding of content standards (e.g., CCSS, state, district, school)	4.48	0.68
TS17	the relationship between conceptual and procedural knowledge	4.47	0.68
TS11	formative assessment (on-going assessment)	4.44	0.74
TS29	motivating students to persevere and take risks	4.36	0.76
TS36	reflection on practice and development as a professional educator	4.35	0.69
TS10	informal assessment (e.g., observation, conversations with students)	4.34	0.79
TS5	choosing and writing instructional goals	4.28	0.8
TS21	positive classroom culture	4.26	0.79
TS37	repertoires of effective mathematical teaching practices and pedagogical tools	4.22	0.74
TS18	pedagogies that address different types of knowledge and skills (e.g., procedural, conceptual, strategic, declarative)	4.17	0.83
TS25	digital tools and technologies (e.g., calculators)	4.14	0.72
TS34	mathematical content knowledge	4.13	0.91
TS30	nature of problem-solving	4.11	0.81
TS23	roles of the mathematics teacher (e.g., teacher as guide, teacher as lecturer)	4.11	0.84
TS26	analog tools and technologies (e.g., manipulatives)	4.05	0.78
TS14	issues of equity, status, fairness, and social justice	4.05	0.93
TS24	mathematical applications or mathematics in context	4.03	0.84
TS12	summative assessment to assess student understandings	3.98	0.78
TS15	needs of underrepresented populations	3.98	0.95
TS1	curriculum vision	3.91	0.86
TS2	knowledge of written curriculum materials	3.81	0.81
TS31	students' metacognitive skills	3.78	0.81
TS27	classroom management that supports cultural and learning goals	3.77	0.95
TS40	learning theories and applications to practice	3.73	0.89
TS22	sociomathematical norms	3.73	0.95
TS19	relationship between participation structures (e.g., pair work, complex instruction) and cultural and learning goals	3.72	0.97
TS33	personal and societal beliefs about teaching and learning mathematics	3.62	1.07

TS13	expectations, purposes, and design of homework	3.61	0.88
TS39	teaching theories and applications to practice	3.54	0.88
TS38	read educational research	3.38	0.9
TS32	history and nature of mathematics	3.09	0.94
TS41	do educational research (e.g., Action Research)	2.78	1.07

touchstones had a mean near the “not important” or “less important” ratings, indicating that the respondents tended to value a large portion of the potential touchstones. Only 32 out of 116 respondents made any suggestion of additional touchstones and most were singular suggestions (e.g., working with parents). Seven additional touchstones were suggested by at least two respondents. One related to reflecting on practice was mentioned in some form 5 times and another related to learning trajectories was mentioned 4 times.

Comparison by Department

Focusing on respondents from colleges of education or mathematics departments, and using an alpha level of 5%, we found statistically significant differences between the valuations of five touchstones (see Table 2). As there were nearly twice as many participants in educational departments (70) to those in mathematics departments (36), assuming equal variance was not possible in every case. Thus we ran independent *t*-test comparisons with equal variance assumed or not assumed as appropriate according to Levene’s statistic ($p<0.05$).

Table 2: Touchstones that varied significantly by respondents’ department

Touchstone	TS3	TS9	TS10	TS14	TS15
Description	understanding of content standards (e.g, CCSS, state, district, school)	enacting mathematical tasks	informal assessment (e.g., observation, conversations with students)	issues of equity, status, fairness, and social justice	needs of underrepresent ed populations
Equal Variance	Not Assumed	Not Assumed	Assumed	Not Assumed	Not Assumed
***t*-score**	3.399	-2.288	-3.258	-3.205	-3.601
df	96.557	62.510	104	53.573	55.199
***p*-value**	0.001	0.026	0.002	0.002	0.001
Education Department Mean	4.429	4.686	4.529	4.271	4.271
St. Dev.	0.627	0.627	0.583	0.779	0.779
Mathematics Department Mean	4.778	4.361	4.028	3.611	3.556
St. Dev.	0.422	0.723	1.000	1.103	1.054

Table 2 shows that understanding content standards (TS3) varied significantly, with participants from mathematics departments valuing it more than participants from colleges of education. On the other hand, enacting mathematical tasks (TS9), informal assessment (TS10), issues of equity, status, fairness, and social justice (TS14), and needs of underrepresented populations (TS15) were valued more highly by those in colleges of education than in mathematics departments.

Discussion

The purpose of this study is to spur conversation amongst mathematics teacher educators about what we value with regard to topics in secondary mathematics methods courses. Based on our survey results, part of this conversation can be the idea of a set of touchstones for secondary methods courses, possibly consisting of those items valued most highly by our respondents. We found the notion of preservice secondary mathematics teachers coming to understand process standards (NCTM, 2000) or the Standards for Mathematical Practice (NGO & CCSSO, 2010) to be valued the most highly. Also valued very highly were the notions of using multiple mathematical representations, attending to student thinking and student ideas, and developing the preservice teachers' mathematical knowledge for teaching. These touchstones and others align with ongoing national efforts focused on mathematics education in general, such as the National Council of Teachers of Mathematics' (2014) *Principles to Actions*, which laid out eight effective teaching practices that are largely consonant with the highest rated touchstones.

Our results also confirm past work (e.g., Taylor & Ronau, 2006) showing that mathematics teacher education as a field places high value on a wide range of topics and activities. Although the set of touchstones covers a vast array of topics, each of which could easily warrant extended attention and development, results show nearly all of it being valued for inclusion in secondary methods courses. In other words, we may as a field be a bit too ambitious, especially considering the issue of limited time with which to address these touchstones in methods courses specifically. This concern was raised several times in the comment section of our survey and thus is worth discussing. One way to address the time constraints is to move certain touchstones to other facets of teacher preparation programs besides methods courses. Another way to address time constraints is to remove some of the lesser-valued touchstones from consideration, perhaps because they are unnecessary, such as having preservice teachers read or conduct empirical research, or because they are better suited for development for inservice teachers, such as the role of sociomathematical norms in classrooms or the applications of learning theories to practice. To be clear, we are not suggesting specific remedies for this dilemma, but the results presented here can form an empirical basis on which to make these decisions rather than relying solely on the idiosyncrasies of individual instructors as critiqued by Ball and colleagues (2009).

This study is a modest effort in Phase 2 of Arbaugh and Taylor's (2008) roadmap for research on mathematics teacher education. We have gathered input from methods instructors from across the country, representing many different teacher preparation programs. Yet, future research can go further to gather more detailed data to allow for the examination of the ways in which respondents interpreted the touchstones presented here. Moreover, future research could bridge the gap between what we value for methods courses in an ideal sense and what is actually occurring in the methods courses. Some of this work is already underway via the Mathematics Teacher Education Partnership (MTEP), which is a consortium of secondary mathematics teacher educators from 30 states and 69 universities formed to coordinate improvement of secondary mathematics teacher preparation. The results presented here can inform MTEP and other similar efforts to identify what we value in the field and what we want to emphasize with preservice teachers in the limited opportunities that we have to interact with them.

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REASONING QUANTITATIVELY TO DEVELOP INVERSE FUNCTION MEANINGS

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Researchers have argued that students can develop foundational understandings for a variety of mathematical concepts through quantitative reasoning. I extend this research by exploring how students' quantitative reasoning can support them in developing meanings for inverse relations that influence their inverse function meanings. After summarizing the literature on students' inverse function meanings, I provide my theoretical perspective, including a description of a quantitative approach in the context of inverse relations. I then present one student's activity in a teaching experiment designed to support her in reasoning about a relation and its inverse as representing the same relationship. The student's quantitative reasoning supported her in developing productive meanings for inverse function, although this required her to reorganize her understanding of various mathematical ideas.

Keywords: Cognition; Teacher Education-Preservice; Design experiments

Researchers have indicated students can leverage reasoning quantitatively to develop meanings for various mathematical topics before developing more formal mathematical understandings (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Johnson, 2012). Researchers using a quantitative reasoning lens have also provided important insights into students' learning of a variety of secondary mathematics topics including specific function classes (e.g. linear functions (Johnson, 2012; P. W. Thompson, 1994) and exponential functions (Ellis et al., 2012)). A natural extension of this body of research is to explore how students' quantitative reasoning influences their notions of relations (or functions) and inverse relations prior to and concurrently with thinking about specific function classes. In this report, I summarize the research on students' inverse function meanings then propose ways of thinking that have the potential to support students in developing productive inverse relation and inverse function meanings. I present important aspects of a student's activities from a semester-long teaching experiment designed to support her in developing such meanings. I conclude with implications stemming from this work and directions for future research.

Research on Inverse Function

Vidakovic (1996) presented a genetic decomposition for inverse function (i.e. a description of how students might learn a concept, including methods for constructing their schemes). She proposed that students develop inverse function schemas in the following order: function, composition of functions, then inverse function. She conjectured students could coordinate all three schemas and develop inverse function meanings through this coordination. Whether implicitly or explicitly, many researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1997) who have examined students' inverse function meanings have maintained an emphasis on composition of functions as critical to students developing productive inverse function meanings. However, these same researchers noted that students often carry out techniques for successfully determining representations of inverse functions (e.g., determining an inverse function analytically) without connecting these techniques to function composition. As a consequence, students (and teachers) hold compartmentalized inverse function meanings, typically related to executing specific actions in analytic or graphing situations (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, in press). Moreover, Paoletti et al. (in press) reported that pre-service teachers, when given a function meant to represent a context, struggled to interpret the contextual meaning of the inverse function they constructed. Collectively, these researchers' findings suggest that current approaches to inverse function have been ineffective in supporting students in developing productive inverse function meanings. Complicating the matter, and as I argue in more detail below, researchers predominately treat students construction of 'inverse function' meanings as distinct from their function understandings (e.g., Vidakovic's genetic decomposition), as opposed to approaching 'inverse function' as developing hand-in-hand with 'function'.

Theoretical Framing

I explore the possibility of supporting students developing inverse meanings via reasoning about quantities and relationships between quantities (i.e. reasoning quantitatively). A *quantity* is a conceptual entity an individual constructs as an attribute of an object or phenomena that allows a measurement process (P. W. Thompson, 1994). As an individual associates two varying (or non-varying) quantities, she can construct *quantitative relationships* (Johnson, 2012; P.W. Thompson, 1994); an individual engages in *quantitative reasoning* as she constructs and analyzes these relationships (P. W. Thompson, 1994).

Using a quantitative reasoning lens, I conjectured that if a student constructed a relationship between two quantities (e.g., quantities A and B) that did not entail some conceived causation between the quantities, the student could choose one quantity to be the input of a relation (e.g., A input, B output) while anticipating that the inverse relation would involve choosing the other quantity as the input (e.g., B input, A output). With respect to graphing relations and inverse relations in the Cartesian coordinate system, a student who understands relations in ways compatible with this could interpret a single graph as representing both a relation and its inverse. Engaging in such reasoning requires the student to anticipate choosing either axis as representative of an input quantity; researchers (Moore & Paoletti, 2015; A. G. Thompson & Thompson, 1996) have defined such reasoning as bidirectional reasoning. Although this type of reasoning may seem trivial, Moore, Silverman, Paoletti, and LaForest (2014) illustrated that students are often restricted to reasoning about the quantity represented along the horizontal axis as the input.

By focusing on a relationship between quantities, the 'function-ness' of a relation and its inverse is not critical. A student can describe a relation and its inverse without (necessarily) being concerned if either represents a function. Moreover, the student understands that the choice of input-output quantities does not influence the underlying relationship that the associated functions or relations describe. Whereas this approach does not foreground function (or composition of functions) as critical to developing inverse meanings, I conjecture a student who develops understandings compatible with those described would have little difficulty making sense of the formal definition of inverse function that relies on composition of function (e.g., understanding if $B = f(A)$ and $A = f^{1}(B)$, then $f(f^{1}(B)) = f(A) = B$ and $f^{-1}(f(A)) = f^{1}(B) = A$).

Methods

I conducted a semester-long teaching experiment with two undergraduate students, Arya and Katlyn (pseudonyms). I focus this report on Arya's activity. Arya was a junior who had successfully completed a calculus sequence and at least two additional courses beyond calculus. The teaching experiment consisted of three individual semi-structured task-based clinical interviews (per student) (Clement, 2000) and 15 paired teaching episodes (Steffe & Thompson, 2000). I used clinical interviews to explore Arya's function inverse meanings without intending to create shifts in her meanings. I used the teaching episodes to pose tasks and questions that I conjectured might perturb Arya's meanings, leading her to make accommodations to her meanings to resolve her perturbations.

I used the combination of clinical interviews and teaching episodes to explore Arya's mathematical activity, to build models of her mathematics, and to investigate the mathematical progress Aryamade over the semester (Steffe & Thompson, 2000).

In order to analyze the data, I used open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998) in combination with conceptual analysis(P. W. Thompson, 2008)to develop and refine models of Arya's mathematics. Initially, I analyzed the videos identifying episodes of Arya's activity that provided insights into her meanings. Using these identified instances, I generated tentative models of her mathematics that I tested by searching for activity that corroborated or refuted my models. When Arya exhibited novel activity that contradicted my models, hypotheses were made to explain this activity including the possibility that this new activity indicated fundamental shifts in her operating meanings. Through this iterative process of creating, refining, and adjusting hypotheses of Arya's meanings, I was able to not only characterize her thinking at a specific time or situation, but I was also able to explain transitions in Arya's meanings throughout the teaching experiment.

Task Design

I focus this report primarily on one task (*Graphing sine/arcsine task*, Figure 1), which a research team designed to support the pair of students in developing productive inverse relation meanings via reasoning bidirectionally. Relevant to this report, the first two parts of this task involve the students creating graphs of the sine (Graph 1) and arcsine function (Graph 2). The third prompt asks the students to consider how they could use Graph 1 to represent the arcsine function. The prompt also asks the students to consider if Graph 1 and Graph 2 represent "the same relationship." I conjectured by asking the students to foreground the "relationship" represented by both graphs, they might engage in reasoning bidirectionally in order to conceive Graph 1 and Graph 2 as representing both the sine and arcsine functions (or relations).

Graph 1:	Create a graph of the sine function with a domain of all real numbers. What is the range?
Graph 2:	Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function.
Prompt 3:	Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not?

Figure 1: Graphing sine/arcsine task

Results

For brevity's sake, I highlight important instances in Arya's activity in order to describe her thinking including shifts in her thinking. I first describe Arya's activity (most relevant to this report) during the initial clinical interview in order to characterize her meanings prior to the teaching episodes. I then provide data from four consecutive teaching episodes in which Arya addressed the prompts in the *Graphing sine/arcsine task* for various relations.

Results from the initial clinical interview

When given a function's graph and asked to determine a graph of the inverse function, Arya switched the coordinate values (e.g., a point (a, b) from the original curve became (b, a) on the inverse curve). When asked to determine the inverse of a function represented analytically, Arya switched the variables and solved for the previously isolated variable (heretofore referred to as switched-and-solved) (e.g., the inverse of $y = x + 1$ was $y = x - 1$). I also gave Arya a function defined analytically that converted degrees Fahrenheit to degrees Celsius (i.e., $C(F) = (5/9)(F - 32)$ and asked her to represent the inverse function. She switched-and-solved obtaining $C^{-1}(F) = (9/5)F + 32$. When asked to interpret the meaning of the inverse equation, Arya considered again switching the variables (e.g., $F(C) = (9/5)C + 32$), but rejected the resulting equation because it defined the same relationship between degrees Fahrenheit and Celsius as the original equation and function. I inferred from Arya's activity that she anticipated a function and its inverse function represented relationships that differed in some way other than a choice of defining input-output quantities (e.g., having different graphs or defining a different relationship).

Also of note from the interview, Arya exhibited activity in multiple problems that I took to indicate she was restricted to reasoning about the horizontal axis as representing a function's input. For instance, when given the graphs in Figure 2 and asked "Are these graphs the same or different?" Arya argued, "[the graphs are] showing the same thing but in a different way… this [*Figure 2a*] is what is happening to distance as time is going on." Then describing Figure 2b, Arya stated, "As you change your distance… the time is moving forward." In this and other cases, Arya maintained considering the horizontal axis as representing a function's input or the independent quantity of a relationship.

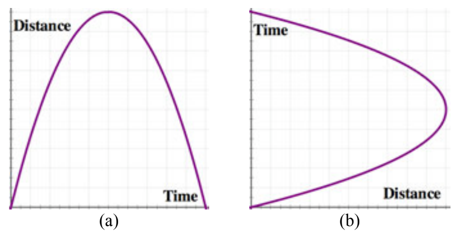
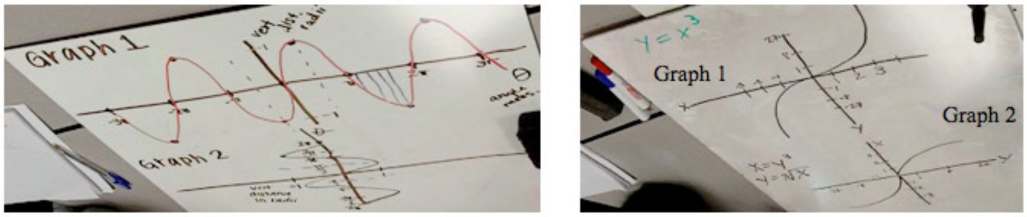


Figure 2: Double parabolas problem: Are the graphs the same or different?

Considering sine and arcsine

Nine days prior to the first teaching episode exploring the *Graphing sine/arcsine task*, Arya and Katlyn constructed the sine function as a relationship between an angle measure (input) and a vertical segment length above the horizontal diameter measured relative to a circle's radius (output) in a circular motion context (see Moore (2014)). Upon my giving the students the *Graphing sine/arcsine task*, they reproduced the graph of the sine function (see Figure 3a, Graph 1). Arya then leveraged her understanding of switching the coordinate values to graph the arcsine relation (e.g., the point ($\pi/2$, 1) became (1, $\pi/2$), see Figure 3a, Graph 2). The students labeled the horizontal and vertical axes in Graph 2 'vertical distance' and 'angle measure', respectively. I then questioned the students about "what [the] two graphs are representing?" Arya responded, "They're showing the same relationship, but this [*pointing to Graph 2, Figure 3a*] shows… if you're changing your vertical distance on your graph, what [*pointing to θ-label on the vertical axis*]radian measure that corresponds to. And this shows [*pointing to Graph 1, Figure 3a*] if you're changing your angle measure what vertical distance that corresponds to." As during the *Double parabolas problem*, Arya *described* the graphs as representing the same "relationship," but her interpretation of each graph relied on the horizontal axis as representing the input quantity (e.g., the quantity that she first envisioned varying or caused the other quantity to vary).

With both students content in their explanations of the two graphs, I asked them to consider the third prompt with the hopes of raising the underlying difference between their understandings of the graphs. Katlyn first wrote the analytic equation $\sin^{-1}(y)= \theta$ near Graph 1 and described interpreting Graph 1 with the vertical and horizontal axis representing the input and output quantity, respectively, of the arcsine relation. Arya responded, "I don't know if that, can you do that?" Katlyn's claim contradicted Arya restricting a function's input to the horizontal axis. As the interaction continued, Arya attempted to refute Katlyn's reasoning. But, as she attempted to do so, Arya continually returned to her understanding that Graph 1 represents the same distance-angle measure pairs regardless of which axis is denoted as a function's input. She concluded, "I don't see anything mathematically incorrect. I don't see it."



(a) (b)

Figure 3: Graphs of (a) the sine and arcsine functions and (b) a cubicand its inverse

By focusing on both graphs as representing, “Vertical distance and angle measure… the relationship between the two,” Arya reorganized her meaning for interpreting graphs. Specifically, as Arya addressed the third prompt in the *Graphing sine/arcsine task* and Katlyn’s claim, she had to consider whether graphs unquestionably represented the input quantity on the horizontal axis or if this was a common practice of graphing. Once Arya understood that graphs could be interpreted with either axes as representing the input (i.e., reasoned bidirectionally with respect to axes), she understood a single graph as representing both a relation and its inverse. By the end of the second teaching episode, Arya exhibited this understanding multiple times with respect to both Graph 1 and Graph 2, leading me to conjecture she had constructed the sine and arcsine relations as representing the same relationship between angle measure and vertical distance.[1] Moreover, she understood that a graph of the sine relationship simultaneously represented a graph of the arcsine relationship, and vice versa.

Considering a decontextualized function

I designed the third teaching episode to explore how Arya might extend her reasoning with the sine and arcsine relations to a relation or function represented by a decontextualized equation ($y(x)= x^3$). For instance, I was unsure if she would continue to reason about a relation and its inverse as representing the same underlying relationship, particularly when graphed, or if she would encounter perturbations due to the different context and the chance that she might use her switch-and-solve technique (on the previous task they maintained a quantitative referent for each variable rather than switching the variables). After graphing $y= x^3$ (Figure 3b, Graph 1), the pair switched-and-solved to obtain the inverse rule $y=x^{1/3}$. They were unsure how to graph this equation so I suggested they recall their activity from the previous sessions. In response, they labeled the horizontal axis y, the vertical axis x, and they constructed Graph 2 (Figure 3b) by considering how y changed (along the horizontal axis) for changes in x (along the vertical axis) so that they maintained the same x-y relationship of Graph 1. That is, when drawing Graph 2, their focus was not on the equation $y=x^{1/3}$, but instead the relationship between the varying values x and y as depicted in Graph 1.Arya argued, “all the same information is in both graphs.” Compatible with the outcome of the prior two teaching episodes, Arya’s (and Katlyn’s) graphing activity indicated that she anticipated that a relation (or function) and its inverse represented the same relationship between quantities that could be represented graphically in multiple ways.

Due to my perceived discrepancy in their Graph 2 and the equation they had determined, I asked Arya to write an equation for Graph 2. She pointed to the equation $y = x^{1/3}$ but quickly noticed Graph 2 (Figure 3b), as labeled, did not represent an equivalent relationship between the varying values x and y. Because of this, Arya relabeled the vertical axis y and the horizontal axis x so that Graph 2 represented, “This equation [*pointing to* $y = x^{1/3}$].” Although Arya’s newly labeled graph represented the equation $y=x^{1/3}$, she immediately experienced another perturbation. Given her new axes labels, she noted that Graph 1 and Graph 2 did not represent the same relationship between the varying values x and y. Hence, Arya realized that her switching-and-solving activity was inconsistent with her activity in the previous teaching episodes where she maintained the relationship between the quantities (i.e., variables).

As Arya was unable to reconcile her perceived inconsistency between switching-and-solving and maintaining the relationship between quantities (or variables) in both Graph 1 and Graph 2, I directed her to address the third prompt believing this may support her in considering the relation in *Graph 1* as simultaneously representing the inverse relation, and that she might then note that her Graph 2 was merely the result of using variables arbitrarily. Consistent with her activity interpreting Graph 1 in Figure 3a as the arcsine relationship, Arya described two ways of interpreting Graph 1 in Figure 3b by considering either the horizontal or vertical axis as her input. Although Arya had no difficulty reasoning bidirectionally with respect to the axes in Graph 1, this did not support her resolving the differences she perceived between Graph 1 and Graph 2 (Figure 3b) due to the discrepancy introduced by her use of variables.

Contextualizing the function

As Arya began to question the validity of her activity when graphing the arcsine relation, and based on my interpretation that she did not realize that switching-and-solving requires using variables arbitrarily, I attempted to give contextualized meanings to the variables in order to support Arya in reflecting on her use of variables. I rewrote the given equation as $V = s^3$ and asked Arya to consider the equation as representing the volume of a cube (V) for a given side length (s) with the caveat that we could have negative side length and volume values. When considering the context, Arya labeled both Graph 1 and Graph 2 in a way that maintained both the relationship between volume and side length and the variable referents; in Graph 1 she labeled the horizontal axis *side length* and the vertical axis *volume* and in Graph 2 she labeled the horizontal axis *volume* and the vertical axis *side length*. Although Arya maintained the relationship between side length and volume in both graphs, this did not alleviate her perturbation as she remained unsure how this related to her switching-and-solving activity.

To support Arya in considering a way to relate her activity maintaining the relationship between quantities (and maintaining variable referents) with her switching-and-solving activity, I raised the idea of using the variables arbitrarily. I wrote the equations $y = \sin(x)$ and $y = \arcsin(x)$ next to two unlabeled Cartesian coordinate systems. I asked the pair to describe how they would label each coordinate system for the given equation and what quantity each variable would represent in each case. Katlyn stated she would use the conventional x-horizontal, y-vertical axis labeling and that for the $y = \sin(x)$, x would represent angle measure but in $y = \arcsin(x)$, y would represent angle measure. Arya questioned, "Why do we do that?… It doesn't make sense… Just because we want this to be our input [*pointing to x in the equation* $y = x^{1/3}$] and that to be our output [*pointing to y in* $y = x^{1/3}$]? I feel like that's really the only…It's [*referring to switching the variables*] just so you can call your input x and your output y."

Although Arya identified that switching-and-solving maintained calling the input quantity x, this did not resolve her perturbation. For instance, Arya leveraged reasoning bidirectionally to question the need of a second graph if a relation (or function) and its inverse were meant to represent the same relationship between quantities, saying, "If they gave me this graph [*Graph 1*] and wanted me to find the information, with this [*the vertical axis*] as the input I certainly could… I could turn my head and look at it [*Graph 1 with the vertical axis as input*] and understand what that means. There's still no reason for this [*pointing to Graph 2*] graph. So… like when you switch then you're saying something new." Arya maintained that her switching-and-solving technique resulted in 'something new', which was incompatible with her anticipation that a function or relation and its inverse maintain the same relationship.

Through much of the remainder of the last two teaching episodes, Arya experienced a state of perturbation as she attempted to relate her quantitative meaning for inverse to her switching-and-solving activity. By the end of the fourth teaching episode, Arya understood using variables arbitrarily (e.g., switching the quantitative referents of variables) to relate these meanings but continued to question why this would be done if a function and its inverse were meant to represent the same relationship. Arya's activity in contextualized situations, along with her reasoning bidirectionally with respect to the axes and her reasoning about variables as arbitrary, supported her in reorganizing her meanings for inverse relations (and functions). In later interactions, Arya maintained that a relation (regardless of if the relation was a function) and its inverse represented the same relationship and that in order to make sense of switching-and-solving she had to use the variables arbitrarily to represent the quantities under consideration (although she continued to question why she was taught to switch the variables).

Discussion and Concluding Remarks

By the end of the teaching experiment Arya understood that a relation and its inverse (regardless whether the relation was a function in the formal sense) represented the same relationship. However, developing this understanding was not trivial; it required Arya to reorganize her meanings for interpreting graphs (e.g., which axis could represent the input quantity), her meanings for variables (e.g., changing the quantitative meaning of x depending on the function under consideration), and her inverse function meanings (e.g., a function and its inverse represent something different). Whereas previous researchers have focused on students' and teachers' developing inverse function meanings via their understanding of function composition, these results provide insights into a different approach to inverse functions (or relations). To develop an understanding of a relation and its inverse as representing the same invariant relationship, the student in this study had to consider and reflect on various meanings she maintained (e.g., graphical conventions, interpreting variables, inverse procedures). As she reflected on her activity including her coordinating relationships between quantities, the student reorganized her meanings for various mathematical ideas so that she could adequately address the prompts in the *Graphing sine/arcsine task* (as well as all previous problems she was able to address). Previous researchers have indicated students can leverage quantitative reasoning to develop foundational meanings for various mathematical topics (Ellis et al., 2012; Johnson, 2012), and these results indicate students can reorganize already developed meanings via quantitative reasoning, although this process is not trivial.

In this report, I focused on Arya's development of inverse relation (or function) meanings via her reasoning quantitatively about relationships. Arya's activity had the potential to influence other meanings as well, including her function meanings. Future researchers may be interested in exploring how quantitative and/or bidirectional reasoning has the potential to support students in developing foundational function meanings. Additionally, this work examines the activity of a student who had already developed meanings for inverse function. Researchers may be interested in exploring how students who have not had formal instruction in functions and inverse functions (e.g., middle school students) could develop meanings for (inverse) function via their reasoning about quantities as these results indicate that such an approach has the potential to support students in developing productive inverse meanings.

Endnote

[1]Arya did discuss restrictions to the graphs in Figure 3(a) such that both Graph 1 and Graph 2 would represent functions regardless of chosen input quantity, but she typically worked with the arcsine relation (a multi-valued function).

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"I DON'T THINK I WOULD TEACH THIS WAY": INVESTIGATING TEACHER LEARNING IN PROFESSIONAL DEVELOPMENT

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Can professional development (PD) have a profound and lasting effect on participating teachers? The purpose of this study was to understand how teachers learn new mathematics content in professional development in order to contribute to the open question of how PD affects teachers' actual instructional choices in the classroom. Teachers were followed from a content-based PD program into their classrooms and their content knowledge was probed in each context. Data was analyzed from the perspective of motivation theory. Findings show compelling links between teachers' motivations, their conceptions of their students, and the nature of their knowledge of algebra. These links have broader implications for conceptualizing in-service teacher learning.

Keywords: Teacher Education-Inservice; Teacher Knowledge

Research on the professional development (PD) of mathematics teachers depends heavily on the question of what kinds of knowledge teachers need in order to teach mathematics effectively. After Shulman's (1986) introduction of the idea of pedagogical content knowledge and its subsequent refinement, researchers interested in both preservice and inservice teacher education have developed a clearer idea of what types of knowledge teachers needed to develop (Grouws & Shultz, 1996). However, defining the types of mathematical knowledge teachers need is only the first step in fostering that knowledge in teachers. Although the field has struggled to define and measure the effectiveness of PD without using teacher self-reports or student achievement data, there is no doubt that such an endeavor relies on understanding how teachers use concepts learned in professional development in their own classrooms. The purpose of this study was to investigate the ways in which teachers participating in content-based professional development made connections between the mathematical content of the PD program and their own mathematics instruction.

Background

The professional development of teachers is just one stage in the overall education of teachers and often comes at a time when teachers have settled comfortably into their practice (Feiman-Nemser, 2010). According to Feiman-Nemser, research has viewed teachers at this stage in two ways, either having already settled into a basic style and resistant to efforts aimed at change or constantly changing (with or without the help of professional development) in order to become more effective with students and to gain professional satisfaction. This study is done with the second view in mind, positing that professional development is one way of guiding or channeling the change teachers already are invested in making. However, according to previous research, the effectiveness of any program aimed at changing instructional practice must rely on how well such programs align with a teacher's preexisting beliefs about teaching, learning and mathematics (Arbaugh, Lannin, Jones, & Park-Rogers, 2006; Chapman, 2002; Thompson, 1984).

Due to the increasing diversity of teachers' mathematical backgrounds, educating inservice teachers produces issues very similar to those faced when educating diverse learners (Adler, Ball, Krainer, Lin, & Novotna, 2005). However, while some facets of professional development reflect the dynamics of a mathematics classroom, such as an instructional triangle consisting of interactions between nodes representing the professional development instructors, the participating teachers and the mathematics content (Borko, 2004), teachers constantly consider their own students–even while attending to their own learning–creating an instructional rhombus with the fourth node representing the teachers' real or hypothetical students (Nipper et al., 2011). However, Nipper et al. found that tension between the teachers and the content of the professional development arose as teachers realized that the mathematics content of the program was not content that they could directly use in their own classrooms.

Previous research has identified particular design features that characterize effective professional development: sustained learning over a period of time, active learning by teachers, examples from classroom practice, collaborative activities, modeling effective pedagogy, opportunities for reflection, practice and feedback, and focus on content (Boyle, White & Boyle, 2004; Hill, 2004). Ross, Hogaboam-Gray, and Bruce (2006) found that a professional development program which incorporated all these features resulted in a significant increase in student achievement on an external assessment. However, Marra et al. (2010) argue that individual design features are not as important the interactions between these features. Thus, they propose the orientations framework for classifying professional development programs. Based on their framework and the meta-study they conducted, they conclude that the most effective orientations of professional development are either completely or partially content-driven. This conclusion is echoed by Sowder, Philipp, Armstrong, and Schappelle (1998), who found that teachers appreciated when attention was paid to learning mathematics content. Sowder et al. found that changes in mathematical knowledge prompted changes in instruction, but this was mediated by teachers' comfort level with the content. Moreover, increased understanding of the mathematics they were teaching prompted the teachers to have greater expectations for their students' mathematical learning and changed their views about the centrality of curriculum materials and the quality of the classroom discourse. Content-driven professional development has also been shown to support student achievement (Hill, Rowan, & Ball, 2005; Saxe, Gearhart, & Suad Nasir, 2001).

However, while much research has been done to identify effective professional development, most of it relies on teachers' self-reporting of aspects of professional development that they liked and what instructional changes they have made (Marra et al., 2010). Less work has been done in understanding how teachers use the knowledge gained in professional development and how that knowledge leads to instructional change.

Theoretical Frameworks and Research Question

This study adds to the existing knowledge by applying the frameworks of motivation theory to analyze teachers' participation in and engagement with professional development. There are four major frameworks that make up the foundation of motivation theory as applied to the classroom: expectancy-value theory, self-efficacy beliefs, goal orientation and attribution theory (Karabenick & Conley, 2011). In the context of education, these describe students' willingness to engage with school tasks. According to expectancy-value theory, the effort a student expends on a task depends on whether he thinks he will be successful at the task and on whether he believes that success on the task will result in a valued reward – either internal or external. A student's self-efficacy determines her assessment of her own abilities, interpreting her past failures and successes in order to set her own personal goals and define success for herself based on those self-determined goals. The character of the goals set by the student for himself also affects the student's level of effort. Two different goal orientations produce different patterns of effort and perseverance: performance goals are goals that rely on affirmation from others and can result in low effort from students with low-self efficacy, while learning (or mastery) goals are goals defined by gaining new skills or new knowledge, potentially prompting students with low self-efficacy to learn from failure and try again. Finally, a student's attribution of failure or success to either effort or ability also affects her motivation to attempt or to persevere in completing a task.

However, as teacher learning is different in character than student learning, applying motivation theory to teacher learning narrows the focus of this framework in order to allow for a more nuanced discussion of how teachers' self-efficacy and subjective task values can influence their participation in professional development. Karabenick and Conley (2011) use motivation theory to investigate teachers' motivation to participate in professional development, situating teacher motivation within the context of a PD program by tying motivation to the ways in which teachers participate in the program and enact the practices recommended by the program. They extend a framework for teachers' choices developed by Watt and Richardson (2007). The studies by Karabenick and Conley (2011) and Watt and Richardson (2007) both rely on a framework for value developed by Wigfield and Eccles (2000), which classifies four different types of value:

> *interest value* is the enjoyment the individual derived from performing the task; *utility value* is how the task relates to future goals; *attainment value* is the importance to the self of doing well on a task, linked with identity (in this case teacher identity); and *cost*, which refers to the accumulated negative aspects of engaging in the task, including anticipated emotional states (performance anxiety, fear of failure), and the amount of effort required to succeed at the task. (Karabenick & Conley, 2011, p. 11).

Expectancy-value theory relies on the intertwined concepts of self-efficacy and value, both of which have been studied extensively with regard to students, but are only beginning to be explored with teachers. This study, like others that apply a motivation theory framework to the processes and concerns of teaching (Karabenick & Conley, 2011; Watt & Richardson, 2007), primarily relies on the lenses of expectancy-value theory and self-efficacy theory, with other constructs in motivation theory referenced if relevant.

This study attempts to contextualize the choices teachers make with respect to their own learning within professional development, as well as their use of content from PD in their own classrooms. This paper will address the findings related to the following research question: what influences mediate teachers' alignment with the mathematics content of professional development and connections they make between that and the mathematics content of their classrooms?

Methods

These qualitative case studies were conducted within a three-year part-time degree for middle school mathematics teachers with elementary certification run by a large research university in the southwestern United States. This program had a heavy focus on mathematical content and included four required mathematics courses: number & operation, algebra, geometry, and probability and statistics, each of which was team-taught through the university by a research mathematician and a high-school teacher. After completing the program, participants are awarded a Master of Arts degree in Middle School Mathematics Teaching Leadership from the university. The general orientation of the program would be classified as content-driven (Marra et al., 2010). Three participating teachers, each of whom had at least five years of teaching experience in either middle or elementary school, were chosen from a single cohort of the professional development program after that cohort had finished the program's algebra course.

Data was collected in four major stages: 1) during the Spring 2012 semester as teachers participated in the professional development algebra course, 2) after the end of the algebra class, 3) during the 2012-2013 academic year as the teachers began their first year of teaching after finishing the algebra course, and 4) during the Fall 2013 semester. Data collected from the first stage of the study was made up of observations of the PD algebra course at the university. Special attention was paid to the instructors' mathematical decisions and perspectives of algebra. Stage two of the study included a post-class task-based interview where teachers were asked to reflect on the experience of the algebra class and their attitudes about the content, as well as the program as a whole up to that point and then to pedagogically unpack five mathematical content questions taken from the algebra course. The third stage of the study asked teachers to open their practice and their classrooms to observation by the researcher. Teachers were asked to identify lessons for observation which they saw as connected to the content covered in the professional development algebra class. Teachers participated in pre- and post-observation interviews in order to chart the teacher's intentions more specifically by focusing on a particular mathematical topic. Also, the teacher was asked to identify explicitly the connections she saw between the lesson she was teaching and the content covered in the professional development algebra class. The final stage of the study was a follow-up interview that asked teachers to contextualize this PD experience with their other PD experiences and within the greater narratives of their careers.

Non-task-based interview data from the second and fourth stages of data collection and relevant tangential comments from the third stage were analyzed within the value framework developed by Wigfield and Eccles (2000) in order to produce a motivational portrait of each teacher. This was done by isolating references teachers made to their reasons for participating in the PD program, statements that revealed their attitudes toward particular concepts, and any other remarks that revealed aspects of their affect. While many of these pieces of data were explicitly prompted by actual interview questions, some telling comments arose amidst other portions of the interviews. The task-based portion of the second stage of data collection and the mathematical observations and interview excerpts from the first and third stages were analyzed for the algebraic perspectives put forward by the teachers or PD instructors. This was done by placing solutions within contrasting frameworks for algebra established by Pimm (1995), Kaput (2008) and Kieran (2007). Teachers' written work from the task-based interviews was used in conjunction with the interview transcripts to provide further clarification.

The importance of motivation theory to this study became apparent after a cursory analysis of the data. Teachers' reasons for participating in the PD program appeared relate strongly with their initial expectations for the program and its usefulness to their teaching, prompting the explicit use of motivation constructs in analyzing the data.

Findings

This paper will focus on one teacher: Felicia. At the start of her progress through the content-based PD program, Felicia was beginning her tenth year in the profession. Her confidence in her mathematical ability was very high, often referring to herself in interviews as a "math person," and saying that, as a teacher, "Math has always been my thing. And I think I'm really really strong in it, so I teach to my strength." Felicia did not think the mathematics in the PD program would be much of a challenge for her, an opinion she retained throughout most of the program. Felicia's motivations for enrolling in the PD program were a mix of personal and professional goals. Her eventual career goal was to move into administration, and she was hoping that a Master's in Middle School Mathematics Teaching Leadership, paired with a Master's in Educational Leadership, would help propel her into more leadership positions. Although her interest in mathematics created some amount of interest value in the PD program, Felicia mostly held utility value for the program, as she felt it would help her career. Felicia also had a high level of self-efficacy as a teacher of mathematics: "I think the level that I'm at and where I am with classroom management and building those relationships with the kids, I think I'm there. There's a lot of things I can still learn from older teachers, but then again too, I have a different – better – different way of doing it that I think is better."

Felicia's overall contentment with her mathematical abilities (especially in algebra) and instructional style led her to expect that she would not be prompted to change as a mathematics teacher due to her participation in the PD program. The nature of the utility value that she held for the PD program – professional, not practical – also signals her expectation that the content of the PD program (and specifically the PD Algebra course) would hold mainly interest value for her. Moreover, her idea of "relevant" content was very narrow: the mathematics had to directly mirror the mathematics her students would be expected to learn. Felicia referred multiple times to the fact that she taught sixth grade and that most of the content of the PD Algebra course was not appropriate for sixth-grade students. Moreover, Felicia viewed her own ongoing mathematical development mainly as an opportunity to develop more and better "tricks" or solution strategies for traditional algebraic tasks. Otherwise, Felicia described her gains from the PD program as mostly personal: "I like math... I mean, honestly, that's what it comes down to. I like math, and I like some of the times it was a challenge and I was like, ooh! Something new let's go find out how to do it, it was for me. Because I'm a math nerd. It's a challenge. Some of the things were challenges."

While the PD instructors emphasized certain perspectives of the algebraic concepts and encouraged teachers to explore those perspectives, during the task-based interview, Felicia fell back on her previous understandings of algebra to solve the given problems. For example, consider the following item from the task-based interview:

> Construct a function with the following properties if possible or explain why it would not be possible.
> 1. One element in the domain and four elements in the range.
> 2. Four elements in the domain and one element in the range.
> 3. Four elements in the domain and four elements in the range.

Felicia's solution to this item relied heavily on her visualization of functions in the Cartesian coordinate plane. She concluded that the first set of properties could not describe a function, since "technically, if you're graphing it, it'd be a vertical line, and it won't pass the vertical line test, so it won't be a function." Similarly, she identified the second set of properties as a horizontal line, so it would describe a function that was not one-to-one. The third set of properties described a one-to-one function, "because for every unique domain there's a unique range." In her pedagogical analysis of this task, Felicia connected her solution strategy with how she thought students would approach the task, asserting that seeing (or knowing) graphical representations of the sets of properties given would make this task easier for students to complete. Although she brought up the "circles" representation of functions (i.e. the map-between-sets representation) that was emphasized in the PD Algebra course as another possible visual representation that might be meaningful for students, Felicia admitted that she did not find much meaning in that representation herself, and that she was unlikely to present it to students. She explained her conception of functions as follows: "When it comes to functions, I automatically think of something that can be graphed or can't be graphed. I don't think of the circles like we were taught in class." Although she said that the more abstract visual representation of functions was new to her, it is clear that this representation held little meaning for her, since she followed the previous statement by saying, "I don't think I would teach it this way because I'm not comfortable with it. I would go directly to the graph, because graphing is an easy way to see it." Felicia referred multiple times to graphical representation being "easier" to understand – either for herself or for her students.

Felicia's work on and reflection about all the tasks in the task-based interview diverged a great deal from the PD instructors' presentation and development of the concepts. For three of the tasks, Felicia used or advocated the use of numerical solution strategies, making her mathematical perspective of those tasks non-algebraic (Kaput, Blanton, & Moreno, 2008). In general, the PD instructors did not encourage the teachers to use numerical solutions, choosing instead to focus teachers' attention on algebraic structure. In fact, Felicia's work only reflected the mathematical perspective of the PD Algebra course on one task, possibly because she had no other solution strategy outside of the one she had learned in the PD Algebra course. Even her emphasis on the Cartesian-coordinate-graph-representation of functions to the exclusion of the more abstract map-between-sets representation placed more weight on a representation that the PD instructors introduced almost as an afterthought in that unit. Felicia's lack of alignment with the development and presentation of mathematics in the PD Algebra course provides an important context for understanding how she viewed algebraic content in the PD Algebra course and its relevance to the algebraic content of her own classroom.

Felicia's conception of her role as a teacher (and especially in her role as an interventionist) was as a provider of different strategies for her students. She considered her job to be finding different ways of teaching the material, stating that the "traditional" methods and algorithms "don't really work for this generation." She also equated "better ways" of teaching the material with showing students different strategies for approaching problems. As a result, Felicia considered the main utility of content-based professional development to be as a way to help her learn or create different strategies and algorithms to teach her students:

> *Felicia*: [The PD courses] have helped me realize the reason why it works, or the reason behind the actual math, the algorithms or the operations or whatever. So it helps me develop a trick, per se, that the kids might get a little bit easier than the traditional here's how you do it type of thing.

In turn, she would then present the alternate algorithm or strategy to her students. Unfortunately, Felicia did not see many opportunities in her classroom to develop different strategies or alternate explanations with respect to the content of the PD Algebra course.

In one episode from the observations of Felicia's classroom, a student asked her for help with the one-step equation: $v - 3.7 = 8.78$. Felicia's interaction with the student around this problem reflected this belief about professional development, since she did not in any way mirror the PD instructors' approach to similar problems, which were discussed extensively in the PD Algebra course:

> *Felicia*: I explained to her: a number minus this number is going to give me this number right? And she's like, yeah. I'm like, okay, so what are we going to do with this number that we're minusing? And at first, she said subtract. And I'm like, so wait a second, if I subtract that number from [the answer], I'm gonna get a smaller number right? She said yeah. And I'm like, but if we have a smaller number over here, for the variable, is that gonna make sense that you subtract something and get a bigger number? She's like, no. And I'm like, so... She said we were going to add it to it, and I'm like, okay why? And she was like, because we want a bigger number over in the variable spot than we do over in the equals spot. In other words, of course.

Notice that Felicia emphasizes the relative sizes of the known numbers, implying that the operation of subtraction should diminish the unknown.

> *Felicia*: So, I'm like, good, good, good. So I'm like, well look at this, it's the opposite of the operation that's happening to that number, to the variable, to other side, right? She's like, yeah. I said, so whenever you do this, you're going to do the opposite of what's in there to isolate the variable…and I'm like, what's negative 3 point blah blah blah plus that same number, and she's like zero. I said, okay, well that's gone. All we have left over here is the variable, and then we add this over here and that's how we get our answer.

Felicia uses this emphasis to highlight the use of opposite operations in order to isolate the unknown. The PD instructors' emphasis throughout the first unit of the PD Algebra course was on the importance of equivalence when working with equations, a concept that Felicia barely uses in her explanation of the problem. In the PD Algebra course, the PD instructors emphasized the *transformational* aspects of tasks involving solving equations (Kieran, 2007). In this transcript, Felicia conceptualizes the task as a *generational* one for the student; although the equation in question is already formed, Felicia focuses the student on the unknown v in order for the student to reason about the relationship between v and the other two numbers. In essence, she prompts the student to reason backwards from the given equation to the formation of the equation. Although the PD instructors encouraged meaning-building with expressions in the PD course, they did not emphasize meaning in the solving of algebraic equations. Moreover, Felicia prompted the student to do some numerical reasoning with respect to the possible size of the unknown. This reflects Felicia's own work in the task-based interviews, where she would diverge from the presentation and development of algebra from the PD Algebra course in favor of numerical reasoning.

Discussion and Implications

These examples, when put into the context of Felicia's motivational structure, have serious implications for understanding how teachers learn new mathematics content in content-based professional development. Sowder, et al (1998) recognized that teachers' retention and usage in the classroom of new content relied in part on how comfortable they were with the content to begin with. However, the example of Felicia introduces a new dimension to our understanding of how teachers learn. Although Felicia's mathematical self-efficacy was very high and she was comfortable with particular approaches to the content, her largely utility value for the PD course seemed to strongly influence how she approached new and unfamiliar content. Her tight focus on her students (which reflects the findings of Nipper, et al (2011)) prompted her to filter all the content in the PD Algebra course through her perception of what would be useful to her students.

The frameworks of motivation theory present us with a different perspective through which to examine teachers' participation in professional development. Further study on professional development done through this lens may illuminate new considerations for the designers and facilitators of content-based PD programs.

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INSIGHTS ON THE RELATIONSHIPS BETWEEN MATHEMATICS KNOWLEDGE FOR TEACHERS AND CURRICULAR MATERIAL

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This study reports teachers' insights and challenges after one year of adopting a curricular material designed to move students through carefully engineered, small steps and encourage learners through success and accessible challenges. The analysis of interviews showed that teachers 'followed' the material in different ways, not necessary in-line with its underlying principles. Two of these principles—bonusing and breaking down concepts into smaller elements—were particularly difficult for many teachers, suggesting the need of a specific teachers' mathematical knowledge.

Keywords: Mathematical Knowledge for Teaching; Elementary School Education; Teacher Education-Inservice (Professional Development); Teacher Knowledge

Introduction

While there is extensive research on both mathematics teachers' knowledge and the quality of curricular materials, the number of studies combining these two factors is limited. In an effort to address this gap, Charalambous and Hill (2012) reported a multiple case study suggesting that curricular materials can increase quality of instruction if they are supported and followed properly. Understanding the relationships between mathematics teachers' knowledge, curricular materials, and student performance would inform policy decisions regarding adoption and implementation of new resources, as well as the design of corresponding professional learning opportunities for teachers.

This paper analyzes one case of an elementary school adopting new curricular material and engaging teachers in corresponding professional learning over the course of one year. The study was conducted as part of a broad, longitudinal project, the Math Minds Initiative, involving a school district in western Canada, researchers from the University of Calgary, and the JUMP Math (2015) organization. The initiative focused on a particular school with a history of low performance in mathematics. The purpose of the initiative was to improve mathematics teaching and learning at the elementary level and to understand the relationship between curricular resources, teachers' knowledge and students' performance. We are interested in what teachers need to know in order to teach mathematics well, and how this knowledge can be supported through access to particular resources and related teacher professional development. As design-based research, this study draws on multiple sources of data informing next steps in the initiative. However, the focus of this paper is on teachers' experience of adopting the JUMP Math program. Specifically, we address the question: what were the insights and challenges perceived by teachers during the first year that all teachers at the school adopted the JUMP material?

Understanding teachers' insights sheds light on teachers' learning through the year, as well as knowledge required to adopt the JUMP Math materials. Teachers' challenges during this project provide information about the knowledge required not only for the adoption of the material, but also for quality mathematics instruction in general.

Curricular Material and Mathematics Knowing for Teachers

Teachers' disciplinary knowledge of mathematics has been a focus of research since the 1970s. With an initial emphasis on formal mathematics content, over the last few decades, the main interest has shifted to more varied aspects of mathematics knowing such as access to a diversity of meanings for concepts, beliefs on the nature of the subject matter, and how knowledge is enacted in the classroom (Davis & Renert, 2014; Thompson, 2015). While there are efforts to measure this knowing through tests, such as the instrument proposed by Thompson, we concur with Davis and Renert's argument that such knowing includes an open disposition and cannot therefore be readily measured with tests and other instruments. Two features of this disposition are relevant for this report. First, teachers have to be responsive to students' mathematical conceptions and misconceptions. They should be continuously aware of students' potential interpretations of a concept. Second, school mathematics is not limited to standard definitions, notations and algorithms such as those reflected in a program of studies. Teachers should be open to enact mathematics as a creative, emergent activity, which involves mathematical explorations and inquiry beyond textbooks that may result in insights not only for students, but also for teachers.

Teachers draw from a variety of resources including textbooks, teachers' guides, online material, electronic devices, and the community (Clark-Wilson et al. 2014; Gueudet, Pepin, & Trouche, 2013; Gueudet & Trouche, 2009). Following Gueudet, Pepin and Trouche, we conceive the adoption of curricular material as a creative act: "teachers' work with resources includes selecting, modifying, and creating new resources, in-class and out-of-class" (p. 1003). Gueudet and Trouche proposed the term *documental genesis* for the evolving process of the manner in which teachers use a resource. A *document,* for a particular teacher in given moment, consists of a resource and a scheme of utilization. As the scheme of utilization changes over time, a document is dynamic, whereas the resource may remain unchanged. The process of document genesis is twofold: "The *instrumentalization* dimension conceptualizes the appropriation and reshaping processes … .The *instrumentation* dimension conceptualizes the influence onthe teacher's activity of the resources she draws on [emphasis added]" (Guedet & Trouche 2009, p. 205). Most recently, Gueudet et al. (2013) considered a collective dimension of document genesis including joint work on selecting and adapting educational resources. We extend the idea of document genesis to a more ecological perspective in which the community includes not only other teachers, but also the research team and professional learning facilitators. The teacher participants are *coupled* with the researchers and facilitators in a process of mutual influence (Preciado Babb, Metz, Marcotte, 2015). In this sense, our perception as researchers of curricular material is also influenced by our interactions with teachers and informed by the data collectively gathered and analyzed for research purposes.

The Math Minds Initiative

The Math Minds Initiative is a five-year project started in 2012. While the school district provided a research school as a main focus for the study, the team from the University of Calgary provided professional support to teachers from this school as well as from other schools in the district. The JUMP Math organization contributed the mathematics program as well as further support for professional learning. During the first year of the initiative, two teachers started mid-year to use the program with no further support. In 2013 all the teachers were required to adopt JUMP Math as official curricular material and to attend the corresponding professional development sessions through the year.

The curricular material provided by JUMP Math consisted of teachers' guides, an assessment and practice book for each student, and access to pre-designed SmartBoard slides. Additionally, students were provided with individual mini-whiteboards—a suggestion from the research team to assist with the continuous assessment recommended by the resource package.

JUMP Math Principles

The Canadian version of JUMP Math is based on both the *Western-Northern Canadian Protocol for Collaboration in Education* (WNCP, 2006), which provides guidelines for the curriculum in several provinces in Canada, and the Ontario program of studies. The teachers' guide (Mighton, Sabourin, & Klebanov,2010) provides lesson plans with references to each particular outcome in the corresponding program of studies—WNCP or Ontario. The lesson plans correspond to the assessment and practice book and include individual and group activities and explanations. The guide shows teachers how to introduce one concept at a time, explore concepts and make connections in a variety of ways, assess students quickly, extend learning with extra bonus questions and activities, and develop problem-solving skills. It also provides support material for each strand.

The JUMP Math program is based on a number of principles, including confidence building, guided practice, guided discovery, continuous assessment, rigorously scaffolded instruction, mental math, and deep conceptual understanding. While the assessment and practice book consists of sequences of exercises, the teachers' guide has numerous suggestions for engaging students in discovery and problem solving. The guide also encourages students' independent thought and work: "When you feel your students have sufficient confidence and the necessary basic skills, let them explore more challenging or open problems" (Mighton et al. 2010, p. A-5). The JUMP material shows teachers how to break the material into steps and assess component skills and concepts. It teaches "fundamental rules, algorithms, and procedures of mathematics for mastery, but students are enabled to discover those procedures themselves (as well as being encouraged to develop their own approaches) and are guided to understand the concepts underlying the procedures fully" (p. A-6).

Despite the seemingly direct approach to instruction, every lesson in the teachers' guide refers to at least one problem solving strategy, including: looking for patterns; changing into a known problem; reflecting on other ways to solve a problem; doing a simpler problem first; making and investigating conjectures; using mental math and estimation; representing; guessing, checking and revising; selecting tools and strategies; using logical reasoning; justifying the solution; and revisiting conjectures that were true in one context. An important component of the program is *bonusing,* which involves extensions of concepts and skills in each lesson. The teachers' guide advises teachers to "be ready to write bonus questions on the board from time to time during the lesson for students who finish their quizzes or tasks earlier" (Mighton et al. 2010, p. A-8). Lessons in the teachers' guide include examples of such questions, and teachers are encouraged to create their own. Strategies to create bonus question include: change to larger numbers or introduce new terms or elements; ask students to correct mistakes; ask students to complete missing terms in a sequence; vary the task or the problem slightly; look for applications of the concept; look for patterns and ask students to describe them.

Method

The Math Minds Initiative is design-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) that includes the implementation of the curricular material, as well as professional development aimed at improving mathematics literacy in a school with a long history of low achievement—as well as other schools in the district. The initiative also aims at further research and theory on mathematics teacher knowledge. The research project includes multiple sources of data such as video-recorded lessons, class observations, longitudinal results of the Canadian Test of Basic Skills (CTBS, Nelson 2014), and interviews with teachers and students. In this paper we present the analysis of six semi-structured interviews with teachers who taught during the school year 2013-2014 at the research school. Examples of the interview questions are: What specific advice would you give to new teachers joining Math Minds? Have you found [JUMP Math] materials to be helpful? Restrictive or difficult? To what extent did you follow the teachers' guide? SmartBoard lessons?

Workbook? In what ways did you improvise / extend / elaborate? Have you found [JUMP Math] principles helpful? Restrictive or difficult? What are your goals or priorities for improving your teaching of math?

Transcripts of the interviews were coded using NVivo with a particular focus on the manner teachers used the resources to capture the documentation process. The initial codes were compared each other, forming broader categories. Four major categories resulted from the analysis, which included the codes with higher prevalence. These categories are consistent with class observations conducted by different members of the research team.

Results

We present the results in four sections, corresponding to each major category. While the first category refers to how teachers used the material in general, the other three are more specific to JUMP Math principles. Excerpts from the interviews are included as evidence to support our findings.

Document Genesis

Teachers claimed that they followed the teachers' guide and used JUMP Math materials consistently at the beginning. Some tried to fully complete all the pages in the assessment and practice books that applied to the official program of studies and to use all of the associated lessons in the teachers' guide and, often, all of the associated SmartBoard slides. This is evident in the following excerpt:

> *Teacher:* Whereas I think when you first begin, you feel like, okay, I've got to go through each one, and it just wasn't working. So again, it's just the experience and sort of knowing, okay—and obviously previewing the slides and saying, okay, we don't—we can skip this one, or this—unless they're really struggling or—and just being able to know where can I stop and how much do I really need to go through all of this.

While it was clear for teachers that some slides or parts of a lesson would have to be selected, the motivation for such decisions varied. The previous excerpt suggests that the teacher made the decisions based on assessment of students' struggles. However, other motivations included both time pressures and a-priori judgments that some steps were not needed, as is evident in the following excerpt:

> *Teacher:* Towards the end, when I was trying to catch up a bit, I was taking the teacher guide and I was looking at the outcomes and what our curriculum outcomes were, and if it was like … number sense … in four lessons, then I would look at those four lessons, see what the big picture was, because then I could condense them maybe to two lessons instead of four.

A third type of motivation identified in the interviews was familiarity with another, previously used resource. One teacher commented that it was easier to use a resource she was already familiar with, as long as it was similar to what was suggested in the JUMP Math materials:

> *Teacher:* Well I have one that's very similar that will still teach the same outcome, but it's a different game in a little bit of a different way…. Taking what I've had from my past as a teacher, because it worked, it was good. Is it the activity in the JUMP lesson? No, but it worked. And so it would save me some time that way, because it does take a lot of time to prep for these, so I would have something like that, maybe use that game instead.

Finally, another manner in which teachers used the material was to select pages from the practice book for bonusing:

Teacher: I try to follow [the teachers' guide] exclusively. The SmartBoard lessons, like I say, some of them—if they're very hands-on, I will use a lot of them. … I just make sure that I've looked through [the material] and then I just pull up those two or three that I need. And the workbook, I look at it: is this going to be for everybody or is it going to be a bonus page?

This last excerpt shows a decision based on two JUMP Math principles: continuous assessment and bonusing.

Continuous Assessment

All teachers mentioned continuous assessment in the interviews. They also consistently referred to the use of the small whiteboards to assess students in-the-moment. Overall, the material seemed to impact teachers' knowledge regarding this fine-grained presentation of concepts and procedures, as well as the corresponding assessment practice. Continuous assessment not only served to break the content into small pieces, so everybody understands the concept, skill or instruction in class, but also to inform decisions about whether to skip parts that might already be mastered. These decisions, however, seemed to be more difficult to make, as suggested in the following teacher's comment:

*Teacher :*I feel like I need to speed up. I don't know. I need to become better at just moving on and not getting hung up on things and being able to recognize when we can move on and at the same time—and so it's not—at the same time, not compromising that in-depth study of things. Like knowing where, hey, they got it, we can go. We don't need to keep doing this.

Breaking down into smaller steps and constantly assessing students was particularly relevant to a teacher who had been a successful mathematics student:

Teacher: I was very successful in math as a student, and I just get it, and I find it difficult to do those microsteps back as to how to make it simpler for the kids and simplify it. And when I taught it that way, I'm like, oh my God, I don't know how to teach it a different way, because I just get it. And so I don't see a different way to get there, and I think that's my biggest challenge because I've never struggled with math. As a student, I was very, very successful, but that makes teaching math harder, because I don't know how to attack a problem from a child's perspective.

This excerpt is consistent with Davis and Renert's (2014) notion of the teacher being an expert who is able to appreciate the struggles of a novice.

Bonusing

All teachers made reference to bonusing. However, all but one claimed that finding and creating bonus questions and tasks was challenging. Although the teachers' guide shows how to create bonus questions and the assessment and practice book has bonus questions, one teacher perceived the need to find bonus questions beyond the material:

Teacher: You need to find bonus questions often from a variety of other sources beyond the JUMP resource in order to find the proper challenge for each individual child.

This comment also highlights a perceived need to personalize bonus questions. The following statement reflects a similar assumption:

Teacher: Coming up with really good ones has taken a lot of time, a lot of effort. But I feel now I've got a better idea of kind of what works for the kids as well and also just realizing not every kid's going to have the same bonus question, right? Like you're going to change the bonus question based on the kid and kind of the extra challenges that they need.

In contrast, the teacher who found it easy to create bonus questions claimed:

> *Teacher:* Everybody is so engaged in the workbooks and so it gives me an opportunity to continually assess their learning and because there—there is generally enough in the workbook that everybody has enough to do, and it's easy. Having said that, it's very easy to create challenges from—because of the way that the questions are structured, because of the way the work is structured. It's very, very easy to just create challenges on the spot for those who need it. And in a lot of cases, the students will create challenges on their own—their own challenges.

For this teacher, bonus questions and tasks were easy to create on the spot by following the structure of questions in the material. The excerpt also suggests a culture of self-bonusing in her classroom.

Inquiry and Problem Solving

Teachers consistently indicated a lack of opportunity for problem solving or inquiry in the JUMP Math approach. However, most of them indicated that going through the mini-steps was necessary, and that the program did this very well. An example of a teacher's perception on inquiry in the material follows:

> *Teacher:* And when I—and as far as inquiry goes, that is our direction in education in the next ten years, and as soon as I heard that, I thought, well, JUMP doesn't lend itself to inquiry. But in thinking about it, it certainly can, it just has to—it's maybe how we're going to start praising things but once again, I still think we—we need the foundation before we can even [missing word?] an inquiry.
> And so my struggle this year is sometimes should it—should I just do like an inquiry lesson or should I stick with my microsteps, but I want to do the microsteps because I'm learning so much about what I missed teaching them. So to me, right now, that's more important and maybe we throw in an inquiry day on Fridays or something. Throw in everything and just give them an open-ended question and maybe change that next year. It's just this year I'm just sticking to my recipe.

In the previous excerpt, the teacher gave second thought to the possibility of including inquiry in the JUMP Math approach. However, the last comment regarding sticking to the recipe suggests that she did not see inquiry addressed in the material.

There was a particular comment regarding students not being used to more complex, or multistep problems:

> *Teacher:* So all of a sudden, when [students] had to do this sort of a—in a way, it was a multistep problem, whereas the vast majority of this program is very one step questions and these microsteps. So as soon as you throw a multistep problem at them, I was very surprised at how many kids were just like, whoa, what am I going—how do I solve this? And there was just no—not even an attempt to work through the problem.

Overall, teachers' perceptions of inquiry and problem solving seem to be contrary to the problem solving strategies included in the teachers' guide.

Discussion

The analysis of teachers' interviews presented in this paper yields several conclusions regarding the interactions of the classroom resources and mathematics knowledge for teaching. First, the analysis of document geneses showed that teachers' interpretation of what it means to follow JUMP

Math were very different. The initial approach of having all students cover all the material contrasts with the approach based on assessing students and selecting pages from the assessment and practice book for bonus. The latter approach seems to be more aligned with the philosophy of the program.

Second, all the teachers made reference to the incremental steps and to continuous assessment. In particular, the use of the mini-whiteboards supported continuous assessment during class. It is particularly interesting that the teachers who claimed having no problem with mathematics when she was a student found it difficult to break concepts into smaller steps. The use of the resources enabled this realization; however, the resource did not seem to enable her to deconstruct concepts appropriately. This suggests that even if teachers know that the resources were designed around microsteps, they may experience difficulty in breaking concepts and skills into smaller elements themselves. This research finding is consistent with a most recent analysis of teachers' perception of scaffolding in the year proceeding the interviews reported in this paper. Sabbaghan, Metz, Preciado Babb, and Davis (in press) found that teachers with less experience in the initiative tended to use traditional strategies for scaffolding—such as modeling and coaching—in contrast to teachers with more than one year in the initiative who considered micro-level scaffolding strategies.

Third, even though the teachers' guide provides advice on bonusing, most teachers found this very challenging. The idea of bonusing has been evolving during the Math Minds initiative. The research team has compiled examples from teachers implementing the program. The team has also identified connections to the literature on intrinsic motivation, shaping the collective understanding of the bonus principle of the JUMP Math program. Moreover, in contrast to the teachers' guide's emphasis on creating bonus for early finishers, Mighton (2007) also advised to consider bonus questions for everyone: "I always make up special bonus questions for the most challenged students, too, so they can feel that they are doing harder work as well" (p. 106). We have come to perceive bonusing as a strategy for both fostering a positive attitude towards mathematics and deepening mathematical understanding.

The longitudinal results for student performance on the CTBS tests—omitted in this paper due to limited space—showed a significant improvement after one year of adopting the JUMP Math program (Metz, Sabbaghan, Preciado Babb, & Davis, in press). This was particularly reflected in students who initially had low performance. However, scores leveled or decreased for some students with initially high performances. Our principal hypothesis for this situation is that it might be attributed to teachers' lack of confidence in creating bonus questions for students. This hypothesis is supported by the fact that the students of the one teacher who reported confidence with bonusing showed significant improvement across the board (Preciado Babb, McInnis, Metz, Sabbaghan, Davis, in press).

Finally, the general agreement that a resource like JUMP Math does not include inquiry contrasts with the problem solving strategies included in each lesson in the teachers' guide. This is probably due to a strong focus on the assessment and practice book instead of the suggested activities in the guide and the SmartBoard slides. The research team considers that both bonusing and the selected sequence of tasks (Metz, et al., in press) in the assessment and practice book afford opportunities for mathematical inquiry. There is, therefore, a need to better understand the mathematical knowledge required for bonusing and for breaking down concepts into smaller elements.

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THROUGH THEIR EYES: EARLY CHILDHOOD TEACHERS AS LEARNERS AND TEACHERS OF MATHEMATICS

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We explore early childhood teachers' lived experiences learning and teaching mathematics with young children, adding finer-grained context and detail to broader research descriptions. We interviewed ten early childhood teachers in a university laboratory school about their mathematics training, classroom mathematics curriculum, who controls their mathematics curriculum, and their mathematics teaching and learning philosophies for young children. Analysis included coding for Bourdieu's (Grenfell, 1996) social field theory. Results highlight social factors that influence what early childhood teachers do and teach, such as standards, regulations, administration, research, university organization, parents, and children. Teachers worked with autonomy and competence; mathematics teacher educators should leverage early childhood teachers' interest in young children.

Keywords: Early Childhood Education; Teacher Knowledge; Teacher Beliefs; Pre-School Education

Purposes

Early childhood education, for children ages birth to age eight (NAEYC, 2013), is changing due to advances in neuroscience (Shonkoff & Phillips, 2000) and greater public awareness of the effects and benefits of early childhood education (Grunewald & Rolnick, 2006). In the past, early childhood teachers taught little mathematics to children, so they received limited or no professional development related to mathematics instruction (National Research Council [NRC], 2009). This left teachers with limited knowledge and experience with mathematics, mathematics pedagogy, and the mathematical processes and thinking strategies of young children (Early, Maxwell, Burchinal, Bender, Ebanks, Henry, et al., 2007; Sarama & Dibiase, 2004). Therefore, limited research exists concerning early childhood teachers' mathematics knowledge as derived from their experiences and the mathematics they plan for and identify in the activities and centers they create for children. While large-scale studies of early childhood teacher characteristics focus on teacher demographics, training, working conditions, and years and experiences in the field, finer-grained studies are required to give context to teachers' lives and work in order to inform teacher educators, policy-makers, and other stakeholders who develop training and policies for the evolving early childhood field (Early et al., 2007; NRC, 2009). In particular, researchers should examine sources that early childhood teachers draw on to make decisions and judgments about their mathematics instruction (Brown, 2005).

In this study, we examine finer-grained interview data that adds detail and context to these broader descriptions of early childhood teachers' training and that captures social and cultural influences on what teachers do and think in their teaching. In addition, we highlight aspects of early childhood teachers' attitudes, knowledge, and practice that are affected by their varied professional development opportunities and activities (Pianta, Barnett, Burchinal, & Thornburg, 2009). Our study attempts to give voice to often unheard or hidden early childhood teachers and addresses gaps regarding the mathematics that they plan for and identify in activities by recording their lived experiences and concerns regarding the learning and teaching of mathematics (Early et al., 2007; Brooks, 2007). We hope to provide a counter-narrative to one that portrays early childhood teachers as lacking in mathematics knowledge and understanding, both in their own thinking about mathematics and in their understanding of the mathematical thinking of the children in their classrooms.

Theoretical Perspectives and Background Literature

Bourdieu's Social Field Theory

Because the knowledge-beliefs – we consider beliefs a form of knowledge – with which early childhood teachers operate emerge from social and cultural influences (Pajares, 1992), it is appropriate to look at the work and characteristics of early childhood teachers through a sociocultural lens (Edwards, 2003). Bourdieu developed a sociological perspective that can be used to objectively study early childhood teachers' relations, actions and the evolution of their dispositions for teaching mathematics (Grenfell, 1996; Grenfell, 2012; Noyes, 2004).We use his constructs of *habitus, field,capital*, *mechanisms of change,* and *structures*. Next, we describe Bourdieu's constructs using examples from early childhood education.

The *field* of early childhood education. A *field* is a social context involving a network of structures, relations, laws for functioning and specific institutions (Grenfell, 1996; Grenfell & James, 2004; Noyes, 2004). Fields exist on a continuum between heteronomy and autonomy based on the degree to which they can generate their own dilemmas rather than unknowingly reproduce repressive power structures or be affected by external forces. The working environment (or *field*) of early childhood teachers includes different groups of collaborators. Social structures are created between teachers, between teacher and parents, and between teachers and the children in their classrooms. Teachers *structure* the *field,* i.e., the classroom environment and activities, for the children, considering what they know about the children and the implicit feedback the children give (Cadwell, 1997).

Several authorities influence the early childhood field. The National Association for the Education of Young Children (NAEYC),professional organization for the field, provides guidance through publications, position statements, conferences, and accreditation. Those who teach young children refer to "developmentally appropriate practice," (Bredekamp & Copple, 1997), a framework developed by NAEYC to guide those in the early childhood field, both nationally and world-wide, for the best research-based practices regarding young children's learning and development. Also, state early childhood standards play a central role in the lessons that teachers plan (FSSA, 2006). These authorities act as *mechanisms of change* (Grenfell & James, 2004) by changing, for example, what early childhood teachers are required to teach.

Ways of shaping early childhood teachers' knowledge (*habitus*). *Habitus* is a set of dispositions or tendencies that are created in and by individuals' social interactions, and shape and orient how they see the social world (Grenfell, 1996; Grenfell, 2012; Noyes, 2004). Early childhood teachers' habitus is influenced by professional development opportunities and their teaching experiences. Training opportunities for both in-service and pre-service early childhood teachers range from no mathematics courses, courses not specifically related to the mathematics of young children (i.e., college algebra), content courses related specifically to the mathematics of young children, methods courses focused on the mathematics of and pedagogy of teaching young children, and general early childhood curriculum courses that include some content and pedagogy for teaching mathematics (NRC, 2009). Professional development previously focused on developmentally appropriate curriculum and the importance of play (Bredekamp & Copple, 1997; NRC, 2009), but now includes research on children's thinking and learning and using technology to provide training and support to full-time teachers who are part of a large, diverse workforce (NRC, 2009; Sarama & DiBiase, 2004). Research-based interventions have also shaped early childhood teachers' knowledge for teaching (Herron, 2010; Jung & Reifel, 2011).In addition, early childhood teachers' practices are influenced by the experiences they have as teachers, such as working with children and families from different socioeconomic backgrounds and under various state requirements (Lee & Ginsburg, 2007).

Knowledge-beliefs (*capital*). *Capital* refers to different types of assets that individuals possess, such as economic, status, position, or knowledge (Grenfell & James, 2004; Grenfell, 2012; Noyes, 2004). Examining teachers' beliefs is often used to learn about what teachers do and think in their teaching (Pajares, 1992). Pajares described beliefs as created through a process of enculturation and social construction, thoroughly intertwined with knowledge. For example, teachers' beliefs changed after implementing constructivist instructional strategies, evidence that beliefs are grounded in social experiences (von Glasersfeld, 1993, as cited in Philipp, 2007, p. 276). Although we can participate in shared physical and social experiences, our individual understandings of these shared experiences are unique to each of us, not consensual (Philipp, 2007). Because teachers operate as if their beliefs regarding mathematics, teaching, and learning are true for themselves, they operate as if knowledge and beliefs are a single construct (Beswick, 2010). Therefore, as knowledge is constructed from experiences, so too are beliefs.

The literature on early childhood teachers' knowledge-beliefs about mathematics includes general survey information as well as small qualitative studies. Early childhood teachers tend to support more social-emotional development rather than the development of children's mathematical thinking (NRC, 2009). Knowledge-beliefs focus on when children are ready to begin mathematics instruction, as early as age 2 (Sarama & DiBiase, 2004). Counting, adding, subtracting, and shapes constitute the most necessary mathematics topics. Teachers ordered mathematics activities with counting first, followed by sorting, numeral recognition, patterning, number concepts, spatial relations, making shapes, and measuring. They use manipulatives, number songs, and games to accomplish their objectives, but not workbooks or software. Teachers preferred that children explore mathematics activities and engage in open-ended free play rather than large-group lessons. Misconceptions that early childhood teachers hold about young children's mathematics teaching and learning focus on which young children are ready for mathematics education and the content of curriculum, classroom environment, and mathematics assessments; and they might be overcome with sustained research-based professional development(Lee & Ginsburg, 2009).

Mechanisms of change. Change mechanisms challenge the status quo and indicate the amount of autonomy a field possesses. Mechanisms can be internal or external. In order to understand what teachers do, we need to understand both the evolving fields in which they are situated and the nature of their evolving habitus. Disjunction is a *mechanism of change* that teachers experience between the structuring of their habitus and current field, causing change in their practice (Noyes, 2004).

Structures. Structures explain how field, habitus, and capital interact, and are both the product and source of tensions, described as "structuring and structured" (Grenfell, 2006, p. 293). Because habitus cannot be seen, the relational structures that underlie practice and knowledge-beliefs must be explored (Grenfell, 2012).

Women, Their Work, and Mathematics

This study joins other studies that document women as participants in mathematical activity in their work within a traditional female context, such as sewing or caring for young children (i.e., Hancock, 2001). A goal of this study is to allow the teachers to make sense of their own thinking, and to correct the invisibility and distortion of their experiences by giving voice to their multiple perspectives and ways of knowing, creating more knowledge and a broader picture, as each woman's experience tells something different and valuable (Brooks, 2007). Lather (1988) cautioned that by attempting to explain others' lived experiences, the others' reality would be violated. In other words, we will only attempt to explain the teachers' lived experience, knowing that this is just our own understanding of the teachers, not a replica of their experience.

Research Questions

Using a portion of data from a larger research study, we investigate the following research questions: What experiences do early childhood teachers have learning and teaching mathematics? What do these experiences mean for their teaching mathematics with young children? What can their stories of lived experiences as learners and teachers of mathematics tell early childhood teacher educators, policy makers, and other stakeholders?

Methods

Participants

A purposive, convenience sample was recruited from an early childhood development laboratory school at a Midwestern state university. Ten early childhood teachers volunteered. Table 1 includes their education, years at the center, and teaching assignments.

Table 1: Participant and Classroom Descriptions

Classroom Team	Teacher	Position	Degree	Years at Center	Classrooms	
					# of Children	Ages (years)
Ducks	**D1**	Head	BS ECE	17	20	3 – 5
	D2	Associate	BS Child Dev	11		
	D3	Assistant	AS EC Dev	6		
Eagles	**E1**	Head	MS C&I/BS ECE	16	20	3 – 5
Squirrels	**S1**	Head	BS Child Dev	22	14	2
	S2	Student Teacher	BS ECE	Just Hired		
Tigers	**T1**	Head	BS Child Dev	5	14	2-1/2
Koala Bears	**K1**	Head	MS Child Dev	19	14	2-1/2 – 4-1/2
	K2	Associate	BS Ed Studies	1		
	K3	Assistant	CDA Credential	6		

Data Collection Process

Here, we report on one of three data collection activities of the larger study, initial semi-structured interviews with classroom participant teams regarding their experiences learning and teaching mathematics. Because teachers collaborate on planning and implementing curriculum, initial interview data was gathered from each teaching team (see Table 1). During one-hour interviews, teams were asked to describe their training related to teaching early childhood mathematics; classroom mathematics curriculum, activities, and experiences; issues related to who has control over mathematics curriculum in their school setting; and philosophy regarding teaching and learning mathematics with young children. Researchers used a semi-structured interview protocol that was developed from literature (Frid & Sparrow, 2009; Lee & Ginsburg, 2009) and prompted teachers for additional information during interviews as needed.

Analysis

Several rounds of analysis were conducted in order to gain a sense of the underlying "web" (Grenfell, 1996) of relations and tensions (Clandinin, Murphy, Huber, & Orr, 2009) that teachers experience as they learn and teach mathematics. First, each transcript from the initial interviews was divided into chunks expressing a cohesive idea focused on the same idea or activity. Each chunk was coded using Bourdieu's constructs of habitus, field, structuring structures, capital, and mechanisms of change, with subconstructs generated from the data (Grenfell, 1996; Grenfell, 2012; Grenfell & James, 2004); teachers' experiences learning and teaching (Early, et al., 2007; NRC, 2009); and mathematics content (FSSA, 2006). Anecdotal snapshots of each classroom were developed, and short narratives of each teacher were created that included a timeline and dominant themes found in their individual transcripts. From these pieces, we developed narratives of the teachers as learners and teachers of mathematics with young children, along with their philosophies.

Results

The data describes autonomous early childhood teachers affected by external forces as they teach and learn, through interactions with children and adults.

Bourdieu's Social Field Theory "Web"

Habitus and experiences. Teachers described experiences in their classrooms that have affected their teaching habitus (Grenfell, 2012). For example, E1 and D2 explained that they did not like geometry as students and had trouble learning it (structured structure), but as early childhood teachers, they have learned about geometry with and from the children and found engaging geometry activities (structuring structure). T1 explained that most of the children in her classroom of 2-1/2-year-olds speak English as a second language (structured structure), so she has adjusted her learning activities to use multiple media to share new words with the children (structuring structure). T1 also noted that most of the teachers at the center obtained their degrees from the university (structuring structure), which she says explains why many teachers sing the same songs and do many of the same activities (structured structure).

Social fields and power. Teachers alluded to social fields that exert power on their practice and in their classrooms (Grenfell, 2012). S1 and E1 noted that the center is part of a university, and therefore, they teach pre-service teachers in practicums and student teaching, in addition to the children in their classrooms. They also noted that the state department of education has made changes in center and teacher licensing regulations that have affected the early childhood education field in which they work; all of the teams reported that standards exert control over their practice, both in what they plan and how they record and communicate their plans. Most teachers referred to components of NAEYC that influence their practice, including accreditation, developmentally appropriate practices, and state and national conferences. K1 and T1 mentioned that parents expect their children will have fun and receive academic instruction, while D1 indicated that parents share their concerns about their children with her. Most of the teachers noted that children exert control in their classrooms in various ways, as their interests and development/age drive curriculum, and through interactions and language.

Capital and knowledge-beliefs. All of the teachers talked about learning about mathematics with young children from their methods courses, conferences, other professional development, and other teachers. They all also discussed their ideas about what constitutes mathematical activity. For example, teachers in the Duck room incorporate patterning into their line up routine. Veteran teachers appreciated the new knowledge that D3 and S2 brought to their classrooms from their recent coursework, as well as knowledge gained from early childhood research. Half the teachers (K1, S1, S2, D2, D3) described instances of gaining knowledge from working with the children. The teachers all use their knowledge of the curriculum and typically developing children to integrate mathematics into the daily schedule.

Mechanisms of change. E1 described a major event in the history of the school that challenged the status quo immensely and has caused a great deal of disjunction for her. Previously, two early childhood settings co-existed on the university campus, one providing child care and the other operating as a laboratory school that provided instructional settings for undergraduate students and research opportunities. Recently, the school and center merged, and the department under which they operated has been subsumed into another department, resulting in less observation time with young children and fewer connections to research, both as participants and implementers. S1 also mentioned fewer undergraduate students working in the center and conjectured that the economics of working in the early childhood field affected the number of students in the program. Degree programs that an assistant (D3) and a student teacher (S2) are completing have affected activities in the Ducks and Squirrel classrooms, such as by adding mathematics to everyday contexts (mathematizing). Several teachers mentioned that they experienced change in their practice after having the opportunity to put theories learned in coursework or training into practice with children.

Interactions between habitus, field, and capital – Structuring & Structured Structures. The data also includes examples of the field of the teachers using their capital in their interactions with the children. The teachers described mathematics activities and materials they created for the children and classroom environment. These practices are the result of teachers' experiences and knowledge of young children, interacting with external forces including center rules, standards, NAEYC accreditation, state regulations, and the university. The teachers acknowledge children's reactions to the activities and materials as they provide the next activity or materials, or introduce the activity to children in a future classroom. This example of a structured and structuring structure can be explained by Maton's (as cited in Grenfell, 2012, p. 51) equation: [(habitus)(capital)] + field = practice. The teacher has power imposed on her by the outside fields, while she exercises power and capital in her classroom, all the while adjusting her practice according to feedback she receives from the children.

The teachers' experiences as *learners* of mathematics were often frustrating, leaving the teachers feeling that they were "not good at math," "math was hard," and "math was not my favorite subject." However, their experiences *teaching* math with young children have been rewarding, leaving the teachers feeling that "this math is fun," and they want the children to continue having "fun" experiences with math so they grow up to enjoy it, develop positive dispositions toward mathematics (unlike some of the teachers), and succeed as students and mathematicians. It is interesting to note that eight of the ten teachers in this study were seasoned veteran teachers, with five to 22 years of experience. However, the two novice teachers in our study reported more positive experiences learning mathematics than the veteran teachers. Perhaps their stories point to improvements in K-12 mathematics since the veteran teachers were in those learning environments.

Women, Work, and Mathematics

Double consciousness (Brooks, 2007) means that women know their own lives, work, and knowledge, but they also know the dominant culture's knowledge as well because they have to navigate between both. For the early childhood teachers, this means that they know their own experiences and thinking about mathematics. They also have a sense of the "finished product," the mathematics created by the dominant male culture the children in their care will eventually be expected to know. They work back and forth between the two different experiences of mathematics, the math that they find "fun" and "enjoyable" with and for young children, and the mathematics they know the children will eventually be expected to learn. Teachers also work between their training and experiences in child development and requirements of the standards, many stating that the standards "validate" the choices they previously made based on their knowledge of children and development.

Conclusions

Although the veteran teachers in our study described many negative and less-than-productive experiences as K-12 and college mathematics learners, all of the teachers in our study currently draw on their experiences with young children and more recent research-based professional development and state standards in their practice as early childhood mathematics teachers. In descriptions of their work, teachers expressed autonomy, confidence, and competence. Their initial habitus upon entering the field of early childhood education has evolved during interactions with the children in their classrooms, their families, professional development activities, and state standards. Despite acquiring a negative habitus while learning mathematics, they all now enthusiastically teach mathematics with young children, planning activities based on their understanding of early childhood mathematics and the children's development and interests.

This study illuminates several social factors that influence what early childhood teachers do and teach, such as standards, center regulations and administration, research, university organization, parents, and children. Some factors affected all teachers at a center in a similar way, such as the use of standards, while other factors only affected individual teachers, such as each teacher's training. Implications are that early childhood teachers may benefit from examining their early experiences learning mathematics and the role those experiences have on their teaching. Teacher educators might ask preservice teachers to examine these early experiences and then provide tasks that support them to reflect on and integrate new information with their knowledge-beliefs developed earlier (Brown, 2005). They might also highlight the fixed nature of structures such as standards and ask preservice teachers to consider ways they could adjust their notions of practice.

The results from our study suggest that both veteran and novice early childhood teachers bring a lot of capital to their work, which should be respected by stakeholders, including early childhood teacher educators and policy makers. In addition to appreciating veteran teachers' capital, teacher educators might emphasize to their students that, although novice teachers expect to learn from veteran teachers, the capital novice teachers bring to teaching is often appreciated by veteran teachers. Our results also suggest that grounding math methods or content training in experiences *with* young children could engage early childhood teachers when they feel less competent about the training subject, such as geometry with young children. This would connect and value the work that these women do with young children with more formal mathematics.

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STANDARDIZED ASSESSMENTS OF BEGINNING TEACHERS' DISCUSSION LEADING PRACTICE: IS IT POSSIBLE AND WHAT CAN WE LEARN?

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This paper examines the possibilities of using standardized assessments to assess elementary and secondary novice teachers' skills with leading problem-based mathematics discussion. The findings reveal that our standardized assessments were able to elicit and reveal variations in skills across teachers; provide fine-grained detail about the performance of particular teachers for feedback purposes; and account for existing classroom norms. This suggests that such tools could be useful in assessing and supporting beginning teachers.

Keywords: Teacher Education-Preservice; Instructional Activities and Practices; Assessment and Evaluation; Classroom Discourse

Theoretical Framing

The preparation of beginning teachers has received increased attention in recent years with calls to ensure that beginners are prepared to take responsibility for all students' learning upon entry to the teaching force. This call has spurred changes towards preparing beginners to <u>do</u> core tasks of teaching. In a number of teacher preparation programs, course content is shifting to focus on high-leverage teaching practices such as eliciting and interpreting student's thinking and leading discussions (Ball, Sleep, Boerst, & Bass, 2009; Davis & Boerst, 2014; McDonald, Kazemi, & Kavanagh, 2013). This shift means that teacher education programs must also develop ways to assess the <u>doing</u> of teaching. Many current assessments of teaching, including observation tools (e.g., Danielson, 2011) and portfolios (e.g., Darling-Hammond, 2010), offer information about beginning teachers' skills with respect to broad domains of work like planning, instruction, or assessment. But we need additional tools to improve estimates of beginners' skill with particular practices of teaching.

One practice that has received considerable attention in teacher education is the practice of leading a discussion in mathematics. Mathematics discussions are important for supporting students in developing conceptual understanding (e.g., Michaels, O'Connor, & Resnick, 2008) and learning disciplinary norms and practices (e.g., Lampert, 2003; Yackel & Cobb, 1996).For these reasons, a number of current efforts are focused on helping novice teachers learn to enact this complex practice (Boerst, Sleep, Ball, & Bass, 2011; Lampert et al., 2013). Given this, teacher educators will also need ways to assess novices' skill with leading mathematics discussions, beyond relying on plans for leading a discussion, reflections, or analysis of others' enactment (e.g., through video analysis).

Many factors beyond sheer skill influence novices' enactment of specific practices. As teachers lead whole class discussions, for example, factors related to students, the content, and the environment, matter for the unfolding of the discussion (Cohen, Raudenbush, & Ball, 2003). Teachers' knowledge of the content, the "discussability" of the mathematics task, students' prior experiences participating in discussions, and teachers' knowledge of and relationships with their students all shape how discussions play out. All of this can make it difficult to appraise novices' skills in ways that are comparable and fair. We sought to investigate whether it is feasible to design a standardized assessment that is capable of: (a) eliciting and revealing variations in skills across teachers; (b) providing fine-grained detail about the performance of particular teachers for feedback purposes; and (c) accounting for existing classroom norms. Below we begin by articulating and specifying what we mean by a "mathematics discussion" then turn to the describing the assessment development.

Work of leading a discussion

We grounded our assessment development in a particular decomposition (Grossman et al., 2009) of the practice of leading a mathematics discussion. We define classroom discussion as "a period of relatively sustained dialogue among the teacher and multiple members of the class" in which students respond to and use one another's ideas to develop collective understanding (TeachingWorks, 2015).Our decomposition is informed by research on practices for orchestrating productive discussions (Smith & Stein, 2011), the concept of talk moves (Chapin, O'Connor, & Anderson, 2013), and research on decomposing practices so that novices can learn them (Boerst et al., 2011). We differentiate *discussion-enabling practices* (Boerst, Moss, & Blunk, 2009), such as anticipating student thinking and monitoring student work, and *discussion- leading practices* used to manage a discussion, such as eliciting student thinking. We distinguish between three stages of the discussion: launch, orchestration, and conclusion. The launch comprises the work a teacher does to frame the discussion (Engle, 2006) and is separate from the actual setting up of the task. Within each stage, teachers engage in particular discussion-leading practices. For example, orchestration includes practices such as *eliciting contributions* and *probing student thinking*. Figure 1 shows our decomposition of discussion-leading, as well as examples of discussion-enabling practices. Within each discussion-leading practice, we further identify techniques (Boerst et al., 2011), including particular talk moves.

Discussion Enabling	Discussion Leading		
Anticipating student thinking Setting up the problem Monitoring student work	**Launching**	**Orchestrating** - Eliciting - Probing - Orienting - Making Contributions	**Concluding**
	Recording		

Figure 1: Decomposition of Leading a Mathematics Discussion

Assessing beginning mathematics teachers' discussion leading practice

Because beginning teachers are more often assessed on their ability to plan, we set out to design an assessment focused squarely on the interactive work of leading a discussion. We also wanted the assessment to be classroom-based, since so much of the work of leading a discussion involves responding to student thinking in the moment. We wanted our assessment to elicit and capture both a range of and variation in performance, as well as provide fine-grained detail about the demonstrated skills of individual teachers. Because we intended to use the assessment in real classrooms, it was important that it account for variations in classroom norms and grade levels.

With these goals in mind, we developed parallel elementary and secondary assessments to be implemented in novice teachers' classrooms. Simultaneously, we sought to design the assessments with as much standardization as possible to allow for comparing teachers' performances. Because discussion-enabling practices are critical to the work of preparing for a discussion (e.g., Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013), we provided supports for teachers to prepare for the discussion so that our assessment could focus squarely on their discussion-leading skill. Both the elementary and secondary level assessments required teachers to lead problem-based mathematics discussions, in which students work on a mathematical task and then participate in a discussion about their work on the task. We selected mathematical tasks that have been extensively piloted in K-12 classrooms and can be found in existing materials (Ball & Shaughnessy, 2014; Mathematics Assessment Resource Service, 2012). Table1 provides an overview of each of the tasks. These tasks offer opportunities to compare and connect a range of solutions, strategies, or approaches, which represents one discussion structure (Kazemi& Hintz, 2014). The tasks were also selected because they focus on high-leverage mathematics content (Ball & Forzani, 2011), can be used across grade bands, and because we conjectured that they could be implemented as a "drop-in lesson" without necessarily being connected to the current instructional focus of a classroom.

Table 1: Elementary and Secondary Mathematics Tasks

Elementary: You will lead a discussion of solutions to the problem, "Make number sentences for 10," a problem with multiple solutions. The students will first work independently or in pairs on the problem. Then, you will lead a discussion of the task. The goal of the discussion will minimally be to elicit several solutions to the problem and to have students explain why they are or are not solutions and to notice similarities and differences among the solutions. The focus for your discussion will vary depending on student solutions. A table of possible student solutions is included to help you organize your discussion.	*Secondary:* You will lead a discussion of solutions to the problem "Proofs of the Pythagorean Theorem?" The students will first work on the task, then you will lead a discussion of the task. The task asks students to evaluate several attempts at proving the Pythagorean Theorem. The goals of the discussion include eliciting several solutions to the problem, ensuring that the merits and limitations of each solution are explained, and identifying features of a strong mathematical proof. It is not necessary to complete a standard proof of the Pythagorean Theorem.

By specifying the mathematics tasks, we sought to control for the variation that would arise if teachers led discussions on different topics. This was important for achieving our goal of designing a tool that could compare performances. Prior experiences teaching methods courses led us to believe that providing a discussable task was a support that could enable the assessment to focus on assessing teachers' discussion-leading skills.

Grades 2-3	
Generate number sentences for 10 using multiple operations determine whether the value of a proposed expression is 10 listen to classmates and provide justification for agreeing or disagreeing with specific explanations begin considering the idea of infinitely many solutions make a claim regarding the number of possible solutions with justification	Students will likely generate number sentences for ten that move beyond integer pairs that sum to 10 (e.g. 1 + 9). Their expressions are likely to contain multiple terms and are likely to incorporate subtraction, although some students may need prompting The types of expressions that students might generate include: 1 + 1 + 1… = 10, 2 + 3 + 5 = 10, 11 – 1 = 10, 83 – 73 = 10 Students might notice number patterns that allow them to generate many expressions quickly, e.g. 12 - 2, 22 - 12, 32 - 22…. Students may also incorporate multiplication and division. If students use subtraction, multiplication and/or division, they are likely to think that there is a very large finite number of solutions to the problem . Students may state that the number of solutions to the problem is infinite.

Figure 2: Support for Differentiation of Learning Goals and Anticipating Student Thinking

We also built in supports around other discussion-enabling practices. For example, the assessments included supports for understanding the mathematics, anticipating student thinking (Smith & Stein, 2011), and adapting the task for different grades. Figure 2 shows an excerpt of the support materials for differentiating the goals of the task and anticipating student thinking. Both assessments also included lesson plans with suggested timing, participation structures, and guidance for setting up the task.

We also developed a scoring tool that could be used to analyze and assess videos of teachers' leading these discussions. This tool was designed to focus on the discussion-leading practices core to our decomposition of the work: launching a discussion, eliciting responses, probing students' thinking, orienting students to one another's contributions, making contributions to the discussion, recording/representing content, and concluding the discussion. Our tools also focused on specific techniques. These techniques were selected for their importance to skillful beginning teaching. For example, "orienting students to the contributions of peers" had five associated techniques: (1) students are prompted as needed to talk to the whole class; (2) the teacher poses questions to students about others' ideas and contributions; (3) students are asked to comment on, add to, or restate other students' ideas; (4) listening is supported by moves that ask all students to respond to one another's work; and (5) students are encouraged to listen, and respond to maintain productive and focused interactions. We included this level of detail in the scoring tool to reflect our decomposition of practice and to meet the goal of developing an assessment that would allow us to capture differences in performance across and within performances. Importantly, we were assessing teachers' skills in responding to student contributions, not whether students responded in a particular way (e.g., students might not respond to a well-formulated question intended to probe their thinking).

In developing the scoring tool, we recognized that there are many ways to score or assess instruction, including rubrics that differentiate levels of performance in particular domains. We chose to use threshold statements to record the presence or absence of particular techniques with respect to a defined threshold. This choice motivated the goal of using of the tool for identifying whether teachers were enacting techniques in the different practices. This format could be particularly useful in identifying patterns across groups of teachers as well as within an individual teacher's performance. Additionally, we built in a "not applicable (N/A)" choice because some specific techniques may not be needed in particular cases. Finally, we developed a section that captures common issues that may arise in discussions. A codebook was developed with definitions of each technique, examples of what it was, and examples of what it was not.

Methods

We piloted the assessment with 17 first-year teachers (9 elementary teachers across grades 1-5 and the 8 secondary teachers across grades 7-12). We recruited a diverse sample of teachers with respect to grade level, school district, and teacher preparation program. The purpose of the pilot was to gather data from a range of first-year teachers and was not intended to be representative of all first year teachers. We provided the plan for the discussion and gave participants 45 minutes to prepare.

The discussion was video-recorded by the research team. One camera was set up in the back of the classroom and focused on teacher and student interactions. Discussions ranged from approximately 15 to 45 minutes in length, although the full lessons were longer. Participants also completed a background survey indicating their prior experience and training in discussion leading practices. All teacher names used in the subsequent sections are pseudonyms.

The analysis of the discussions was conducted by the research team through independently watching and scoring each video using the tool described previously. Then the team, comprised of members with expertise in both elementary and secondary mathematics teaching, met together to discuss the scoring and reach a final consensus. When there were discrepancies across scores, the full team examined the video and resolved the issue, referencing the codebook as needed. A subset of the videos (>20%) was additionally coded by a trained rater, yielding an inter-rater agreement of 85% (Miles & Huberman, 1994).

Findings

We analyzed the 17 discussions to examine how well the assessment was able to: (a) elicit and reveal variations in demonstrated skills across teachers; (b) provide fine-grained detail about the performance of particular teachers for feedback purposes; and (c) account for existing classroom norms. We address each of these in the following sections.

Eliciting and revealing variation in demonstrated skills

In looking across all performances elicited by the assessment, we found that the scoring tool was able to reveal a range of skill with respect to all of the discussion leading practices. For example, within the domain of orchestration (including the practices of eliciting, probing, orienting, and making contributions), the assessment and tool revealed skills that varied from a minimum of three demonstrated techniques to performances that revealed use of all 15 techniques associated with orchestration. Additionally, the assessment and tool was able to elicit and reveal a range of performance within each discussion leading practice. For example, Table 2 shows a range of performance on each of the four discussion leading practices used during the orchestration stage of the discussions.

Table 2: Range of performance within discussion leading practices

	Eliciting	*Probing*	*Orienting*	*Contributing*
Range of techniques	1/3 to 3/3	0/4 to 4/4	0/5 to 5/5	1/3 to 3/3

Our study suggests that the assessment and scoring tool are capable of revealing variation in performance. We found that the tool was capable of revealing variation in skills at multiple levels, including higher-level variations between discussion leading practices and more fine-grained variations in performance within particular discussion leading practices. As an example of this variation, within this sample, teachers demonstrated greater skill at eliciting student thinking than probing student thinking or orienting students to the contributions of peers. All of the elementary teachers demonstrated skill with every eliciting technique/move, and six out of eight secondary teachers did as well. In contrast, only two out of 17 teachers engaged in all aspects of orienting, many of them engaged in only some aspects, and a few teachers engaged in almost no orienting. This means that the tool enabled us to discern differences in the skills of groups of novice teachers with respect to discussion-leading practices.

The assessment and scoring tool were also able to capture fine-grained variations of performance within particular discussion-leading practices. For example, with the practice of *probing student thinking*, which included four different techniques, 14 out of 17 teachers engaged in probing students' mathematical processes, while only nine teachers probed students' understanding of key mathematical ideas. For example, one secondary teacher, Ms. Mason, first elicited student ideas about which of three attempted proofs of the Pythagorean Theorem is most valid and complete. A student replied "In attempt number one, you have to find the area of the squares in order to get the triangle in the middle". Ms. Mason responded by probing the student's process, "Okay, how do I find the area of a square?" The tool showed that almost all of her probing focused on process. In contrast, another secondary teacher, Mr. Jacobs, probed both for student process and understanding. In one case, as students were defending their choice of one potential proof as being the best, a student said "I said that number two…it explains things. There's a lot of information". Mr. Jacobs responded, "What do you mean by the most information?" probing the students' understanding of what "information" meant in the context of a proof. This suggests that the structure of the scoring tool used with the discussion assessment data was able to highlight differences in teachers' demonstrated skill with particular practices.

Providing fine-grained detail about the performance of particular teachers

The assessment and scoring tool were able to elicit and capture variations of performance within teachers' discussions. To illustrate this capability, we describe what the scoring tool revealed about the performance of one fourth grade teacher. Mr. Weber launched his discussion efficiently and framed the mathematics to be discussed, although he did not have all students' attention before beginning. Within the practice of eliciting, Mr. Weber enacted all of the techniques that we sought to capture, including eliciting multiple solutions/strategies. He demonstrated less skill in the practices of probing and orienting, posing no questions that probed students' processes or understanding. Mr. Weber also did little to orient students to each other's contributions. He occasionally reminded students to listen to each other and asked for signs of agreement/disagreement with a contribution, but he did not ask students to comment on, add to, or restate others' ideas. Mr. Weber did enact techniques within the practice of making contributions. He revoiced and asked questions to engage students in substantive discussion of the key ideas; however, the mathematical contributions he made did not enrich the core ideas of the discussion nor did they keep the discussion focused on the learning goals. Throughout he demonstrated skill with recording and representing content as he recorded students' number sentences on the board, although he did not always attend to the mathematical accuracy of his records. Mr. Weber concluded by taking stock of the discussion and worked to support students in remembering that they had determined that there could be infinitely many solutions to the task. This highlights how the assessment and scoring tool allowed us to capture detail and nuance within individual teachers' practice.

Accounting for classroom norms

Across the performances the scoring tool appeared to be able to account for variations in existing classroom norms. In some classrooms, students added on to another's thinking and spoke to the whole class without prompting from the teacher. The designation of N/A allowed us to recognize that students were doing these thing while still acknowledging that the teacher did not visibly employ moves that supported students in participating in this way. Yet, only two techniques received a substantive number (>3) of N/As. This indicates that the assessment and tool were able to elicit and capture demonstrated skill with the majority of the techniques. The two techniques that were frequently coded as N/A were (1) supporting students to speak to the whole class and (2) supporting students to listen to contributions of peers. Both techniques are used in the service of *orienting students to the contributions of peers*. The high frequency of N/A suggests that the standardized assessment and scoring tool may not be suitable for systematically eliciting and capturing teachers' skill in this area.

Discussion and Implications

We examined the utility of using a standardized assessment to assess novices' discussion-leading practice. The assessment and tool prompted and captured a range of performance and revealed variations across groups and within individual discussions. We also found that it was able to account for existing classroom norms.

To consider the implications of these findings, we begin by acknowledging the limitations of how we designed and piloted the assessment and scoring tool. One limitation is that the assessment and tool are grounded in a particular decomposition of the practice of leading a discussion. A different decomposition might reveal different capabilities of the assessment and scoring tool because if teachers are more familiar with other decompositions of the practice, they may perform differently. A second limitation concerns the design of our scoring tool, which used threshold statements, yielding information about the presence or absence of each technique within a practice. This meant that the scores did not always distinguish the quality or quantity of teachers' enactments above the threshold. Finally, these analyses did not consider the validity of the assessment with respect to whether it accurately reflects a teacher's typical practice. Despite these limitations, our findings suggest that a standardized assessment of discussion-leading practice can reveal important information about teachers' skill. As we were not intending to make claims about this sample of teachers or a larger population of novice teachers, the small sample was appropriate for our goals. However, the variations in performance that are used to illustrate the capabilities of the assessment and tool cannot be interpreted as representing the skills of a larger population of teachers.

This standardized assessment accomplished many of our design goals. Its ability to capture a range of skill and to distinguish patterns across groups and within individual teachers' performances could make the assessment and scoring tool useful in teacher education. Teacher educators and programs could use such assessments to track teacher candidates' growth over time and to identify areas of strength and weakness with respect to the practice, which would allow for targeted support and program-level curriculum design. The use of scaffolds for particular parts of a lesson also showed promise for allowing the assessment to focus on a single instructional practice.

Additionally, the subject matter standardization allowed for comparison across performances, as well focusing the assessment squarely on discussion leading practices through providing standardized supports. Standardized mathematics content supported the efficiency and manageability in using both the assessment and scoring tool, as scorers did not need to familiarize themselves with the content and task of each discussion. Even with standardization of content and content supports built into the materials, there were instances in which subject matter knowledge appeared to be a factor in teachers' discussion-leading practice. In these cases it was unclear whether we were accurately capturing a teacher's skill with discussion leading or whether we measured indirectly his/her use of content knowledge in teaching. Subsequent work will need to take this into consideration.

We found that the practice of recording and representing content was challenging to assess using this assessment. Although the assessment materials asked teachers to record student contributions and other relevant mathematics, a number of elementary and secondary teachers involved students in the recording or relied completely on student-generated records. This may have been a reasonable choice given existing norms and routines and in some cases it appeared to support students in providing explanations to the class, but this choice made it difficult to see evidence of how these teachers record content. The materials may not have specified clearly enough that while students often contribute to recording, the assessment was asking teachers to record contributions. Future versions of the assessment will need to keep this challenge in mind; we see several different possibilities for addressing this challenge.

Future research could consider how well the assessment of novices' discussion-leading practice corresponds to their typical practice when leading discussions. Another important direction will be to further take up the questions about the role of subject matter knowledge in assessing teachers' skill with discussion-leading practices. Finally, further research could also investigate the impact of scaffolding materials for novices with different conceptions of what makes a mathematics discussion.

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NOTICING STUDENT MATHEMATICAL THINKING IN THE COMPLEXITY OF CLASSROOM INSTRUCTION

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Noticing students' mathematical thinking is recognized as a key element of effective instruction, but novice teachers do not naturally attend to and make sense of student thinking. We describe a design experiment in which prospective teachers were engaged in analysis of minimally edited classroom video in order to support their ability to notice important student mathematical thinking within the complexity of classroom instruction. We discuss evidence of prospective teachers' learning in five iterations of the intervention, including the extent to which they developed a focus on students' mathematics, changes in the ways they discussed that mathematics, and the extent to which they focused on instances of student mathematics that had potential to be capitalized on to support student learning. Aspects of the intervention that seemed to support teachers' noticing are discussed, as well as future directions for the work.

Keywords: Classroom Discourse; Design Experiments; Teacher Education-Preservice

Research has shown that a key distinction between novice and expert teachers is their ability to notice what is important in complex classroom situations (Berliner, 2001). Because teachers' use of student thinking has been identified as a key element of effective instruction (e.g., NCTM, 2014) and has been linked to increased student learning (Fennema, Carpenter, Franke, et. al, 1996), one particularly important focus of teacher noticing is student mathematical thinking. However, research suggests that teachers, particularly novices, do not naturally attend to and make sense of student ideas (Jacobs, Lamb, & Philipp, 2010). Fortunately, noticing student mathematical thinking is a skill that can be learned (e.g., Jacobs, et al., 2010; Sherin & van Es, 2005) and has thus become a focus in many teacher preparation programs.

Many recent prospective teacher noticing interventions, focused on what has been termed *professional noticing of children's mathematical thinking* (Jacobs, et al., 2010), have supported noticing through analysis of student written work (e.g., Fernández, Llinares, & Valls, 2013; Haltiwanger, & Simpson, 2014) or short video excerpts of one-on-one student interviews (Schack, Fisher, Thomas, et. al, 2013). This work has also typically focused on student thinking related to specific mathematical content, for example, proportional reasoning or early numeracy. Evidence across these interventions suggests that prospective teachers' noticing skills can be successfully developed in "environment(s) in which the number of salient features was limited and, therefore, a manageable focus for discussions" (Schack, et. al, 2013, p. 395). Mathematics classrooms, however, are not limited in the scope of what might be noticed. This raises questions of whether prospective teacher noticing can be supported in a more complex context that simulates that of classroom noticing, and whether prospective teachers' professional noticing skills can be developed using classroom artifacts that include a range of mathematical foci.

Evidence suggests that using classroom video as a medium to promote professional noticing might be enhanced by providing a way to scaffold teacher noticing, such as targeted questions or a framework (e.g., Roth McDuffie, Foote, Bolson, et al., 2014; Santagata, 2011). Santagata (2011), for example, found that posing targeted questions to focus teachers on the relationship between a teacher's actions and students' learning of mathematics supported teachers in providing more in-depth analyses of these interactions. Similarly, Roth McDuffie and colleagues (2014) found that providing carefully designed prompts supported prospective teachers in higher level noticing, including making connections between key components of teaching and learning. In short, research suggests that specific prompts support the noticing of what is valued within classroom interactions. This study builds on this work by examining the use of an explicit analytic framework to scaffold prospective teachers' mathematical noticing.

In this study, we examine the effects of an intervention designed to promote prospective teachers' noticing of student mathematics that could be used by a teacher to support students' understanding of the mathematics. In particular, we focus on the following research questions: (a) In what ways does prospective teacher noticing change as a result of the intervention?; and (b) How do variations in the viewing framework appear to affect prospective teacher noticing?

Theoretical Framework

Consistent with Jacobs and colleagues (2010), we aim to promote the *professional noticing of [student]'s mathematical thinking*. We follow their definition of this practice to include three interrelated skills: (a) attending to student thinking, (b) interpreting what students are saying mathematically, and (c) deciding how to respond. In this study, we focus on just the first two components of the practice. Our choice of this noticing focus stems from its close connection to our goal of helping prospective teachers learn to enact *ambitious teaching* (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010)—"deliberately responsive and discipline-connected instruction" (p.130) that supports all students in developing deep understanding of mathematics.

Although our work is firmly grounded in the noticing of student mathematics, we also take the perspective that not all instances of student mathematical thinking have the same potential to enhance student learning. Thus, we focus specifically on noticing instances of student thinking that have significant potential to be used during the lesson to support the learning of important mathematics. In particular, we draw on Leatham, Peterson, Stockero, and Van Zoest's (2015) definition of **M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking [MOSTs] as occurring at the intersection of three characteristics: (a) student mathematical thinking, (b) significant mathematics, and (c) pedagogical opportunity. The MOST analytic framework uses two criteria to determine whether an instance of student thinking embodies each of these characteristics. For student mathematical thinking the criteria are that there is sufficient evidence to reasonably infer the student mathematics and that one can articulate a mathematical point closely related to this student mathematics. The significant mathematics criteria are that the mathematical point is appropriate for the mathematical development level of the students and is central to mathematical goals for their learning. The pedagogical opportunity characteristic requires that the student mathematics creates an opening to build on student thinking to help develop an understanding of the mathematics and that the timing is right to take advantage of the opening. Instances that satisfy all six criteria, and thus all three required characteristics, are MOSTs (see Leatham et al., 2015 for more details). In this study, the MOST analytic framework was used as a tool to focus participant noticing.

Methodology

The Intervention

The participants in the study were 17 prospective mathematics teachers (PTs) enrolled in an early field experience course between fall 2011 and fall 2014. They participated in the study in five cohorts, each with three to four PTs. Each PT was assigned to observe a local, experienced secondary mathematics teacher's classroom. Participants recorded videos of mathematics lessons in these classrooms on a rotating basis; over the course of each semester, efforts were made to collect video from a range of grade levels with varied mathematical topics. The instructional portions of the classroom video were left mainly unedited for analysis, although portions in which students could not easily be heard were removed. The PTs and researchers used the Studiocode (SportsTec, 1997-2015) video analysis software to individually analyze one video each week, marking mathematically important moments a teacher should notice in the classroom. The PTs included a description of why they chose each moment. The researchers met weekly to agree on instances that were MOSTs in the video and to discuss the instances PTs had identified as important, including which instances would be discussed at a weekly group meeting among the PTs and researchers. These group meetings were facilitated by the first author and focused on building PTs' skills in noticing mathematically important moments.

Because the intervention was conceived as a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), the specific scaffolding activities varied by semester. All of the PTs were initially prompted to identify *mathematically important moments that a teacher should notice during a lesson*, with the definition of this construct left open ended to provide baseline data for PT noticing. All of the cohorts except cohort 5co-developed labels in the weekly group meetings to describe and categorize types of mathematically important moments. These labels drew on Stockero and Van Zoest's (2013) pivotal teaching moment categories, but the categories were not made explicit to the PTs. After the creation of these labels, PTs assigned labels to moments they marked in subsequent videos. Each cohort was introduced to a variation of the MOST framework at some point during the semester. This framework evolved over the four years of the study, so cohorts 1 and 2 used a general version of the framework in which the MOST criteria were only loosely defined in terms of student thinking, important mathematics, and pedagogical opportunity. Cohorts 4 and 5 used the most explicit, analytical version of the framework as defined in Leatham and colleagues (2015). The MOST framework was introduced to the first four cohorts five to six weeks into the semester and to cohort 5 two weeks into the semester. After the introduction of the framework, PTs were expected to focus on moments in the videos that were MOSTs, as defined by the version of the framework available at the time. Cohort 5 was pushed to most explicitly discuss each characteristic of the framework.

Data Collection and Analysis

The study data was the PTs' video analyses and video recordings of the weekly meetings. To analyze how the PTs' noticing changed during the intervention, we analyzed the video instances the PTs had marked and their reasoning. APT's written description of an instance received the most weight. When available, aPT's label and what (s)he said about the instance in the meeting were also considered. To minimize the possibility that PTs influenced each other's thinking, the meeting commentary received more weight if a PT was the first to speak about an instance.

The unit of analysis was a PT-identified instance. Drawing on frameworks used in previous research (Stockero, 2008; van Es & Sherin, 2008), each instance was coded for the agent of the PT noticing and the level of specificity with which the mathematics was described. If there was any student focus in an instance, an additional code was assigned to describe the nature of the PT's noticing of the student(s). Figure 1 gives the coding categories, definitions and codes.

Coding Categories	**Description**	**Codes**
Agent	Who or what was the focus of the noticing*	Teacher (T), Teacher/Student (T/S), Student/Teacher (S/T), Student Group (Sg), Individual Student (Si), Math (M)
Math Specificity (MS)	Whether and how the mathematics is discussed	Non-math (NM), General Math (GM), Specific Math (SM)
Nature of Noticing (NoN)	For instances with some student focus, what about the students was attended to	Affective Interaction, General Understanding, Mathematical Interaction, Noting Student Math (NSM), Analysis of Student Math (ASM)
*Student-teacher interactions are coded as teacher/student if the teacher is the primary focus and student/teacher if the		

student is the primary focus

Figure 1:Coding Framework

To give a sense of some of the key codes that we will discuss, consider the three instances in Figure 2. These all focus on the same instance of an individual student's thinking so were all coded as Si for the agent. Although instance 2 is focused on a student mathematical question, we have no idea what the related mathematics is, so it was coded GM for math specificity. The mathematics is discussed in a much more specific way in instances 2 and 3, however, so these were coded as SM. Instances 1 and 2 describe the important student mathematics, but do not go beyond this, so were coded as a NSM focus. Instance 3 on the other hand, is making sense of why the student asked the questions she did, so was coded as an ASM focus.

	Description of Instance	**MS**	**NoN**
	I think that this is a very good question that is necessary to further learning (making examples similar). The student describes her question in depth to infer her thinking.	GM	NSM
	Student is curious as to why they need to divide by the determinant to find the inverse [matrix] in this example, but not in the previous example.	SM	NSM
	[S]he does not know why we are multiplying by 1/2 in one example and not the other. The student does not understand that they have to multiply by 1/det[A], probably because the multiplication of 1/det[A] was not shown in the previous example because the det[A]=1. We {we]re multiplying the matrix [d -b –c a] by 1, which is trivial. The students should understand that if two matrices multiplied together give the identity, that they are inverses of each other.	SM	ASM

Figure 2:Coding Examples

In the analysis process, each PT-identified instance was individually coded by either two or three researchers. The researchers then met to reconcile the coding; when there was disagreement about one or more codes for an instance, the instance was discussed among the group until agreement was reached. In cases where two researchers were unable to reconcile their coding, the third researcher was brought into the discussion to help resolve any coding differences.

Results

The goal of the intervention was to promote the noticing of instances of student mathematics that have the potential to be capitalized on during a lesson to support student learning. Thus, our analysis focused on the extent to which we were able to promote such noticing. We discuss both changes in the PTs' noticing as a result of the intervention, as well as the extent to which the PTs met our "target" for noticing: individual students, specific mathematics, and noting or analyzing student mathematics. In this discussion, *baseline* refers to the PTs' noticing in the first two videos each semester, before there was any attempt to focus their noticing. *Final* refers to the noticing in the last four videos each semester—an indication of the PTs' most refined noticing. We use four videos to report the final noticing because most of the PTs noticed significantly fewer instances in these later videos. In fact, PTs noticed an average of 8.8 instances per video early on, and less than half that amount (3.75 instances/video) in the final four videos, an indication of becoming more selective about the instances deemed important to notice.

Agent

Table 1 shows the percent of PT noticing focused primarily on students (coded Sg, Si, or S/T), primarily on the teacher (T or T/S), and on the mathematics itself in the baseline and final data. At the start of the intervention, a significant percent of the PTs' noticing was focused primarily on the teacher in the video, and to a lesser extent, on the mathematics itself. This non-student focused noticing accounted for between 25% and 81.2% of the baseline noticing. Except for cohort 1, 51.0% or fewer of the instances noticed by the PTs in the baseline data had a primary student focus. At the end of the intervention, however, the majority of each cohort's noticing was focused on the students in the video, ranging from 85.5% to 100% of instances.

Table 1: Participant Noticing by Primary Agent

		Cohort 1	Cohort 2	Cohort 3	Cohort 4	Cohort 5
Primary Student Focus	Baseline	75.0%	46.7%	51.0%	18.8%	28.4%
	Final	88.9%	100.0%	90.0%	98.3%	85.5%
Primary Teacher Focus	Baseline	22.7%	33.3%	42.9%	59.4%	62.5%
	Final	9.5%	0.0%	6.0%	1.7%	14.5%
Math Focus	Baseline	2.3%	20.0%	6.1%	21.9%	9.1%
	Final	1.6%	0.0%	4.0%	0.0%	0.0%

Although the primary focus on students was encouraging, we were particularly interested in the extent to which PTs focused on individual students and their mathematics. Thus, we further analyzed the subset of instances with a primary student focus to determine whether the PTs' focus in these instances was on individual students, groups of students, or student-teacher interactions (Table 2). At the start of the intervention, each cohort's student-directed noticing was either focused mainly on groups of students (cohorts 1, 2, and 5) or on student-teacher interactions (cohort 4); cohort 3's noticing was evenly split between the two. With the exception of cohort 1, the PTs' final noticing was primarily focused on individual students (79.7% to 90.3% of instances), indicating that the intervention was successful in focusing the PTs on what individual students were saying or doing during the lesson.

Table 2: Primary Student Noticing

		Cohort 1	Cohort 2	Cohort 3	Cohort 4	Cohort 5
Individual Students	Baseline	39.4%	21.4%	16.0%	38.9%	4.0%
	Final	37.5%	90.3%	84.4%	79.7%	85.1%
Groups of Students	Baseline	51.5%	57.1%	44.0%	16.7%	68.0%
	Final	8.9%	9.7%	11.1%	6.8%	8.5%
Student/Teacher	Baseline	9.1%	21.4%	40.0%	44.4%	28.0%
	Final	53.6%	0.0%	4.4%	13.6%	6.4%

Focus of Noticing

Instances with any student focus (coded Si, Sg, S/T and T/S) were also coded to characterize what about the students the PTs had noticed. As a reminder, the goal was to focus PTs' attention on what students were saying mathematically, so the target was noting or analyzing the student mathematics, with analyzing deemed to be the higher-level of the two foci. Due to space constraints, we discuss just these two noticing foci.

There was a wide difference in the percent of the student-focused instances coded as either noting or analyzing student mathematics for each cohort in the baseline data (Table 3). The first two cohorts had a total of 47.5% and 46.4% of such instances coded with one of these foci, while the latter three cohorts had a significantly lower percent, ranging from only 2.3% to 20.4%. In the baseline data, only cohort 2 demonstrated any analyzing; this was all attributable to just one PT. All of the cohorts increased their noticing with these two foci. In fact, with the exception of cohort 1, the final data shows that 85.5% or more of the PTs' student-centered noticing was coded as noting or analyzing student mathematics, with noting instances accounting for the majority of the instances for cohorts 2, 3 and 4, and a more even split between noting and analyzing (47.3% and 38.2%, respectively) for cohort 5. Cohorts 2 and 5 demonstrated the highest percentage of analyzing, but again, cohort 2's analyzing was largely attributable to one PT. Cohort 5's percent of analyzing was not only the highest among all cohorts, but was also more consistent among the PTs in the cohort. Collectively, the data suggests that the intervention was successful not only in focusing the PTs' attention on students, but also on the important mathematics that came from these students in the lessons. More specific versions of the framework were generally more effective in supporting noticing. The higher percent of analysis by the last cohort also suggests that providing a framework earlier in the intervention and engaging PTs in a more structured use of the framework better supported noticing than earlier iterations of the intervention.

Table 3: Participant Student-Centered Noticing Focus

		Cohort 1	Cohort 2	Cohort 3	Cohort 4	Cohort 5
Noting Student Math	Baseline	47.5%	32.1%	20.4%	15.6%	2.3%
	Final	59.0%	76.7%	78.0%	89.8%	47.3%
Analyzing Student Math	Baseline	0.0%	14.3%	0.0%	0.0%	0.0%
	Final	1.6%	20.0%	8.0%	8.5%	38.2%

Specificity of Mathematics

The specificity of the PTs' noticing indicates whether and in what level of detail the mathematics in an instance was discussed. Non-mathematical noticing was present in the baseline data for all cohorts and was most prevalent for cohort 5, with 39.8% of their noticing focused on non-mathematical features of classroom interactions (Table 4). There was no non-mathematical noticing present in the final data, however. Perhaps surprisingly, most cohorts discussed the mathematics in a specific way more than half of the time from the start (between 52.5% and 75% of instances), although cohort 5 only did so 29.5% of the time in the baseline data. All of the cohorts maintained or became more specific in their discussion of the mathematics through the intervention, with the final three cohorts most consistently discussing the mathematics in a specific way in the final data and cohort 5 showing the most significant change in specificity. This again suggests the benefit of a more explicit analytic framework.

Table 4: Specificity of Participant Noticing

		Cohort 1	Cohort 2	Cohort 3	Cohort 4	Cohort 5
Non-math	Baseline	15.0%	10.7%	2.0%	3.1%	39.8%
	Final	0.0%	0.0%	0.0%	0.0%	0.0%
General Math	Baseline	32.5%	14.3%	38.8%	26.0%	30.7%
	Final	24.6%	26.7%	12.0%	13.6%	5.5%
Specific Math	Baseline	52.5%	75.0%	59.2%	70.8%	29.5%
	Final	75.4%	73.3%	88.0%	86.4%	94.5%

Noticing of MOSTs

An additional measure of PT learning was whether they became better able to identify instances of student mathematics that had potential to be used to support student learning of mathematics—that is, instances that were MOSTs. Table 5 gives the percent of the instances that PTs marked that coincided with instances the researchers identified as MOSTs. As seen in the table, all of the cohorts increased in the percent of moments that aligned with the researchers' moments, but the data needs to be interpreted with caution. Ongoing analysis is focused on determining whether the PTs talked about these instances in a way that indicates they were focused on the student mathematics, rather than for some other reason.

Table 5: Participant Noticing of MOSTs

	Cohort 1	Cohort 2	Cohort 3	Cohort 4	Cohort 5
Baseline	60.0%	32.1%	44.7%	25.0%	17.0%
Final	86.9%	66.7%	68.0%	67.2%	67.3%

Discussion and Conclusions

The findings of this study add support to a growing body of research demonstrating that it is feasible to develop prospective teachers' professional noticing skills. In fact, we developed these skills at the start of a teacher preparation program, when the prospective teachers had very little knowledge of students to draw upon. Key differences between this and many other interventions are that the noticing intervention used longer unedited classroom video recorded in local teachers' classrooms and focused on a range of mathematical topics, depending on the topics of the recorded lessons. Thus, these findings suggest that there may not be a need to narrow either the scope or length of classroom artifacts used to develop noticing skills (i.e., use short transcripts or video clips), nor the mathematical focus of the noticing activities.

The data also showed a general trend that when a more explicit analytic framework was provided, the PTs came to discuss the mathematics in a more specific way, and more often noted and analyzed what the students in the video were saying mathematically. Although the total percentage of instances in which the PTs noted or analyzed student mathematics was a bit lower for cohort 5, they and cohort 4 showed the greatest increase in these foci from beginning to end of the intervention. Furthermore, cohort 5 reached the analyzing level in 38.2% of all instances in the final videos—significantly more than any other cohort. These findings suggest that, although using even a loosely defined framework can support prospective teacher noticing (cohorts 1 and 2), using a more structured framework can significantly improve the outcomes of a noticing intervention. The fact that cohort 5 was given the framework earlier and was prompted to use the framework in a more structured way likely also enhanced their learning, although additional analysis is necessary to confirm whether this is the case.

Although the findings are noteworthy, there are still open questions that need to be addressed. For instance, we do not yet know how the participants' interactions with each other during the weekly meetings or the facilitation of these meetings supported the PTs' noticing, nor have we yet examined the PTs' proposed teacher responses to identified moments. Ongoing work is also focused on understanding how the noticing skills developed during this intervention transfer to noticing student ideas during the PTs' own instruction. Addressing questions such as these holds potential for helping novice teachers learn to enact the type of student-centered instruction the field has been striving to achieve (e.g., NCTM, 1989; 2014).

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A TOOLBOX FOR SUPPORTING EARLY NUMBER LEARNING IN PLAY: MOVING BEYOND "HOW MANY'?

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This paper explores the ways preschool teachers orchestrate instructional environments to promote mathematical play related to early number and, how they intervene during play to promote children's engagement with early number. We highlight these practices to identify resources for the growing numbers of early childhood teachers. This is important as many prospective and practicing teachers do not have access to the knowledge of teaching that supports young children's math learning because of the constraints of mathematics methods courses and the dearth of research on early childhood mathematics in mathematic education journals – particularly in-depth attention to early number and teaching mathematics in play.

Keywords: Early Childhood Education; Number Concepts and Operations; Teacher Education-Preservice

One of the most important challenges for young children in mathematics is gaining fluency with foundational concepts related to number, which include developing an understanding of the number word list, 1-to-1 correspondence, cardinality and related concepts like subitizing, and relationships of more, less, and the same (National Research Council, 2009). Generally, children develop understandings of these number concepts from ages 2 to 6 (Ibid.), meaning that the early years of public schooling are important to helping children master more advanced concepts, such as conceptual subsidizing (decomposing larger sets into easily identifiable subsets), and to helping children who come to school without mastery of early skills related to number quickly catch up. Increasingly, research has demonstrated that early and successful mastery of these early number concepts is predictive of later success in mathematics in particular and in schooling more broadly (e.g., Duncan et al., 2007; Jordan Kaplan, Locuniak & Ramineni, 2007). For example, in a large-scale longitudinal study, Duncan and colleagues (2007) found that early math skills are better predictors of later school success than measures of reading, attention skills, or socio-emotional behaviors, with little difference across gender and socio-economic status. More specifically, in a study of low-income families, Ramani, Row, Eason and Leech (2015) found that caregivers' engagement of children in activities aimed at teaching about number and caregivers' use of advanced number talk predicted children's understanding of cardinality, ordinal relationships, and early arithmetic. Additionally, Ramani and colleagues found no relationship between demographic variables, such as parent education, home language, or family income and children's knowledge of the number core.

Other research has shown that adults' intentional engagement of children around the number core, particularly in relation to constructs, such as cardinality and subitizing, improves children's understandings of these concepts. For example, Gunderson and Levine (2011) found that children's understanding of cardinality increased when adults counted sets and labeled them simultaneously (e.g., "1,2,3. That's 3.") in comparison with counting or naming sets in isolation. In a study of three-year-olds, Mix and colleagues (2012) found similar results. This body of work, generated primarily by developmental psychologists, demonstrates the importance of attending to early number and suggests that there are particular strategies that early childhood teachers could draw on to promote children's mastery of the number core. Yet, relatively little attention is paid to early number in mathematics education research, particularly in relation to teacher education practices for prospective elementary teachers (Parks & Wager, 2015).

Indeed, none of the studies cited above about the importance of early number concepts for predicting later mathematical success were published in mathematics education journals. In a recent review of 20 years of mathematics education and early childhood education journals, we found that only 28 of the 239 articles published about early childhood mathematics attended to the number core at all. In addition, the mention of early number concepts in many of these 28 articles was quite cursory – often an aside to a broader discussion. In fact, cardinality was explored in depth in only one article; 1-to-1- correspondence was explored in two articles, and subitizing was examined in four. (As a point of comparison, 66 articles mentioned fractions, 11 in some depth, and this was within a dataset of articles identified as attending to pre-K-Grade 3 contexts). In addition, only three articles concerned with early number concepts focused on strategies for helping preservice or practicing teachers support children's early number learning (Schack et al, 2013; Tirosh, Tsamir, Levenson & Tabach, 2011; Tsamir et al, 2013), which suggests a need for greater attention to developing strategies to help teachers promote early number understandings.

Recent increases in publically funded prekindergartens and staffing of these programs with teachers who have been certified in elementary education makes this need more pressing. In the 2012-2013 school year, 28 percent of four-year-olds in the US were enrolled in state-funded preschool programs (Barnett, Carolan, Squires & Brown, 2013). More than half of the states require that teachers in these prekindergarten (pre-K) programs have four-year degrees in education, which means they have likely learned to teach mathematics in university methods courses. Given current interest in universal pre-K (Barnett et al., 2012), these numbers are likely to increase, resulting in greater numbers of future pre-K teachers in elementary mathematics methods courses across the country. Additionally, given the lack of attention to early number in research (and anecdotally on methods course syllabi), future kindergarten teachers almost certainly also need more support in attending to early number than they have been getting in typical mathematics methods coursework.

In addition to requiring more in-depth attention to early number (and geometry), these future pre-K and kindergarten teachers also need strategies for helping their children maximize opportunities to learn mathematics in play settings in the classroom. Recommendations for best practices in early childhood settings almost universally include calls for time for play, which promotes cognitive development, language learning, and social growth (Copple & Bredekamp, 2009). Typical classroom play materials, such as blocks, games, and toy collections, offer potentially rich opportunities for children to learn and practice early number concepts (Wager & Parks, 2014; Seo & Ginsburg, 2004; van Oers, 2010); however, without teachers who intentionally intervene to mathematize play by attaching mathematical language to play, scaffolding more complex play, and directing student attention toward potential mathematics, children are unlikely to get the full benefits from their mathematically oriented play (Ginsburg, 2006; Graham, Nash & Paul, 1997). Yet, there is also little attention to teaching strategies for mathematizing play in the research literature. In the review described above, we found only one study (Eberly & Golbeck, 2001) that focused on understanding ways that practicing or preservice teachers learned about or implemented strategies to mathematize play.

In order to address these gaps in the literature, this study draws on a rich library of data from two broader studies in order to identify productive strategies that practicing preschool teachers used in order to promote mathematical play in their classrooms and to mathematize that play for children. The research questions guiding our analysis were: How do preschool teachers orchestrate their instructional environments to promote mathematical play related to early number? And, how do these preschool teachers intervene during play to promote children's engagement with early number concepts?

Theoretical Framework

In this study, we take the perspective that early number learning develops not only as a result of maturation, but is significantly influenced by context and engagement with knowledgeable others (Vygotsky, 1978). Although Piaget is best known for his stage theories of development, which argue that children move through predictable stages of learning (e.g., going from more concrete to more abstract understandings of quantity) as they mature (Piaget, 1962), Piaget also did not believe that these stages operated independently of the environment or of social interactions with others (Piaget, 1964). Similarly, Vygotsky argued that adults can and should provide instruction that "marches ahead of development" (Vygotsky, 1962, p. 104) to foster children's learning.

This perspective is important because it means that rather than waiting until a child is "developmentally ready" for a particular mathematics concept, teachers must take responsibility for guiding children toward new understandings. In relation to early number, this means that primary grade teachers should not simply wait until understandings of the number core develop before beginning instruction in early arithmetic nor should they assume that children who come to school without competence with cardinality, 1-to-1 correspondence, and subitizing will develop these skills by engaging with instruction aimed at early arithmetic. Rather, they must both create instructional environments that foster children's growth in relation to these early number understandings and engage in intentional instructional practices that will move children along the developmental continuum (Parks, 2015). Both Vygotsky (1968) and Piaget (1962) saw play as an important context for these kinds of instructional interactions, both because play is motivating for children and because it encourages exploration with physical objects, which often leads to more sophisticated understandings of quantity.

Mode of Inquiry

The analysis presented here is drawn from data collected in two broader studies, both aimed at understanding mathematical learning and teaching in early childhood classrooms. Both studies were designed within a tradition of interpretive research that values attention to meaning-making by participants and understanding the role of context in human interactions (Erickson, 1986; Graue & Walsh, 1998). The first study provided an intervention of sorts as teachers engaged with professional development explicitly designed to support their understanding of mathematics learning in play-based pre-K classrooms, whereas the second study observed existing phenomena – what math was happening. Thus, we have examples from teachers who were supported to think about math in pre-K and from a teacher who was not.

The first study involved a Professional Development (PD) program designed to promote culturally and developmentally responsive early number teaching. The project team designed, facilitated, and studied the PD for three cohorts of pre-K teachers who were teaching in a local districts' new 4-year-old kindergarten. Each cohort took four graduate courses over a two-year period. Data included audiotaped group discussions and artifacts from the course, interviews with teachers, and bi-weekly observations in a subset of participants' classrooms. For the current analysis, we focused on artifacts – in particular, 51 learning stories (narrative assessments of young children's learning, Carr, 2011) teachers wrote to identify what they noticed and how they responded to children's mathematics engagement in play.

The second study was a longitudinal examination of the mathematical experiences of a cohort of children as they moved from preschool to Grade 1 in a rural public school. Data included video of classroom events, assessment interviews, and out-of-school engagements, as well as interviews with teachers and parents. For the current analysis, we focused on video data collected in the preschool classroom of mathematics-related interactions in both formal and informal settings. The teacher in this pre-K was an experienced, white, female teacher with more than 15 years of experience in pre-K and kindergarten. The majority of children in the cohort were African American (13 out of 16) and all children in the cohort came from low-income families.

For this analysis, we identified episodes in both sets of data that had been previously coded as involving cardinality, 1-to-1-correspondence, and subitizing as well as adult interactions. We then coded these episodes as relating to organization of the instructional environment or intervention in the moment. Then, each of these sets of data was analyzed (by each researcher and through conversation and writing with each other) to identify strategies used by the teachers in both studies to promote students' understandings of early number.

Findings and Discussion

We found that teachers engaged (or had the opportunity to engage) children in rich math interactions in three instructional spaces: (a) activities purposefully planned for whole class engagement in math; (b) play with math-like materials in the room or in specific math centers (e.g. board games, counting manipulatives) and; (c) during play that might not obviously be seen as mathematical (Wager, 2013). For each of these instructional spaces we provide vignettes from classroom observation (either ours or reported by the teacher) and identify strategies teachers used.

Planned activities

Teachers in both projects planned activities that were purposefully designed to engage children in counting related activities, although teachers in the professional development project were more likely to design activities where children routinely got to handle objects as they practiced counting. In the classroom observed without intervention the most frequent counting activities involved reciting the number sequence, singing a song prompted with cards to practice number recognition, and counting during calendar time, when only the teacher or a single child handled objects while counting.

Activities when children could handle objects and count typically occurred during transitions, meals, or other routines such as morning meetings. Sometimes these were whole group activities while other times individual children 'led' the activity for the class. A common practice among teachers was to start each class period by having a child count how many children were present. In the following examples we see how a teacher supports a child who is still working on 1-to-1 correspondence and another who is moving toward problem solving. In Vignette 1, Betty reflects on her observation of Bob counting his classmates during circle time. The children had been doing this activity for about six weeks.

> *Vignette 1:* When I asked [Bob] to count how many children were sitting in the circle, he was not exactly sure where to start. With a little guidance he started with the child that was sitting next to me. He counted with 1:1 correspondence tapping each child on the head, however, he skipped one girl and she said, "hey you missed me". He turned to me and looked confused. He started again and I encourage him to slow down. He skipped another child and then the aide assisted him so he could finish counting all the way around. *(Betty, Counting Children)*

In the next example, Marley shares how she supported Sam to make counting his classmates a more challenging activity. She had previously noticed that Sam would write addition problems on a dry erase board and ask, "how much is this?"

> *Vignette 2:* I asked Sam to first count the girls and then count the boys as I wrote the numbers on the easel. I used this opportunity to also explain what the + and = signs mean. After I recorded the numbers I had all the children stand up to be counted showing that 5 girls plus 6 boys equals 11 children all together. *(Marley, Math problems…)*

Snack time is another space that teachers engaged children in counting activities. Sadie had a snack helper in her class whose job was to count out 14 plates and put 4 orange slices on each plate.

> *Vignette 3:* I helped Clara with the first plate, working to count out 4 orange slices. She put 3 slices on the second plate, then counted the number on the first plate and the number on the second plate and asked me, "that the amount?" When I asked her what she thought, she put on more on to make 4. Then on the 4th plate she put 5 slices and asked again. I helped her take one orange slice away, using the words more, less, and take away, until she discovered the plates had the same amount. *(Sadie, Clara's Snack Task)*

The teachers in these vignettes used a variety of strategies to support children's early number development. In vignette 1, Betty recognizes that Bob can count using 1-to-1 correspondence to about 3 or 4 before he switches to rote counting. Her strategy was to encourage Bob to slow down so he could match a number to each child that he taps, and ultimately have an adult count with him. In vignette 2, Marley modeled a joining problem by using a practice the children regularly engaged with in the class. In vignette 3, Sadie does not directly tell Clara whether she is right or wrong but rather asks her what she thinks. Sadie then modeled how to compare sets. Both Sadie and Marley modeled strategies for children using math vocabulary.

Math games and materials

Preparing the environment for children to engage with learning is a critical aspect of any early childhood classroom, and certainly true for mathematics learning. Yet providing materials and organizational structures that encourage interactions around math is not enough. We found in both of our studies that teachers provided varying levels of support to children as they played with math games and activities. All the classrooms had several math games both teacher-made and commercial. In addition, teachers made decisions to highlight the mathematical nature of materials. For example, Sarah organized collections of small toys, such as cars and animals, on a shelf along with a balance. By placing these objects in proximity, she encouraged her students to engage with them in a mathematical way. Vignette 4 describes the impact of Sarah's decision to organize unifix cubes in sticks of 8, rather than loose in a bucket.

> *Vignette 4:* Clay sat down among a pile of unifix cubes that other children had been playing with and began, without prompting to organize them into sticks of 8. He began by touching each cube as he counted it to ensure he had the correct amount. After he made a few sticks of 8, he started using his previous sticks as points of comparison, holding up a new stick and then removing two cubes when he noticed that it was too long. After watching Clay for a moment, two other boys began to make sticks of 8, touching each cube as they counted.

Sarah intentionally asked the students to keep the unifix cubes in sticks of 8 (a number chosen because it was the number of spaces on their bingo cards) because she knew it would provide opportunities to practice meaningful counting whenever the cubes were cleaned up; however, she did little to deepen students engage with these cubes across the year, such as changing the goal number for the size of sticks or asking students to articulate the strategies for composing sticks from smaller groups of blocks or comparing finished lengths. Sarah, who did not receive professional development around early mathematics and who reported that she'd had little preparation around the counting core in her preparation program, was not as adept as the teachers receiving PD as recognizing mathematics in play and intervening to highlight it for children, as demonstrated in the next vignettes.

In analyzing teachers' interactions when playing games with children or observing children play the games, we were particularly interested in those examples of teachers finding alternative ways to ask ‘how many’ or go beyond asking ‘how many’. The ways in which teachers did this included general comments about sets that encouraged the children to count such as “wow, that is a lot” or asking which one has more when children had more than one set of objects. In vignette 5, Betty observes John at the math center working on a Counting Book.

> *Vignette 5:* On the first page, he wrote the numeral 4 and then stamped 4 stamps. On the second page he wrote 1-10 saying each number to himself as he wrote. I asked him, “what did you write”, he told me the numbers up to 10. I asked, “how many will you stamp” and he started stamping and counted each one up to 10. On the next page he stamped 7 stamps, counted them and wrote the number 7. Then he stamped on more and stopped. I asked, “what now”. He crossed out the 7 and asked me how to write 8. I showed him a picture of the numeral 8 and he tried it. *(Betty, How Many Stamps?)*

In this next vignette, Chela describes how Kiki used the balance scale to count and compare various objects in the room. Kiki had filled the cups with different objects from the science table, putting insect and spider counters in one and one-gram yellow weights in the other. Then using a chart that Chela had made for children to compare the number of objects in different sets, Kiki lined up the insects on one side and weights on the other.

> *Vignette 6:* In a very precise manor, Kiki had lined up the yellow grams and commented, “wow this is lots!” At this point I prompted him to also line up the insects and spiders, which Kiki thought was a great idea. Kiki had a difficult time knowing for certain which one had more, but thought that it was the yellow grams because there were lots. I asked him how he could know which was more and he counted each. *(Chela, Measurement and Balance Activity)*

In Vignette 5, Betty asks the question ‘how many’ but does so in a way that encourages action to count out rather than how many are already in a set. She also asks John ‘what now’ so that he could choose between crossing out a stamp or changing the numeral. In Vignette 6, the nature of the materials Chela has in class encourage the practice of counting. By asking the child how he could find out which set had more, she encouraged him to think of counting rather than directing him to do so.

Mathematizing Play

Perhaps one of the most challenging responsibilities that early childhood teachers face with respect to supporting math learning is mathematizing the activities children engage with in free play. To do this effectively, teachers must recognize the mathematics that children are engaging with and make in-the-moment decisions to support further learning. We found examples of situations in which teachers were able to recognize and respond to math in a variety of classroom environments. In Vignette 7, Sadie observes Clara in the housekeeping area as she was making food for her friends. Clara was the mom and her friends were her babies.

> *Vignette 7:* They all started talking about birthdays and her babies said they needed 4 ‘cupcakes’ because they were 4 years old, or 5 ‘cupcakes’ because they were 5. She counted 4 cupcakes for one of her babies and 5 for the other. When I was invited over to share in the birthday festivities, I was given a plate and a pile of food. I asked how many cupcakes I had and Clara counted 8 of them and stopped (there were more pieces on my plate). When I said I want 2 cupcakes because I am 2, Clara removed some pieces of food off my plate and left me with 2 and said “there, now lets sing”. *(Sadie, Clara’s Counting Kitchen)*

In Vignette 8, Sadie observed Sam using 3D shapes to build “a huge castle”. Although play with shapes could be coded as mathematical materials, in this situation we are examining the interaction around counting and thus we coded this as play. There was considerable conversation about the different shapes Sam used for his castle. Here we pick up where Sadie's reflection shifted to a discussion of counting.

> *Vignette 8:* While the conversation continued we talked about the various heights of his towers. He counted with 1:1 correspondence up to 4, which was the tallest tower. He noticed that the lower tower was only 2 shapes tall and compared them saying "this one is shorter than that one." We talked about how the tallest tower did not have the most shapes. *(Sadie, The Castle King)*

Vignette 7 provides a wonderful example of how teachers can enter children's play to support mathematical thinking. Clara was already counting and showing evidence of 1-to-1 correspondence and cardinality but Sadie was able to push on this without disrupting the play by having Clara think about how to remove enough 'cupcakes' to leave Sadie with two. Sadie could have taken this further by asking how many cupcakes Clara took off her plate. In vignette 8, Sadie joined Sam in his castle building and raised questions about the height of his towers, which encouraged him to compare heights, count, and ultimately see that more blocks did not always mean a higher tower.

Not surprisingly we found more instances when teachers recognized the opportunity to mathematize but did not do so in the moment or did not engage with mathematics at all, despite the opportunity to do so. For example, students in Sarah's classroom routinely built tall towers of blocks and argued with each other about whose was the largest, but the teachers in the room never intervened to ask the children to find out. Similarly, while mathematical games were available, teachers did not intervene to teach children the rules of the games, which would have supported mathematical thinking, but instead let them play however they wanted, which often resulted in play-acting with the pieces (intellectually valuable, no doubt, but unlikely to lead to development of the skills highlighted in the number core).

Implications

The vignettes provide examples of how teachers who have had PD explicitly focused on early number were able to provide activities and materials, and respond to children's engagement in these activities, materials, and play in order to support early number development. In interviews and classroom discussion about the PD, each of these teachers shared that they would not have attended to math in these ways without the PD. The purpose of the above is not to argue for the effectiveness of the particular PD but to see the possibilities of early number engagement when teachers are supported to develop the skills for these interactions. Many of our prospective teachers do not have this opportunity because the constraints of programs are such that mathematics methods courses are intended to cover a broad grade band and the early counting is often left out. Without more research in mathematics teacher education journals on the practices that support early math and the content knowledge required, our future early childhood teachers will not be able to engage in these kinds of interactions and the children in their classrooms will not interact with the math thinking that research shows is important.

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LEARNING INSTRUCTIONAL PRACTICES IN PROFESSIONAL DEVELOPMENT

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In this paper, we describe teachers' learning of an instructional practice related to launching a cognitively demanding task. Using a multi-case study design, we analyze eight teachers' participation in a rehearsal of the instructional practice in professional development and the ways these teachers enacted this practice in their classroom instruction. Our findings suggest that what teachers attended to in their rehearsal related to their classroom practice and the degree to which the cognitive demands of the task were maintained.

Keywords: Teacher Education-Inservice

Introduction

As a model of teaching, student-centered mathematics instruction supports students in engaging in tasks that build conceptual understandings, communicating and evaluating mathematical reasoning, and making connections among mathematical concepts. Such instruction has been shown to lead to increased student learning (Fennema et al., 1996; Boaler & Staples, 2008) and may address issues of equitable instruction for traditionally marginalized students (Myers, 2014). Though a consensus among researchers suggests that expert teaching involves understanding and using students' thinking to guide instruction (Bransford, Brown, & Cocking, 2000; Darling-Hammond, 2008; Kilpatrick, Swafford, & Findell, 2001), mathematics teaching in the US remains largely teacher directed, focused on procedures, and marked with few opportunities for intellectual engagement (Stigler & Hiebert, 2004; Wiess et al., 2003).

Professional development (PD) that relates new learning to teachers' practice can influence their instructional strategies (Darling-Hammond et al., 2009; Goldsmith, Doerr, & Lewis, 2013), yet many teachers have difficulties implementing new instructional strategies in their teaching. In PD, mathematics teacher educators (MTEs) have found that guiding frameworks for instructional practices are often insufficient to support teachers in enacting new practices in their classrooms (Boston & Smith, 2009; Munter, 2014). Recently, work around *rehearsal* in pre-service teacher education is showing promise as an approach for assisting novice teachers in learning and enacting student-centered instructional practices (Lampert et al., 2013). Our research adopts rehearsal as a pedagogy for supporting teacher learning of new instructional practices for a PD setting. In this paper, we aim to describe the ways a group of practicing secondary mathematics teachers attended to instructional moves for one student-centered instructional practice through rehearsal in PD. Specifically, our research is guided by the question: *in what ways does teachers' participation in rehearsal of an instructional practice in PD relate to their enactment of the practice in their classrooms?*

Background

A critical foundation for student-centered mathematics instruction is students' engagement with cognitively demanding mathematics tasks. Stein, Grover, and Henningson (1996) described the cognitive demands of a mathematics task as "the kind of thinking processes entailed in solving the task" (p. 461) and offered a framework for categorizing tasks as low and high cognitive demand. Their research showed that though teachers may begin instruction with a cognitively demanding task, the demands of the task often decline during implementation. Stein and Lane (1996) reported that the greatest gains in student learning occurred in classrooms where cognitively demanding tasks were implemented and maintained during instruction.

MTEs have developed frameworks for assisting teachers in maintaining the cognitive demands of tasks (e.g. Smith, Bill, & Hughes, 2008; Smith & Stein, 2011). Recently, Jackson and colleagues' (2013) investigated the relationship between task setup at the beginning of a lesson and the quality of the culminating discussion. They identified four factors that related to its implementation which were the degree to which: key contextual features were made salient; key mathematical concepts were highlighted and examined; a common language that supported contextual and mathematical elements of the task was developed; and the mathematical integrity of the task was maintained. We consider the practice of launching cognitively demanding tasks as paramount to teachers success in maintaining the cognitive demands of a task.

Theoretical Perspectives

We conceptualize PD as a boundary encounter (Sztajn et al., 2014; Wenger, 1998) where both the teaching and MTE communities bring distinct practices and identities. In a boundary encounter, teachers and MTEs come together to exchange knowledge by interacting around representations of knowledge that carry meaning in both communities called boundary objects. Members from differing communities introduce, negotiate, and integrate elements of their own practice as they interact and make new meanings of the boundary object together. From this perspective, teacher learning is taken as changes in their participation in the boundary encounter and the presence of new aspects of practice in their classroom teaching.

In PD, MTEs design professional learning tasks (PLTs) to facilitate participation around boundary objects (Edgington et al., 2015). Grossman and colleagues (2009) characterized PLTs as representations, decompositions, and approximations of practice. Representations, such as video or model lessons, refer to the ways MTEs make visible particular aspects of teaching. Decompositions refer to partitioning practice for in depth study, such as introducing a lesson or responding to students' thinking. Approximations of practice refer to opportunities for novices to engage in a particular practice, such as analyzing written work. Rehearsal is a particular kind of approximation of practice that supports novices by providing opportunities to learn about, practice, and reflect upon important aspects of practice while receiving in-the-moment feedback. Emerging research on the use of rehearsal has demonstrated its effectiveness in supporting teacher candidates in enacting particular instructional practices (Lampert et al., 2013;Tyminski, Zambak, Drake, & Land, 2014).

Building from this research, we design learning opportunities around boundary objects for instructional practices in our work in PD. We use a sequence of PLTs that begins with a representation of practice (e.g. experiencing a model lesson) followed by a decomposition of a particular aspect of that representation (e.g. debriefing the facilitation of a task). Next, we design a PLT that allows teachers to approximate the specific practice (e.g. rehearsing that practice with peers). One such PLT is a rehearsal. In our work, rehearsals have three interrelated components: teachers *rehearse* a particular instructional practice with their peers; receive in-the-moment *feedback* to support learning the practice; and *reflect* on their rehearsing and observations.

Methods

Our multi-case study investigates how teachers' participation in rehearsal relates to their classroom enactment of one student-centered instructional practice – launching cognitively demanding mathematics tasks. Case study research is useful for understanding a phenomenon bounded by a particular context and aims for its detailed description (Stake, 1995). To answer our research question, we first examined case teachers' participation in rehearsals and subsequent enactment in their classrooms. Next, we conducted a cross-case analysis to identify trends across case teachers rehearsals and enactments of launching cognitively demanding tasks.

Context and Participants

Our study is part of a multiyear PD and research project investigating secondary mathematics teachers' learning of student-centered instructional practices. The 108-hour PD was designed for a 10-month period, beginning with a 60-hour summer institute followed by 20 hours of face-to-face meetings and 28 hours of online work throughout the school year. In the summer institute, we shared several research-based frameworks for instructional practices that served as boundary objects through sequences of PLTs described above. As a part of the school year meetings, teachers were asked to plan and teach several student-centered lessons in their classrooms. This study focuses on teachers' learning the practice of launching cognitively demanding tasks.

Building from Jackson et al.'s (2013) four factors, we articulated a framework for the practice of launching cognitively demanding tasks. For us, the purpose of the practice is to ensure that students understand the mathematical goal of the task and can engage productively when the task is implemented without lowering its cognitive demands. The framework describes five instructional moves for launching: allowing think time (TT); checking for understanding (CU); addressing barriers for engagements (AB); sharing approaches (SA); and ensuring students can begin the task (BT). Allowing time for students to think about what the task is and what mathematical approaches they might take enables students to formulate a plan and identify additional information or questions they may have. Prompting students to share their interpretations of the problem ensures that students understand the task's mathematical goal. Addressing barriers provides an opportunity to resolve issues that may prevent students from engaging with the mathematics of the task during implementation, such as contextual questions, uncertainties about terminology, or mathematical skills not directly related to the goal of the task. Sharing approaches encourages multiple strategies and representations to be made public and allows students who may not have an approach to hear other's ideas. Ensuring that students can begin the task gauges whether enough students are confident about their approaches for the class to productively and collectively engage.

In the summer institute, each teacher completed one rehearsal to approximate launching. For each, three participants simulated "students" based on profile cards that outlined particular understandings, strategies, or barriers to engagement. One participant served as "teacher" and rehearsed launching the task. A MTE served as facilitator and provided in-the-moment feedback to the teacher while they rehearsed. Upon conclusion of each teacher's rehearsal, all participants reflected on what they learned by rehearsing and observing. During the school year, participants planned and taught student-centered lessons in their classrooms. These enactments served as a basis for continued reflection and discussion during the school year PD meetings.

Seventeen teachers from four suburban and rural school districts in the Southeastern United States volunteered and received a stipend for participation in the PD. Of the twelve teachers completing the PD, ten completed the activities used for this research, with eight teaching tasks of high cognitive demand. These eight teachers served as cases for this study.

Data Sources and Analysis

Data consisted of transcribed video recordings of the rehearsals and classroom enactments and written reflections following the rehearsals. We specified our unit of analysis as a teacher's talk turns in the transcripts and their written responses to reflection prompts. For the within-case analysis, we first coded each of these units in relation to the boundary object (TT, CU, BA, SA, and GS) to identify moves from the launching framework. Four researchers collaboratively coded the units for one of the eight case teachers, discussed discrepancies, clarified code definitions, and reached consensus. The remaining cases were double coded independently by members of the research team who then met and resolved discrepancies. This procedure allowed us to characterize each teacher's participation in the rehearsal in relation to classroom enactment.

For the cross-case analysis, we first used constant comparative methods (Strauss & Corbin, 1998) to identify noticeable trends relating teachers' participation in the rehearsal with their enactments. Next, we used a subset of the Instructional Quality Assessment (IQA) rubrics (Junker et al., 2006) related to Academic Rigor to determine the quality of the instructional task (AR1) and its implementation (AR2). After reaching agreement on three classroom videos with the IQA rubrics (IRR 88%), we used the IQA to rate the remaining lessons. To assess whether the cognitive demands of the task were maintained in the lessons, identical scores on AR1 and AR2 were taken to indicate that the cognitive demands of the task were maintained; a decreased in scores indicated a decline. Together, these two approaches allowed us to understand the relationships across teachers' participation in the rehearsal, their enactments in classrooms, and the maintenance of the cognitive demands upon task implementation.

Findings

Our cross-case analysis identified three trends in teachers' launches of cognitively demanding tasks. During rehearsal, teachers' launches varied in attention from addressing barriers to engagement to ensuring students' understanding. This variation corresponded to the degree to which their classroom launch attended to students' thinking and the degree to which the cognitive demands of the task were maintained. Instructional moves made in each teacher's rehearsal and enactment, and the maintenance of cognitive demands are summarized in Table 1.

Table 1. Instructional moves and related maintenance of cognitive demands.

	Rehearsal					Classroom Enactment					
Case	**TT**	**CU**	**AB**	**SA**	**BT**	**TT**	**CU**	**AB**	**SA**	**BT**	**Cog. Demand**
Eli		X	X	X	X	X	X	X	X	X	Declined
Mia		X	X		X	X	X	X		X	Declined
Cal	X	X	X	X	X	X	X	X		X	Declined
Tea		X	X	X		X	X	X	X	X	Maintained
Pat	X	X	X			X	X	X	X	X	Maintained
Ema		X	X	X	X	X	X	X	X	X	Maintained
Amy			X			X	X	X	X	X	Maintained
Ann		X	X			X	X	X		X	Maintained

Removing Barriers

Cal, Eli, and Mia used a variety of the instructional moves from the launching framework during rehearsal. Though they demonstrated the ability to use the moves as a tool to support them in launching, their launches had a marked emphasis on addressing barriers of context and language. During rehearsal and feedback, teachers' focus on barriers prevented them from the ultimate goal of launching– ensuring student understanding so that students may begin the task productively. Similarly, their reflections centralized the importance of addressing barriers without attention to the overarching goal of launching. For example, Eli began his rehearsal with the question, "Are there any words you do not understand?" He continued to focus on seeking out contextual and language barriers throughout his rehearsal despite feedback that encouraged him to move forward. The facilitator attempted to redirect Eli to move beyond barriers by saying, "So let's assume that issue is resolved and move on from there." He persisted with moves to clarify context through the end of the rehearsal. In his reflection, Eli stated, "What do you do when the issues don't go away? Redirect them with another question?" His primary focus on removing barriers overshadowed the ultimate goal of ensuring understanding and allowing students to engage with the task.

In their classroom enactments, Cal, Eli, and Mia demonstrated most of the launching moves. They proactively sought to resolve ambiguous features they anticipated would prevent students from making progress on the task. Though they were successful in addressing contextual and language barriers, this preemptive approach directed students' attention toward the mathematics central to the task, endorsed particular strategies, or supplied particular mathematical tools that resulted in a clear path for students to take to solve the problem. Additionally, in all three of these cases, the cognitive demands of their task declined. For example, Cal's lesson was organized around a task about landing an airplane with a goal of using trigonometric ratios to solve problems. Prior to introducing the task to his students, he discussed the context of plane flight and drew students' attention to mathematical features important to the task.

C: So what do you think as a pilot flying a plane or making a landing – what kind of math is going through a pilots head or should he be thinking about?
S1: A lot. The angle that the plane has to – the angle that you have to have the plane at when you're landing it.
C: The boy mentioned angles. What have we been talking about for 7 days? [S: Angles] Great. What else?
S2: Triangles.
C: How does a triangle work when you're flying a plane? How would it relate to flying a plane? How would it relate to flying a plane?
S2: Cause you're using the degrees.
C: Okay. Degrees.
S3: How high it goes?
C: Alright. How high you're going.
S4: How fast you gotta be going.
C: Speed. Okay, what else?
S5: How far you have until you gotta be up in the air.
C: Okay. So we've heard angles, triangles, speed, height. Okay. When you get this, keep your pencils down. Take a few seconds to read over it. Read over this, see what questions you might have after you read it.

In his attempt to contextualize the task prior to introducing it, Cal's attention to barriers he anticipated included the mathematical ideas central to his learning goals. His launching moves directed students towards a specific approach without allowing them the opportunity to consider other possibilities. During implementation, his students used triangles and trigonometric ratios as discussed in the launch to solve the task. The lack of ambiguity along with teacher-endorsed mathematical strategies led to a clear path for students to solve the task.

Though the rehearsal provided opportunities for them to try most of the launching moves, there was a marked emphasis on addressing contextual and language barriers. In their classrooms, their launching practice removed the ambiguities of the task that they anticipated and supplied students with the key mathematical concepts needed for the task. We infer that their focused attention toward removing barriers to engagement and supporting students in accomplishing the task successfully related to the lowering the task's demands.

Emphasizing Understanding

Ema, Pat, and Tea also demonstrated many instructional moves from the launching framework during the rehearsal, but their participation differed in significant ways. Their rehearsals centralized the importance of ensuring students' understanding by integrating requests to re-voice the problem with responses to contextual or language difficulties that emerged. In their reflections, they noted the importance of students understanding the problem, addressing barriers, and making sure students could begin the task. For example in Ema's rehearsal, the first five questions she posed focused on ascertaining whether the "students" understood the problem. When a contextual barrier emerged from this questioning, Ema paused and the facilitator questioned her thinking. She responded, "in my head, I'm thinking I have to go back to what the problem is, but it was a good time to go ahead and get that [context barrier] out of the way." In her reflection, she noted the struggle of ensuring understanding while not overemphasizing aspects that were peripheral to the mathematical goal of the task in a timely manner.

In their classroom enactments, Ema, Pat, and Tea used all of the launching moves. Their emphasis on ensuring student understanding in the rehearsal led to moves that elicited and responded to their students' thinking about the task as it arose in the launch discussion. They worked to refine students' understanding of the task through questions like, "in your own words, what is the problem asking us to do?", clarifying terminology and contextual information, sharing different ideas about how to approach the task, and checking to make sure that students felt confident to begin working on the task. As an example, Pat began her lesson by allowing students time to think and discuss the problem. While listening to this discussion, she noted different students' questions and then strategically introduced them to the whole class.

P: Okay, I was asked a couple of good questions so I want to make sure that everybody hears the answers to them. Roman, will you ask your question first?
S1: Do you have to find out how much, how many toothpicks it takes to build all of the figures combined or just 1 individual?
P: Okay, do you all understand what Roman's question is?
S2: Yea, like the 8th or all of the 8 together.
P: So he is asking, are you trying to find the number of toothpicks in general for just the 8th or the 8th, the 7th, the 6th… the 2nd, and the 1st all together. Which one do you think that we're trying to find?
Ss: [multiple responses aloud] "The 8th," "all of them", etc.
P: Just the 8th? [multiple responses aloud, "No," "Yea"] So from the… let's come to a consensus of what we think overall. I've heard a lot of people say just the 8th. Which one do you think would make more sense? When you're normally trying, would you be finding, if you're trying to find your square, do you really care about all of the squares that came before it? [Students "no!"] Or just the specific square?
S3: Just the specific square.
P: Just the specific square. So let's go with the assumption that we're only trying to find that specific square. Ok Zack, what was your question?

Pat's launch continued by introducing other questions for whole class discussion, addressing contextual barriers as they arose, and ensuring that one member from each small group was confident to begin. Her integrated use of the launching moves suggests her attention to ensuring students understood the problem so that they could productively engage with the task.

The rehearsal provided opportunities to use the launching moves with a goal of ensuring understanding. In their classroom enactments, they addressed context and language barriers as they arose in discussion, had students share a variety of strategies for approaching the task, and verified that students could engage with the task. We infer that their attention toward ensuring students could engage with the mathematics of the task by eliciting and responding to students' ideas supported the maintenance of the cognitive demands of the task.

Removing Barriers to Emphasizing Understanding

Amy and Ann did not demonstrate many of launching moves during the rehearsal. Similar to Cal, Eli, and Mia, they focused almost exclusively on addressing barriers of context and language. Yet in their reflections, both noted the importance of not leading students through a launch and to leave the essential mathematical question open for students to resolve. In Ann's self-reflection, she stated that she learned, "how difficult it is to not directly lead the student to the solution." Similarly, Amy's reflection emphasized the difficulty of addressing "student misconceptions without leading their thinking too far into the task."

In their classroom enactments, Amy and Ann used most of launching moves to both respond to students' thinking as well as highlight anticipated barriers related to context and language. They provided time for students to think about the problem, probed to ensure students' understanding, and verified that all students could begin the task. Amy prompted students to publically share the approach they were going to use for the task.

For Amy and Ann, their attention when launching shifted from addressing barriers in rehearsal to emphasizing students' understanding in enactment. The rehearsal provided opportunities to reflect on the importance of listening and responding to students rather than leading to a particular approach. Their rehearsal was similar to Cal, Eli, and Mia – both teachers responded to students' thinking as well as highlighted anticipated barriers related to context and language. However, their enactments were more like Ema, Pat, and Tea. Their launches allowed students to engage with the mathematics of the task and maintained its cognitive demands.

Discussion

Our findings describe three ways secondary mathematics teachers participated in rehearsal of launching cognitively demanding tasks, enacting the practice in their classrooms, and whether the practice maintained the cognitive demands of the task. For teachers focused on removing barriers during their launches, their participation in the rehearsal related to instructional moves that lowered the cognitive demands in their lessons. For others focused on ensuring students' understanding, their participation led to a use of the framework in ways that maintained the demand. For Amy and Ann, participation in the rehearsal sensitized them to the importance of attending to students' thinking and led to launches that maintained task demands.

Our findings suggest that rehearsals are a viable pedagogy for MTEs to address the "problem of enactment." Opportunities for teachers to rehearse, receive feedback, and reflect on instructional practice may assist them in enacting the practice with students. Our results underscore the importance of clearly communicating both the moves and the purpose of a particular instructional strategy in PD. We urge PD designers and facilitators using rehearsals to carefully consider how they communicate the purpose of an instructional practice and align feedback and reflection opportunities with this purpose.

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PRE-SERVICE TEACHERS' UNDERSTANDING OF FRACTION OPERATIONS: PROVIDING JUSTIFICATION FOR COMMON ALGORITHMS

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This study examined pre-service elementary teachers' change in their understanding of fraction operations while taking a mathematics methods course. Specifically, their explanations and justifications for common algorithms for multiplication and division of fractions were coded using an existing framework (SOLO; Biggs, 1999) for the assessment of understanding. Results indicated that most students made improvement in terms of their level of understanding around fraction algorithms. Implications for mathematics teacher educators are discussed.

Keywords: Elementary School Education; Teacher Education-Preservice; Mathematical Knowledge for Teaching; Reasoning and Proof

Currently, the amount of conceptual understanding pre-service elementary teachers hold around common algorithms is weak (Ball & Bass, 2002; Simon, 1993); however, a push for conceptual understanding is being deemed necessary for all students (CCSSO, 2010; NCTM, 2000; Stylianides, Stylianides, & Philippou, 2007). This means that pre-service elementary teachers need to hold a deeper understanding of common algorithms if their own elementary students are expected to understand the meaning behind each algorithm (Ball & Bass, 2002; Ball, Thames, & Phelps, 2008). Although previous studies have focused on elementary teachers' conceptual understandings (Simon & Blume, 1996; Yackel & Cobb, 1996) more work is needed around fraction algorithm understanding and how teachers justify those algorithms for their students. The current study aims to begin to fill that void.

The purpose of the study was to examine how pre-service elementary teachers provide explanation and justification for algorithms around fraction operations. Specifically, the two research questions were: (1) What are pre-service elementary teachers' levels of understanding of algorithms for the multiplication and division of fractions *before* and *after* experiencing instruction focused on fractions in a mathematics methods course?; and (2) How do pre-service elementary teachers *change* in their level of understanding of the algorithms?

Theoretical Framework and Related Literature

This study draws on the work of Skemp (1976) who defined the ideas of relational versus instrumental understanding. According to Skemp (1976), relational understanding is equivalent to conceptual understanding such that it is knowing *why* something happens, whereas instrumental understanding is similar to procedural understanding such that it is taking a rule and using it without understanding. In the past twenty years, there has been much attention to developing relational understanding in pre-service teachers (e.g., Ball, 1990; Ball & Bass, 2002; Simon, 1993;). Researchers have noted the specialized content knowledge that is specific to the work of teaching (Ball, Thames, & Phelps, 2008) and have emphasized that in order for teachers to be able to answer students' questions of "why," they must have a more robust relational understanding than that of their students.

One such study, performed by Eisenhart et al. (1993), followed Ms. Daniels, a student teacher who tried explaining the "invert and multiply" rule to a student in her class and abandoned the explanation midway through when she realized the example she was using pertained to multiplication, not division. Another study, performed by Ball (1990), gave the example of Allen, an elementary education major who also struggled with the "invert and multiply" rule for division of fractions and could not generate an example that did not reference multiplication. Much of the challenge for our work as mathematics teacher educators is to prepare pre-service teachers to answer these types of questions effectively and not give the reasoning "because it's the rule". To do so requires the development of a deep conceptual understanding of the mathematics for pre-service teachers.

To measure understanding in mathematics, researchers have typically used open-ended assessments and interviews that then must be analyzed. One analysis tool, and the tool used in this study, is the Structure of the Observed Learning Outcome Taxonomy (SOLO; Biggs, 1999) as displayed in Figure 1. The nature of the SOLO Taxonomy is one such that student learning is examined as they move from a lower level of understanding to a higher, more abstract, level of understanding. As students progress through the levels, they retain the traits from the previous level; in other words, each level builds on the previous. According to a study performed by Ball (1990), pre-service teachers' mathematical understandings typically are found to be at the Unistructural level where they are simply reciting algorithms in their explanations to students.

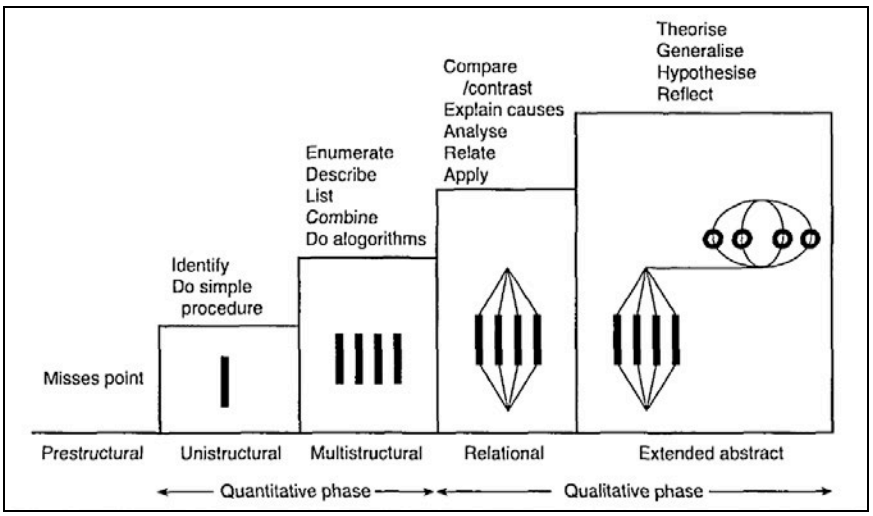


Figure 1: SOLO Taxonomy(Biggs, 1999, p. 67)

With the continued emphasis on conceptual understanding in documents such as the National Research Council's *Adding it Up* (2001) and in the Common Core State Standards for Mathematics (2010), there is a need to better understand pre-service teachers' conceptual understanding so we can better prepare them for their future work. Furthermore, historically, fractions have been difficult for both children and adults in the United States (Lamon, 2005). This speaks to the need for further research focused on fraction operations and namely pre-service teachers' understanding of them, as is in the case in the current study.

Methods

Participants

The participants in this study were forty-eight juniors in an elementary education program who were enrolled in a mathematics methods course, the second in a two-course sequence and focused on multiplicative reasoning in grades 3-5. Data was collected from two sections of this course, which were taught by a total of four instructors, two teaching one section and the other two teaching the other section. All lessons were created as collaboration between the four instructors; therefore, all students experienced the same tasks and activities during the class sessions. Of the students enrolled in the methods course, 96% had taken one or two Calculus courses, either in high school or at the college level, or they had taken *Calculus for Elementary Teachers,* which focuses on a conceptual understanding of Calculus-related topics. The mathematical background of these students is important to note because the focus of this paper is on the pre-service teachers' conceptual understanding and level of justification they can provide to students. One can see from the level of mathematics achieved by these students that they have experienced the content they will be expected to teach; however, they may or may not be able to explain the ideas conceptually.

Intervention

After the pre-service teachers completed the pre-assessment, they participated in a five-week unit of instruction during which they learned about algorithms for fraction operations and also had opportunities to examine student work. Examples of tasks completed during the instruction period are shown in Figures 2 and 3.

Misconceptions with Fraction Multiplication

1. Maisy draws a picture like the one shown to depict3 $\cdot \frac{4}{5}$. Maisy concludes from her picture that

$$3 \cdot \frac{4}{5} = \frac{12}{15}$$

Because 12 pieces out of 15 are shaded. Is Maisy right? If not, where is her reasoning flawed?

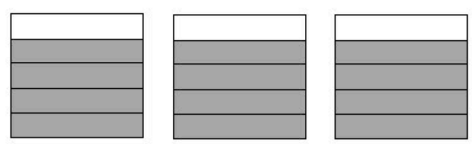


Figure 2: Misconceptions with Fraction Multiplication. Adapted from Sybilla Beckman "Mathematics for Elementary Teachers" fourth edition.

Popcorn Problem #1 **Name** ______________________________

At the Carolina Popcorn Shoppe, the managers direct their employees as to how much popcorn constitutes a serving when a customer orders one. There are currently 6 cups of cheese popcorn at the serving counter. How many servings can be made if the customer is given:

a. ½ cup of cheese popcorn?

b. ¼ cup of cheese popcorn?

c. 1/6 cup of cheese popcorn?

d. 1/3 cup of cheese popcorn?

e. 2/3 cup of cheese popcorn?

f. 3/4 cup of cheese popcorn?

g. 5/6 cup of cheese popcorn?

Solve this problem using pictures or another tool.

Can you write a mathematical equation to represent your model?

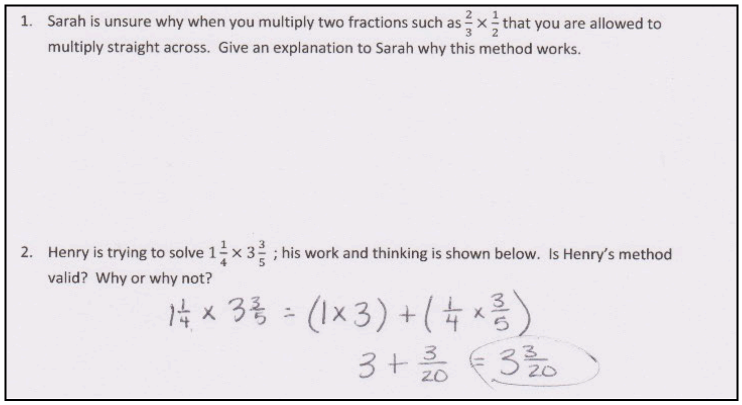
Describe the pattern that you notice, if any, across the sub-questions a-g.

Figure 3: Building Towards the Fraction Division Algorithm.

The activities were selected to build pre-service teachers' conceptual knowledge of fraction-based algorithms as well as their ability to recognize common student errors relating to fraction multiplication and division. Class readings were also given to help further pre-service teachers' understanding related to the given topics.

Measure

Before and after the instructional sequence, participants were asked to complete an assessment (see Figures 4 and 5) related to algorithms for fraction multiplication and division. Prior to instruction, the participants had not worked with multiplication and division of fractions in either of their methods courses.

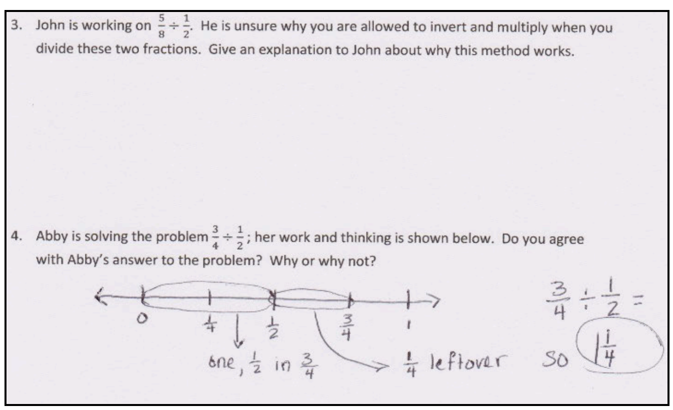


1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.

2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not?

$$1\tfrac{1}{4} \times 3\tfrac{3}{5} = (1 \times 3) + \left(\tfrac{1}{4} \times \tfrac{3}{5}\right)$$
$$3 + \tfrac{3}{20} = 3\tfrac{3}{20}$$

Figure 4: Questions 1 & 2 from Pre/Post Assessment regarding fraction multiplication.



3. John is working on $\frac{5}{8} \div \frac{1}{2}$. He is unsure why you are allowed to invert and multiply when you divide these two fractions. Give an explanation to John about why this method works.

4. Abby is solving the problem $\frac{3}{4} \div \frac{1}{2}$; her work and thinking is shown below. Do you agree with Abby's answer to the problem? Why or why not?

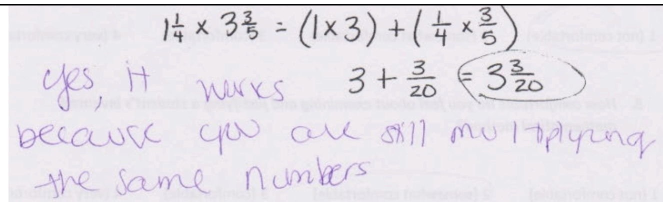
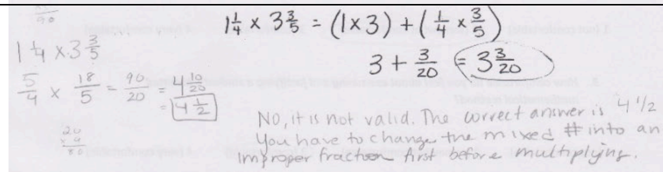
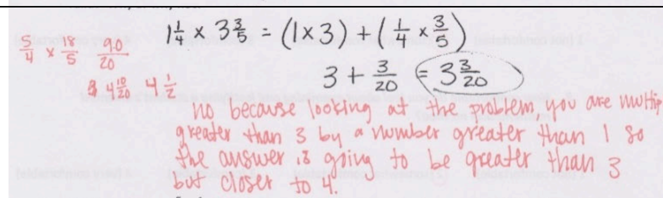
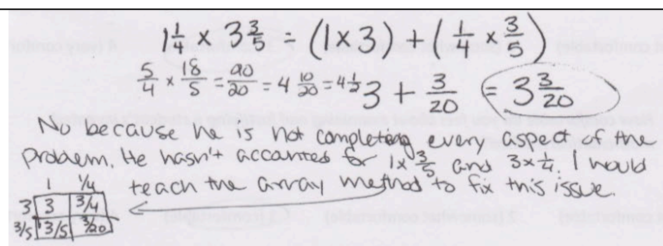
Figure 5: Questions 3 & 4 from Pre/Post Assessment regarding fraction division.

Analysis

Answers were coded according to the levels of the SOLO taxonomy with numbers from 0 to 4 assigned to each of the levels, with Preoperational being a 0 and Extended Abstract being a 4. Two independent raters coded 25% of the pre- and post-assessments in order to check for inter-rater consistency. 98% of the two raters' codes were either exact matches or within one scale point of each other (with 76% exact match agreement). Table 1 provides an example for each level in the SOLO taxonomy as well as a justification for the assigned code.

Beyond the example in Table 1, we now provide a general overview of the coding scheme. Responses coded as the Prestructural level meant the pre-service teacher misinterpreted the question or did not provide an answer. The Unistructural level meant the pre-service teacher did not explain, but instead recited the algorithm to the student as a means of justification. In terms of examining student work, they identified something was wrong, but could not pinpoint exactly where the mistake was occurring. The Multistructural level meant the pre-service teacher recited the algorithm, but gave a further justification for the specific example; however, the justification was not complete or was

Table 1: Examples of Student Responses for each Level.

SOLO Taxonomy Level	Example of Student Response	Justification for SOLO Level
Preoperational (0)		Incorrectly answered the problem.
Unistructural (1)		Recited the rule of turning the mixed numbers into improper fractions.
Multistructural (2)		Recited the rule, but began to display signs of conceptual understanding in the reasoning of estimating the answer.
Relational (3)		Understood the numbers were not fully decomposed and wanted to teach Henry how to view the problem in terms of an area (array) model.
Extended Abstract (4)	N/A – an example might include generalizing the problem to that which includes variables instead of numbers.	No examples of student work provided.

incorrect. In terms of examining student work, the pre-service teacher described what had been done and identified the student's mistake. Within the Relational level, the pre-service teacher provided an explanation in which the idea was fully explained conceptually in terms of the specific example. In regards to examining student work, they correctly identified the problem and explained what it meant in terms of the particular example. Finally, the highest level of justification was the Extended

Abstract in which the pre-service teacher gave a generalized proof in terms of variables, and explained the student's method in terms of a generalized approach.

Results

For question #1 on the assessment, the average score was a 0.50 on the pre-assessment and a .96 on the post-assessment (change of 0.46) on a scale of 0-4. Question #2, resulted in a 0.63 for the pre-assessment and a 1.15 for the post-assessment (change of 0.52) on a scale of 0-4. Question #3 resulted in a 0.40 for the pre-assessment and a 0.77 for the post-assessment (change of 0.37) on a scale of 0-4. Question #4 resulted in a 0.71 for the pre-assessment and a 1.46 for the post-assessment (change of 0.75) on a scale of 0-4. A total score for the entire pre-assessment was .56 and a 1.08 for the post-assessment. A paired t-test was performed for each question with 95% confidence, and all questions showed statistically significant improvement from pre to post assessment (all p-values < .05). A breakdown of the count of students for each level and question for the pre and post assessment is shown in Table 2:

Table 2: Count of Students in each Level and Question.

SOLO Taxonomy Level	**Question #1**		**Question #2**		**Question #3**		**Question #4**	
	Pre	**Post**	**Pre**	**Post**	**Pre**	**Post**	**Pre**	**Post**
Preoperational (0)	34	20	19	13	30	28	25	19
Unistructural (1)	7	13	28	19	17	9	17	7
Multistructural (2)	4	12	1	12	1	5	1	3
Relational (3)	3	3	0	4	0	6	5	19
Extended Abstract (4)	0	0	0	0	0	0	0	0

As one can see from Table 2, no student was classified as Extended Abstract. This may have been because the course did not focus on formal proof and the questions were worded in such a way that they did not suggest giving a formal proof. Before instruction, students typically fell between Preoperational and Unistructural. In other words, pre-service teachers either did not know how to explain the problem to the student or they just simply recited the algorithm. After instruction, most students were between Unistructural and Multistructural; therefore, many students were still reciting an algorithm, but several were also making sense of the algorithm conceptually. The most improvement occurred on question #4 regarding Abby's invented solution to dividing fractions. Many students on the post-assessment recognized the answer of $1\frac{1}{4}$ was not correct because the leftover $\frac{1}{4}$ piece referred to a whole of 1 and not a whole of $\frac{1}{2}$. Therefore, the leftover $\frac{1}{4}$ piece was actually $\frac{1}{2}$ of the whole of $\frac{1}{2}$.

Discussion

In terms of the course, the students were not asked to work formal proofs, but instead use an example to explain a rule in mathematics; therefore, it was expected that no student would fall in the extended abstract category. Second, the questions were given to the student in terms of a particular example and they were asked to explain to the student why the algorithm worked. There might have been students who fell into the Extended Abstract category if the questions were reworded more generally, such as: "explain why the algorithm for multiplying fractions works".

Additionally, the results found from the study hold true with the study performed by Ball (1990), such that prospective teachers' "notions of mathematical explanation seemed to mean restating rules" (p. 138). Although there was a positive increase in levels, work needs to be done to better develop pre-service teachers' conceptual understanding. Finally, some of the increase in scores may have been attributed to fraction division being discussed during the same day of the post-assessment due to time restraints in the course. If the study were performed again, a period of time should be given between the instruction and the post-assessment to check for continued understanding beyond the day of instruction. .

Elementary teachers need to have a deep understanding of the material in order to "handle certain mathematical issues that may arise in the classroom and recognize rudimentary versions of mathematical [proof] in their students' arguments" (Stylianides et al., 2007, p. 148). By strengthening pre-service teachers' conceptual knowledge of mathematics and helping them form solid justifications as to why procedures work, the difficulties secondary students face when they are abruptly introduced to proof in the upper grades may be diminished (Stylianides, 2007). Therefore, in order to improve our education system, we need to improve teachers' knowledge of mathematical content as well as their overall concept of justification (Ball & Bass, 2002; Toluk-Uçar, 2009). The ways in which this can be achieved were highlighted throughout this paper such as: providing more courses in explanation and justification, spending more time on difficult topics, explaining in detail why algorithms work, and providing examples of teachers who teach conceptually through videos or classroom observations. Additionally, more courses involving various elementary mathematics topics are needed to improve pre-service teachers' conceptual understanding.

This paper only touched upon instruction given to students for a fraction-related unit. More research needs to be done to see how students provide justification for other mathematical algorithms, not just for fractions. Also, further research is needed to investigate whether higher-level mathematics courses are a factor in pre-service teachers' ability to justify solutions to students or if there are other factors involved. Finally, interviews of students who performed at the Relational level would be beneficial to see if those students could move into the Extended Abstract level with more time and guidance. Overall, instruction in this study was successful in helping to improve students' scores between levels of the SOLO taxonomy. Therefore, instruction seems to be one stepping stone in helping pre-service teachers in their endeavor to become proficient in explanation and justification.

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TEACHERS' POSITIONING IN PROFESSIONAL DEVELOPMENT: THE CASE OF AGE AND GRADE

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In this mixed methods study, we use positioning theory to analyze teachers' professional discussions about students' mathematical work over the course of 60 hours of professional development focusing on a learning trajectory. After identifying a category of speech actions that suggested student learning is limited by their age or grade level, a quantitative investigation revealed that teachers' discussions grew to include elements of the learning trajectory over time. A subsequent qualitative analysis revealed changes in the structure of these discussions, where teachers came to recognize and use students' prior experiences in instruction, and student learning was more influenced by prior experiences than their age or grade level.

Keywords: Learning Trajectories (or Progressions); Teacher Education-Inservice (Professional Development)

Introduction

Teachers' professional conversations about students often focus on what students cannot do (Franke & Kazemi, 2004) and take evaluative perspectives (Visnovska, Zhao, & Cobb, 2006). Such a focus may lead to limited expectations of students (Rosenthal & Jacob, 1968). In this mixed methods study, we examine teachers' discourse in a professional development setting to consider the ways their learning of a framework for students' mathematical thinking can foster changes in their discourse. We use positioning theory (van Langenhove & Harré, 1999) to frame an analysis of 60 hours of professional development discussions among 22 elementary teachers and examine how certain discourse patterns related to students' ages or grade levels changed during the yearlong professional development. Specifically, our research answers the question: *In what ways does teacher learning of a mathematics learning trajectory relate to changes In their discursive patterns about students as mathematics learners ,and themselves as mathematics teachers, I na professional development setting?*

Background and Theoretical Framework

Professional development (PD) based on students' thinking results in changes in teachers' discourse (Horn, 2007; Kazemi& Franke, 2004). As teachers learn details of students' mathematical strategies, their discussions shift from a focus on students' struggles to more nuanced discussions that attend to students' strategies and levels of sophistication in students' mathematical thinking (Kazemi& Franke, 2004). Teachers' use of more refined language to describe the complexities of student mathematics supports teachers in incorporating student thinking into their model of practice (Horn, 2007). Some researchers have explicitly used student thinking to structure the discourse in a professional development program. For example, Battey and Chan (2010) worked to counteract metanarratives about race and mathematics learning by drawing teachers' attention to student thinking and what students can do—as opposed to what they cannot do. A focus on student thinking led to changes in teachers' discourse about students as they began to base their claims about students in evidence of students as mathematical thinkers instead of assumptions formed by other characteristics.

van Langenhove and Harré (1999) proposed positioning theory as an approach to understanding how psychological phenomena are constructed through social interactions in conversations. In contrast to the “roles” people take in interaction which may be static and limiting, positions in conversations are dynamic, negotiated in the moment, and may be accepted or contested. They argue that conversations can be understood through an examination of three mutually constitutive constructs called positioning-triads: the intention of a speech action (speech acts), positions, and storylines. Whereas speech actions are the actual words that one says, speech acts are the meanings intended by the speaker and are taken up in conversation. Speech acts in conversations tend to follow particular patterns called storylines, which refer to narratives that exist within a culture. Storylines provide a socially constructed image which participants in the conversation use to interpret each other’s actions and positions. Speech acts relative to a particular storyline lead to positions, or the ways these speech acts may be heard by other participants. Positioning can be interactive, where speech acts along with a particular storyline frame another person as competent/incompetent, powerful/powerless, etc., or reflexive, where a speech act frames one’s self in a particular away, such as unknowledgeable, skilled, agentic, etc. Together, speech acts, storylines, and positions describe a structure of conversations and provide a way to understand shared meaning in social interactions.

Researchers in mathematics education have used positioning theory in a variety of ways, including studies of student interactions (e.g., Langer-Osuna, 2011) and classroom interactions (Herbel-Eisenmann and Wagner, 2010). Suh, Musselman, Herbel-Eisenmann, and Steel (2013) used positioning and storylines to study nine teachers’ talk in PD and revealed how teachers’ speech acts positioned students and themselves in relation to teaching and learning. They identified two particular storylines present in the PD discussions related to “low students.” First, teachers’ discussions followed an “institutional tracking” storyline, where students in less advanced mathematics tracks were unlikely to move to take advanced courses. Second, discussions followed an “individual maturation” storyline, where students’ lack of maturity prevented them from being successful in mathematics. In both storylines, students were positioned with characteristics that were beyond the teachers’ control, thus limiting teachers’ self-positioning as agents of change. They report that when teachers were introduced to a new storyline, they positioned students differently and themselves with more agency.

Suh et al.’s (2013) work both demonstrates the utility of positioning theory to examine teacher discourse in PD as well as the possibility of changing discursive patterns about students through the introduction of new storylines. Our research aimed to understand the degree to which teacher learning about a learning trajectory (LT) affected the discursive patterns in PD. Specifically, we focused on speech actions, speech acts, storylines, and positions in teachers’ discussions about students’ mathematical work to understand the ways their learning of a LT affected their conceptualizations of students as learners and themselves as teachers.

Methods

Our study used an exploratory sequential mixed methods design to investigate the changes in teachers’ discursive patterns about students as mathematics learners (Teddlie & Tashakkori, 2009). We followed a two-phase approach. First, we investigated whether teachers’ learning of the LT affected their discussions of students’ mathematical thinking through a quantitative analysis of their speech actions. Next, we investigated qualitatively teachers’ discussions by examining changes in the speech acts, positioning, and storylines over time to understand the ways teachers’ discourse patterns incorporated the LT.

Context

Our study is a part of the Learning Trajectory Based Instruction (LTBI) project, a multiyear PD and research project that investigated teacher learning of students' mathematics LTs and an instructional model where LTs provide guidance for teachers' instructional decisions (Sztajn et al., 2012).LTs are mappings based on empirical research and represent the ways student thinking within a specific mathematical domain evolves over time (Daro, Corcoran, & Mosher, 2011). They outline the partial understandings, common alternative conceptions, and expected tendencies of how learning proceeds in relation to particular forms of instruction (Confrey 2009).

In the first implementation of the PD, we shared Confrey's (2012) equipartitioning LT with teachers and sought to understand how their learning of the LT affected the ways they conceptualized students as mathematics learners and themselves as mathematics teachers. The equipartitioning LT describes how students' informal understandings of fair sharing might evolve to an understanding of partitive division, including students' strategies and common errors related to fairly sharing collections, wholes, and multiple wholes to produce equal-sized groups or parts. The 60-hour PD was designed for a 12-month period, beginning with a 30-hour summer institute during which participants engaged in professional learning tasks (PLTs) that included video analysis of students' working through mathematical tasks, videos of classroom instruction, and analysis of students' written work. Throughout the year, teachers and researchers met monthly after school to discuss the implementation of instructional tasks with their students and refine their understanding of the LT. The PD ended with a two-day follow-up summer meeting.

The research team partnered with one elementary school in a mid-sized suburban school district in the Southeastern United States. The school had approximately 600 students, 35% Caucasian, 29% Hispanic, 25% African American, 7% Asian, and 4% other; 54% of the children qualified for free or reduced lunch. Twenty-two K-5 teachers completed the project at years end.

Data Collection and Analysis

Data sources included video and audio recordings of teachers' discussions while engaging in 21 selected PLTs that focused teachers on students' mathematical thinking. During the summer, teachers engaged with various practice-based artifacts to learn about students' thinking of equipartitioning. During the school year meetings, teachers discussed various classroom-based activities aimed at eliciting and understanding their own students' thinking. Recordings for these 21 PLTs, totaling 41 video and 55 audio files, were transcribed.

During the PD, patterns in teachers' discussions about students' mathematical work began to emerge, often attributing students' lack of success to their grade level or age. Statements such as, "we don't do that in third grade" or "he is low," revealed some of the narratives about students as learners that were accepted and used in the group. We conceptualized these statements as categories of speech actions related to storylines that were used to position students as mathematics learners and themselves as mathematics teachers. Upon completion of the PD, we developed a codebook to identify these speech actions based on the field notes collected during implementation (Wilson, Edgington, Sztajn, & Decuir-Gunby, 2014). Four codes defined the categories and described teachers' speech actions as suggesting student learning is dependent upon their: Ability/Achievement; Age/Grade; Effort; and Luck. In this paper, we focus exclusively on speech actions related to Age/Grade.

Phase one. For the quantitative analysis, we specified the unit of analysis as a speech action. Four independent coders first identified speech actions related to Age/Grade (85% inter-rater reliability). To understand if learning the LT resulted in changes in their speech actions, the research team revisited the coded units for evidence from the LT. Evidence was taken to be both explicit use of LT terminology (e.g., direct reference to the LT structure, specific student strategies described within the levels) as well as implicit where teachers used less formal language to describe ideas from the LT. In both cases, these units were assigned an additional code of LT, resulting in three variables: Age/Grade with LT, Age/Grade without LT, and total Age/Grade. Next, we organized the total speech actions within each of the 21 PLTs chronologically (see Table 1). We hypothesized that as the PD progressed, teachers' speech actions would increasingly include references to the LT and thus subjected each variable with its associated time to Spearman rank-ordered correlation tests of significance.

Phase two. To understand the ways and the extent to which teacher learning of the LT resulted in changes in speech acts, positioning, and storylines in the PD, we identified episodes in teachers discussions, that is segments of discussion around one idea that begins when a speech action is made and taken up by other participants in the discussion (Harré & van Langenhove's, 1999). We then used a constant comparative method (Strauss & Corbin, 1998) to discern themes across all episodes, meeting regularly to discuss emerging patterns, deviations from those patterns, and reconcile them with the data. We considered the resulting themes as storylines teachers used to conceptualize students as mathematics learners and themselves as mathematics teachers. Next, we applied the positioning-triad as an analytic tool to the episodes to understand the ways, and extent to which, teacher learning of the LT resulted in changes in the storyline and positions across the PD. For each PLT, two members of the research team summarized the speech acts, teachers' positioning of themselves and students, and variations in the storyline for all episodes occurring during the PLT. The entire research team then looked across the summaries to mark shifts in teachers' discussions as the PD proceeded and understand the ways the LT affected teachers' discursive patterns.

Findings

Results from the two phases of analysis indicate that teacher learning of a LT affected the discursive patterns about students as mathematics learners in a PD setting. First, we show how the speech actions related to Age/Grade changed quantitatively over time. We then qualitatively examine how the speech acts, storylines, and positions changed.

Phase one analyses indicated that teachers' speech actions related to Age/Grade remained present throughout the PD ($n = 143$; $\rho = -0.052$; $p = 0.411$). Further, speech actions related to Age/Grade that also used the LT significantly increased over time ($n = 40$; $\rho = 0.352$; $p = 0.059$). For the subset coded no LT, no significant decrease was found ($n = 103$; $\rho = -0.079$; $p = 0.367$). These findings confirm our hypothesis that, as the PD unfolded, teachers learned about the LT and came to use LT-language in Age/Grade discussions of students as mathematics learners. These analyses show that teachers' speech actions changed, yet they do not characterize the changes in positions or storylines used to conceptualize students as mathematics learners.

Phase two analyses focused on the episodes related to Age/Grade in order to examine positioning and storylines. As seen in Table 1, a total of 46 episodes related to Age/Grade occurred in the focus PLTs. Table 2 summarizes the speech acts, storylines, and positions across these episodes. Whereas the initial storyline of the discussions indicated that age and grade level are key influences of student learning, the storyline changed such that students' prior experiences influenced learning. Though teachers' speech actions still referred to age and grade, these actions were intended more as descriptions of the student. Teachers began the PD by positioning themselves as one who had experiences with students and expertise to make sense of students' mathematics but had no instructional recourse if a student did not meet their age or grade level expectations, because students' capacities to learn were limited. Over time, teachers began to note that students had experiences, beyond their age or grade level, that they may bring to instruction, and began to indicate that they could include these experiences in their instructional decisions. Eventually, teachers positioned students as having resources that support their mathematics learning and could tailor their instruction to make use of these resources. In what follows, we provide two episodes to illustrate these shifts.

Table 1. Number of Age / Grade speech acts and episodes by Time and PLT.

Time	PLT	Speech Actions			Episodes	Time	PLT	Speech Actions			Episodes
		LT	No LT	Tot.				LT	No LT	Tot.	
Summer Institute	1	0	31	31	5	Dec.	12	0	0	0	0
	2	0	1	1	1	Jan.	13	3	1	4	2
	3	0	13	13	7	Feb.	14	5	7	12	5
	4	0	0	0	0	Mar.	15	2	2	4	2
	5	1	3	4	2	Apr.	16	3	2	5	1
	6	0	0	0	0		17	0	2	2	1
	7	1	1	2	1	May	18	0	1	1	1
	8	2	11	13	5		19	0	0	0	0
Sept.	9	3	2	5	1	June	20	2	5	7	1
Oct.	10	9	7	16	6		21	6	4	10	3
Nov.	11	3	11	14	2		Total	40	104	144	46

Table 2. Number of Age/Grade episodes & related speech acts, storylines, and positions.

PLT	N	Speech Act Examples	Storyline	Teacher Positioning	Student Positioning
1-3	13	"I would have expected that from the third grader"	Learning is limited by a student's age or their grade level	Teachers have expertise in understanding students and have expectations based on age / grade level	Students' age or grade level are expected to be successful at certain mathematics
5-12	17	"That radial cut has not appeared yet…kids have the idea from [their] experience that it's supposed to look like a triangle"	Learning is influenced by age or grade level and prior experiences	Teachers have expertise and expectations but recognize the role of prior experiences	Students have prior experiences beyond their age or grade level that influence learning
13-21	16	"We think that because kids are older, they've had experiences, but that's not necessarily true"	Learning is influenced prior experiences	Teachers can use students' prior experiences to support learning	Students' prior experiences are resources that support their learning

As an example of initial positioning during the summer institute (PLTs 1–3), the following example occurred during the first task (PLT 1). Teachers viewed two clinical interviews of two students completing a series of equipartitiong tasks where they were sharing 24 coins among four and three people (Wilson, Edgington, & Confrey, 2010). The following episode was part of a group discussion about what they described as surprising about the students' work:

G1: Well, the kindergartner did what I thought starting back over with a pile in order to move from four to three. I thought that. But she surprised me with what she did after that.
A1: Well, the most surprising thing was that she seemed to know quickly that it was eight. And I was trying to decide if that was just a guess or if it was an intuitive understanding because she did seem to even recognize that if you took the six coins that the fourth person had, you could distribute those three ways and have eight. She almost seemed to know that intuitively –[D5 agreeing] – which I would never have expected.
D5: No. Me neither.
A1: I would have expected that from the third grader and yet I felt dismayed that formalized classroom mathematics education had stifled her, so that she [G1, D5 agreeing] didn't see it at all that way and immediately suggested that you couldn't do it because three was an odd number.
G1: Right - and there would be some leftover.
A1: You know? Certainly the five year old wasn't encumbered by that. [D5 agreeing] She probably -- she might know odd from even, but that wasn't part of the discussion.
G1: Yeah, that –
A1: was a real stumbling block for the third grader, I thought.

As teachers explained students' mathematical approaches in this discussion, they used the grade levels or age of the students to support their expectations and understanding of what the student did. Speech acts, such as "The kindergartener did what I thought…but surprised me when", "which I never would have expected…I would have expected that from the third grader", and "the five year old wasn't encumbered," positioned themselves as competent in inferring students' mathematical understanding based on their expectations of students at a particular grade level or age. Simultaneously, students were positioned according to these expectations in comparison to students of other age or grade levels, resulting in teachers' surprise. Taken together, the teachers' speech acts and positions follow a storyline that students' mathematical understandings are defined by their age or grade level.

As the PD progressed through the fall (PLTs 5-12), PLTs were based on students' written work and videos of whole classroom instruction. Although teachers' speech acts still referred to students' ages or grade levels, they began to include terminology and ideas from the LT to describe student thinking in more detail. For example, in an episode where teachers were examining a set of student work, one teacher commented, "Developmentally, that radial cut has not appeared yet…kids have the idea from [their] experience that it's supposed to look like a triangle. But when they're dividing, we only had one that actually tried to use those radial cuts." The teacher attributed students' lack of success on the task to their age (development), but also recognized that students' experiences influence their thinking. In such episodes, teachers continued to position themselves as competent in understanding students' thinking but began to expand their explanations beyond age or grade level to include other students' experiences. In turn, students were positioned as having experiences that may support them in learning. These nine episodes followed a storyline that shifted from learning being limited by age and grade to include opportunities to learn and prior experiences also influencing student learning.

By January (PLT 13),a stable positioning-triad emerged in the episodes related to Age/Grade and persisted for the duration of the PD. At this point, speech acts incorporated ideas from the LT and referred to age and grade level not as a limiting factor for learning, but as a descriptor. These 16 episodes followed a storyline where students' prior experiences affected learning, teachers positioned themselves as able to use those experiences as resources for instruction, and students were positioned as bringing resources to instruction. To illustrate, the following example episode from the summer follow-up in June occurred as teachers revisited the two clinical interviews from the example above.

A4: If we believe learning is along this trajectory and it's this path way and they need to have these things earlier on before they can get to these things, then it does seem that age is less relevant. Yes, they have more experiences so hopefully they're getting there.

D4: Yeah, isn't it both? Because they don't necessarily have to have- it's not really linear. I mean, they can jump certain things, you know, levels.

[one speech act omitted from the episode]

D3: I liked what D4 was saying about like with age, we often link that to experiences, that because they're older, they've had more experiences. And so I think a lot of what we do is that we do learn from our experiences and we often make the assumption that because kids are older, they have consistently had experiences and they've learned through that. But I don't think that's necessarily true when you look at home backgrounds that kids come from or look at being in a different classroom or a different school, different things like that. That's going to come into play. So each child comes to the table with a different set of experiences. And so that's where like sometimes the high kids do surprise us because they don't know something. Because maybe we've made that assumption before and not provided them with those experiences, or those experiences have always been done for them at home.

Teachers' speech acts incorporated ideas from the LT (e.g. "trajectory," "levels") and referred to age and grade level, not as a limiting factor for learning, but as a descriptor. The storyline suggests that student learning is influenced by prior experiences and opportunities and is not limited to their age and the expected school experiences for students at particular grade level. Teachers positioned themselves with agency – they could provide learning opportunities for students that draw upon, or even provide, such experiences as resources to foster learning. Students were positioned as having these resources that could be used when learning mathematics.

Discussion

Learning the LT changed teachers' discursive patterns about students as mathematics learners and themselves as teachers. Similar to Suh et al.'s (2013) individual maturation storyline, we found a storyline with elementary grades' teachers that a student's age or grade level limits their learning. Yet in our study, teachers grew to acknowledge and use prior experiences as a resource for supporting learning rather than viewing students' learning as bounded by their age or grade level. Thus, we conclude that learning a framework for students' thinking may led to changes in the ways teachers position students from a more strengths-based perspective. Further, such learning may support teachers' agency.

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A MULTI-YEAR STUDY OF THE IMPACT OF A PROBLEM-SOLVING FOCUSED PROFESSIONAL DEVELOPMENT ON TEACHER LEADERSHIP

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This study explored the connections between teacher leadership and a professional development model focused on problem-solving, Math Teachers' Circles (MTC). Surveys were completed by 213 MTC participants resulting in three years of data across multiple sites. A mathematics education leadership framework provided a data analysis tool. Findings suggested MTCs did impact participant teacher leadership, as survey results attended to all three areas of leadership from the analysis framework.

Keywords: Middle School Education; Teacher Beliefs; Teacher Education-Inservice; Teacher Knowledge

Objectives & Purpose of Study

With the implementation of the Common Core for State Standards in Mathematics (CCSSI, 2010), the need for mathematics teachers to be learners and leaders in mathematics education is stronger than in the past. Mathematics teachers are being called upon to not only improve their own mathematical content knowledge and how to teach that content, but also to lead students, parents, colleagues, administrators, and communities in understanding and meeting the new standards presented in the CCSS. Math Teachers' Circles (MTCs) are one form of professional development that may help teachers reach these standards for teaching and leading.

In this study, the researchers highlight the potential of MTCs to develop mathematics teacher leaders. Both are experienced MTC leaders, and they use their knowledge of MTCs, of teacher leadership, and of teacher professional development—augmented with quotes from national surveys of MTC participants—to highlight the potential of MTCs to develop mathematics teacher leaders.

After preliminary background on MTCs, their history, and sample problems, the authors use the PRIME Leadership Framework (NCSM, 2008) as a lens with which to view the work of MTCs. The paper ends with some concluding remarks on growing MTCs as a means for teacher leader development.

Math Teachers' Circles

MTCs are a form of professional development, aimed primarily at middle school math teachers, which tap into the instinct that mathematicians have to share their love of mathematics with others. This makes MTCs an accessible entry point for mathematicians interested in starting to work with teachers and for teachers interested in working with mathematicians to learn more mathematics through problem solving. With the advent of the CCSS, this partnership is particularly timely for both parties.

The canonical model for an MTC involves a leadership team consisting of two mathematicians, two middle school math teachers, and one administrator. After attending a summer training workshop, teams spend a year securing funding and then launch their own MTC the following summer, generally with a 4-5 day residential workshop. Academic year sessions follow, generally about three per semester, with each 2-2.5 hours in duration. Each MTC tailors this basic model to meet the needs of their local setting.

Sessions typically begin with a leader presenting a rich problem that fosters exploration involving multiple levels of deep mathematical content. As the session develops, problems lend themselves, for example, to discussions of symmetry, algebraic representations and functions, and arithmetic and algebraic properties like the commutative, associative, and distributive laws. From a problem-solving perspective, problems may lend themselves to the techniques of "ask a simpler question" and "work backwards." Related to the CCSS, it requires teachers to use most, if not all, of the Standards of Mathematical Practice as they explore the various facets of the problem.

A final key feature of some problems is the "low-threshold, high-ceiling" property. That is, an entry-level problem requires minimal mathematics background to understand and begin to explore it, yet it can be connected to research-level mathematics. In addition to in-depth explorations, participants also investigate shorter problems. For example, the following two problems can be discussed in a meaningful way in about 30 minutes: (1) What is the last (units) digit of 7^{503}? (2) What are the last two digits of ^{503}?

Computing either of these by hand would be quite cumbersome, and plugging them into a calculator provides an estimate in scientific notation, but does not help directly with finding the last few digits. Thus, one needs to use mathematical reasoning, in particular concepts of place value and the process of multiplication, to both find and justify a pattern for the last tens and units digits of the powers of 7 that allows one to answer each question. In working through these problems, teachers work directly with content relevant to their students, but at a level that develops their own mathematical reasoning skills. Fernandes, Koehler, and Reiter (2011) and Donaldson, et al. (2014) provide a more in-depth discussion of a problem unfolding in a MTC session, while White (2015) provides a more in-depth description of one MTC.

In addition to working on engaging mathematical problems, many MTCs include sessions or portions of sessions that directly connect to the classroom. These comprise diverse topics such as effective questioning strategies, how to translate lessons learned in MTCs to the classroom, and how to implement the CCSS. Two valuable texts the authors have found to help lead some of these sessions are Boaler and Humprey's (2005) *Connecting Mathematical Ideas: Middle School Video Cases to Support Teaching and Learning* and Burago's (2010) *Mathematical Circle Diaries, Year 1: Complete Curriculum for Grades 5 to 7.*

A somewhat unexpected (at least to the original organizers) consequence of the MTC program has been the development of teacher leadership. This was first discovered on a national survey of MTC participants in 2010, where a surprising number of respondents made comments that indicated that they had emerged as informal or formal leaders in their schools or districts. Moreover, they attributed this to their participation in MTCs. In the remainder of this paper, we discuss how MTCs connect directly with the PRIME Leadership Framework (NCSM, 2008), augmenting our narrative with illustrative quotes from participants.

Literature Review & Theoretical Framework

Need for Teacher Leaders

Reform documents emphasize K-12 students should be learning mathematics through problem solving (National Council of Teachers of Mathematics [NCTM], 2000; CCSSI, 2010).Mathematics Standards call for students to discuss, collaborate and justify their thinking through engaging tasks (NCTM, 2000; CCSSI, 2010) and for teachers to support such work (NCTM, 2014).

Despite this push for reform, middle school mathematics teachers still often teach content in traditional teacher didactic manners emphasizing textbooks and lecture (Grouws & Cebulla, 2000; Kent, Pligge, & Spence, 2003; Weiss, Pasley, Smith, Banilower, & Heck, 2003). With so many barriers to instructional change (Anderson, 1996; Roehrig, Kruse & Kern, 2007), teachers need to feel empowered to make changes to their instruction that benefit student learning (Fullan, 2001).

Professional Development

Professional development is one avenue for empowering teachers to make changes. Effective professional development should build teachers' content knowledge, immerse them in authentic experiences, address beliefs, involve collaborative communities, and provide long term support (Darling-Hammond, Chung Wei, Andree, Richardson, & Orphanos, 2009;Johnson, 2006; Loucks-Horsley, Hewson, Love, & Stiles, 2003). Flowers and Mertens (2003) described a need for professional development specific to middle school teachers' content and student learning needs. Teachers who attended professional development experiences longer than 8 hours and connected to other school-based initiatives reported the experiences improved their teaching while shorter unconnected experiences did not (Flowers & Merten, 2003). Other studies show long-term professional development of more than 80 hours is needed for teachers to enact inquiry-based practices (Porter, Garet, Desimone, & Birman, 2003). Professional development focused on mathematical problem solving increases teachers' content knowledge and improves pedagogical strategies (Anderson & Hoffmeister, 2007). MTCs are designed using these research-based characteristics: focused on building teacher content knowledge through problem solving strategies, was long-term, and involved building collaborative communities between teachers, mathematics educators, and mathematicians. This study further investigated the connection between high quality professional development and teacher leadership (Yow & Lotter, 2014).

Teacher Leadership

Due to teacher influence on student learning (Darling-Hammond, 1999), developing teacher leadership in mathematics education is one of the most important factors in schools (Pellicer & Anderson, 2001). Dozier (2004) defined teacher leaders as "good teachers who influence others." Before the phrase teacher leadership (Barth, 2001; Lieberman & Miller, 2004), teacher empowerment was the phrase used to describe a similar construct: teachers believing they had the "skills and knowledge to act on a situation and improve it" (Graham & Fennell, 2001; Short, 1994). Subject specific teacher leadership is a recent phenomenon; professional organizations define educational leaders in teaching (NBPTS, 2010; NCTM, 1991). Characteristics specific to mathematics teacher leaders are described in the literature (Langbort, 2001; Miller et al., 2000; Yow, 2007); however, few empirical studies exist (Webb, Heck, & Tate, 1996; Yow, 2010). Therefore, our research question asked *How does teacher involvement in a Math Teachers' Circle impact their enactment of teacher leadership?*

Theoretical Framework

The National Council of Supervisors of Mathematics (2008) developed the *Principles and Indicators for Mathematics Education Leaders* (PRIME) leadership framework responding to a lack of attention to the importance of school leadership in improving teaching and learning (Spillane, Halverson, & Diamond, 2004). The framework "aims to describe actions for mathematics education leaders across all settings, preK-12, in all its complexity" (NCSM, 2008, p. 2). We see teachers as educational leaders closest to student learning and therefore choose to focus on their data.

The framework lists specific actions that fall along a continuum of three stages of leadership growth. The continuum includes knowing and modelling leadership (leadership of self) as stage 1, collaborating and implementing structures for shared leadership on a local level (leadership of others) as stage 2, and advocating and systematizing improvements into the wider educational community (leadership in the extended community) as stage 3 (NCSM, 2008, p. 2). With this focus on leadership in mathematics education, we employed this framework as our data analysis tool.

Methods

Data was analyzed from a national survey distributed to all MTCs in 2010 and to the authors' local MTCs in 2013 and 2014. Collectively, 213 participants completed the survey measuring MTC teacher impact. The survey contained Likert scale and open-response items asking teachers to rate and address their gains in mathematical content knowledge, attitudes and dispositions toward mathematics, classroom instructional practices, and professional activities. Data was analyzed through the three areas of leadership put forth in the PRIME Leadership Framework (NCSM, 2008). As shared earlier, the framework consists of: Stage 1 Leaders, considered ***Leadership of Self*,** "know and model" good teaching practices in their own classrooms; Stage 2 Leaders, considered ***Leadership of Others***, "collaborate and implement" these good teaching practices with all students and other teachers and administrators; and Stage 3 Leaders, considered ***Leadership in the Extended Community***, "advocate and systematize" these good teaching practices on a larger scale helping to create instructional change at the district, state, national, or international level. At each stage, leaders are respected for the self-knowledge, influence of others, and advocacy on a larger scale, respectively.

Data was coded using the above three foci. For example, when teachers mentioned impact of MTCs on their own practice, those comments were coded as *Leadership of Self.* Comments that reflected MTC impact on collaborative work, such as "we all worked as a team to try and learn together," or encouraged teachers to invite others to learn with them (e.g., "I've started bringing my colleagues from the department to the Circles as well") were coded as *Leadership of Others*. Lastly, comments were coded as *Leadership in the Extended Community* when they spoke to the impact of MTCs on teachers expanding their sphere of influence to a larger community: "I have started giving presentations at meetings and conferences, have become the mentor for new math teachers, and am peer reviewer of math activities for the classroom at my level for the region."

Results, Discussion, and Conclusions

Findings showed MTC participant responses attended to each of the three stages of leadership put forward in the PRIME Leadership Framework.

Stage 1: Leadership of Self

MTCs allow teachers to develop their content and pedagogical expertise, change practice, and take risks (Yow, 2007), all part of leadership of self. A prerequisite to knowing and modeling good teaching practices is knowing and understanding the content and the disciplinary practices of mathematics. With the advent of the CCSS, many teachers are being asked to teach in ways that are often quite different from how they were taught. MTCs provide an opportunity for teachers to take on the role of learners of mathematics, and to revisit what that role looks and feels like. In fact, in initial end-of-workshop surveys of the MTC program, comments about being back in the role of a learner of mathematics were so prevalently cited as a benefit of MTCs that it was incorporated as a question on the most commonly used end-of-workshop survey form, and has been widely reported at conferences as one of the key outcomes of MTCs.

Math Teachers' Circles focus on building teacher content knowledge through problem-solving while also strengthening teachers' problem-solving skills and fluency with implementing the CCSS Standards of Mathematical Practice. They also learn the habits of mind and the disciplinary practice of mathematics, whereby the "answer" in not always known and uncertainty followed by exploration is the norm. As noted by one participant, "I have not participated in a workshop where I as a person have to struggle through, and the presenter did not share the correct answer." Often times, "answers" to the problems are not immediately given to participants at the end of problem-solving sessions so they can continue to grapple and discuss the problem. This directly addresses the first mathematical practice of perseverance in the CCSS (2010).

Teachers see what is modeled in the MTC and develop their pedagogical expertise in their own classrooms:

> In the meetings, they deliberately have us talk about our techniques and approach to the problem. When we think out loud as a group, we start to see patterns, and together we think about tools we can use. When possible, I like to integrate this approach in my classroom, giving my students the tools they need to figure out an overarching problem and then letting them chip away at it collectively until they figure it out.

MTCs provide an opportunity for teachers to be Stage 1 Leaders by allowing them to improve their own content knowledge as well as problem-solving skills. MTCs allow teachers to work in community to learn mathematics and solve problems that they can then implement in their own classrooms. "My classroom teaching has become more student-centered and engaging. Students are working together and discussing problems in groups, or exploring individually before sharing with a larger group."

Stage 2: Leadership of Others

MTCs focus on building community and learning how to solve problems alongside colleagues which lends itself to Stage 2 Leadership of Others by "collaborating and implementing." One teacher noted

> We all worked as a team to try and learn together…As different participants have different backgrounds and learning styles, the group work revealed to be very effective because we used our previous knowledge and our personal ways of seeing, interpreting and solving each problem. The support and the conversation with other members of the circle were always positive and enriching.

Teachers become a part of a community of learners within the MTC and then use this structure to begin similar communities at their schools. Seeing the model of mathematical discussion among the mathematicians, mathematics teacher educators, and teachers helps teachers learn how to facilitate mathematical discussions with their own colleagues. They gain a broader sense of what it means to be a part of a mathematics education community: "I feel that working on mathematics with my colleagues gives me a wider perspective on how to view mathematics and what it means to teach mathematics."

MTCs provide the opportunity for teachers to be Stage 2 Leaders by their returning to their schools and sharing what they learned with their fellow teachers and administrators. For example, following the intensive summer workshop, teachers invited colleagues from their schools to join us at our monthly Saturday meetings so they could see for themselves the types of mathematics and problem-solving in which we engaged. Some of these guests came to more than one Saturday meeting to extend their own learning and take what they were learning back to their schools and classrooms. In addition, one of the mathematics teacher educators invited her preservice teachers to come to the Saturday sessions. Their visit introduced them to current practicing teachers and offered them an opportunity to see what types of professional development are available to teachers and how they may incorporate such problem-solving approaches into their own future classrooms. One teacher wrote, "I've started bringing my colleagues from the department to the Circles as well, and that helped them realize that there really is something to this approach. It's also helped us cohere as a group back at school, and our students benefit immensely."

Stage 3: Leadership in the Extended Community

MTCs provide an opportunity for teachers to be Stage 3 Leaders by their sharing what they have learned on a larger scale. For example, teachers from MTCs have presented at state mathematics conferences. During these presentations, teachers advocate for strong instructional practices by explaining what happens during MTC meetings and then present rich problems that they themselves have spent time solving with their students. They discuss the rich mathematical content of the problem and the mathematical directions they or their students may take to solve the problem. One teacher shared, "I have started giving presentations at meetings and conferences, have become the mentor for new math teachers, and am peer reviewer of math activities for the classroom at my level for the region."

Because MTCs often span several schools and school districts, teachers are able to interact with an extended community of math teachers within a region and increase their ability to systematize the practices used in MTCs. This leads to teachers building a wider network of colleagues with whom to learn and collaborate: "Participating in the MTC meetings has encouraged me to network with others and attend conferences with them." This newfound agency of being a member of a professional community also builds confidence in MTC participants, which results in taking on more of a leadership role in areas outside of their classrooms as well: "[Being a part of an MTC] has given me the confidence to step into more of a leadership role and a role in developing curriculum and lesson plans." Another participant who has become an instructional coach credited MTCs with helping in her job transition. "My leadership role in our math circle has correlated directly with my role change in my district from classroom teacher to teacher leader (instructional coach)." Strengthened by belonging to a professional mathematics education community empowers teachers to be mathematics teacher ambassadors (Yow, 2007) charged with telling others about the work they are doing in the MTCs and in their classrooms with students.

Concluding Remarks

MTCs provide a valuable professional development experience for mathematicians, mathematics teacher educators, and mathematics teachers. A powerful outcome of MTCs is the sense of community and broader understanding teachers gain of mathematics and the mathematics education community as a result of their participation. Even more, they provide an opportunity to develop mathematics teacher leaders by offering teachers the opportunity to develop in the areas of leadership of self (e.g. developing their content and pedagogical expertise, changing practice, and taking risks), leadership of others (e.g. becoming a part of a learning community), and leadership of the extended community (e.g. teaching their colleagues).

MTCs have the potential to impact mathematics teacher leadership on a national level given there already exists a national community and network of MTCs. Preliminary research has already been published on MTCs (White et al., 2013) and their relationship to mathematical knowledge for teaching. Also, Marle, Decker, and Khaliqi (2012) report that at one MTC, after a year of participation, classroom observations showed increases in the use of inquiry-based learning and in pedagogical content knowledge. More research is needed among long-standing MTCs to measure how extended participation in an MTC aides the continued evolution of a mathematics teacher leader from Stage 1 leadership to Stage 3 leadership (NCSM, 2008).

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SUPPORTING NEW K-8 TEACHERS OF MATHEMATICS TO BE CULTURALLY RESPONSIVE USING A LESSON ANALYSIS TOOL

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This paper briefly describes a lesson analysis tool, and its embedded conceptual framework, designed to help teachers analyze their mathematics instructional practice from a culturally responsive standpoint. The tool consists of six dimensions: cognitive demand, depth of knowledge and student understanding; mathematical discourse; power and participation; academic language supports for English learners, and cultural/community funds of knowledge. These six dimensions have theoretical underpinnings in culturally responsive pedagogy and pedagogical content knowledge.

Keywords: Equity and Diversity; Teacher Education-Preservice; Teacher Education-Inservice

This brief report describes a lesson analysis tool, and its embedded conceptual framework, designed to help teachers analyze their mathematics instructional practice from a culturally responsive standpoint. The Culturally Responsive Mathematics Teaching – Lesson Analysis Tool (CRMT6-LAT) consists of six dimensions: (a) cognitive demand, (b) depth of knowledge and student understanding, (c) mathematical discourse, (d) power and participation, (e) academic language supports for English learners, and (f) cultural/community funds of knowledge and social justice (Aguirre, et al., 2012). These six dimensions have theoretical underpinnings in culturally responsive pedagogy and pedagogical content knowledge, and an empirical basis in research that has documented positive effects on mathematical learning and identity development. The tool can be used to analyze real-time, videotaped, or written lessons, or to support lesson planning, reflection, and refinement. The power of this tool is its combined multi-dimensional focus on traditionally separate constructs of equitable mathematics instruction.

Theoretical Framework

Culturally responsive mathematics teaching (CRMT) is an equity-based instructional approach that integrates attention to mathematics, mathematics thinking, culture, language, and power to comprehensively support and strengthen student mathematics learning, engagement, and positive mathematics identity development (Aguirre & Zavala, 2013; Leonard et al, 2010). CRMT reflects the scholarship intersection between culturally responsive pedagogy and pedagogical content knowledge. Gay (2000) argues that culturally responsive pedagogy "simultaneously develops, along with academic achievement, social consciousness and critique, cultural affirmation, competence and exchange; community building and personal connections; individual self-worth and abilities and an ethic of caring (p.43-44)." Pedagogical content knowledge is the specific knowledge needed to teach content, including knowledge about the purposes of teaching mathematics, children's content understandings or misunderstandings, curriculum, and instructional strategies and representations (Grossman, 1990). Both literatures are robust in teacher education yet rarely connected to help support instructional practices needed to advance the mathematical learning of today's youth.

CRMT6-Lesson Analysis Tool

To help mathematics teachers teach from a culturally responsive approach this tool supports the comprehensive examination of teaching practice including lesson design, implementation, and reflection. The full tool, available from the TEACH MATH (Teachers Empowered to Advance Change in Mathematics) website, includes a reflection prompt for each dimension as well as rubric-style descriptors with progressive scores 1-5 reflecting a range of practices.

Cognitive Demand

From a culturally responsive stance, providing all students with access to high cognitive demand tasks is an equity issue. Research consistently finds that students who are placed in "lower" mathematics classes, and who are members of non-dominant groups, are less likely to be exposed to mathematical tasks that demand higher order thinking (Mosqueda, 2010; Oakes, 2005).

Depth of Knowledge and Student Understanding

This dimension highlights how knowledge and understanding are expressed in mathematics lessons, through activities such as reasoning, explaining, justifying, and engaging in complex problem-solving. Research shows that when teachers support mathematical learning by attending, interpreting, and responding to students' mathematical thinking, student understanding deepens (Carpenter, Fennema, Peterson, Chang, & Loef, 1989).

Mathematical Discourse

Mathematical discourse is an important component of deepening and communicating mathematics understanding. Research shows that all children, including young children and children learning mathematics in another language can showcase their mathematical knowledge when given opportunities to participate in mathematical discussions (Celedón-Pattichis & Turner, 2012; Herbel-Eisenmann & Cirillo, 2009; Turner & Celedón-Pattichis, 2011; Yackel & Cobb, 1996).

Power and Participation

Important considerations for understanding and supporting student participation are issues of power, status, and authority in the classroom. The way (whether intended or not) a teacher positions students as valuable contributors or "uninvited guests" can determine who has legitimate mathematical authority in the classroom (Spencer, 2006; Turner et al, 2012; Yoon, 2008). Developing worthwhile mathematical tasks that promote mathematical thinking, have multiple entry points, and tap into a wide range of knowledge can help to minimize status issues and foster a shared authority over mathematics (Featherstone, et al, 2011; Horn, 2012).

Academic Language Supports for ELLs

Academic language is vital to the way students learn and communicate their understanding of mathematics (Moschkovich, 2010). Therefore it is important to consider the instructional supports needed to facilitate student development and use of academic language. This is particularly true for students who are learning mathematics in an additional language (Celedón-Pattichis & Ramirez, 2012; Civil & Turner, 2014). This dimension focuses attention on lesson structures and language scaffolding strategies (e.g. revoicing, use of cognates, graphic organizers, realia) that support academic language development especially for students learning English as another language.

Cultural/Community Funds of Knowledge and Social Justice

This dimension highlights the important mathematical knowledge and practices that occur outside of school in children's homes and communities, specifically cultural funds of knowledge (Gonzalez, Moll, & Amanti, 2005). Examples of connections to children's funds of knowledge include mathematical investigations of family/community gardens, sewing techniques, family budgeting practices, as well as practices of community businesses such as auto painting, flower shops, fire stations, and *mercados* (Civil, 2007; Civil & Kahn, 2001; Taylor, 2004; Turner, Varley Gutiérrez, Simic-Muller, & Díez-Palomar, 2009). Mathematics is an analytical tool to make sense of socio-political issues that impact their world and life experiences including issues of civic engagement, overcrowding, and fairness (Gutstein, 2006; Turner & Strawhun, 2007).

Lessons Learned From Project Uses of the Tool

The central aim of TEACH MATH is to transform mathematics teacher education so that teachers will be equipped with powerful tools and strategies to increase student learning and achievement. We have used the CRMT6 tool with new teachers in several ways:

Analysis of lessons in mathematics methods classes. Prospective teachers have used the CRMT6-LAT tool to analyze a lesson they designed and/or taught during the mathematics methods course. Based on their analysis, prospective teachers wrote a critical reflection describing the strengths and limitations of the lesson and specific ideas for improvement supporting teachers to simultaneously synthesize different dimensions of culturally responsive mathematics teaching.

Observation of practice. The tool was also used when observing lessons during student teaching. Used in this way, the tool enabled the mathematics teacher educator to understand which dimensions of culturally responsive mathematics teaching prospective teachers felt confident with, which dimensions might warrant further exploration, and which dimensions were still quite challenging for them to attend to in lesson planning and implementation.

Providing targeted feedback to early career teachers. As part of our longitudinal study, we also observed and provided targeted feedback to early career teachers using this tool. For example, to support professional learning and growth, we encouraged teachers to select two dimensions from the CRMT6-LAT. The most prevalent choice was mathematical discourse (30% of observations) followed by cognitive demand (17%), power & participation (16%), depth of knowledge (14%), academic language (12%); funds of knowledge (11%), suggesting teacher hesitancy to focus on the equity-oriented dimensions addressing language, culture, and power.

Conclusion

There is a need for pedagogical tools, such as the CRMT6-LAT, that help teachers to be culturally responsive with content area teaching. These tools need to be analytical, applicable, and flexible in order to maximize pedagogical discourse about mathematics teaching and learning and support a robust and equitable instructional practice in mathematics. The CRMT6-LAT is such a tool.

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DEVELOPING TEACHERS' KNOWLEDGE OF CONTENT AND STUDENTS FOR TEACHING CATEGORICAL ASSOCIATION

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Students come to the learning of categorical association with many misconceptions. The purpose of this study was to determine the effectiveness of novel curriculum materials to improve mathematics teachers' knowledge of students' conceptions regarding categorical association. Results showed that prior to use of the materials, teachers' knowledge was mostly limited to variations on one misconception. Following use of the materials, they were more broadly aware of a number of different misconceptions and improved their ability to analyze categorical data for association.

Keywords: Teacher Knowledge; Data Analysis and Statistics

Background

Statistical association is one of eight big ideas in statistics (Garfield & Ben-Zvi, 2004). This is reflected in the inclusion of the study of categorical data for association in curriculum documents such as the Common Core State State Standards for Mathematics (Standards 8.SP.4 and HSS-ID.B.5) (CCSSI, 2010) and the Guidelines for Assessment and Instruction in Statistics Education Report (Franklin et al., 2007). Researchers (e.g., Batanero, Estepa, Godino, & Green, 1996; Watson and Callingham, 2014) have established that students have great difficulty and many misconceptions when analyzing bivariate categorical data presented in contingency tables for association. For students, Batanero et al. (1996) and Watson and Callingham (2014) found the following misconceptions to be most prevalent:

- Deterministic (D): Students conceive of association from an 'all-or-none' perspective: all cases must show an association with no exceptions in order for an association to exist. These students believe that the cells in the two-way table that do not agree with the association should have zero frequency.
- Unidirectional (U): Students believe association only occurs when it is direct, and do not recognize that inverse association is possible. These students tend to give more relevance to positive cases than negative cases that confirm a given hypothesis.
- Localist: Students examine only a part of the data-often the cell with the highest frequency (L-S) or only one conditional distribution (L-RC)-to determine if an association exists.
- Ignore the data (I): Students ignore the data at hand and instead use their previous theories/knowledge about the variables under investigation to decide if they are associated.
- Lack of Proportional reasoning: Students compare frequencies rather than percents in their analysis (P-S), or claim that one cannot compare two groups of unequal sizes (P-M).

Students come to the learning of categorical association with conceptions, including the misconceptions described above. Teachers tasked with teaching students the topic of categorical association need to make their teaching responsive to students' various conceptions of it. This can be supported through knowledge regarding what students may think about particular content topics and how this thinking can progress, known in the Mathematical Knowledge for Teaching theoretical framework (Hill, Ball, & Schilling, 2008) as Knowledge of Content and Students (KCS). Thus, the purpose of this study was to determine the effectiveness of novel curriculum materials to improve mathematics teachers' KCS for teaching categorical association. This manuscript presents preliminary results from a larger study concerning the effectiveness of the materials to develop mathematics teachers' knowledge for teaching statistics more broadly.

Method

Forty-two teachers in statistics and education courses at three large universities in the United States during the Fall 2014 semester participated in the study. Eight were in-service teachers and the remaining were pre-service teachers. All participants were enrolled in a course that utilized new curriculum written by the first two authors of this report. The curriculum materials were designed to develop the knowledge teachers need to teach categorical association, including KCS. One section that focused on KCS in particular was based on Batanero et al. (1996) and Watson and Callingham's (2014) identification of common misconceptions in interpreting contingency tables. It presented real excerpts of students' reasoning about contingency tables and asked the teachers to do three things: describe each student's misconception, look for relationships between pairs of students' misconceptions, and respond to the student as his or her teacher. After the teachers discussed the misconceptions, the instructor provided the formal terms for the identified misconceptions (e.g., unidirectional). The homework for this section asked the teachers to (a) write their own definitions of three of these misconceptions and (b) create a short classroom lesson plan (not including homework or exam assessments) to address at least two of the misconceptions.

This manuscript reports results from the first item on a pre- and post-assessment designed to evaluate teachers' knowledge for teaching categorical association (Figure 1). It required participants to write a correct response and distracters for a multiple choice question (adapted from Batanero et al., 1996) about association between categorical variables. The directions emphasized the idea that distracters should reflect common student misconceptions about an item, allowing participants to exhibit their knowledge of an array of empirically-observed student tendencies when dealing with the content. This empirical grounding helped make the item an authentic means for eliciting participants' KCS (Hill, Ball, & Schilling, 2008) in regard to categorical association.

Analysis of the responses began with three of the authors independently coding teachers' answer choices as Correct (C) (corresponding to the correct answer on the item, which they were directed to write as an answer choice) or one of the misconceptions included in the curriculum materials. Any that did not fit these categories were classified as Other (O). The use of the coding scheme was piloted with a subset of six teachers' answer choices. Based on the pilot results, the coding scheme was expanded to include new categories for common responses: 1 –Students treat the problem as a one variable problem, looking at the marginal distribution for one of the variables only; N- Students believe that there is not enough information to determine if there is an association. All of the coding categories except N are grounded in research regarding common student misconceptions. In the next phase of analysis, all teachers' pre-and post-assessment answer choices and associated reasoning provided on the assessment task were coded. After comparing codes, any discrepancies were resolved through discussion by the coders and a final code was determined.

Now, imagine that you are writing the answer choices for an eighth-grade multiple choice test item. Th item itself is shown below, without any answer choices. Write several answer choices so that one is correct, and so that the others are distracters that reflect common student misconceptions about the assessment item. Explain the erroneous student thinking that would lead to each distracter, as well as the sound student thinking that would lead to the correct answer choice. Keep listing distracters until you have exhausted your knowledge of student thinking patterns about this type of item:

In a medical center 500 people have been observed in order to determine whether the habit of drinking caffeinated beverages has some relationship with high blood pressure. The following results have been obtained:

	High blood pressure	No high blood pressure	Total
Drinks caffeinated beverages	180	120	300
Does not drink caffeinated beverages	120	80	200
Total	300	200	500

Using the information contained in this table, would you think that, for this sample of people, high blood pressure is associated with drinking caffeinated beverages? Select the best explanation from the choices below.

Figure 1: Assessment Item 1

Results

The analysis of the responses provided to represent the correct answer choice for the item revealed that 69% of the teachers provided a correct answer with sound reasoning on the pre-assessment. This rose to 95% on the post-assessment, providing evidence that the curriculum materials improved teachers' abilities to analyze a contingency table for association.

Table 1 presents the results of an analysis of the responses meant to be distracters for the item, representing common student misconceptions identified by the teachers in the study. The percent reported in the table identifies the percent of the total responses each misconception was referenced for that assessment (pre or post). Multiple copies of the same code per student are counted multiple times.

Table 1: Identification of Common Student Misconceptions on Assessment Item 1

Misconceptio n	P-M	P-S	I	U	L -S	L -RC	I	1	N	O	s um	c ount
Pre	6 %	3 %	4 %	1 %	9 %	4 8%	2 %	1 8%	2 %	8 %	1 00%	1 17
Post	8 %	3 %	6 %	0 %	1 4%	3 4%	1 0%	9 %	1 %	1 6%	1 00%	1 14
Change	2 %	0 %	2 %	- 1%	5 %	- 14%	8 %	- 9%	- 1%	8 %		

The L-RC misconception was predominant on the pre-assessment, identified in nearly half (48%) of the 117 distracters written by the participants. In fact, 22 of the 42 participants applied the L-RC misconception in different ways (i.e., different rows and columns were isolated) to write multiple distracters on the pre-assessment. Interestingly, for those 13 teachers who provided an incorrect response for the correct answer choice on this item, 7 of them had the L-RC misconception themselves. A related misconception, 1, was the second-most common misconception identified on the pre-assessment, accounting for 18% of the distracters.

L-RC was still the most commonly identified misconception on the post-assessment, but it declined to be identified in approximately one-third (34%) of the distracters and was only used multiple times by 9 of the participants. Instead, the teachers on the post-assessment shifted to increase their identification of I, L-S, D, and P-M misconceptions. Few identified the U misconception on either assessment, but that is as expected given that the item displays a table with no association. A preliminary paired analysis showed essentially the same results as those presented above.

Discussion

This study found that prior to instruction many teachers understood that students commonly mistakenly analyze a contingency table for association by focusing on a single row or column-whether in the body of the table (L-RC) or in the margin (1). In addition, that misconception was also held by 17% of the participants prior to use of the curriculum materials and they generally were unaware of other misconceptions. Thus, an implication of this work is the need for teachers to have learning experiences specific to their knowledge for teaching categorical association. The curriculum materials evaluated in this study were effective at providing such learning experiences, facilitating teachers' learning of how to correctly analyze a contingency table for association and increasing their KCS with respect to their recognition of student difficulties with the topic (particularly the L and I misconceptions). Future work will involve more detailed analysis of the responses classified as O in this study (which accounted for 16% of the post-assessment responses) and connecting the results regarding this first item on the assessment to the results of the other three items to give a more complete picture of the participants' KCS for teaching categorical association before and after use of the curriculum materials. Also, based on this study's results, adaptation of the curriculum materials will be made to draw teachers' attention toward less commonly recognized student misconceptions.

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WHAT TEACHERS NEED TO KNOW: NOVICE TEACHERS' VIEWS OF USING CONTENT KNOWLEDGE IN TEACHING MATHEMATICS

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This qualitative study examined how preservice secondary teachers with strong content backgrounds reported using their content knowledge in beginning teaching. Data included 3 interviews with 4 preservice teachers enrolled in a one-year master's program designed for content experts. Interviews spanned their teacher education program and first year of teaching. We mapped how participants described using their content knowledge to the construct of mathematical knowledge for teaching (MKT). We found that participants used various aspects of their MKT in making pedagogical decisions and providing explanations.They struggled to use their knowledge to make content accessible to their students. Participants' confidence in their content knowledge obscured the need for them to develop knowledge of content and students.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice

In recent years, several agencies have offered certification programs to content experts in STEM fields to become high-quality teachers of mathematics (e.g., Robert Noyce Scholarship, National Science Foundation's STEM-TP). The assumption underlying these programs is that, given a strong content background, this population of preservice teachers needs only coursework in pedagogy and a teaching practicum to be prepared to be highly qualified teachers (Selke & Fero, 2005). Our research is directed towards understanding how prospective teachers' strong STEM background contributes to their teaching. This population is of interest because of their in-depth content knowledge and how they might call on that knowledge in teaching. Our institution has created an intensive one-year master's program leading to secondary mathematics teacher certification for those individuals who have earned an undergraduate degree in a STEM discipline. We are reporting how the prospective mathematics teachers in one cohort of program candidates described using their content knowledge in teaching.

Framework

Content knowledge is a necessary but insufficient prerequisite for effective mathematics teaching (Ball, Thames, & Phelps, 2008; Monk, 1994). Effective teachers also need pedagogical knowledge (Brown & Borko, 1992; Goos, 2013).Research tells us that both extensive subject-specific content knowledge and pedagogical knowledge are needed for teaching mathematics(e.g., Ball et al., 2008). Rather than considering teacher knowledge as a distinct knowledge of content *or* of pedagogy, a critical component of teacher knowledge is the interaction between content *and* pedagogy, what Shulman (1986) termed *pedagogical content knowledge* (PCK). PCK goes beyond *knowing how to teach*. It involves creating and analyzing multiple representations; making the subject accessible, yet challenging; increasing critical thinking skills; and aiding students in their understanding of broad mathematical concepts (Shulman, 1986).

Ball (2003) expanded Shulman's construct in context of teaching mathematics. What she termed *mathematical knowledge for teaching* (MKT) is composed of both subject-matter knowledge and pedagogical content knowledge. Subject matter knowledge is subdivided into *common content knowledge* (CCK), *specialized content knowledge* (SCK), and *knowledge of the mathematical horizon*. CCK refers to the ability to do mathematics accurately and efficiently, whereas SCK involves the mathematical skills that are unique to teaching, such as assessing the generalizability of students' nonroutine strategies. Pedagogical content knowledge contains subcategories of *knowledge* *of content and students* (KCS), *knowledge of content and teaching* (KCT), and *knowledge of content and curriculum*. KCS includes the ability to anticipate student thinking and misconceptions, and KCT includes the development, selection, and sequence of content to achieve a particular goal. In this study we considered how participants' called on different aspects of their MKT in beginning teaching.

Program

Our teacher preparation program is for STEM professionals and for outstanding recent college graduates with degrees in STEM fields, who want to earn a master's degree in teaching and middle (Grades 4–9) or secondary (Grades 7–12) teacher certification in mathematics or science. During one full academic year teacher candidates complete coursework on curriculum, learning, adolescent/child development, content-specific methods, and making connections between STEM disciplines and Grades 4–12 school mathematics and science. Three distinctive program features provide context for this paper:1) courses focused on all STEM content areas and the connections among them, 2) a year-long student teaching placement, and 3) meetings of candidates, mentors, and program faculty throughout the year to build relationships, discuss candidates' progress, and engage in professional development.

Methods

We collected data on 4 mathematics teacher candidates (hereafter participants)from the same cohort and 3 of their mentor teachers. All participants were seeking initial licensure for Grades 7–12 mathematics, had earned their bachelor's degree within the previous two years, and had prior informal teaching experience (e.g., tutoring, teaching assistants). Data were collected through 3 interviews: at the outset of their teacher preparation program(first), at the end of the year-long internship (second), and at the end of their first year of teaching (third). We asked participants to describe the connections among their backgrounds and work experiences in relation to their STEM content knowledge and current teaching. We interviewed their mentor teachers once about their mentees' backgrounds, teaching styles, and educational philosophies. At least two researchers independently open-coded transcripts. From the coded transcripts, we identified salient themes. We also examined the mentor interview transcripts for confirming or disconfirming evidence of these themes.

Results

Participants called on various aspects of their MKT in several ways and with a range of success: to make pedagogical decisions about what and how to present content, to provide multiple explanations and in-the-moment adjustments, and to unpack content to make it accessible for Grade 4–12 students.

Content Informs Pedagogical Decisions

Several participants described both large-scale curricular decisions and small-scale pedagogical decisions that were informed by their aspects of their CCK and KCT. The participants spoke positively of their decisions concerning both what was taught and how it was taught. For example, Alex described how his CCK affected what he taught:

> Being able to understand, like understand where everything comes from, is really important. I think it's only helped me, because you have to pick and choose what you're going to spend the most time – or … perceive to be most important, right. And you also have to weigh does it matter if … the students understand this to a certain level? (Alex, third interview)

Gerald used KCT to determine how he supplemented the textbooks saying, “My textbook will break [a problem] down differently [than I will]. I’ll show both ways [to the students]” (second interview). In other instances, Gerald, drawing on CCK to inform his KCT, explained:

> Are you going to mark off points if it’s not in standard form, like exponents going from greatest to least? I said, for Algebra I it’s not important. It’s really only important when you’re dividing polynomials. I was like if [students] get it in any form, I’ll take it. (second interview)

Others used their KCC to provide a better sense of “what was the most important thing that [students] had to understand” and their KCT to consider “how to make that [concept] clear in many different ways” (Jennie, second interview). Similarly, Alex used CCK to explore his own understanding, saying, “It's always important for me to understand what's going on [with particular content],” and KCC to determine “what do they really need to understand right now?” (second interview). Jennie credited her CCK for providing the ability to see the connection between disciplines, but we say that she relied on her *knowledge of the mathematical horizon*.

Participants used KCT and KCS to make decisions about how they taught. All participants developed long-term projects and/or connected applications of mathematics to student interests. Jennie used CCK and KCC to create a project-based lesson integrating mathematics and science. To develop students’ conceptual understanding, Reagan called on KCT to create lessons with real-world applications that required students to discover, explain, and justify common mathematical algorithms. Using KCS, Alex drew on students’ interests by using the popular game, Angry Birds, as a context for mathematical modeling of flight paths with parabolas, applying concepts of physics, and comparing paths with lines of best fit; stating “all of[creating the project] came through my math background” (Alex, second interview**).**

Content Informs Explanations and In-The-Moment Adjustments

Participants also gave positive reports of specific ways that their CCK helped them make in-the-moment pedagogical adjustments and provide alternate explanations. Reagan referred to the power of her CCK in managing the unexpected questions and providing multiple explanations:

> I think that majority of teaching is about the content and the fact that I’m a content expert…I can explain concepts in multiple ways. Kids can ask me questions that are not entirely related to what we're talking about and I can give them answers or let them know that I’m going to look them up….I can answer questions students ask all the time..… Also, I’ve had poor teaching moments where there has not been enough planning and prep time put into classes. It is easier to on-the-fly talk about things as well. (second interview)

Jennie agreed saying that because of her knowledge of mathematics “I think I am better able to explain things and give different examples” (second interview). Additionally, Reagan used her CCK to elaborate off-the-cuff on topics of interest to students: “There’s a lot where they’ll say, ‘Hey is that related to this?’ And I know whether or not it is and I can talk to them a little bit more and I can give them more applications and when is this used” (second interview).

Making Content Accessible to Students

What the participants referred to as “present[ing] [content] to the kids in a way that was at their level” (Jennie, third interview), Reagan described as the “biggest stumbling block related to my education” (third interview).It was in this pedagogical skill that we found their CCK was an inadequate substitute for their limited KCS. Gerald was challenged by “breaking the math down” and was forced to “rethink what I did in high school and what I've learned throughout” (second interview). Gerald’s mentor noted these struggles as well: “the only thing is with him being at such a high level sometimes it has been very difficult to bring it back down to the level that some of these kids are at." Similarly, Jennie's mentor observed that though "the content for her was somewhat simple," she struggled in "bringing learning [the content]…down to the eighth grade level. And display[ing] the content in an understandable way to the students." Jennie continued working on this struggle in her first year of teaching,"I found myself at the beginning of the year kind of teaching above them," and noted the dissonance between their knowledge and her own, "I assumed going into it that they knew more than they actually knew, because of where I was at" (second interview). Reagan explained:

> I'm struggling with ways to explain what seems elementary to me … I think that that's been my biggest frustration because I am so familiar with the content that it doesn't make sense to do it any other way. (second interview)

Though participants became aware that "not everybody see[s] it like I see it," they still focused on making content accessible by breaking down problems "into extremely small steps" (Gerald, third interview).They began to recognize that they needed to consider the mathematics they were teaching from the perspective of their students. At this early time in their teaching career, their KCS was underdeveloped and their only pedagogical strategy for making content accessible was decomposing content into smaller, manageable pieces. In taking this approach, they may have inadvertently overemphasized a procedural view of mathematics while underemphasizing the development of conceptual understanding.

Conclusions

In programs tailored to prospective teachers with strong STEM backgrounds, it is commonly assumed that they have in-depth understanding of content and, hence, their programs focus primarily on pedagogy. However, our participants' general pedagogical focus did not fully support them in making connections between their content knowledge and pedagogical preparation. We suspect that our participants' struggles (e.g., making content accessible) were the result of a lack of PCK, in particular KCS. One remedy is to provide teacher candidates who have content expertise with early experiences that make them aware that CCK does not provide all that is needed for MKT. There is, however, an underlying challenge in implementing this remedy: their initial confidence in CCK obscured the need to develop their PCK. Hence, there is a need for experiences that disturb their complacency regarding content to motivate them to attend to their KCS.

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MATHEMATICS STUDENT TEACHERS' DEVELOPMENT OF PEDAGOGICAL CONTENT KNOWLEDGE: AN INTEGRATIVE-TRANSFORMATIVE PROCESS

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How mathematics student teachers (MSTs) develop their pedagogical content knowledge (PCK) during their preparation is an important question to teacher educators. This proposal is based on the first author's Ph.d dissertation on the changes In Chinese MST's PCK over two years to further discuss the changes in the interrelationships among components of PCK. The PCK in this study was defined as three components--- knowledge of students (KOS), knowledge of teaching (KOT) and content knowledge (CK). The focus of this proposal is the changes regarding the interrelationships between KOT and KOS, and KOS and CK. The types of these changes are reported by examples, and a discussion is made on a hypothesis model on interpreting MSTs' PCK development from an integrative and transformative perspective.

Keywords: Mathematical Knowledge for Teaching; Teacher Education-Preservice

Research Background

Pedagogical content knowledge (PCK), first proposed by Shulman (1986), as an important indicator of teaching competency, continues to attract attention of researchers from different disciplines. PCK has been described as the unique teaching knowledge that distinguishes teachers from scholars in a given subject. A number of studies in PCK have focused on its definition and components, and its development within individual cognitive domains. These studies suggested that PCK composes of several major knowledge components, for example, knowledge of learners, teaching, curriculum and content (e.g., Tatto et al., 2008). However, for the development of PCK of mathematics teachers, there remains to be further explored on a specific model to explain its change. In the field of science teacher education, integrative and transformative perspectives were adopted to interpret science teachers' PCK and PCK development (e.g., Gess-Newsome, 1999).The transformative model views PCK as the transformation of different constructs into a unified form, while the integrative model assumes that PCK does not exist as an epistemic entity, rather it is an incorporation of accessible knowledge components such as knowledge of students. This proposal tries to provide some insights on a possibility of a hypothetic model- an integrative and transformative process to explain (pre-service) mathematics teachers' PCK development.

Theoretical Framework

The general ideas of integrative and transformative point of views can be considered as promising in applying the field of mathematics teacher education. It can be found that for those, who classified the PCK into different components, might have the tendency to adopt integrative perspectives to understand PCK where it is understood as a mixture of different knowledge components. In addition, another strand tending to interpret mathematics PCK as a derivation of mathematics subject matter knowledge (SMK). For example, in knowledge quartet (KQ) model (e.g., Rowland, et al, 2005) described the different levels of melting between mathematics knowledge and teaching .The integrative and transformative perspectives also have the potential in explaining the process of PCK development especially among mathematics pre-service teachers. Since there existed studies to document the difficulties and dilemmas that pre-service teachers met in mathematics teaching in relationship to their teaching knowledge (e.g., Inoue, 2009; Kinach, 2002). These difficulties and dilemmas, in some extent, reflect pre-service mathematics must experience a process from possessing unified and unstable PCK (i.e., separate PCK components) to an integrative and mature PCK unit. For example, Even (1993) found that insufficient SMK might lead to MST's adoption of teaching strategy for an emphasis on procedure mastery rather than on conceptual understanding. Some isolated literatures documented pre-service or novice teachers' teaching explanation and representation reflecting fragmentary and unstable teaching knowledge (Charalambous, Hill, & Ball, 2011; Inoue, 2009). Inoue (2009) reported that the MSTs under studied basically possessed SMK for solving mathematics problem, but a majority of them could not give pedagogically meaningful representation to support students to understand the concept. However, few studies in the past documented the process of MSTs' PCK development, in particular, how the different components of PCK becomes a unified one. PCK in this study is consisted of *knowledge of teaching*(KOT), *knowledge of students*(KOS) and *content knowledge*(CK). This proposal tries to interpret the identified types of interrelationships between KOT and KOS, and between KOS and CK through integrative and transformative perspectives.

Methods

This proposal is built on a portion of the first author's PhD dissertation on tracing changes in the PCK of a group of Chinese MSTs during the final two years of their four –year teacher education. The selected preparation program aimed to nurture future secondary mathematics teachers. To capture MST's PCK change, both quantitative and qualitative approaches were adopted. In this proposal, only the data collected from three video-based interviews (the qualitative approach) were focused. Three video clips were selected from Hong Kong TIMSS video study to serves as prompts for stimulated recall. Three teaching topics were selected: Three-term ratio (R3), Distance formula (DF), and Pythagoras theorem (PT). The video clips were edited with research purpose, including a coherent teaching process on teaching the key content of the topic, and students' responses reflected from either blackboard exercises or questions. The detailed content and major snapshots representative of student blackboard work of the video clips, and the interview questions about KOS, KOT and CK categories will be elaborated during the presentation. All 36 video-based interviews (6 PSTs* 3 topics* 2 stages) were recorded and transcribed verbatim by the first author. Data collected from the three video-based interviews were examined and analyzed. A total 20 questions (10 KOT questions, 7 KOS questions and 3 CK questions) across three topics were discussed in this proposal. The MSTs' response to each question in each stage was analyzed in terms of a level (either one among Low, Intermediate and High). In this proposal, the interrelationships between KOT and KOS, and between CK and KOS were examined and major types of these changes are presented.

Results

Changes in Interaction between KOT and KOS

In general, for KOT and KOS interaction, the changes reflected in Stage Two indicated the development of a degree of awareness of student learning trajectory in designing teaching strategies. The KOS revealed in the six STs' KOT are mainly general knowledge about students' characteristics and interests, and their perception on students' effective learning including developing learning habits, standardized ways of solving mathematical problems. The three types of changes are illustrated below and the specific examples will be shown in presentation.

Type 1: Regarding students' comprehension as an important measurement of the effectiveness of teacher instruction. The commonality in MSTs' commentaries on teacher instruction in Stage Two was that the MSTs showed a more careful analysis of the teacher's instructions in the video clips to strategize on how teacher instruction can match student comprehension. This confirms a transition from a more teacher-directed teaching approach shown in Stage One to a more student-centered teaching approach that was evident in Stage Two where student comprehension was regarded as an important measurement of the effectiveness of teacher instruction.

Type 2: Facilitating students' development of learning and thinking habit through teaching. Five STs displayed this type of change which refers to STs' concern about students' development of a learning habit. Phrases such as "experiencing by themselves", "learning habit" and "standardized methods for solving mathematics problems" were used in STs' responses to KOT questions. All examples of this type of change reflect MSTs' concern about students' examination performance which can be seen as a kind of integration between KOT and KOS.

Type 3: Taking students' interest and prior knowledge into account when designing teaching strategy. This type of change was mainly identified among MSTs whose paired responses improved from Low to Intermediate level in Stage Two. In general, in Stage One, these MSTs showed little or no awareness of the students' interests or prior knowledge. In Stage Two, they began to integrate student concerns into their teaching approaches like encouraging student participation, offering fewer abstract exercises and more real-life examples, activities with manipulatives and sensing student diversity. In addition, the idea of creating real-life examples or scenarios was widely used by the MSTs in considering their strategies in Stage Two.

Change in the Interaction between CK and KOS

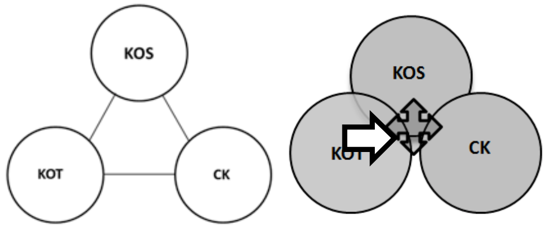
For the interaction between KOS and CK, the two types of KOS-CK change reported related to the changes in how MSTs attributed students' mistakes, difficulties and confusions. Compare to the interaction between KOT and KOS, the interaction between KOS and CK seemed to plays a more important role in terms of promoting these MSTs' PCK development in a more advanced level. With the involvement of CK, MSTs' KOS became more content-specific and tend to focus on students' mathematical thinking. In particular, MSTs were able to elaborate their thinking of students' thinking which was a KOS-CK interaction.

Type 1: A shift from a focus on the correctness of students' answers to the interpretation of students' mathematical thinking. Two STs displayed this type of change. In the Stage One, they focused on correctness of students' answers, but paid more attention to why the student thought that way from a mathematical perspective in Stage Two. For example, one MST tried to analyse why a student made a mistake by examining the sequence of content topics in the curriculum, while the other more able to interpret explicitly students' thinking process such as what caused student confusion.

Type 2: Developing a more sophisticated interpretation of student's thinking from a mathematical standpoint. Three STs showed this type of change. In Stage One, these MSTs inclined to either characterize the student's method as simple trial and error or make a quick conclusion of students' thinking based on their observation from video clips. In Stage Two, these MSTs were able to reinterpret students' methods from a mathematical perspective, or evaluate the students' thinking in terms of mathematical learning trajectory.

Discussion

The analysis above in identifying major types of changes through the paired responses supports the emerging of connections between KOS and KOT and between KOS and CK in the MSTs' PCK development during the period under studied. It was found that both the relationships between KOT and KOS and between KOS and CK became more integrated. The MST's awareness of relationships among teaching strategy, student perspective and mathematics increased and the boundaries between these knowledge domains became less rigid. The current evidences focused on relationships between KOT and KOS and between CK and KOS, these types of change suggest an integrative-transformative PCK model to explain these MSTs' PCK evolution process from a relatively integrative compartmentalized PCK to a transforming PCK with its components interacting and merging, which can be visualized as Figure 1.



MSTs Stage One PCK MSTs Stage Two PCK

Figure 1: An integrative compartmentalized PCK with three components evolves into a transforming PCK with several components interacting and merging

The tentative model proposed in this study highlights the interactive relationship among three PCK knowledge domains, and suggests the possibility of PCK transformation trajectory. This model has heuristic value in terms of helping teacher educators to think deeper about potential developmental trajectories for a better design of training courses. This study is an initial attempt to build a PCK model and evidences were provided to support its soundness. More discussion will be made in the presentation.

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THE COLLECTIVE EFFECTS OF TEACHERS' EDUCATIONAL BELIEFS AND MATHEMATICAL KNOWLEDGE ON STUDENTS' MATHEMATICS ACHIEVEMENT

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Research suggests that teachers' knowledge and beliefs about teaching and learning mathematics are among the key factors for effective teaching. This study explores the extent to which K-12 mathematics teachers' educational beliefs and mathematics knowledge for teaching (MKT) have an impact on students' math achievement. The effects of students' prior math achievement and teachers' years of experience and mathematics degrees earned were also examined. Hierarchical regression analysis results indicated that prior achievement was a significant student-level predictor of mathematics achievement. Teachers' MKT and teaching experience also had a significant effect on the relation between prior achievement and current achievement. Results may have implications for teacher professional development programs as well as education policies at both district and state level.

Keywords: Mathematical Knowledge for Teaching; Teacher Beliefs; Teacher Education-Inservice; Teacher Knowledge

Purpose of the Study

A significant body of research highlights the integral role that teachers' domain-specific knowledge for teaching and their educational beliefs about teaching have on their knowledge development, decision-making and planning, and instructional practices (e.g., Pajares, 1992; Philipp, 2007). Adding to this line of research, this report extends our findings from a larger study that examined both antecedents and outcomes of teachers' beliefs about teaching and learning mathematics (Ekmekci, Corkin, & Papakonstantinou, 2015) by connecting teachers' beliefs and their mathematical knowledge for teaching (MKT) to student outcomes. Specifically, the current study is guided by the following research questions: (a) to what extent does a student's prior mathematics achievement relate to their subsequent mathematics achievement, (b) to what extent do teacher-level characteristics (e.g., experience, beliefs, and MKT) relate to students' math achievement, and (c) to what extent do the effects of student-level factors on math achievement vary by teacher-level characteristics?

Literature Review

Teachers' personal and domain-specific educational beliefs should not be overlooked in the evaluation and development of effective instruction (Stipek, Givvin, Salman, & MacGyvers, 2001). There are various types of educational beliefs that math teachers possess such as self-efficacy beliefs, locus of control beliefs, and epistemic beliefs about mathematics that influence their instructional approaches (e.g., Stipek et al., 2001). Teachers' self-efficacy can be defined as the degree to which teachers believe they can successfully perform teaching-related tasks within a particular domain or context (Enochs, Smith, & Huinker, 2000). Teachers' locus of control may be defined as the extent to which teachers attribute student outcomes (i.e., achievement) to themselves or other (external) factors (Hofer & Pintrich, 1997). Epistemic beliefs can be perceived as beliefs about the nature of knowledge—i.e., where it comes from, its essence, and how one comes to know (Muis, 2004).

Prior studies have found a strong association between teachers' beliefs and students' achievement-related outcomes (e.g., Goddard, Hoy, & Woolfolk-Hoy, 2000; Love & Kruger, 2005). However, the vast majority of these studies focused on only one type of belief (e.g., self-efficacy) and failed to scrutinize the collective impact of different types of beliefs on student achievement. Moreover, the relation between teachers' domain-specific beliefs and student outcomes has not been adequately addressed in previous research.

In addition to teachers' educational beliefs, MKT, defined as "the mathematical knowledge that teachers use in classrooms to produce instruction and student growth" (Hill, Ball, & Schilling, 2008, p. 374), has been found to positively relate to student performance (Hill, Rowan, & Ball, 2005). While previous findings indicate that each of the aforementioned beliefs and knowledge are associated with student achievement, no studies were identified that examined the varying effects of each of these beliefs on students' mathematics achievement. Questions remain as to whether certain educational beliefs have stronger effects on students' mathematics achievement compared to other educational beliefs. Preliminary findings suggest that certain beliefs may play a more important role in student achievement in mathematics given that Ekmekci, Corkin, and Papakonstantinou (2015) found that among these three beliefs, a teachers' epistemic beliefs about mathematics is the strongest predictor of a teachers' MKT, which is a reflection of their instructional practices (Hill et al., 2008).

In terms of teachers' professional background, years of teaching experience has been positively associated with teacher quality (see Rice [2003] for review; 2010). A second teacher background variable that has been linked to student achievement is teachers' educational background in the subject matter that they teach. The majority of the research that examines the influence of educational background in a teaching discipline assesses its impact on student-related outcomes (Barry, 2010). Given the significant relations that have been found between teachers' educational and experiential background with student achievement, the current study will examine the extent to which teachers' beliefs and MKT explain the variation in student performance after accounting for these variables.

Conceptual Framework

The conceptual model in Figure 1 provides a representation of our multilevel research design. Arrow A displays the direct link between the student's prior math achievement (level-1) and math achievement as the outcome variable. The main effects of the teacher-level variables (level-2) on math achievement are depicted by arrow B. Arrow C represents the effects of teacher-level variables on the relation between students' prior math achievement and current math achievement (e.g., does the predictive value of prior achievement change by level of teacher experience?).

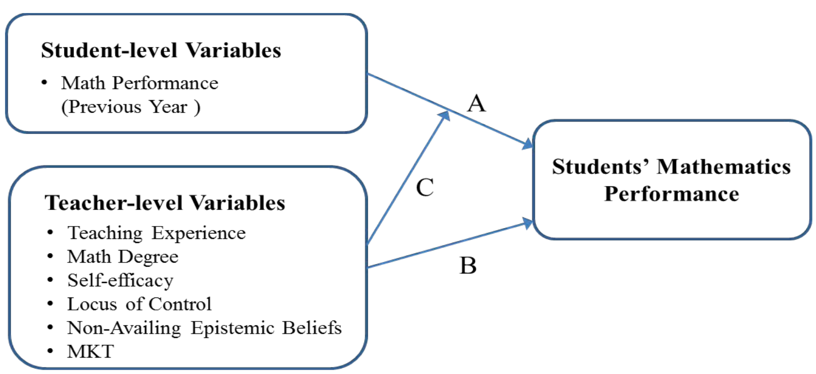

Figure 1: Conceptual Model of the Study

Methodology

The teacher-level data for this study has been collected as part of a project that was partially funded by Teacher Quality Grants Program at the Texas Higher Education Coordinating Board under Grants #496 and #531. For the past two years, consistent measures were administered to assess teachers' educational beliefs after a summer campus program (SCP)—a three-week intensive professional development program aimed at improving mathematics teachers' MKT. Participating teachers took a post-survey on the last day of the SCP assessing teachers' self-efficacy for teaching mathematics, internal locus of control, and non-availing epistemic beliefs about mathematics. The teachers also took the Learning Mathematics for Teaching (LMT) assessment, a standardized assessment that measures MKT (Hill, Ball, & Schilling, 2008), on the last of day the program. More specifically, elementary teachers took El NCOP 2008 scale and middle and high school teachers took MS PFA 2007 scale.

In addition to these measures, teacher level variables included professional background variables such as years of teaching experience and whether teachers had earned a mathematics degree. Student level data requested from the school district included student scores on a standardized mathematics test that was administered at the end of the academic year. The ongoing study will continue to collect data from 2014 teachers and students. Although 2014 teacher data is readily available, since the student achievement data for this cohort is not available yet, this brief only reports the findings for the 2013 cohort.

Among the 51 K-12 teachers who participated in the 2013 SCP, 76% were female; 39% were African American, 30% Hispanic, 20% White, 10% Asian, and 2% other. Students' ethnic background breakdown in the school district is as follows: 29% are African American, 58% Hispanic, 9% White, 10% Asian, and 2% other. About 25% were high school teachers; 25% were middle school; and 50% were elementary teachers. We used SPSS to conduct Linear mixed effects (multilevel regression) analyses. The complete between-teacher model is as follows:

Level-1 (student level): $Y_{ij} = \beta_{0j} + \beta_{1j}(\text{PriorMathAchvmt})_{ij} + r_{ij}$

Level-2 (teacher level): $\beta_{qj} = \gamma_{q0} + \gamma_{q1}(\text{MathMajor}_j) + \gamma_{q2}(\text{TEACHEXP}_j) + \gamma_{q3}(\text{Self}-\text{eff}_j)$
$+\gamma_{q4}(\text{LocusOfControl}_j) + \gamma_{q5}(\text{Epis. Beliefs}_j) + \gamma_{q6}(\text{LMT}_j) + u_{qj}$

Results and Discussion

The results showed that teacher-level variation accounted for 22% of the variation in math achievement (Model 1: Unconditional Model). Table 1 displays the results for the within-teacher (Model 2) and between-teacher (Model 3) models predicting the student achievement outcome. Findings showed that the only significant stand-alone predictor was students' mathematics achievement in the previous year (corresponds to Arrow A in Figure 1). None of the three types of teachers' beliefs were significantly associated with student achievement in math (Arrow B). Whether teachers' had a mathematics degree or not was also not significant (Arrow B). However, teachers' LMT scores and years of experience both had a statistically significant effect on the relation between prior achievement and current mathematics achievement (Arrow C). This implies that the predictive value of prior achievement on students' mathematics achievement varies by teachers' MKT and teaching experience. More specifically, higher MKT (i.e., LMT scores) and greater years of teaching experience strengthens the relation between students' prior and current mathematics achievement. Although statistically not significant, teachers' beliefs were positively associated with students' math achievement.

These results suggest that teacher educators should pay close attention to developing teachers' math knowledge, especially MKT. Urban school districts may consider hiring more experienced teachers to boost their students' academic achievement. Lastly, teacher preparation programs should look for ways to offer more mathematics content and methods courses to improve their MKT.

Table 1: Linear Mixed Effects Model Results

Independent Variable	Model 2 Coeff.	Model 2 *SE*	Model 3 Coeff.	Model 3 *SE*
Intercept	92.01**	16.08	72.10**	42.51
Prior Math Achievement	0.83**	0.02	1.05*	0.45
Math Degree			115.47	91.10
Years of Teaching			3.58	2.92
Self-Efficacy			23.95	51.93
Locus of Control			30.70	51.55
Non-Availing Epistemic Beliefs			-29.06	47.53
LMT			38.09	20.57
Prior Math Achvmt * Math Degree			-0.23	0.13
Prior Math Achvmt * Years of Teaching			0.01**	0.00
Prior Math Achvmt * Self-Efficacy			-0.07	0.06
Prior Math Achvmt * Locus of Control			0.06	0.07
Prior Math Achvmt * Non-Avail. Epistemic Beliefs			-0.01	0.06
Prior Math Achvmt * LMT			0.07**	0.02

* $p < .05$. ** $p < .01$

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STANDARDIZED ASSESSMENTS OF DISCUSSION LEADING PRACTICE: ARE THEY VALID MEASURES?

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Leading mathematics discussions is an important practice for beginning teachers. Assessments designed to determine skill with this practice are critical. But it is important to determine if such assessments yield performances that correspond with teachers' typical discussion leading practice. In this paper, we investigate the validity of one standardized assessment that was designed to assess novices' skill with discussion leading practices, providing evidence of its alignment with typical practice. Analyses suggest that a standardized assessment has the potential to accurately reflect teachers' typical discussion leading practice and provide opportunities for teachers to use additional practices.

Keywords: Instructional Activities and Practices; Classroom Discourse; Assessment and Evaluation; Teacher Education-Preservice

The assessment of teacher candidates' skills with core practices of teaching, such as leading a discussion, is crucial as teacher education shifts toward preparing beginners to do these core tasks of teaching (Ball & Forzani, 2009; McDonald, et al, 2013). As part of a larger project, we are developing a standardized assessment of beginners' skill with leading mathematics discussions. We use "standardized" to refer to having teachers lead a discussion with a provided mathematics task, and an accompanying lesson plan, including instructional goals. However, one may wonder if this standardized assessment will yield performances that differ from "typical" practice when leading discussions. This study investigates the concurrent validity of the assessment by examining how novices' performances on the standardized assessment corresponded with how they lead discussion in regular classroom practice.

Leading a discussion is challenging and complex with many interacting practices at play. Given the complexity, we sought to standardize some of these practices in order to assess others. To determine where to standardize, we turned to a particular decomposition of the work of leading a mathematics discussion that draws on research around practices for orchestrating discussions (Smith & Stein, 2011) and talk moves (Chapin et al, 2013). As illustrated in Figure 1, our decomposition distinguishes between three stages of discussion: launch, orchestration, and conclusion. Within each stage, teachers engage in discussion-leading practices using particular techniques (Boerst et al., 2011), including talk moves.

Discussion Enabling	Discussion Leading		
Anticipating student thinking Setting up the problem Monitoring student work	**Launching**	**Orchestrating** - Eliciting - Probing - Orienting - Making Contributions	**Concluding**
	Recording		

Figure 1: Decomposition of Leading a Mathematics Discussion

We chose to focus the assessment on discussion-leading practices and to standardize the discussion-enabling work by providing scaffolds similar to those teachers might encounter in curricular materials including the task (Figure 2), lesson plan, and anticipated student thinking. We then developed an assessment of beginning elementary teachers' skill with leading mathematics discussions that asked teachers to do the following:

> *You will lead a discussion of solutions to the problem "make number sentences for 10" with an eye to the similarities and differences among presented solutions. Your goal is to elicit several solutions to the problem and to have students explain why they are or are not solutions and to notice similarities and differences among the solutions. Depending on the level of your students, your discussion may also focus on the number of solutions to the problem. You may modify the lesson as necessary for your students.*

The task was selected to engage students across grades and skill levels with opportunities to generate and compare solutions.

The assessment includes a scoring tool designed around the stages of work and practices outlined in Figure 1. This tool further decomposes the practices into related techniques, which represent a sample of possible techniques that we selected as crucial for beginning practice. For example, *orienting students to the contributions of peers* occurs during the orchestration stage of the discussion and is specified by five techniques such as "the teacher poses questions to students about other's ideas and contributions." In developing this tool, we recognized that there are many ways to score or assess instruction, including rubrics that differentiate levels of performance. For the purposes of identifying correspondence of techniques used by teachers in their typical practice with those used in an assessment, we chose to record the presence or absence of each technique in relation to a given threshold. A choice of N/A (not applicable) was built into the tool to recognize that some techniques may not be needed depending on the norms of the classroom. To investigate the degree to which a standardized assessment of discussion leading practices yields performances that correspond with the same teachers' typical practice, we piloted this assessment with 9 first through fifth grade elementary teachers and also documented a mathematics discussion that they led as a part of their regular classroom practice.

Methods

The participants in this study were nine first-year elementary school teachers who served as a proxy for candidates at the end of their program. We sampled teachers from a range of grade levels (1-5), school districts, and teacher education programs. While we sought a diverse sample to elicit variation in performance, this sample is not meant to be representative of all first-year teachers. Data sources for this study include video recordings of a discussion led as a part of regular classroom practice (observation discussion) and the assessment discussion. Observations were captured in order to compare typical discussion leading practice with teachers' assessment performance. Teachers were given the assessment described above and were encouraged to spend no more than 45 minutes preparing. Observations were conducted prior to the assessment discussion so that teachers' typical practice would not be influenced by the assessment.

To investigate the concurrent validity of the assessment, we used the developed scoring tool to analyze both the observation and assessment discussion videos. Multiple members of the research team scored all of the videos. For this study, scoring refers to recording the presence or absence of particular techniques at a particular threshold. Any discrepancies were resolved through discussion and reference to the codebook. A subset of the discussions (> 20%) was coded by a trained rater, yielding an inter-rater agreement of 85%. All nine individual teachers' scores were compared for consistency between the observation and assessment discussion both at the level of discussion leading practices and at the level of the techniques associated with them. We coded the techniques and the discussion-leading practices for whether teachers received the same rating in both contexts or not. Changes in use of techniques from "not present" on the observation to "present" on the assessment will be referred to inconsistent favoring the assessment. All teacher names used in the subsequent analyses are pseudonyms.

Findings

To examine the correspondence between the assessment discussion and teachers' typical discussion leading practice, we first describe the patterns of consistency that we found across the sample and then use cases of three teachers to illustrate some of the nuances of these patterns. Overall, when comparing individual teachers' use of particular techniques across observation and assessment, we found that many techniques were used across both contexts. Additionally, many of the techniques that were not used on the observation were used during the assessment. When examining performances at the practice level, we found that *eliciting student thinking* and *concluding* the discussion were well matched with most teachers matching or showing more skilled performance during assessment on all techniques. To get a clearer picture of the correspondences between performances on the assessment and observation, we consider the practice of *orienting students to one another's thinking.* When we compare the use of particular orienting techniques, we find that the techniques were often different across performances; in other words teachers used different orienting techniques across contexts, lowering the number of matching techniques used. However, Figure 2, which shows the number of orienting techniques that received a score of N/A and present for both instances of instruction, highlights that at the level of the discussion leading practice, teachers were fairly consistent in the number of different orienting techniques that they were using with inconsistencies favoring use of more techniques in the assessment. For example, T25 used four orienting techniques in both the observation and assessment discussions, and also earned one N/A in both instances, showing high levels of consistency across discussions.

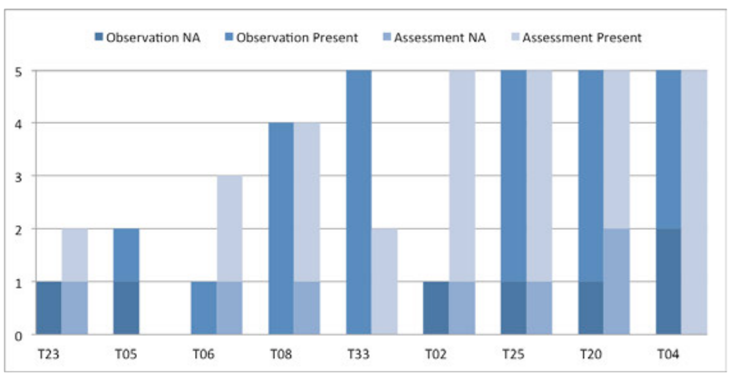

Figure 2: Orienting techniques across observation and assessment contexts

Cases of comparing teachers' discussion leading practice

To further illustrate the findings, we selected three teachers representing different cases of technique use: consistent (T08), inconsistent, favoring the assessment (T02), and inconsistent, favoring the observation (T33).

Consistent. Ms. Bryan (T08) presents a case of a teacher whose performance was consistent across contexts. In both discussions, her *launch* engaged students in the key mathematical work, though in neither case was she efficient. In both instances she consistently *elicited student thinking* and *probed* for both process and understanding. One area of her orchestration that varied slightly was around *orienting*. Though she did the same number of orienting techniques in both instances, the techniques she used varied slightly.

Inconsistent, favoring the assessment. Mr. Dawson (T02) performed significantly better on the assessment than the observation. His *launch* and work around *eliciting student thinking* was quite similar in both instances but in all other areas he did demonstrated more skill on the assessment. Most notably, his work around the practice of *orienting students to one another's thinking* increased from no orienting during the observation to use of all five techniques during the assessment discussion. We noted that his treatment of the content in his observation, recording the mass of objects in different units, did not seem discussion-worthy. We wondered if his choice of task and content impacted his performance.

Inconsistent, favoring the observation. Ms. Jordan (T33) performed significantly better in some areas during the observation that she did on the assessment. In both cases she launched the discussion efficiently with a tight mathematical focus and was able to elicit and probe student thinking. However, she did much more work orienting students to the contributions of others during the observation. We noted that in her assessment discussion, Ms. Jordan made critical errors while attempting to respond to a student contribution. This prompted us to wonder if her content knowledge impacted her use of particular techniques.

Discussion and Implications

In this investigation of the concurrent validity of a standardized assessment of beginning teachers' discussion-leading practice, we found that the assessment was reflective of teachers' discussion-leading practice in their typical instruction in many cases. When this was not the case, inconsistencies favored better performance on the assessment. This may mean that the scaffolds built into the assessment could support beginners and show their potential to utilize discussion-leading practices even if they are not yet consistently using them in their typical practice.

The findings suggest that the standardized assessment could reflect teachers' typical discussion leading practice and could reveal better performances when provided with scaffolds, (e.g. the lesson plan appeared to support beginners' ability to conclude a discussion). This indicates that the assessment may make visible teachers' potential to use the discussion leading practices given the right supports and content. Although we did not set out to account for differences in performances, we did observe a number of interesting features of lessons with discrepancy including the role of the task, subject matter knowledge, and learning opportunities built into the assessment. Future research could examine systematically the ways in which these and other factors contribute to differences in performances.

Although there were several limitations to the study including the sample, the limited number of observations, and the design of the scoring tool around a particular decomposition of discussion leading practice with threshold statements, the findings suggest that a standardized assessment has utility for assessing beginners' skills and should be further investigated.

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PROSPECTIVE TEACHERS' SENSE MAKING

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This study explored how prospective elementary teachers described their past experiences engaging in the first of the Common Core Standards for Mathematical Practice, which requires students to "make sense of problems and persevere in solving them." Through a process of grounded theory we found that prospective teachers' responses varied between whether or not they described the problem solving process as one that relied on non-routine thinking, or as a process of following prescriptive steps. Just under one third of the 41 students gave examples of the practice that demonstrated an understanding of solving problems as a sense making process, and many cited that this happened in the contexts of "geometry" and "word problems". These findings suggest that teachers' past experiences may have afforded few opportunities to engage in this practice, and teachers may need support in developing new understandings.

Keywords: Teacher Education-Preservice; Standards; Elementary School Education

Objectives

The Common Core State Standards (CCSS) (NGA, 2010), adopted by the majority of U.S. states and territories, include Standards of Mathematical Practice (SMP) that describe practices students should develop when learning mathematics. A challenge currently facing many prospective teachers is that the Standards for Mathematical Practice (SMP) require them to learn to teach mathematics in ways distinct from how they learned mathematics. We sought to examine how prospective teachers interpret the SMP through the lens of their previous experiences with mathematics. In this study we focused on the first SMP, which requires students to "make sense of problems and persevere in solving them." More specifically, we investigated the following question: What are prospective teachers interpretations of the CCSS.MP.1 based on their past experiences?

Perspectives

In this study we focus on the first SMP because it is a core practice with broad application to all levels of mathematics. This practice emphasizes student thinking and persistence through solving non-routine problems, which contrasts traditional ways of learning mathematics through rote memorization and an emphasis on skill over understanding.

One enduring challenge in mathematics education is how teachers (and teacher educators) can be responsive to the various conceptions of mathematics held by their students. The literature of the field supports that an emphasis on reasoning and sense making should be inherent in all mathematics learning (e.g., NCTM, 2009; Schoenfeld, 2009), but in order to teach in this manner teachers themselves need a conceptual understanding of mathematics (e.g., Ball, Hill, & Bass, 2005). Prospective teachers in the U.S. may not be adequately prepared to teach the demanding mathematics curriculum required by the Common Core State Standards (Education Policy Center, 2011), and many of them may hold conceptions of mathematics that do not easily align with the SMP.

Prior to entering teacher education programs, prospective teachers have spent many years as students in mathematics classrooms, often learning mathematics in ways that do not align with current standards. Lortie's (1975) theory of "apprenticeship of observation" suggests that a teacher's past experiences observing and engaging in learning as a student are a major contributor to how they learn to teach. Prospective teachers' understandings of mathematics and teaching are strongly influenced by their past experiences as students of mathematics. If they did not develop a conceptual understanding of mathematics, they may have difficulty interpreting the Mathematical Practices, and struggle to support their future students in developing them.

In this study we explored prospective teachers understandings of how they had engaged in the practice described in CCSS.MP.1 in their past experiences as a mathematics student. We took a grounded theory approach (Glaser and Strauss, 1967) in this study because there were no existing frameworks with which to analyze the data.

Methods

Context

This study was conducted in two sections of a Number Theory content course designed for prospective elementary and middle school teachers at a private university in the Midwest. Prospective teachers were in their freshman or sophomore year and this was their first mathematics course at the university. The population consisted of 41 prospective teachers: 38 females and 3 males.

Data

The data collected for this study were from an online survey given prior to the first day of class. We asked demographic questions as well as items pertaining to attitudes and beliefs. The focus of this study was on how pre-service teachers make sense of the first Common Core Mathematics Standard through the lens of their past experiences as prospective teachers. More specifically, the survey asked prospective teachers to read the description of CCSS.MP.1 and think about what it meant to them. "Describe any past experience you have had as a student of math, in which you feel like you participated in a way that met this standard." The description of CCSS.MP.1 that was provided to the prospective teachers is available at http://www.corestandards.org/Math/Practice/.

Analysis

Qualitative data pertaining to prospective teachers' understanding of the CCSS.MP.1 were imported into an Excel spreadsheet with all identifiers removed. The two co-authors began the grounded theory process (Glaser and Strauss, 1967) by reading and rereading all prospective teachers' responses, and engaging in constant comparison to search for possible themes. We met to discuss our initial observations and concluded that an observable variation between responses was whether or not the individual described the problem solving process as one which relied on non-routine thinking, or as a process of following prescriptive steps. After discussing this variation, the co-authors separately returned to the data and categorized responses according to these two categories. This categorization was done by reading and rereading responses and underlining words or phrases that indicated an understanding of problem solving as either a thinking process, or a process of relying on prescriptive steps. After both authors conducted this analysis separately, we met to discuss our categorization and to examine together the responses that were particularly difficult to classify. This process of categorizing, sharing, and discussing helped us to further clarify our understanding of the criteria that differentiated responses in each category.

The prospective teachers' responses that we categorized as demonstrating an understanding of the practice were those who indicated that thinking and reasoning was an integral part in coming to a mathematical solution. Some examples of phrases used by prospective teachers that indicated the importance of their own reasoning to help them arrive at solutions were: "thinking about the problem critically and from different angles", "support our answers", "problem solve and explain our reasoning to how we got our answers". Responses that we did *not* categorize as demonstrating an understanding of the practice fell into one of two cases: a) they indicated a reliance on rules, steps, and procedures to arrive at a solution rather than their own thinking, or b) they were vague and lacked enough detail to make a determination. When responses emphasized prescriptive steps and did not indicate the use of their own thought processes in arriving at a solution, it fell short of offering enough evidence to indicate understanding of CCSS.MP.1. Once all responses were categorized according to whether or not they indicated an understanding of the practice, we conducted further analysis by reading and rereading the responses within each category, and noting themes that emerged within each group. In the following sections, we describe the findings.

Results

Of the 41 responses, 13 prospective teachers demonstrated an understanding of CCSS.MP.1 as representing a thinking process. All 13 of the responses that we categorized as "understanding" indicated that the prospective teacher saw this standard as being about a thinking process in which the ideas and strategies come from the student (i.e., "I had to figure out…"). Student-centered thinking was indicated through verbs and phrases such as: *plan, questioning, pondering, discovering, think out, thoroughly considers the question, determine the best way to solve.* In 9 of the 13 responses, prospective teachers mentioned the existence of "different methods" to solve. Four of those responses also mention a goal of trying to come up with the best solution or determine the best way to solve the problem. Three of the responses included a reference to a geometry course, and 4 responses referenced "word problems" as contexts in which they had participated in the practice in the past.

Of the 41 prospective teachers who completed the survey, the remaining 28 did not show ample evidence to demonstrate an understanding of CCSS.MP.1 as a thinking process. In 15 of these 28 cases, the prospective teacher restated phrases directly from the standard or gave a vague response and did not provide any elaboration on what the standard meant to them or how they had experienced it. In 13 of these 28 cases, we categorized the response as "procedural," because they described the problem solving process as one that attempted to minimize it to a procedure. Another theme that surfaced from our analysis was that 7 of the 28 responses emphasized the importance of checking and rechecking their work using multiple methods. In the presentation, we will support findings with examples of prospective teachers' responses.

Discussion/Conclusions

Just under a third of the prospective teachers were able to show evidence of understanding that CCSS.MP.1 was about a student-centered thinking process, by giving examples and drawing on their past experiences. Of those prospective teachers who showed evidence of understanding CCSS.MP.1, about half of them referenced either geometry or word problems as contexts in which they had had opportunities to participate in this standard in the past. These findings suggest that prospective teachers' past experiences may have offered them little opportunities outside of those two contexts to engage in the practice promoted by CCSS.MP.1.

Current emphases on problem solving across the field of mathematics education focus on "reasoning and sense making" (NCTM, 2009) and "make sense of problems and persevere in solving them." These ideas clearly delineate the need for students to develop their own thinking strategies for working through non-routine problem solving, and necessitate that these practices become a part of "every mathematics classroom every day" (NCTM, 2009, p.5) instead of limited to narrow contexts such as geometry and word problems.

There is risk that prospective teachers who do not conceptualize mathematics as a discipline of reasoned sense making may not understand the importance of developing students' thinking strategies, and may attempt to reduce problem solving to procedural steps. Close to one third of the prospective teachers we surveyed interpreted CCSS.MP.1 as describing a problem solving process that could be minimized to a procedure. They emphasized activities such as circling, highlighting, underlining, and double checking rather than activities that have the potential to scaffold the development of students' own thinking strategies.

This is a time of transition for teacher educators as we develop our programs to prepare teachers for new expectations established by the Common Core. The findings from this study are significant as they illuminate prospective teachers' early understandings of the first SMP, and identify a need for teacher educators to support further development of their understandings of the SMP that they will be responsible for promoting in their future classrooms.

Without extra support, teachers (both prospective and practicing) are limited to the examples provided within the Common Core documents. Past experiences may not have provided them with the relevant conceptions and understandings needed to enable them to generalize the meanings of each of the SMPs. Thus, mathematics teacher educators should explore ways to help prospective teachers understand these ideas through engagement in mathematical inquiry and subsequent reflection aimed to help them understand the thinking processes that should be an inherent part of doing mathematics.

Further research is needed to explore ways of supporting the development of prospective teachers understandings of these ideas, and to examine the conditions that help prospective teachers come to see mathematics as a sense making process.

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REFLECTING ON A DECADE OF CURRICULUM DESIGN: THE IMPORTANCE OF SETTING THE TONE

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For over a decade I have led a collaborative effort to develop, enact and revise curriculum materials for a Number and Operations course for prospective elementary teachers (PTs). A main focus of this course is on justifying one's thinking by relying on elementary-age appropriate meanings of numbers and operations, rather than rules. This article summarizes some of the key aspects of the curriculum development project, changes in the style and content of the curriculum, and recommendations for the importance of tone setting guidance for PTs and their instructors.

Keywords: Teacher Education-Preservice; Number Concepts and Operations; Curriculum; Mathematical Knowledge for Teaching,

Framing the Issue

Although the issue of what knowledge teachers need to teach has been debated since teacher education programs began, there has been a surge in the intensity and depth of the conversation in the last few decades. Most notable are two recent efforts, both based on a systematic examination of practice. Ball and colleagues have been working on developing a theory of the mathematical knowledge need for teaching by analyzing the work of elementary mathematics teaching (e.g., Ball, Bass & Hill, 2004; Hill, Rowan & Ball, 2005). Other efforts are geared at designing mathematics education courses for PTs by adopting a collaborative, systematic and iterative process of lesson development, analysis and refinement (e.g., Ball, Sleep, Boerst & Bass, 2009 and Berk & Hiebert, 2009). While some consider this to be the work of teaching, Berk & Hiebert (2009) highlight the similarities between this work and design research.

For over a decade I have led a similar project focused on developing and revising the Number and Operations course for PTs that facilitates the ability to justify one's thinking in ways appropriate for elementary students. This work began in 2003 when I was one of the main authors on the NSF-funded *Understanding Mathematics Deeply for Teaching (UMDT)* project (Flowers & Rubenstein, 2003 – 2007). A main product of this project was a set of instructor notes including a 1-page overview of the lesson (Brief Notes), and a set of Detailed Notes of varying length, but typically 10+ pages. Interspersed among activity descriptions were stand-alone essays called Teacher Notes (loosely modeled after a similar feature from the *Investigations in Number, Data and Space* curriculum, TERC, 1998). These included activity-specific notes, such as examples of student work and related mathematical issues, and more general notes on issues like social norms, sociomathematical norms (Yackel and Cobb, 1996), and tone setting for new topics. [Although I have found no literature defining tone setting, practitioner articles discuss suggestions for tone setting activities. The idea is to find an activity that conveys and motivates the goals of a course and introduces students to class norms.]

Following this project, I led my institution's efforts to continue to develop these materials in an incremental way. Our cyclical process included weekly meetings and impromptu conversations about the lessons that all instructors would use, and suggestions for future alterations. This work was also informed by research conducted with colleagues on PTs' understanding mathematics (e.g., Lo, Grant & Flowers 2008, Lo & Grant, 2012).

Setting the Tone for a Meaningful Approach to Mathematics

Over the years I have come to more fully appreciate the difficulties faced by PTs in both embracing the course goals and making the necessary shifts in thinking. Thus in the summer of 2012, I abandoned the idea of accomplishing tone setting in the first class (or two), and instead created a new 4-week unit to address the multi-faceted issue of setting the tone for making this fundamental shift from rule-based to meaning-based thinking. The new unit was designed to support PTs as they built a new foundation for doing mathematics, while simultaneously working on their understanding of number. Foci included: recognizing when rules are being invoked; generating and exploring age-appropriate meanings; and learning to focus on meanings by generating and analysing diagrams of problem situations before solving them. As we experimented with the new unit over four semesters, the Instructor Notes remained relatively sparse. With the new unit solidified, I devoted a portion of my sabbatical to generating detailed Instructor Notes and reflecting on how the curriculum had changed over the last decade.

Methodology

I structured my work on Unit 1 in three phases: 1) immerse myself in the activities; 2) refine the activities and create detailed Instructor Notes; 3) compare the resulting unit with the first few weeks of the *UMDT* curriculum. Since I was not teaching during my sabbatical, I invited an experienced mathematics education instructor to participate in more intense variation of our curriculum design process – thus providing a means of immersing myself in the activities. I chose an instructor who was essentially new to the course (she had not taught it in over 10 years), but not new to teaching PTs, nor new to working collaboratively on curriculum design for PTs. My goal was to: 1) reflect on the activities and their function in the unit; 2) identify sticking points of both instructors and PTs as they navigated the first unit, and 3) use this information to refine activities and create detailed Instructor Notes. For every lesson in Unit 1, I offered pre-lesson planning, observed the lesson, and had post-lesson discussions with the instructor. During and immediately after each observation I recorded my own reflections.

The process of creating detailed Instructor Notes and analyzing field notes from observations, interviews, and my own reflections, was a cyclical one. For each identified theme, I considered how to most effectively address it in the Instructor Notes; and in the course of writing up each activity, I reflected on its relation to larger themes. At several points I consulted with the instructor about sticking points and developing themes. Drafts of the Instructor Notes were shared with several instructors for feedback. Once satisfied with the newly designed Instructor Notes, they were compared to those from the original *UMDT* project, both for content and style.

Results and Discussion

The Instructor Notes: An Overview

Many of the suggestions raised by the instructor for inclusion in the Instructor Notes were not surprising: expected student responses; suggestions for what ideas to pursue and which ones not to pursue; a clear sense of the goals of each individual activity, and how this related to overall unit and course goals. Although this kind of information had been included in some Instructor Notes, particularly for well-established activities, it did not necessarily exist for every individual activity, particularly those in the new first unit. Furthermore, in instances where such information was included, the format of this information greatly impacted its usefulness. For example, there were some activities for which I provided a written description of expected student work. In discussing one such activity, it was clear that this description had little impact or staying power, however when I drew pictures to match these descriptions, the instructor quickly understood the important issues raised by the different approaches. Finally, it was also clear that a better organizational system was needed facilitate the quick location of particular information while teaching.

After making these and other alterations to Unit 1, I compared the notes to those from the original *UMDT* project. The new Instructor Notes were of similar length and maintained many of the original features of the notes, with some new features added including activity-specific goals. Also notable was a difference in the number and type of activities. In the original course, there were typically fewer activities in one 100-minute class. For example, Day 1 had been one activity: grappling with the reasoning behind the divisibility rule for 4. This was not an easy task for PTs and required a great deal of understanding and skill to facilitate in such a way that students stayed engaged without becoming discouraged. The current content focus of the first lesson is on even and odd numbers – something that is more accessible to many than proving divisibility rules. The lesson is also divided into three related, though separate activities. In addition, there is a distinct shift from generally open-ended activities with broad goals, to open-ended activities that are more focused on achieving specific goals, particularly in the first unit. This transition was due to a recognition of PTs need for more focused activities to make issues more clear, as well as instructors expressed needs for more clear and specific goals.

Broader Issues: A More Nuanced Approach to Norms, Goals and Tone Setting

Two broader sticking points emerged from the analysis of the data, particularly for classes 1 through 4: the unproductive use of generic teaching moves; and a need for a "big picture" view of the first unit. Consider the following generic teaching moves that could be considered standard moves in a stereotypical "student-centered classroom":

- A reluctance to tell anything, accompanied by the assumption that generalizations/insights/conclusions will always be articulated by students.
- Encouraging the sharing of many ideas, rather than delving into a few in greater depth.
- An overreliance on general why questions (e.g., *What do others think?* or *Can anyone add to this*?), rather than questions designed to help focus PTs on particular issues.

The result of these moves was that important mathematical ideas were not always pursued <u>and</u> many activities wound up taking twice the amount of time intended. My conversations with the instructor indicated that the richness of individual tasks, and the lack of prior knowledge of how PTs react to the task, led her to lose focus on the goal of a particular task. Which brings me to the second, related sticking point: the critical role of having clear goals for individual tasks, how these relate to the goals for the lesson/unit/course as well as the progression PTs go through as they learn to do mathematics meaningfully.

As I worked on the cyclical process of writing Instructor Notes and considering these broader sticking points, I returned to the impetus for creating the new Unit 1: the need for a more detailed and multi-faceted approach to setting the tone for PTs. I realized that the Instructor Notes needed to provide explicit suggestions for doing so, thus better setting the tone for instructors. In the original *UMDT* notes for the first day, there had been a Teacher Note titled *Expectations and Initial Tone Setting*, containing a sample script of how an instructor might explain the goals of the course and their relation to expected norms. In the newly created Instructor Notes, tone setting was supported in a multitude of ways: 1) all goals for activities were refined, and activity descriptions were altered to more clearly reflect goals; 2) whenever possible, summary suggestions were included pointing out the relationship between the activities on a particular day and both course goals and PTs future work as teachers. These additions were designed to help set the tone for both PTs and their instructors as they worked on individual activities. In addition, more general Teacher Notes were created to deal with bigger issues, most notably, two general Teacher Notes addressing the two main sticking points: *Confronting the Taboo of Teacher Telling* and *Confronting the Sanctity of Student Sharing*. As the titles suggest, the notes were aimed at confronting extreme views of a "student-centered" classroom and highlighted the struggles PTs face in moving from rule-based to meaning-based mathematics in the area of number and operation, thus setting the tone for instructors.

Conclusions

One of the enduring challenges of mathematics education is preparing PTs to teach K-12 students to reason mathematically. Many things contribute to this dilemma, including a lack of agreement on what it is that PTs need to know. While progress is being made, a recent national survey of those teaching mathematics content courses for PTs concludes that as a group, these individuals "likely have not had opportunities to think deeply about important ideas in elementary mathematics, and most institutions do not provide training and/or support for these instructors" (Masingila, Olanoof, & Kwaka, 2012, p.357). Thus it is imperative that we create educative materials for this varied population. I propose that in addition to features common to the university-based curriculum projects cited earlier (e.g., specific learning goals, rationale for the activities, expected student responses, etc.), careful attention must be paid to providing the kinds of tone-setting guidance necessary for students (PTs) and their instructors alike to navigate the complex process of basing mathematical work on reasoning rather than rules.

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SITE-BASED MATHEMATICS METHODS COURSEWORK: THE DEVELOPMENT OF ATTITUDES AND THEORY-PRACTICE CONNECTIONS

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This paper reports on the development of attitudes about mathematics, as well as the construction of theory-practice connections among elementary preservice teachers engaged in site-based mathematics methods coursework. Using the attitudes scale from the Mathematics Experiences and Conceptions Surveys, preservice teachers engaged in site-based methods coursework had more negative entering attitudes than control groups, but demonstrated higher growth rates at two subsequent time points. Further, preservice teachers were able to saliently describe theory-practice connections during reflections on the site-based experience.

Keywords: Affect and Beliefs; Teacher Education-Preservice

This paper focuses on the designs and outcomes of field-based mathematics methods coursework organized around opportunities for elementary preservice teachers (PST) to make solid, lasting theory-practice connections using an apprenticeship model. The design of such courses is predicated on the notion that live demonstrations, engagements and reflections on work with real students in classroom settings provide critical experiences on which theory practice connections are constructed {c.f. \Cochran-Smith, 1999 #457}. When mathematics methods courses are taught onsite in elementary schools, PSTs are offered ongoing opportunities to explore strategies that support children's mathematical thinking under the guidance of their methods course instructor and the classroom teacher {Hodges, 2014 #534}. After working directly with children or witnessing a live teacher demonstration, PSTs return to their methods classroom with their methods instructor to engage in immediate reflective conversations that help them learn to theorize from practice.

Objectives of the Study

Little is known about the impact site-based courses have on PSTs conceptions of mathematics teaching and learning, nor the types of theory-practice connections PSTs are able to construct through their participation in such contexts. Consequently, we undertook an exploratory mixed-methods design aimed at unpacking PSTs development as a consequence of their experiences in the site-based methods course. The Mathematics Experiences and Conceptions Surveys [MECS] (Jong & Hodges, in press) were administered to (n = 38) PSTs at three time points: (a) at the beginning of mathematics methods coursework; (b) at the end of mathematics methods coursework; and (c) at the conclusion of the student teaching semester. The MECS measure, in part, PSTs attitudes towards mathematics. Since attitudes are felt less intensely, and are more apt to change than beliefs {Philipp, 2007 #160}, we focused our attention on the attitudes scale as growth might be more readily visible over the short duration of a methods course and/or the student teaching experience. Additionally, the MECS contain contextualized sets of experiences items which help to explain any changes in attitudes as a consequence of prior and/or ongoing experiences within teacher education. Additionally, we drew upon artifact collection and analysis to explore the types of theory-practice connections PSTs developed as they participated in the site-based design.

The overall aims of this research were to (a) determine what, if any, changes occur in attitudes towards mathematics over the duration of the site-based mathematics methods course and into student teaching; (b) compare changes in attitudes among those engaged in the site-based course to populations of students which participate in a more conventional campus-based coursework; and (c) explore the types of theory-practice connections PSTs are developing within the site-based design. This focus contributes to the knowledge base on mathematics teacher development by observing both affective and cognitive aspects of experimental site-based PST mathematics methods course designs.

Perspectives

The MECS measurement of attitudes focuses on participants' ways of feeling and thinking about mathematics, maintaining a focus on PSTs enjoyment of and inclination to see mathematics as a worthwhile activity from both teaching and learning perspectives. Many elementary PSTs enter teacher education coursework with negative attitudes towards mathematics {e.g. \Connor, 2011 #387}. Our own work has suggested that elementary PSTs attitudes towards mathematics are strongly linked to their experiences in K-12 classrooms, and can be positively influenced through attention to methods experiences centered on reform practices in mathematics education (Jong & Hodges, in press). Ultimately, subcomponents of the affective domain, including attitudes, influence PSTs orientations to their own classroom in relation to students' mathematical thinking, use of curriculum materials, and student achievement (Philipp, 2007).

Given the link between attitudes and attention to reform practices in methods coursework, we wished to maintain a dual focus on the development of attitudes alongside the types of theory-practice connections PSTs were able to make. Theory-practice connections were considered in two ways: (a) those that analyze the instructional routines present in mathematics classrooms; and (b) those that investigate students' thinking about mathematics within classrooms. As an example, observations of instructional routines often focused on Smith and Stein's {, 2011 #446} *5 Practices for Orchestrating Productive Mathematics Discussions*, as these practices have shown significant promise in their ability to attend to critical features of inquiry-based mathematics instruction. Examples of the types of frameworks that investigate students' mathematical thinking include students' (a) views of the equal sign {Falkner, 1999 #442}; (b) levels of geometric thought {Wu, 2005 #565}; and (c) placement along an equipartitioning learning trajectory {Confrey, 2009 #564}.

Methods

The data presented here include thirty-eight PSTs enrolled in mathematics methods courses at one university in the United States during fall 2013. MECS-1 was administered during the first week of class while MECS-2 was administered during the final week of class. MECS-3 was administered following full time student during the spring 2014 semester. The Rasch Rating Scale Model was selected to create a common metric and rating scale. MECS-1 was selected for anchoring and subsequently the rating scale thresholds from MECS-1 were anchored using additional coding in a Winsteps control file. The attitudes scales from MECS-2 and MECS-3 were reanalyzed with the item and rating scale in place. Once quality control checks for stability, reliability, and fit were completed, the anchored items scales for MECS-2 and MECS-3 were deemed valid. Subsequently, more traditional parametric tests could be completed on the logit values. Data from the university of interest were then compared to data from two other institutions where MECS were administered at each time point. The other institutions, however, offered their mathematics methods coursework in a conventional campus-based course. We followed quantitative analyses with a qualitative thematic analysis of PSTs narrative reflections on theory-practice connections constructed during methods coursework.

Results

Results of the mixed methods design are organized sequentially by data type.

Quantitative Analyses

Results of an RM-ANOVA analysis indicated significant changes ($p < .001$) in all pairwise comparisons from MECS-1 (.522), to MECS-2 (1.296), to MECS-3 (1.928) among those enrolled in the site-based methods courses. Similar growth occurred in those enrolled in campus-based methods coursework at comparative universities. Furthermore, using Levine's test of homogeneity, there were no statistically significant differences between site-based and campus-based attitudes at any of three time points. Mean values for each university are presented in Table 1.

Table 1: Mean Attitudes Scores by University

University	MECS-1	MECS-2	MECS3
Site-Based	.522	1.296	1.928
Campus-Based 1	.540	1.012	1.322
Campus-Based 2	.711	1.303	1.979

Despite a lack of statistical significance between groups, the site-based methods mean values were lowest among the three institutions, yet exceeded the attitudes scores of campus-based 1 and were nearly identical to campus-based 2 at the end of the methods course. Further, the site-based attitudes maintained high levels of growth, along with campus-based 2, at the conclusion of the student teaching semester.

Qualitative Analyses

During observations of the experienced classroom teacher and their methods course instructor, PSTs maintained a five practices evidence log detailing the extent to which the classroom teacher and their instructor attended to each practice. These evidence logs provided a shared context to discuss mathematics lessons during debriefing sessions. Analyses suggest that PSTs were able to attend to the five practices in their observations, and could note "missed opportunities" in the experienced classroom teacher's attention to each practice. Observing these "missed opportunities" resulted in discussions of lesson modifications that would lead students to important mathematical understandings. Further, PSTs were able to generalize effective teacher moves that encourage the development of productive sociomathematical norms {Yackel, 1996 #227} towards argumentation. The following quote from a preservice teacher is indicative:

> Discussions should focus on students' thinking (both correct and incorrect) to develop instruction and help make applicable connections through representations. I learned using these resources that structure is essential in scaffolding appropriate math discussion. This discussion is important in supporting students in taking risks and asking questions to further their own understanding.

Three themes were developed through analyses of PST's responses to experiences focused on students' mathematical thinking: (a) differentiating instruction, (b) the use of curriculum materials; and (c) attention to multiple representations. The frameworks often highlighted misconceptions in core prerequisite knowledge, which supported PSTs in seeing that instruction must address holes in conceptual understanding, even if instructional guides and materials assume such knowledge. The following quote is illustrative of students' attention to representation of prerequisite knowledge:

> All of the students I interviewed seemed to have a clear understanding on moving from symbol to picture. They all knew how to shade in an area model. One pattern I noticed was that all three students had difficulty moving from picture to symbol. This made me think they were better at partitioning than unitizing fractions.

Discussion

The use of MECS indicated significant growth in PSTs attitudes over the duration of the mathematics methods course and into student teaching. The growth in attitudes among those engaged in site-based coursework was not statistically different from that of comparative campus-based courses. This finding is in and of itself important. Much like many comparative studies of reform mathematics curricula, which provide a "do no harm" comparison to traditional curricula, this study indicates that experimental site-based methods courses provide analogous growth across in attitudes across the teacher education program. Furthermore, we find that although not statistically significant, those engaged in site-based methods courses did show higher growth rates than comparative institutions.

The link between attitudes development and reform mathematics experiences in methods courses is clear. Given PSTs' direct attention critical aspects of reform instruction, the development of those attitudes might well be linked to certain critical experiences embedded in the site-based design. Although only an initial step, unpacking the design of methods courses, particularly carefully crafted theory-practice connections provided here, continues to be a fruitful area of research in mathematics teacher education.

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MAINTAINING QUANTITATIVE COHERENCE: PRESERVICE ELEMENTARY TEACHERS' EXPLANATIONS USING CONCRETE REPRESENTATIONS

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In this study, we asked 19 preservice elementary teachers who had completed content courses and were enrolled in a methods class first to solve multidigit addition and subtraction problems and then to demonstrate how they would explain the same solution to a child using base ten blocks. We found that even those with conceptual understanding of place value and multidigit arithmetic faced challenges providing explanations using base ten blocks that children would be able to understand. In particular, only some gave explanations with base ten blocks that had quantitative coherence—that is, the meanings associated with concrete objects and their transformation remained stable across the explanation. Additionally few PSTs' explanations corresponded with their initial solution because they did not coordinate the sequence of representation transformation with the sequence of their initial solution.

Keywords: Teacher Knowledge; Elementary School Education; Instructional Practices

Providing instructional explanations that students can understand is a critical part of teachers' work. Teachers who know what a clear explanation involves and can perform one themselves are better able to orchestrate a discussion that functions as a student-voiced explanation. Leinhardt (1987) contrasted the instructional explanations of novice and experts and provided a description of the key features of expert explanations. High quality explanations address a specific student audience by using representations that students understand, featuring strategic examples, avoiding errors, and highlighting important connections. Existing studies of preservice teachers have documented generally poor facility in explaining a wide range of mathematical ideas, including standard multidigit algorithms (e.g., Ma, 1999; Thanheiser, 2009).

In this study we identified features of preservice teachers' (PSTs) explanations involving base ten blocks that would prevent or impede student understanding. By interviewing PSTs who had already completed a sequence of content courses and were enrolled in a methods class, we were able to interview many participants who had conceptual understanding of place value yet still offered problematic explanations. Our study addressed this research question: *What challenges do preservice elementary teachers face providing instructional explanations with concrete representations?* We found that PSTs may not maintain quantitative coherence in their use of concrete representations and also may not coordinate their actions with the concrete representation and the sequence of the solution they aim to explain.

Theoretical Framework

In this study we envision teaching that is responsive to various conceptions of mathematics. In particular, we assert that instructional explanations that use representations are accessible to a wide range of students only if the underlying quantitative meanings of the representations and of the ideas that are being explained are made explicit and connected. According to Smith and Thompson (2008), quantities are constituted in people's conceptions of measurable attributes of objects, events, or situations. For example, in a situation where a girl started with 187 seashells, picked up more, and finished with 400, using *quantitative reasoning* (Thompson, 1994) could involve thinking about the situation as a starting amount, an unknown change, and the final result. Starting with 187 and keeping track of how many must be added on before arriving at 400 is a solution that might be generated from such quantitative reasoning. In contrast, we use *numeric reasoning* to refer to calculating and either not selecting calculations based on quantities or not explicitly connecting the component values and results of calculation to corresponding quantities. To extend the seashell example, someone reasoning numerically might subtract 187 from 400 to find the answer but not be able to provide a quantitative meaning for subtraction (e.g., subtraction as take away) that makes sense in the problem situation. While the differences may seem subtle, numeric reasoning without an understanding of the associated quantitative relationships can result in difficulties selecting appropriate calculations and in difficulties interpreting and using the results of calculations (Lobato & Siebert, 2002). We consider explanations to be *quantitatively coherent* if the quantitative meanings associated with concrete objects and their transformation remained stable across the explanation. Additionally, we argue that explanations must follow an appropriate sequence to be understandable for children.

Methods of Inquiry

In Fall 2014, we recruited 20 PSTs at a large public university in the Midwestern United States enrolled in two sections of a methods course. They had completed one content course in number and operation and a second in geometry and measurement. The interviews took place before multidigit addition and subtraction was discussed in the methods class. The first author was the instructor of one section of the methods course, and conducted semi-structured (e.g., Bernard, 1994) hour-long interviews with each pair of PSTs. We present data from 17 PSTs work on two tasks and 2 PSTs worked on only one task. One PST arrived late and did not work on either task. The first task was "187 plus what number is 400?" The second task was "Find 120 plus 96." Participants were first asked to explain their solution to the problem and then to demonstrate how they would explain the same solution to a child using base ten blocks. A main goal of the interviews was to examine how teachers' coordinated their initial solutions with their explanations of these solutions using base-ten blocks. The authors watched the videos and reviewed the transcripts many times. In each pass, we examined talk, gesture, and inscription for evidence of what the PSTs were thinking.

Analysis and Findings

For each task, we report representative cases and summarize findings for all participants.

Task 1: 187 Plus What Number Is 400?

To solve the problem, 16 of the PSTs used the standard algorithm for subtraction and 3 PSTs used different methods. When using blocks to explain their solutions, 12 of the PSTs maintained quantitative coherence and 7 did not. None of the explanations, however, corresponded in sequence with the initial solution supposed to be explained.

Erin did not maintain quantitative coherence. To solve the first problem for herself, Erin wrote the missing value equation 187 + __ = 400 and then used the standard algorithm for subtraction. She crossed out the digits 4, 0, and 0 writing 3, 9, and 10. Then she subtracted by place value starting with the ones to find the answer 213. She checked her work by adding 213 and 187 using the standard algorithm for addition to get 400. When asked to explain her strategy with base ten blocks, Erin counted out a collection for 187 using one flat, eight rods, and seven cubes. Erin then asked if she should "show 400 and then breaking it down" to which the interviewer agreed. She made a pile of 4 flats and stated she would show subtracting the 187 from the 400. She then removed 2 flats from the 400 pile to form a new pile and then took one rod and three cubes from the bank of extra material and placed them on top of the two flats she had moved. Two flats remained in her original pile, and she placed these back in the bank. She stated that the pile with 187 and the other with 213 would combine to make 400.

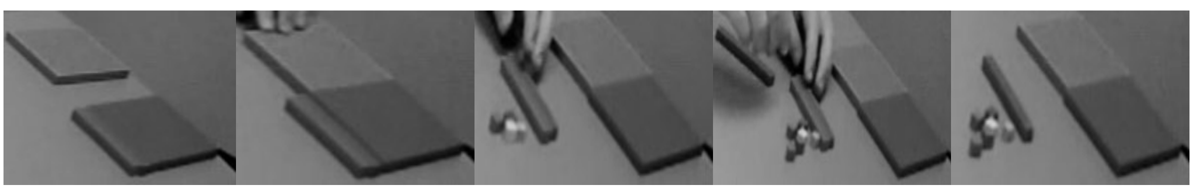
Abby maintained quantitative coherence. Abby initially solved the problem using the same method as Erin. To explain her solution using blocks, Abby laid out 4 flats to represent 400 and she stated she wanted to "take away 100, 80, and 7". To do this she removed 1 flat, traded a second flat for 10 rods and removed 8 of them, and finally traded a rod for 10 cubes and removed seven of them. She described her result as "200, 10, and 3 which would be 213".

Interpretation. Erin's explanation with blocks did not have quantitative coherence because the single rod and three cubes she used to form 213 came from the bank of extra material rather than from the collection representing 400. Abby's explanation did have quantitative coherence in that she used equivalent trades with the bank and constructed 213 by removing 187 from the initial collection of 400. However, her explanation using blocks did not correspond with her initial solution because she reversed the sequence of ones, tens, and hundreds. Thus, we judged that neither PST used blocks to explain her initial solution in a way children could understand.

Task 2: Find 120 Plus 96.

Of the 17 who worked on the task, 14 used the standard algorithm and 3 used different methods. When using blocks to explain their solution, 14 PSTs maintained quantitative coherence and 3 did not. Only 7 PSTs gave explanations that corresponded in sequence with their initial solutions.

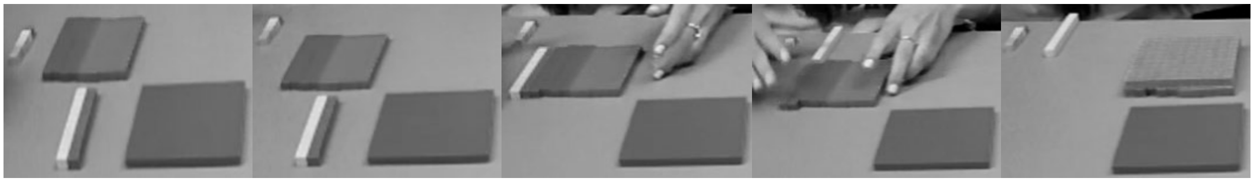
Alan did not maintain quantitative coherence. To solve the second problem, Alan said he first "rounded up" 96 to 100, added 100 to 120 to get 220. He then subtracted 96 from 100 to get 4 and finished by subtracting 4 from 220 to get 216. To explain his solution using blocks, Alan formed a collection of 1 flat and 2 rods. He placed 1 more flat below this (Figure 1a). He pointed to the single flat, held up 4 cubes, and said he "could not really get 96" so he added 4 to it and showed 100. He put the 4 cubes in his hand back in the bank of extra material. Next, he moved the single flat next to the collection of 120 (Figure 1b). He then moved the 2 rods from the collection to the side and put 4 cubes from the bank next to them (Figure 1c). He said, "I would take the 20 minus 4 to just get 16." He then moved in 2 cubes and 1 rod from the bank (Figure 1d) and removed 2 rods to leave a collection of 16 next to the two flats (Figure 1e).



a. b. c. d. e.

Figure 1: Alan's Representation of 120+96 with Base Ten Blocks

Cara maintained quantitative coherence. Cara used the standard algorithm to solve the problem initially. She wrote 120 above 96, wrote 6 in the ones place on the third line, noted 1 above the 1 in 120, wrote 1 in the tens place on the third line, and finally wrote 2 in the hundreds place. When Cara was asked to explain her solution, she formed a collection for each addend, one positioned above the other. The upper collection had 1 flat and 2 rods, and the lower collection had 9 rods and 4 cubes. In each collection, she organized the different blocks in sub-groups ordered from left to right and vertically aligned (Figure 2a). She then said 0 plus 6 would be 6 and moved 6 cubes down (Figure 2b), and that 2 tens plus 9 tens would be 110 and moved 11 rods together (Figure 2c). She moved one rod down, and slid the remaining 10 rods left and underneath the flat from the first collection, covering them with a flat from the bank (Figure 2d & e). She said the final answer was 216.



a. b. c. d. e.

Figure 3: Cara's Representation of 120+96 with Base Ten Blocks

Interpretation. Alan's explanation did not have quantitative coherence because he said "twenty minus four" and then used the same 4 blocks (originally representing "minus four") with another 2 cubes and 1 rod from the bank to form a collection representing 16. Cara's explanation had quantitative coherence because the meanings of the blocks in her original collections remained stable throughout her explanation. The new flat from the bank covered up 10 rods, effectively replacing them. Unlike Abby on the previous task, Cara's explanation directly corresponded to her initial solution. We judged that Cara's explanation of her initial solution could be understood by children but that Alan's explanation of his initial solution could not.

Conclusion

We found that PSTs who have completed content courses still struggle to provide understandable explanations using concrete representations. To be understood, instructional explanations that involve concrete representations require that transformations of the concrete representation correspond to the sequence of the solution that is being explained. Moreover, quantitative coherence must be maintained to provide instructional explanations that are numerically accurate, grounded in representations, and build from students' intuitive and informal strategies. Only 6 explanations of the 36 we analyzed maintained quantitative coherence and corresponded with the sequence of the initial solution that was supposed to be explained. Our finding suggests that an important goal for elementary mathematics methods classes is supporting PSTs in maintaining quantitative coherence when using concrete representations and may require explicit focus on the sequence of representation use.

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ELEMENTARY TEACHERS' PERSPECTIVES ABOUT THE TENSIONS OF TEACHING MATHEMATICS THROUGH ART AND MUSIC

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In order to help more children to be successful in mathematics, teachers are searching for new ways to leverage children's out-of-school knowledge and experiences while teaching mathematics, specifically with regard to contexts of art and music. The following paper details a subset of a larger phenomenological pilot study about seven elementary teachers as they worked with three artists to craft mathematics lessons that incorporate art and music. The presentation will detail how Kara and Miranda, two third grade teachers, discussed the tensions of teaching elementary mathematics through art and music. Implications for future research and practice regarding the practicality these lessons will be discussed.

Keywords: Elementary School Education; Instructional Activities and Practices; Teacher Education-Inservice

Objectives and Background for the Study

Research in mathematics education suggests that many students, particularly those in traditional classrooms, feel disconnected from their in-school mathematical learning (Greer, Mukhopadhyay, Powell, & Nelson-Barber, 2009). To help more students be successful in mathematics, teachers can explicitly incorporate students' out-of-school knowledge and experiences into their practice (Turner, Gutierrez, Simic-Muller, & Diez-Palomar, 2009). Research frameworks such as those on culturally responsive mathematics teaching (Bonner & Adams, 2012; Gay, 2000), children's multiple mathematical knowledge bases (Turner et al., 2012), and STEAM education (Science, Technology, Engineering, Art, and Mathematics) has provided concrete examples of how teachers might contextualize their practice in the familiar experiences of their students. Research-based frameworks that leverage children's familiar experiences and knowledge for teaching mathematics are particularly important for children who are not typically successful in traditional classrooms (Greer et al., 2009). When teachers use more authentic, non-traditional contexts of art and music, as related to the diversity of children's particular cultures and native languages, more students might learn mathematics more easily (Courey, Balogh, Siker, & Paik, 2012). The purpose of this study was to explore the perceptions of seven elementary teachers as they learn to craft tasks, related to mathematical concepts of fractions and geometry, within Latin@ children's knowledge and experiences with art and music. (It should be noted that the author is using "Latin@" specifically to honor those who do not identify as male or female.) The guiding research questions for this pilot study ask: (1) Prior to the Te'ALaMO (Teachers, Art from Latin@ cultures, and mathematical MOdeling), how did teachers utilize art and music as a context for teaching elementary mathematics, and (2) during the Te'ALaMO, what affordances, tensions, and/or implications for future practice do the teachers describe when considering art and music as a context for teaching elementary mathematics? For this particular brief research report, the authors will focus on the experiences of two third grade colleagues, Kara and Miranda.

Methodology

Te'ALaMO was designed as a qualitative, phenomenological study (Creswell, 2007) that explored the experiences of seven in-service teachers as they learned from a musician, an actress, and a muralist about how to integrate elements of art and music in their mathematics instruction during the summer of 2014. The setting for the study was in a large city in a south-central area of the U.S. where the author is a mathematics teacher educator at a large urban university. Three pre-K, three third grade, and one fifth grade teacher were purposefully recruited for the study from schools that serve large populations of Latin@ children and non-native English speakers and that were nearby a Latin@ community cultural arts center, where the workshop was housed. Prior to the workshop, the teachers discussed their beliefs and prior experiences about using art and music to teach mathematics and then worked on activities with the artists that situated mathematics within playing mariachi music, acting, and designing murals. The research team conducted focus group interviews and collected observational field notes of the activities, teachers' quick reflections at the conclusion of each day regarding their experiences, and an optional demographic survey about the teachers' backgrounds. The researchers also created transcriptions of video and audio recordings of the participants during the workshop.

Because phenomenological research is described through the perspective of the participants, findings based on the data analysis emerged from the teachers' experiences during Te'ALaMO. Data analysis began by reading the transcripts and observational field notes and identifying the ways in which teachers discussed how and why (or why not) they might use art and music as a context for teaching mathematics. Memos (Creswell, 2007) about these emerging themes were created for each day of the workshop and across all three days. The researchers compiled and compared their detailed memos about the teachers' experiences regarding the affordances and tensions of using art and music to teach mathematics as a way of conducting member checks of their findings.

Preliminary Findings

Over the course of the three-day workshop, the teachers learned about mariachi music, performance art, and murals with respect to teaching elementary mathematics concepts such as rational number operations (e.g., equivalent fractions, ratios and proportions) and transformational geometry (e.g., shape attributes, patterns, reflections). As an example, the musician and teachers compared how the structure of music note values (e.g., whole notes, half notes, and quarter notes) was similar to how elementary teachers might introduce the meaning of creating and operating on fractions equivalent to one whole (e.g., one whole note is the same duration as two half notes or four quarter notes when played at the same tempo; 1 = 2/2 or 4/4). Later, the Chicana actress described the nine equal sectors of a performance stage and how this these sectors referred to the actors' relative stage position. Finally on the last day, the muralist extended the discussion about rational numbers by helping the teachers to use proportions and scale to replicate a small picture on a larger poster board.

Many of the teachers like Kara and Miranda, two third grade teachers in the workshop, already utilized some elements of art and music in their daily practice as a mathematics teacher (e.g., children illustrating mathematical representations through drawings, singing songs to help memorize basic facts, students learning about area and perimeter by creating their name in block letters on graph paper). Throughout the workshop, Kara and Miranda were two vocal participants who discussed at length about how the challenges and tensions that they might face if they wish to situate more of their mathematics teaching within the contexts of art and music.

Preparing Students for Standardized Testing While Developing Their Flexible Mathematical Thinking

Although Kara and Miranda discussed many ways that they used art and music to address elementary mathematics concepts with their students, they still recognized that the pressures of standardized testing could be in conflict with their goal to help children develop a flexible understanding of mathematics. Because Kara and Miranda taught in a grade level that was subject to statewide standardized testing, they assumed that other colleagues might not agree that art and music could be a viable context for helping children to learn mathematics. Specifically Kara stated:

> It's hard to sell something like this [teaching mathematics through art and/or music] to other people if you don't have, like, evidence. You know? Especially, when you teach a [name of the standardized test] grade. Oh my God. It's…it's, well 'how's it going to affect the data?' That's like [laughs], always on everybody's mind. But, just with, especially with the first day Te'ALaMO, the [mariachi] music, I could see how much higher order thinking is involved when you're having to count and clap…

In this moment, Kara concluded that she felt the pressure of using art and music to teach elementary mathematics when each moment of her practice as a teacher needed to be tied to some measurable outcome or goal as it related to the state's standardized test. Kara continued her thought by claiming that even though she felt pressure of the standardized test, she still maintained her belief that art and music could help her students to develop a flexible understanding of mathematics. She stated that

> I think a lot of, well, even what we're trained on is that sometimes it's you're teaching the child how to think in ways that they're not used to, and sometimes that's hard, you know? They're being assessed in a certain way, but we're training them to think this one way, and then it's hard for them to think differently. So, giving them all these avenues to express the same sort of idea, you know, I think is important. And I think, you know, as teachers when we come to things like this, we have to be advocates for it at our own campus, you know? I think it's hard to do at third, fourth, and fifth grade, because… You can get other teachers on board with this sort of model [of teaching mathematics through art and music].

Kara concluded that she could address this tension of creative teaching while preparing her students for the yearly standardized test if more of her colleagues adopted non-traditional methods of teaching mathematics by leveraging elements of art and music.

Finding Resources for Implementing Mathematics Lessons that Incorporates Art and Mathematics

Miranda, Kara's colleague at the same school, agreed with Kara's belief that art and music could serve as a flexible context and proposed that she could also address the required elementary standards across multiple content areas. At first, Miranda suggested that art and music " might open some avenues for some of them [the students], you know, and show them how the art can be integrated into math, and engineering, and science, and you know, you can find it everywhere. I think it's, it's fantastic for the kids." Then after the muralist concluded her session on using proportions and scale to create murals from smaller pictures, Miranda reiterated her perception on the final day of the workshop when she stated that

> And I find that I use art, I integrate it a lot with social studies because it's easier…like when we're studying, let's say Cinco De Mayo You know, where we can talk about the history and then we can make something. They make it, a piñata. And so I have two grades. I have my social studies, you know my actual… about the history, and what they wrote in their journals, you know, that grades then I also have an art grade. But now with this [the mural lesson in the workshop], I can integrate it with math.

Miranda described how she already leveraged her Latin@ students' particular students' out-of-school knowledge and experiences (e.g., Cinco De Mayo and piñatas) while teaching social studies, history, art, writing. Now after the mural activity, Miranda could see how she might use a mural about Cinco De Mayo and piñatas that could help her address her mathematics standards as well.

Shortly after the muralist's activity, Kara and Miranda mentioned yet another challenge to implementing similar lessons like the mural activity with their students. Specifically, Kara stated "materials and supplies. Some schools are, you know, really, uh, considered wealthy, right? You get to go into the supply room and get whatever, and then, not all campuses are like that." Because all of the teachers who participated in the workshop worked at schools that served children from mostly economically challenged communities, Kara in particular recognized that although she could teach a lesson that addressed multiple state standards by utilizing elements of art and music, she would have a challenge of finding enough supplies and resources for these lessons.

Implications and Conclusion

As teachers find new ways to support their students' mathematical thinking, they are exploring new contexts by which to connect their lessons to children's out-of-school mathematical knowledge and experiences (Gay, 2000; Wager, 2012). Findings from this study suggest that elementary teachers are working to negotiate both the inherent challenges and potential opportunities for situating their instruction within the contexts of art and music. More research is needed to explore the potential tensions and challenges that teachers might face if they elect to use art and music as a context by which to teach mathematics, particularly with the growing popularity of STEAM education. Furthermore, more research is needed explore how teachers can successful negotiate the challenges with the affordances of using art and music to teach mathematics so that more children develop a flexible understanding of mathematics that is connected to their out-of-school knowledge and experiences.

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PROBLEM-SOLVING: ANALYZING NARRATIVE GENRE ASPECTS OF PROSPECTIVE MATHEMATICS TEACHERS' DISCOURSE

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This paper reports on an empirical study that examined the language and word choices of three prospective teachers (PTs) of secondary mathematics as they facilitated group discussions with elementary PTs. In the context of a university elementary mathematics content course, groups of PTs collaborated to solve problems involving whole-number and fractional operations. This paper reports on the participation in one such activity that asked PTs to find fractional parts of pattern-block designs. Through a narrative-genre framework, I analyzed how particular language choices of the facilitators functioned to help PTs make sense of problem-solving tasks.

Keywords: Teacher Education-Preservice; Classroom Discourse; Problem Solving

Purpose of the Study

Communication process standards, published by the National Council of Teachers of Mathematics (NCTM, 2000), have emphasized that students should communicate their mathematical thinking clearly and use mathematical language to express ideas precisely. From these ideas we can construe that teachers need to understand how language is used to diagnose student thinking, compare that thinking with desired mathematical understandings, and give appropriate responses through mathematical discourse. However, according to Schleppegrell (2012), using aspects of mathematical language necessary to help all students learn and construct meaning is a key challenge for prospective teachers (PTs). One reason may be that PTs lack classroom experiences and opportunities to converse with students about mathematical ideas. Additionally, PTs may not yet see a need for care in their expression and word choice.

In response to the above and a concern for issues of equity in mathematics education, I draw on the social nature of learning mathematics and argue that language is a primary tool that structures participation discourse practices (Halliday, 1978). And if language can be viewed as the principal resource for making meaning in the classroom, in what ways does this happen? How do particular choices of words function to communicate meaning, negotiate understanding, and invite participation in mathematical discussions? The purpose of this paper is to contribute to the scarce literature on the preparation of secondary mathematics PTs as related to awareness of their language choices and the impact of those choices on student understanding and sense- making. Through a narrative-genre perspective of language, I report findings that explicate ways in which this structure of language is appropriate to the facilitation of mathematical discourse.

Theoretical Perspectives

One way that studies have illuminated important facets of language-use in mathematics is with a semiotic perspective. Language can be described as a semiotic system because it involves sets of meaningful choices in human learning and meaning-making (Halliday & Matthiessen, 2004). According to Eggins (2004), when making a choice of language, what someone says gets its meaning by being interpreted against the background of what could have been said in a particular context – but was not.

While much of mathematical meaning is communicated through the expression of signs and symbols, the use of words and grammatical structures associated with these representations remains the fundamental processes for a *shared construction* of mathematical meaning (Halliday &

Matthiessen, 2004). Different language structures trigger different resulting behaviors, or *functions*, based on the particular choices made. This has been described as a functional-semantic approach to language which explores both how people use language in different contexts, and how language is structured for use as a semiotic system (Eggins, 2004). In alignment with a semiotic perspective, I have drawn on Labov and Waletzky's (1997) six stages of narrative-genre for discourse (see Table 1) in an analysis of PTs' language choices as they facilitated problem-solving discourse in group discussions. Here I focus my attention on the question: In what ways do secondary PTs facilitate and structure mathematical discourse through their choices of language and words?

Table 1: Six Stages of Narrative Genre Adapted for Mathematical Discourse

Stage	Name		Narrative Speech Function	Associated Examples of Mathematical Speech Function
1.	Abstract	A	Here, the facilitator cues the listeners that there is a story to tell.	"This task is about…" "You will be asked to…"
2.	Orientation Or Launch	O	Here, the facilitator presents necessary preliminary information needed to understand the problem.	"In this task, we know about…" "Think about this task based on your own experiences with…"
3.	Complication	C	Here, the facilitator introduces one or more problems that need to be solved.	"But we don't know for sure if…" "I didn't think we needed to…"
4.	Evaluation	E	Here, the facilitator makes clear what needs to be accomplished, why the audience should keep listening.	"I have a feeling that you are thinking…" "We still need to continue to…"
5.	Resolution	R	Here, the facilitator helps release the tension and sort out the problem.	"What you say makes sense to the problem…"
6.	Coda or Re-launch	C / R	Here, the facilitator refers back to the theme of the first stage and restates the problem.	"If you remember, we also need to…" "This only addresses part of what we need to do…"

Data Sources and Analysis

This study took place in a content course for PTs of elementary mathematics. Four secondary PTs volunteered to facilitate group discussions because they wanted to strengthen their abilities to communicate effectively with others whose perspectives of what it means to 'do mathematics' might differ from their own. Despite apparent differences in elementary and secondary curriculums, there are overlapping algebraic and geometric concepts, and novice teachers in high schools are often asked to teach beginning algebra and geometry courses. The primary source of data came from audio-recorded mathematical dialogues between 16 elementary PTs in the course (hereafter referred to as students) and the four secondary PTs (hereafter referred to as *facilitators*). Four students and one facilitator comprised each group.

Audio-recordings of six problem-solving activities were distributed over 15 weeks, resulting in 24 transcripts total. Additional data (interviews/debriefs) helped me pay specific attention to each facilitator's personal recollection of the problem-solving events. As the instructor, I chose tasks specific to the course's curriculum involving whole-number and fractional operations.

T his paper highlights an analysis of three transcripts of audio-recordings in which three groups (each with a different facilitator) discussed the task: *The yellow hexagon pattern tile is 3/2 of the area of a second pattern tile design. Use pattern tiles to make what could be the second pattern design.* These three transcripts were selected from a larger set of classroom observations collected over one spring semester. Using Labov and Waletzky's (1997) six stages of narrative-genre analysis, I systematically coded excerpts of each facilitator's language choices, which were then analyzed as to how closely they followed the narrative-genre scheme of *abstract (A), orientation (O), complication (C), evaluation (E), resolution (R), coda (C/R)* (see Table 2). Subsequent interviews and debrief sessions provided insight into the ways that particular excerpts of language choices functioned to direct and structure meaningful discourse.

Table 2: Order of Narrative-Genre Codes Used by Facilitators

Facilitator																						
Brice	A	O	C	C/R	E	C	C	E	C/R	C	C/R	E	C/R	E	C	C/R	E	C	E	R	C	R
Beatrice	O	O	O	E	E	R	E	C	R	E	R											
Linda	O	C	E	E	R	C	C/R	C	E	C	R	E	R	E	R							

Findings

The codes revealed that the facilitators structured discourse in different ways. Overall, Brice was more successful helping his group realize that the second shape would consist of two blue rhombi (or four green triangles). The number of coded excerpts revealed that Brice had more verbal interaction with his group, and that he structured those interactions in a way that restated the task (*coda/re-launch*, C/R) for his group five separate times. Each time, he used a different choice of words which functioned to draw attention to various aspects of the task's goals–that is, needing the yellow hexagon to be 3/2 of the area of the second shape. This refocused the group's thinking and helped to put a spotlight on what still needed to be accomplished. The following excerpt demonstrates one of Brice's C-E-C/R cycles; Brice restated the task's goal in this excerpt because his evaluation did not connect in a way that made sense for the student:

> *Student:* Okay, so we broke that [the yellow hexagon] into three [equal parts] with the blue ones (rhombi), right? Now there's 3/2s, so don't I need-? Now what? [*Complication* because the PT does not connect her idea to the new design asked for in the task.]
>
> *Brice:* So you're saying that this [the PTs' new design of two blue rhombi] would be the whole…and that we're going to make three halves, so if I were to add (two blue rhombi) plus half of this again, it would equal this (the yellow hexagon)? [*Evaluation*]
>
> *Student:* What do you mean?
>
> *Brice:* So remember, you are to make sure the yellow hexagon is three of the half-parts that make up the new design. [*Coda/Re-launch]*.

Beatrice and Linda did not structure their discourse as closely to the narrative scheme as Brice did. The codes revealed little use of the *re-launch* stage (one *re-launch* used by Beatrice and none used by Linda). Consequently, the group members lost focus and conversation slowed. Linda's evaluations were vague (she self-reported not wanting to give out too much information) and included such language as "you're getting close" and "I like your idea". While encouraging in nature, those words did not structure the discourse towards promoting further thinking about the task. Beatrice self-reported that she felt at times "unconnected" to the group's ideas and therefore, felt safer in telling them how she was thinking. Beatrice told her group "what they're trying to tell you is that the hexagon is the larger part of a whole. So the whole is going to be smaller than the hexagon, right?" These words served to inform the group members how to think and consequently stopped any further thinking about the relative sizes of the two designs.

Discussion and Conclusion

Mathematics education researchers and linguists (e.g., Halliday & Matthiessen, 2004; Herbel-Eisenmann & Otten, 2011) have brought increased attention to the ways that language functions to make meaning in mathematical activities. For this study I defined *successful facilitation* of mathematical discourse not necessarily by the lengths of conversation or whether groups found solutions, but more by the ways in which the facilitators' words helped the students make sense of the task and take up peers' ideas. In analyzing discourse patterns revealed in this study, I determined that a facilitation cycle of *complication,* followed by *evaluation* and then by *coda/re-launch* [C-E-C/R] was more successful than other cycles of discourse. Additionally, this study suggested that the C-E-C/R cycle may not only be a use of language, but also a way of thinking. That is, this cycle reflected the facilitators' thinking about the task and also served to re-focus the group's thinking about the task. Brice's use of this cycle ("let's go back and think about how the yellow hexagon has to compare to the new shape we're trying to build") functioned to lead the students to find a solution that made sense. Mathematical discourse plays an important role in this way by shedding light on the functions of various language choices and how those choices structure discussions and thinking towards making meaning (Barton, 2008).

Moschkovich (2010) has maintained that teachers need to create and sustain environments for learning and what takes hold across tasks and practices in these environments. As both elementary and secondary PTs endeavor to use reform pedagogy and communicate effectively with students, an understanding of how their own language choices structure discourse and shape meaning-making aspects of understanding mathematics becomes important. Through an analysis of data from a linguistic perspective, I was led to posit that a narrative-genre analysis could help both PTs and teacher educators identify structures of language-use that facilitate discourse effectively. Further research into how discourse structures connect with and promote students' thinking about mathematical tasks can help teacher educators better prepare PTs to facilitate mathematical discussions in the classroom. Analyzing word choices through Labov and Waletzky's (1997) narrative framework has the potential to shed light on why some mathematical conversations stall and others result in deeper connections of mathematical ideas.

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EQUITY IN MATHEMATICS EDUCATION: WHO IS AN ALLY?

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In trying to promote equitable mathematics education systems, we see a need to aim at developing the role of allies when we work with mathematics teachers and students. With the goal to better understand what being an ally *means and what* ally-work *involves, we review literature outside of mathematics education that describe how allies support more equitable systems. In this report, we synthesize this work in order to engage the mathematics education community in discussing how we might do this work.*

Keywords: Equity and Diversity, Teacher Education-Inservice, Affect and Beliefs

Although the term "stakeholder" has been common when referring to those that have a vested interest in education, such as parents, teacher educators, and community members, we expand the function of a stakeholder by focusing on "allies" in our work in schools. Mathematics education research clearly shows that the system of schooling and mathematics education result in inequitable practices and experiences for many students (e.g., Martin, 2009), aspects of education that the idea of stakeholder does not address. Thus, this brief research report focuses on a conceptual exploration of *ally* and *ally-work* to address issues of equity in mathematics education. As Hand (2003) has argued, mathematical teaching and learning for students from dominant versus nondominant racial, ethnic, linguistic, and socioeconomic backgrounds have resulted in visible "participation gaps" in mathematics classrooms. We believe these disparities must be understood in terms of the distribution of opportunities to learn (OTL) (Gresalfi & Cobb, 2006).OTL are not dependent upon students alone, but on an equitable system that may ensure powerful mathematics learning for minoritized student groups (Martin,2009;Moschkovich,2010). *Fair distribution* of OTL for diverse students is promoted through an "equitable system", which refers to intersecting levels of synergistic support for these students (Hand, Penuel, & Gutiérrez, 2012). This systemic support can be augmented through the engagement of *allies* in disrupting systems of privilege and oppression in education (e.g., Swalwell, 2012; Tatum, 1994).

We review literature outside of mathematics education to explore what being an *ally* means and what *ally-work* involves. We synthesize this literature in order to engage the mathematics education community in discussing how we might do this work. We do this not only because we think the idea of allies deserves conceptual attention in mathematics education but also because we are hoping to negotiate and study an equitable system in an urban district with the goal of promoting a fair distribution of OTL for students who have experienced longstanding marginalization in mathematics education and for the teachers who teach them.

Being an Ally and Ally-Work

We synthesize a series of definitions and stances. First, we describe the roles and functions of an ally. Second, we present some challenges and tensions described in the consulted literature.

Ally-Work is Cyclical (Inward-Outward) and Expansive (Collaborative)

Jenkins (2009) and Clark (2010) define who an "ally" is by comparing similar ideas like "advocates," "anti-work," and "agents," juxtaposing the levels of social justice commitment, empathy, and action. Accordingly, these authors argue that allies are people strongly oriented at supporting oppressed groups. Allies' stances stem from an awareness of the systemic forces that maintain oppression and how these systems inherently disadvantage others. Allies' stances first work inwards by recognizing their "unearned privileges" in a community; then their work and stances expand outwards by embracing their "obligations of privilege" (Clark, 2010, p. 707).

Contrastingly, Jenkins (2009) describes an advocate as a defender, one who pleads for a cause. An advocate may speak up but may or may not engage in action. Similarly, Clark (2010) characterizes "the mere silencing of hate-talk" as "anti-work" (p. 712). Advocates' work represents a type of anti-work through the process of interrupting acts of oppression against marginalized people, actions that are effective in the presence of advocates. These changes, however, are situational and temporary. Jenkins also contrasts ally-work with an agent's work. An agent is a person with power to act and force change. "Agents may or may not identify with the community or group on whose behalf they are acting. Agents orient themselves toward action and go beyond developing empathetic relations or vocal oppositions; they work to create change within dimensions of society in which they may or may not have power (p. 28). Furthermore, Philip et al. (2014) define *ally* as a discursive position that educational scholars of color take in their work with teachers by a humble approach of learning from teachers, addressing issues of racism with teachers at school, and helping them understand the impact on students of neoliberal reforms. Additionally, scholars with teaching experience are *allies* with nuanced understanding of the teachers' work through own experience, or *teacher solidarity.*

As a result, ally-work encompasses the work of an advocate and an empathic co-worker, an agentive work that is cyclical and expansive. As a cyclical process, it starts through an inward questioning and internal transformation of dispositions. Then, this renovated perspective expands outwardly to others in joint action to support change. Ally-work addresses a twofold change, first by inviting others to engage in a "critical dialogue and discussion, interrogating perceived lines of difference and [then] inquiring into the possibilities for creating productive alliances across these lines" (Clark, 2010, p. 705). Through this dialogue participants have "an opportunity to come into contact with likeminded others and find social support for their struggles" (DeTurk, 2006, p. 44). Then a mutual and intercultural understanding is built through "perspective taking and corresponding changes in participants' beliefs, attitudes, and behavior" (p. 47). A relational identity nurtures a genuine empathy that naturally leads to action and to effect changes. Clark argues that ally-work implies a conscious decision to combat disadvantages and inequities by strategically combining self-awareness, empathy, commitment, and action to promote change.

Ally-Work Encompasses an Identity of Tensions and Relative Power

Being an ally represents "an identity that is achieved by acting on the moral imperatives of pursuing social justice and validating differences" (DeTurk, 2011, p. 575).In their support of marginalized groups in relation to sex, race, or any other social identity, allies make use of their social and cultural capital to effect change and influence others. This means that their social identities matter in their context of action. In the U.S. context, DeTurk (2011) asserts that" being White, male, heterosexual, able-bodied, fluent in English, or otherwise equipped with such capital, are relatively powerful in their capacity to influence others" (p. 577). For example, an African American man described how a Black ally needs the support or "validation" of a White person because a White person's social and cultural capital is privileged by the system (DeTurk, 2011). Thus, allies' successful efforts to confront racism and sexism depends" on how much power they had in a given situation" (DeTurk, 2011, p. 578).

Nevertheless, DeTurk (2011) complicates simplistic perspectives on power, arguing that power is fluid, multifaceted, and contingent. So, in-group members like African American persons "could act as allies to Whites by interrupting anti-White prejudice" (p. 584). Likewise, an ally self-identifying as a gay man could promote closer connection as an ally to other gay men. Furthermore and equally strong might be when a heterosexual man is an ally of a gay-men group. But overall the matching of these different power dynamics highlights the contextualized, fluid, and yet systemic nature of power. Membership to a social and cultural group does not determine an absolute power or lack thereof. Allies aware of these dynamics need to develop tactics and alliances that crisscross different groups to support each other.

A tension in being an ally concerns preventing allies' risk (i.e., personal safety, relationships, or status) because, at times, allies can feel their personal wellbeing is threatened (DeTurk, 2011). A greater ethical and moral tension is related to speaking for others or overprotectiveness. Allies asserted that taking actions, of speaking for others, might disempower those who are to be empowered and thus, reinforce oppression. DeTurk (2011) warns that overemphasis on dialogue might highlight mainstream values as well, and the demanding of such procedures might reinforce a laden procedure that only serves "enlightened self-interest" (p. 584). Thus, negotiations need careful procedures.

Finally, Kelly and Chapman (2015) introduce the term "adversarial ally" to identify an issue in ally-work for those in disadvantage. An ally might "unambiguously harm a client and in another moment unambiguously help that same person" (p. 48). Specifically to the context of health services, people with disabilities have limited options for health care. So though patients might experience these services as helpful and necessary, they also may experience them as harmful and oppressive. The adversarial aspect of being an "ally' in this situation is that despite health professionals' stance of empathy and caring action towards people with disabilities, their uncritical support for and work in an oppressive system that limits health care options for people in need, perpetuates an inequitable system.

Being an Ally in Education and Mathematics Education

Although an increasing percentage of school children in the United States are children of color, poor, and from homes where family members speak languages other than English—all potential sources of privilege and oppression—mathematics teachers (MTs) and mathematics teacher educators(MTEs) remain fairly homogeneous along these demographic lines (Hollins & Guzman, 2005). Thus, thoughtful attention to how these differences matter in mathematics classrooms and professional development is necessary. Actions of ally educators in their classrooms oppose the disadvantages that some students may have by providing "extra time and attention towards those in need" (p. 708) and including curricular content that challenges normative notions. Only through explicit and critical action will ally-work promote just, equitable, and safer spaces for diverse students. For example, many teachers feel that it is necessary to take corrective action in order to help students of color to acquire mainstream American English, without acknowledging these student's identities and where they come (Clark, 2010).Ally-work actions need to be critically assessed through equity lenses as the OTL mathematics include not only what students learn, but also how they learn it (Esmonde, 2009).

MTEs seeking to promote equitable systems in mathematics education would benefit from ally-work beyond roles as advocates or agents. Developing an intercultural communication with MTs and students could help with constructing an alliance built on mutual understandings and promoting and sustaining generative action. We would need to become *allies with* MTs, students, and each other.

Our goal would be to learn from one another through participatory action research, and accordingly adapt our work so that MTs work with us, and each other, as agents of change in their own classrooms by making curricular adaptations responsive to their students' interests and needs. Building dialectic relationships and transferring them into collective action are central points in ally-work.

In our presentation, we expand on these ideas in order to prompt discussion about the role of allies and ally-work in mathematics education. For this, we will provide transcripts of our team's reflections. As MTEs, we believe acknowledging power is crucial. Being aware of the relative power of our social and cultural capital can guide us to examine and contextualize our actions. In our work with MTs and students, the success of our roles and relationships must be re-evaluated on a regular basis and contrasted with the expectations and goals of students and MTs. Tensions must be embraced as learning sources and opportunities to develop stronger connections with others. If tensions do not emerge, an exhaustive examination of our stances and actions is required. Likewise, diversity should be constantly addressed not only related to race, culture, language, and gender, but also academically in how we define what counts as mathematics.

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THE MATHEMATICS NEEDED BY ELEMENTARY TEACHERS: DO TEDS-M AND MET II AGREE?

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We investigated the relationship between two representations of mathematical knowledge for teaching: the Mathematical Education of Teachers II (MET II) report and the Teacher Education and Development Study in Mathematics (TEDS-M) knowledge assessment. Expert raters matched MET II specifications to TEDS-M items, finding a suitable match for almost every item. Inter-rater agreement was high for mathematical content expectations, but much lower for mathematical practices. Ratings indicated that TEDS-M released items mapped onto some but not all MET II content domains. The findings suggest that the two representations are well aligned in some respects though not in others.

Keywords: Teacher Knowledge; Standards; Elementary School Education; Assessment and Evaluation

Purpose

A longstanding argument in the educational research literature holds that teachers' knowledge is related to students' achievement in mathematics. However, there has been a lack of detailed examination of the ways in which and the extent to which different specifications agree or disagree about the mathematics knowledge that teachers need in order to be effective. Such examination is essential to identifying commonalities and sharpening points of disagreement, both of which are critical to moving the field forward both in research related to teacher knowledge and in the development of programs to prepare novices to be effective teachers.

In this study we examined the alignment between two representations of essential knowledge for mathematics teachers: the Teacher Education and Development Study in Mathematics (TEDS-M) knowledge assessment for elementary teachers (using the released items) (Tatto, 2013) and the Mathematical Education of Teachers II (MET II) report (including the essential ideas (EIs) for elementary and middle grades teachers and the Common Core Standards for Mathematical Practice (SMPs)) (CBMS, 2012). Understanding the ways in which and extent to which these two representations align should benefit researchers and practitioners alike.

Conceptual Foundation

Our study rests on and draws from two distinct research literatures: scholarship on mathematical knowledge for teaching (MKT) and studies of alignment between and among key elements of educational systems. Each is briefly summarized in this section.

Teachers' Mathematical Knowledge

Shulman (1986, 1987) is widely credited with drawing attention to teachers' knowledge and how the ways teachers know content might differ from the ways it is known by other professionals. As it has become known in mathematics education, MKT has been theorized and studied from a variety of perspectives including efforts to specify the nature and structure of the knowledge needed for effective teaching of mathematics (e.g., Ball & Bass, 2000; Davis & Simmt, 2006), and the relationship among teachers' mathematical knowledge, their instructional decisions and practices, and student achievement (e.g., Hill, Rowan & Ball, 2005). This corpus of scholarship illustrates the value of having both theoretically derived conceptions of knowledge for teaching and empirically validated assessments that reliably test such knowledge.

Alignment

The concept of aligning standards and assessments gained momentum in school mathematics with the advent of the NCTM standards (NCTM, 1989). Webb (2002) outlines four criteria that can be used in alignment studies between state academic standards and assessments and are a subset of his criteria of content focus: categorical concurrence, depth-of-knowledge consistency, range-of-knowledge correspondence, and balance of representation. The focus of our analysis in this study was categorical concurrence.

Methods

Expert Raters

In order to explore the alignment between the TEDS-M elementary knowledge assessment and the MET II EIs and SMPs we solicited the judgments of four experts in mathematics and mathematics education. We chose two who had been involved in creating and/or reviewing items for the TEDS-M assessment and two who were among the authorship team for the MET II document. Each pair included one mathematician and one mathematics teacher educator (MTE). The selection of expert raters was purposeful in that it allowed us to probe the possible influence of the raters' prior familiarity with one of the representations (TEDS-M vs. MET II) or their professional training (mathematician vs. MTE).

The Rating Task

Each expert rater was given the released TEDS-M assessment items, the MET II EIs for elementary and middle grades teachers, a description of the SMPs, and a recording tool. For each item the expert raters were asked to identify, if possible, an EI and a SMP that best fit the item. There was an "Other" option for instances when a rater judged that an item pertained to a particular domain but did not fit with any of the specific EIs; raters also had an option to identify primary and secondary EIs or SMPs in cases where more than one applied. We conducted follow-up interviews with individuals and pairs of raters to probe the raters' thinking for selected items and to gather their impressions of the rating procedures.

MET II Essential Idea and SMP Agreement

Expert raters could be paired based on their document affiliation (TEDS-M or MET II) or their professional training (MTE or mathematician). This resulted in four different pairs: TEDS-M, MET II, mathematicians, and MTEs, with each rater belonging to two pairs. Agreement between pairs regarding the EIs was calculated in three mutually exclusive ways: Identical Agreement (IA) was the percent of cases where rater pairs assigned exactly the same EI to a test item; Domain Agreement (DA) was the percent of cases where two expert raters disagreed on the exact EI that fit a test item, but the EIs they assigned were within the same domain; and Secondary Agreement (SA) was the percent of cases where expert raters disagreed on both the primary EI and MET II domain that fit an item, but one rater in a pair identified the primary EI of the other expert rater as a secondary EI. We similarly calculated instances of rater agreement for SMPs, except that the DA category was not applicable to SMPs because they are not organized by domains. Because the agreement categories were mutually exclusive, we were also able to sum across categories for each pair of raters to obtain a percent of Total Agreement.

Representation of MET II EIs in the TEDS-M Knowledge Assessment

Following the agreement calculations, we examined the representation of MET II domains within the released TEDS-M items by grouping all of the EIs by domain to give a more manageable picture of the representation of the kinds of content included in the assessment

Results

The Rating Task

In 99% of cases expert raters were able to assign either a specific MET EI or a rating of "Other" within a domain to TEDS-M items. Additionally, expert raters were able to assign SMPs to TEDS-M items in 90% of the cases.

Agreement Between Raters on MET II Essential Ideas and Mathematical Practices

Table 1 shows the types of agreement for different pairings of expert raters when assigning EIs and SMPs to TEDS-M assessment items. The percent of Identical Agreement was near 50% for all rater pairs. Given that there are 42 different EIs, assigning the exact same EI to TEDS-M assessment items around half of the time is a high level of agreement. Two rater pairs had exceptionally high Total Agreement, and the other two had acceptably high agreement of this type. On the other hand, the extent of Identical Agreement and Total Agreement for SMPs was considerably lower.

Table 1: Inter-rater Agreement for EIs and SMPs by Rater Pair and Type of Agreement

Agreement Type	MET II Pair	TEDS-M Pair	MTE Pair	Mathematician Pair
MET EIs				
IA	56%	44%	44%	47%
DA	32%	24%	27%	15%
SA	3%	6%	21%	18%
Total	91%	74%	92%	80%
SMPs				
IA	21%	35%	29%	24%
SA	21%	0%	9%	3%
Total	42%	35%	38%	27%

Representation of MET II EI Domains in the TEDS-M Knowledge Assessment

There was variation in the degree to which MET II domains were represented in the released items. EIs in the domains of "Measurement and Data," "The Number System" and "Expressions and Equations" were assigned more than 25 times. In contrast, none of the EIs in "Counting and Cardinality" and "Number and Operations-Fractions" were assigned to any TEDS-M released item. Additionally, EIs in "Statistics and Probability" were assigned in only four instances.

Discussion

Through this examination of the relationship between the TEDS-M knowledge assessment and MET II expectations we found that experts were able to assign MET II EIs to TEDS-M assessment items with high inter-rater agreement. This suggests that the MET II document and the TEDS-M assessment are viewed by experts as depicting a similar view of the mathematics content knowledge needed by elementary teachers. The highest levels of agreement were found for the MET II and the MTE pairs.

Though the expert raters also mapped SMPs onto almost all TEDS-M items, they exhibited far less agreement. In fact, the percent of Total Agreement for SMPs was lower than the percent of Identical Agreement for EIs for all four rater pairs. As there are only eight SMPs one might expect the levels of agreement to be higher for SMPs than for EIs. On the other hand, the SMPs are not described in either the MET II or the CCSSM documents at the same level of detail that characterizes the description of content expectations. In follow-up interviews the raters noted both that many of the assessment items seemed to align with multiple SMPs and that it was more challenging to identify a primary SMP than was the case for EIs. The highest level of agreement was found for the MET II pair, and the lowest was for the mathematician pair. The low agreement between the mathematicians is surprising given the presumption that SMPs are fundamental to doing mathematics.

Another noteworthy finding was the unevenness of the correspondence of TEDS-M items to MET II content domains. Raters judged very few items to be related to EIs in the domain of statistics and probability, and there was a complete absence of items tied to EIs in the domain of fractions. Because these two domains are foundational in elementary mathematics education, this finding needs further examination to see if it is attributable simply to unfortunate omissions in the selection of publicly released items or if it reveals a limitation in the content coverage of the TEDS-M assessment.

Given the array of different conceptualizations and assessments of knowledge needed to teach mathematics, a careful examination of similarities and differences seems wise to inform both research and teacher education. A clear understanding of different conceptualizations of what teachers need to know to be successful will support researchers to study this knowledge and teacher educators to nurture its development. The study reported here is a first step in what we hope will be a larger scale effort to interrogate and integrate across disparate views of the mathematical knowledge needed to teach elementary school mathematics.

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SIMULATING TEACHING: NEW POSSIBILITIES FOR ASSESSING TEACHING PRACTICES

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This paper argues that successful practice-based teacher education requires innovations in assessment that can better inform preservice teachers and those who prepare them. Such assessments must focus directly on specific teaching practices of novice teachers, as well as offer opportunities to assess the use of content knowledge for teaching. Simulations, an assessment type used in other professional fields, hold promise as one means for gathering data about and providing feedback on teaching. To explain how this could work, we describe an assessment that focuses on preservice teachers' ability to elicit and interpret a student's mathematical thinking, and we appraise what it makes possible.

Keywords: Assessment and Evaluation; Instructional Activities and Practices; Teacher Education-Preservice

The Need for Assessments of Practice

The increasing focus on specific instructional practices in initial teacher preparation means that there is a need to develop ways to assess preservice teachers' teaching in new and more precise ways. Assessing preservice teachers' ability to describe, analyze, or reflect on practice does not provide sufficient insight into their development. Further, novice teachers need specific feedback about their practice (Grossman, 2010).

Assessment of teaching practice is not new in teacher preparation. Approaches have focused on appraising preservice teachers in real context of practice, such as in field placements and during student teaching, and have included microteaching, field-based performance tasks, and systematic field observation of lessons (e.g., Hamerness, Darling-Hammond, & Bransford 2005; NCATE, 2003). In field-based assessments, however, contextual factors affect preservice teachers' performance. Although context is a reality of practice, field placement contexts are unique for each preservice teacher, which makes it more difficult for teacher educators to obtain reliable estimates of their preservice teachers' teaching capabilities. For example, our teacher preparation program, like many, used interviews conducted in their field placements to assess our preservice teachers' skill with eliciting and interpreting student thinking. They probed children about their mathematical thinking and then they later analyzed the interviews to make claims about the children's understandings (Sleep & Boerst, 2012). Using video records, instructors were able to see and provide feedback on the types of questions posed, how well they attended to and used children's mathematical ideas, as well their manner with the children. We were also able to assess the quality of their interpretations of the children's thinking. However, issues of fairness arose because some children were less forthcoming with their thinking than others and required different sorts of probing questions to elicit their thinking. Further, because instructors did not know the children themselves, they could not determine whether the preservice teachers were accurately uncovering children's thinking. As a result, it was also not possible to detect patterns in preservice teachers' skills overall within the program. This paper describes work we have been doing since 2011 to develop and investigate the use of simulations as a complementary form of assessment that can address some of the shortcomings of previous approaches.

The Use of Simulations in Professional Preparation

Simulations are used in many other professional fields. In many medical schools, doctors in training engage in simulations of physical examinations, patient counseling, and medical history taking by interacting with "standardized patients," adults who are trained to act as patients who have specified characteristics. Evaluation of medical students' interactions with standardized patients makes possible common appraisal of candidates' knowledge and skills. In medicine, simulations have been used for formative assessment for over 40 years and are currently used in high-stakes medical licensure examinations (Boulet, Smee, Dillon, Gimpel, 2009). In nursing schools, simulations are used to develop and practice clinical skills, including skills difficult to develop and practice on an actual patient. Many of these simulations make use of mannequins (robots) that can be programmed to behave in particular ways and to exhibit particular symptoms. Simulations have not been widely used in education, for either learning opportunities or assessment purposes. There are beginning to be examples of programs using them to support the learning of skills such as managing a classroom (Dieker, Straub, Hughes, Hynes, & Hardin, 2014*),* conducting a parent conference (Dotger & Sapon-Shevin, 2009), and school leader development (Dotger, 2014); however, the use of simulations for assessment has been limited.

A Simulation Assessment of Skill with Eliciting and Interpreting Student Thinking

To concretize ideas of simulations in assessing teaching practice, we turn now to describe an assessment that we developed and are now using of preservice teachers' ability to elicit and interpret student thinking. The assessment makes use of a standardized student (i.e., someone playing the role of a student) and takes about 25 minutes to complete.

In the first part of the assessment, preservice teachers are given a copy of the standardized student's work on a problem (see Figure 1) and they have 10 minutes to prepare for an interaction with one standardized student about her work. Because the student's work produces the correct answer, the task is to determine how the student reasoned about the problem and what she understands. Students can use an array of methods different from those familiar to adults, and an important task of teaching is to probe and make sense of students' mathematical processes and understanding, both when they seem obvious and when they do not. This is particularly demanding for novice teachers who are likely to know less about non-standard approaches.

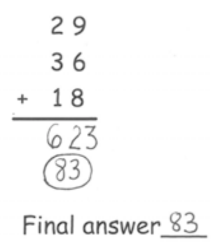
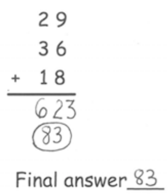


Figure 1.A Student Work Sample on an Addition Problem

In the second part of the assessment, preservice teachers have five minutes to interact with the standardized student. Preservice teachers are told that they should elicit and probe the standardized student's thinking to understand the steps she took, why she performed particular steps, and her understanding of the key mathematical ideas involved. To ensure consistency, the role of the standardized student is guided by carefully articulated rules for reasoning and responding, including responses to questions that are commonly asked, referred to as the "student profile" (see Figure 2). In this case, the student uses an alternative algorithm to solve the problem. The student added the digits in each column, starting with the tens. The student interpreted the 623 in the written work as 6 “tens” and 23 “ones” and produced the final answer of 83.

In the third part of the assessment, the assessor asks a series of questions to elicit the preservice teacher’s interpretation of the student’s process and her understanding. Further, the preservice teacher is asked to predict, based on what the interaction revealed, how the student would solve 27 + 48 and what she would understand about several key mathematical ideas.

Student work:



General orientation to responses:
- do not make basic facts errors
- give the least amount of information that is still responsive to the preservice teacher’s question
- if a question is confusing, say something like, “What do you mean?”
- do not write unless you are asked to write

The student:
- uses the column addition method, except that the student is working from left to right
- correctly applies the column addition method for solving addition problems
- can use the same process to solve addition problems with more than two digits, understands when/why/how to “combine”
- has conceptual understanding of the procedure (e.g., “six hundred and twenty-three” is not the answer because we have to combine the tens)
- is thinking about 623 as 6 tens and 23 ones
- can tell that the sum of 83 is reasonable

Specific responses (*a subset of them*):

Preservice teacher prompt	*Response*
What did you do first?	I added the tens: 2 + 3 + 1 and I got 6.
How did you get from 623 to 83?	I had to combine the 6 and the 2.
Why did you need to combine those numbers?	Because they’re both tens.

Figure 2.An Excerpt from the Standardized Student Profile

The assessment is scored using checklists with the criteria for proficient performance, including both mathematical and pedagogical aspects as it is being completed. Criteria for eliciting are keyed to specific parts of the task (e.g., probes for why the student combines the 6 and the 2) as well as how the preservice teacher takes up specific things that the student did or said. The interpreting criteria focus on the accuracy of the explanation of the student’s thinking and the use of evidence to predict the student’s performance on a similar problem.

Features of the Assessment Situation

In designing assessment scenarios, we make choices about the authenticity and familiarity of the context. Although the teaching context itself is a simulation, some features are nonetheless authentic. For example, the student work is ambiguous with respect to the student’s process (e.g., How did the student get from 623 to the final answer of 83?) and understanding of the core mathematical ideas (e.g., What does the student interpret the “623” to mean?). This means that preservice teachers cannot know in advance how the interaction will unfold. In addition, just as in actual classroom practice, the interaction occurs in real time, which requires teachers to generate questions on the fly in response to the student.

The inauthentic aspects of the assessment actually enable more systematic evaluation of skills. First, all preservice teachers elicit student thinking about the same mathematics content, which avoids the issue of some content being easier or harder to elicit student thinking about. Second, that our preservice teachers are all interacting with a standardized student means that we are able to see their skills under the same conditions. Third, we are able to focus this particular assessment squarely on eliciting and interpreting student thinking, not on other aspects of teaching — also important — such as building relationships with students.

What Simulation Assessments Can Offer

Simulation assessments present one promising possibility for improving assessment of teaching practice. We use these in our program to learn about how our preservice teachers are developing and how we can support improvement in their skills. But we also assess their skills on entry to the program and this, too, has been useful. One year, for example, we found that almost all our entering preservice teachers asked about the student's process for solving a mathematics problem; however, fewer than half of them asked about the student's understanding. Further, they rarely posed follow-up problems (6%) to confirm the student's process or understanding. When asked to produce a problem that could be used to confirm the student's approach, only 54% of preservice teachers were able to generate a numerical example that would present the same conditions as the original problem. These baseline data provided crucial information as we set out to develop our preservice teachers' skills. We are also finding that these assessments support us in providing more detailed and specific feedback to our preservice teachers that can help them improve their practice. In our current work, we are continuing to investigate the validity and practicality of simulation assessments, including exploring their feasibility and design entailments. We are also conducting validation studies that examine the relationship between performances in simulations with performances in classroom contexts.

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SUPPORTING TEACHERS USING APPROPRIATE TOOLS STRATEGICALLY: A PRACTICAL FRAMEWORK FOR SELECTING AND REVISING DGS TASKS

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Given the implications of high cognitive demand tasks for student learning, we sought to determine how to support teachers in selecting or designing tasks that use Dynamic Geometry Software to support students' high-level thinking. We discuss the development of a practical framework for supporting teachers in this important work of teaching. The framework integrates Sinclair's Design Principles (2003) and the amplifier and reorganizer metaphor for technology use (Pea, 1987) in order to assess the role of technology in relation the thinking requirements of a task. Following the description of the framework and its development, we discuss results regarding how pre- and in-service teachers used the framework to evaluate and revise tasks found in current Geometry textbooks.

Keywords: Technology, Teacher Education-Preservice; Teacher Education-Inservice; Teacher Knowledge

The Common Core State Standards in Mathematics identify students' ability to use appropriate tools strategically as an important mathematical practice. We believe a similar principle applies to teachers: that they must be discerning users of technology in considering how to effectively achieve certain mathematical goals with their students. However, there are few practical tools available to assess the role of technology in a given task. One lens for analyzing a technology task is the way it supports students' mathematical thinking. Research on student learning shows that students who engaged with tasks requiring high levels of mathematical thinking demonstrated a greater ability to problem solve, make connections, and communicate their thinking than those who did not (Stein & Lane, 1996). Thus, it is important to support teachers in selecting high-level tasks to implement with their students. This paper describes a framework for selecting and revising technology tasks to support students' high level mathematical thinking, and efforts to implement it with teachers.

Theoretical Framework

Dynamic geometry systems (DGS) support the learning of important mathematics (Hollebrands & Dove, 2011) and support students' mathematical thinking (Sherman, 2014; Cayton, 2012). We have analyzed numerous mathematical tasks that use DGS in order to better understand the role of technology in supporting students' mathematical thinking. In this section we describe the theoretical and research bases guiding our development of a framework to help teachers judge the potential of a DGS task to encourage students' high-level thinking.

Sinclair's Design Principles

An important principle of using technology strategically for mathematics instruction is that the inclusion or exclusion of technology within a given task should depend on mathematical goal(s). Only then can a teacher decide which tools, technological or otherwise, may be most effective in accomplishing that goal. Sinclair (2003) describes design principles related to how the *sketch*, i.e., technological representation of mathematical objects, and associated prompts for students' activity, should depend on the goal of the task. We use these principles to identify three overarching goals for students' mathematical activity: making mathematically meaningful observations, mathematical exploration, and fostering curiosity/modifying thinking. Along each dimension, technology plays a role in achieving the goal, and the metaphor of using technology as an *amplifier* or a *reorganizer* provides a way to describe the role that technology plays.

Amplifier and Reorganizer Metaphors of Technology Use

Pea (1987) introduced the metaphors of amplifier and reorganizer to distinguish two ways in which technology may be used to support students' thinking. As an *amplifier*, technology can make a task more efficient by performing computations and generating representations quickly and accurately, but the focus of students' activity is not essentially changed. As a *reorganizer*, technology can transform students' activity, supporting a shift in students' mathematical thinking to something that would be difficult to achieve without it. With respect to the goals for students' mathematical activity above, technology use as a reorganizer is interpreted to mean that DGS plays an essential role in achieving these goals. We integrated the design principles with the amplifier/reorganizer distinction into a single framework (Figure 1) to evaluate the role of technology with respect to each goal (row) (for more detail regarding the development of the framework and an example of how it may be used to analyze and revise a task, please see Sherman & Cayton (in press)). The framework is intended to serve as a practical tool to support teachers in assessing how the use of technology supports particular goals for students' thinking and reasoning, and to suggest ways in which a task might be revised in order to accomplish certain goals more effectively.

		Technology Used As	
		Amplifier Students could achieve the same goal without the technology	**Reorganizer** The mathematical goal of the task would be difficult to achieve without DGS
Goals	**Question**	**Use of Design Principles**	
Make mathematically meaningful observations; look for invariant relationships.	Do the sketch and prompts use the dynamic affordances of DGS in a way that would be difficult or impossible to replicate without it?	Students create multiple static examples, either by construction or dragging, and reason from those static examples, e.g., students are prompted to make observations or generalizations based on a table or static measurements without reference to the sketch.	The sketch allows for continuous dragging, and students are guided to examine measurements or relationships dynamically. Students are required to make or explain observations or generalizations dynamically in terms of the sketch.
Mathematical exploration; use appropriate tools strategically.	How does technology support mathematical exploration?	Sketch and prompts guide students to investigate the same example or set of examples to explore mathematical connections or invariances. Freedom with respect to dragging does not provide alternative paths if students are all investigating the same example.	Sketch and prompts allow students to explore based on individual observation of mathematical concepts, connections or invariances within the sketch. The sketch supports students' mathematical exploration by providing alternate paths.
Foster curiosity and modify thinking; make and test conjectures; use appropriate tools strategically.	Does the sketch provide feedback? Do the prompts encourage or require students to use feedback?	Experimentation is limited by restrictive construction or does not provide feedback to allow students to explore their own conjectures; or prompts do not explicitly guide students to test conjectures.	Sketch and prompts support experimentation; sketch provides feedback or allows students to test and refine conjectures. Prompts explicitly guide students to use the sketch to test conjectures.

Figure 3: DGS Task Framework

Implementing the Framework with Teachers

In order to better understand how mathematics teachers make sense of the framework, and how it supports their ability to critically examine and describe various uses of technology, we introduced the framework to teachers and had them use it to analyze three tasks and make suggestions for how the tasks might be revised. The contexts for this study include a teacher education course and two professional development courses at three different universities. Data was collected in spring and summer of 2014. Participants included a total of 25 pre-service and in-service teachers with a range of teaching experience.

Results

The way in which teachers' used the framework to evaluate the three tasks we presented them with are summarized in Table 1. (The tasks can be found at the following links: Intersecting Chords: http://tinyurl.com/lhb7acw, Exploring Properties of Polygons: http://tinyurl.com/mkhho55,and Trapezoid Midsegment: http://tinyurl.com/k9vw3d2) The expert evaluation reflects the authors' assessment of each of these tasks along the dimensions represented in the framework in Figure 1.

Table 1: Teachers' Evaluations of Three Tasks

Task: Intersecting Chords		**Expert**	**April**	**July**	**August**
Mathematically Meaningful Observations	Amplifier	x	5	4	11
	Reorganizer		3	0	2
Mathematical Exploration	Amplifier	x	7	4	13
	Reorganizer		1	0	0
Foster Curiosity and Modify Thinking	Amplifier	x	7	3	13
	Reorganizer		1	1	0
Task: Exploring Polygons					
Mathematically Meaningful Observations	Amplifier		1	1	5
	Reorganizer	x	7	3	8
Mathematical Exploration	Amplifier		0	2	2
	Reorganizer	x	8	2	11
Foster Curiosity and Modify Thinking	Amplifier		3	1	0
	Reorganizer	x	5	3	13
Task: TrapezoidMidsegment					
Mathematically Meaningful Observations	Amplifier		4	1	3
	Reorganizer	x	4	3	10
Mathematical Exploration	Amplifier	x	5	3	3
	Reorganizer		3	1	10
Foster Curiosity and Modify Thinking	Amplifier	x	5	2	3
	Reorganizer		3	2	10

Discussion

In our discussions with teachers, we found that the differences in how they categorized a task along each dimension related mostly to interpretation of the prompts. For example, the Midsegment Trapezoid task includes the prompt *Drag point A or point B to change the shape of trapezoid ABCD. Do not allow segment AD to intersect BC. What do you notice about EF and (AB+DC)/2?*Some teachers interpreted this prompt as requiring students to drag continuously, and thus called it a reorganizer along the first dimension of the framework. Others, however, thought the prompt was asking students to engage in a "drag and stop" action, which would merely generate another example, and categorized it as being an amplifier. Since the prompt is not explicit, we believe either interpretation is valid. From a teacher education perspective, we believe that this ambiguity provides a pedagogical opportunity in understanding this dimension of the framework, i.e., that continuous dragging leverages the dynamic affordances of DGS.

Other discrepancies between teachers were also noted. For the first dimension, practicing teachers with more years of experience often categorized a task as an amplifier, citing the fact that they had taught that particular concept without technology for many years. Thus, teachers did not necessarily focus on if DGS was needed to accomplish the goals for students' mathematical thinking within a task. Along the second dimension, teachers tended to focus on the open-endedness of the task rather than the role of DGS in supporting the open-ended features of the task. In response to this issue we revised the framework to be more explicit about the criteria for this dimension. For the third dimension, teachers assumed students would check their conjectures without an explicit requirement to do so leading them to categorize tasks as reorganizer even if an explicit requirement to check conjectures was not present. As students rarely check their conjectures spontaneously, it is included in the framework as an explicit requirement in the prompts.

Implications for Teacher Education

The value of this practical framework is directly related to its effectiveness in supporting teachers' use of DGS to engage students in mathematical reasoning and sense making. Our current results demonstrate that the framework was useful in helping teachers understand distinctions in how DGS might support student thinking, and that teachers were able to use it to critically evaluate tasks that incorporated the use of DGS. Raising teachers' awareness of the idea that DGS can be used in a variety of ways that may or may not promote students' mathematical thinking is an important step in helping them to use DGS more strategically. Fostering this awareness and providing sustained opportunities to move beyond evaluation to revising and creating DGS tasks that support students' mathematical thinking represents an important element of mathematics teachers' technological pedagogical content knowledge (TPACK) (Niess et al., 2009) that this framework has the potential to address.

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PRESERVICE TEACHERS' SELECTION OF MATHEMATICAL TASKS FOR ENGLISH LANGUAGE LEARNERS

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This study examined preservice elementary teachers' selection of mathematics tasks for English language learners (ELLs) following a three-week field experience. For each of the first three weeks of the field experience, the preservice teachers were provided with cognitively demanding tasks for which to plan a lesson for a one-on-one session with an ELL. In the fourth week, the preservice teachers were asked to select their own tasks. Through the use of a qualitative document analysis methodology, we found that the preservice teachers attended to their ELL's culture, language, and mathematics to varying degrees when selecting tasks. Further, they included a number of visual representations to accompany the tasks. The findings suggest that further preparation is needed to help preservice teachers select appropriate tasks for ELLs.

Keywords: Teacher Education-Preservice; Equity and Diversity; Curriculum

English language learners (ELLs) comprise the fastest growing demographic in U.S. public schools (Wolf, Herman, & Dietel, 2010). ELLs are a culturally, economically, and linguistically diverse group of students who can benefit from particular instructional strategies in the mathematics classroom. Many teacher preparation programs in states with historically small proportions of ELLs do not address strategies for teaching ELLs. In fact, though it has been estimated that nearly every teacher in the U.S. has at least one ELL student, less than one third of teachers have been prepared in effective teaching strategies for ELLs (Ballantyne, Sanderman, & Levy, 2009). These realities have contributed to the persistent achievement gap (Fry, 2008) between ELLs and their native English-speaking peers.

Shulman (1986) described particular aspects of knowledge needed for teaching, including curricular knowledge. He referred to these aspects collectively as pedagogical content knowledge (PCK). Magnusson and colleagues (1999) later expanded on the notion of curricular knowledge. This expansion described two categories of this knowledge: knowledge of goals and objectives and knowledge of specific curricular programs and materials. Knowledge in these areas is an important aspect of practice for which teachers must be prepared. Because tasks determine students' learning opportunities (Kloosterman & Walcott, 2010), the selection of appropriate curricular resources for students is a crucial component of a teacher's work.

Although there is some literature related to mathematics curricula and ELLs (e.g., Campbell, Davis, & Adams, 2007; Pitvorec, Willey, & Khisty, 2011), a review of the literature on preservice teachers (PSTs) did not reveal any studies related to their selection of mathematics tasks for ELLs. In this study we examined four PSTs' selection of tasks for ELLs following three weeks of an ELL-focused field experience. The following question guided the study: *What are the characteristics of the tasks PSTs selected for ELLs following a specialized field experience?* As we were interested in the particular tasks selected, we focused on the first of Stein, Grover, and Henningsen's (1996) three phases through which tasks transition: the tasks as represented in the curriculum materials. Attending to this phase is important because teachers' decisions to select particular tasks can limit or expand students' learning opportunities in the classroom.

Methods

The data for this study were from a larger study investigating a four-week field experience for four, elementary PSTs. This field experience paired each PST with an ELL student in weekly meetings. For the first three meetings, the research team provided the PSTs with a cognitively demanding task. The PSTs were then asked to write up a lesson plan for implementing the task with their ELL. The PSTs were allowed to modify or adapt the task in any way they saw fit. In the fourth meeting the PSTs were asked to select their own task to use with their ELL. For each meeting we had video recordings of the meeting, video recorded pre and post interviews, the PSTs' lesson plans, and the PSTs written reflections.

In this study we focused on the fourth and final meeting of the PST and ELL pairings because we were particularly interested in the features of the PSTs' selected tasks. We conducted a document analysis (Bowen, 2009) in conjunction with an inductive coding scheme to analyze the data. This began with open coding (Strauss & Corbin, 1990) the lesson plans and interview transcripts for each PST to identify initial themes related to the features of the PSTs' selected tasks. Axial coding (Strauss & Corbin, 1990) was then used to further refine these themes.

Findings

The PSTs were asked to adapt an existing task or to develop a task of their own for the fourth week. Two of the PSTs elected to create their own tasks and the other two adapted an existing task. In the following sections we discuss the common features of all four tasks to better understand the factors to which PSTs attended to in support of ELLs.

Connection to Culture and Context

Within three of the four tasks, there was evidence that the PSTs attended to the task context for their ELLs. This occurred in one of two ways: focusing heavily on the ELL's culture in selecting a task or on a context that the PST thought would be familiar to the ELL, but not necessarily tied to their particular cultural background. Morgan and Hannah adapted their tasks based on their ELLs' cultural background. Morgan's task was situated within an apartment context, which she believed Ho-Min would be familiar with, saying "I did try and figure out what kind of living arrangements he may have been used to [in S. Korea]." Hannah took a different approach and included a Korean name in the problem and made use of her knowledge of Hwa-Young's upbringing in Japan by selecting a common Japanese dessert, imagawayaki, for the task. This was very successful in gaining Hwa-Young's interest in the task. Hwa-Young even exclaimed, "Imagawayaki is my favorite dessert, too!" Fiona focused on common contexts with which she thought Jin would be familiar, saying "[I] tried to do common things, using food and like soccer and basketball, which I know he liked." Morgan and Hannah seemed to select culturally relevant contexts for their students based on their knowledge of their student's, while Fiona selected contexts devoid of Asian culture. In contrast to these cases, the fourth PST, Kimberly, did not attend to Kyong-Tae's cultural interests or contextual familiarity. The task she selected was familiar to her because it was going to be on her methods course exam the next day. She described her opportunity to work with Kyong-Tae on the task as "totally beneficial."

Attention to Mathematical Content

In three of the four tasks selected, there was evidence that the PSTs attended to the ELLs' mathematical levels when selecting tasks. They did this in two ways. First, they selected tasks appropriate for the grade level of the student. For example, Hannah focused her task on 4th grade fraction ideas for her 4th grade ELL. This approach of matching the task to grade level standards was different than the approach Morgan and Kimberly used. When explaining her choice of task, Morgan stated that she focused on area "because he [her ELL] said he liked area, I'm thinking he, at least, has some experience with it or knows how it works." Similarly, in light of Kimberly's prior interactions with Kyong-Tae and his seemingly advanced mathematics level, Kimberly selected a task related to middle grades geometry. Kimberly and Morgan's selections seemed to be influenced heavily by their knowledge of their students, while Hannah instead relied on grade level expectations.

In contrast to these cases, Fiona did not appear to attend to the mathematical content in choosing her task. Fiona's task consisted of mathematical activities that, during implementation, did not pose much of a challenge for Jin. Although Jin was in the fourth grade, Fiona selected tasks that consisted largely of single step addition, subtraction, and multiplication. Her task selection was driven by her desire to have an interaction that was "more fun than the other tasks when he was like sitting down and working through it. I hope he finds it fun."

Attention to Language

There is evidence that all the PSTs attended to language within their tasks. Morgan paid substantial attention to language, saying she "tried to keep the wording pretty short and simple" while adding in certain words like "dresser" in the prompts to help "expand his [English] vocabulary." Further within the lesson plan, Morgan identified how she would provide linguistic support for words if they became problematic. For example, she stated "if he is struggling with the word 'entire,' [she would] motion that the carpet has to be placed throughout the whole room." Kimberly's attention to language was evident in her choice of an entirely oral task. She explained her decision to do this in her pre-interview: "I would be verbally giving him directions, that this might be more challenging for him to pick up on the cues, but it is probably a lot more similar to what he sees in school." She also anticipated linguistic challenges by preloading vocabulary at the start of the task.

In contrast, Fiona and Hannah's attention resulted in problematic language in their written tasks. Fiona did not anticipate challenges that Jin may face with terms such as digit and number, and therefore did not have a plan for how to support Jin. Hannah's task had multiple spelling errors, multiple verb tenses used within a single problem, and overly complex or incoherent sentences. For example, within one activity, Hannah used eat, uneaten, ate, eaten, and left over to describe what happened to a dessert. In all cases, the PSTs paid attention to language; however, in some instances, this attention did not support their ELL's language development or lessen linguistic demands.

Inclusion of Visuals

Three of the PSTs included visual representations in their task. Hannah included pictures of desserts, identified as a modification intended specifically for Hwa-Young, and stated, "instead of having her draw her own pictures, I make sure that I had a picture for her." Morgan's inclusion of a bedroom diagram was also seen as a support for her ELL, identifying in her lesson plan that she would "walk him through what each picture represents" and point to various places in the diagram to assist with linguistic challenges (e.g. unfamiliarity with 'carpet'). Morgan further adapted the visual presentation of the task by increasing the amount of white space. Kimberly's supports included drawing a table and diagram of a circle on the whiteboard prior to Kyong-Tae's arrival, which she saw as supporting him during her orally administered task. In contrast to these three PSTs, Fiona included no visual supports for Jin. It is not known why she chose to do this and is in contrast to her behavior during the first three weeks.

Discussion

We examined PSTs' task selections for ELLs with whom they were familiar. Each PST used their knowledge of their ELL to select tasks that positively impacted their student's success. However, at times, these selections seemed to be problematic or misguided. When selecting mathematically appropriate tasks, the PSTs primarily selected activities based upon their student's grade level, and not from their prior knowledge of the student's mathematical ability. In all cases, the selected task did not pose substantial mathematical challenges or exhibit characteristics of high cognitive demand tasks (Stein et al., 1996). Although the PSTs had classroom instruction on cognitively demanding tasks, this was insufficient for impacting the level of cognitive demand of self-selected tasks for students with whom they were familiar.

When adapting language for their selected tasks, two of the PSTs were successful at anticipating and planning for linguistic challenges. Although all the PSTs had been adapting language during the prior three weeks, this, in conjunction with their coursework, was insufficient at fully developing their skills in anticipating and planning for language. This is evidence that PSTs need further practice at anticipating and planning for language within curriculum materials for ELLs.

Our findings indicate that completing a four-week field experience with an ELL in conjunction with coursework is insufficient to fully prepare PSTs to work with ELLs. Further research is needed to better understand how to support and prepare PSTs for their work with ELLs in their future classrooms. It is our hope that this study will encourage others to further investigate how teacher educators can better prepare PSTs, and potentially in-service teachers, for working with and supporting ELLs in the mathematics classroom.

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EXAMINING METHODS FOR STUDYING MATHEMATICAL KNOWLEDGE FOR TEACHING

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Over the past 20 years researchers have made progress in understanding mathematics teachers' knowledge. Although the field has made advances in measuring mathematical knowledge for teaching (MKT), much is left undiscovered about the nature of this knowledge. This is due in part to the many challenges inherent in studying MKT, which is multifaceted and inextricably tied to practice. This paper investigates methods around the study of MKT by looking at tasks from three cognitive interview studies. It focuses on how the tasks and the pedagogical contexts within them influence what can be learned about the different facets of MKT. This paper has implications for future studies that seek to investigate the nature of MKT.

Keywords: Mathematical Knowledge for Teaching; Research Methods; Teacher Knowledge

Introduction

Over the past 20 years, researchers have made progress in understanding the knowledge entailed by the work of teaching mathematics and in measuring mathematics teachers' knowledge. Although these studies have conceptualized such knowledge and found valid ways of measuring teachers' knowledge, there is still much left to learn about the nature of this knowledge. This is because this knowledge is multifaceted, involves multiple layers of reasoning, and inextricably tied to practice (Putnam & Borko, 2000). Many researchers have sought and created measures of teachers' knowledge, yet few have looked closely at the nature of this knowledge. One line of research that has looked at the nature of this knowledge stems from Ball and colleagues' work on *mathematical knowledge for teaching* (MKT). Their research group has used extensive records of practice to build and test hypotheses about the nature of mathematics teachers' knowledge (Ball & Bass, 2003). This work has led to an understanding of some of the mathematical demands of teaching mathematics, yet many facets of MKT are still not known. Moreover, this is complex work that cannot be easily replicated without copious data from mathematics classrooms, and a group of researchers with broad, multidisciplinary expertise in teaching, mathematics, and cognition. What is needed is a better understanding of how to study MKT. Such an understanding would allow more researchers to analyze the complex mathematical knowledge and work that occurs in teaching, leading to a better understanding of the unique knowledge demands of teaching mathematics. This paper investigates methods around the study of MKT by looking at tasks from three studies that investigated MKT using cognitive interviews. The differing types and structures of tasks across the three studies affect teachers' responses and the types of knowledge that are made visible. This affords the ability to look at how the tasks themselves and the pedagogical contexts within them influence what can be learned about the different facets of MKT from each task.

Theoretical Framework

This study draws on the theory of *mathematical knowledge for teaching* (MKT), the knowledge needed to carry out the work of teaching (Ball, Thames, & Phelps, 2008).This notion of MKT builds on Shulman's (1986) conceptualization of pedagogical content knowledge. Pedagogical content knowledge, which resides at the core of expert teaching, is a teacher's ability to turn content knowledge into pedagogically powerful forms that can be adapted to students' varying abilities, prior knowledge, and backgrounds (Shulman, 1987). MKT includes both subject matter knowledge and pedagogical content knowledge. Included within subject matter knowledge is a category of content knowledge unique to teachers. Other mathematics professionals use mathematics in a compressed and finalized form, but teachers must interpret, understand, and share the uncompressed versions of mathematics knowledge that their students learn and use. Knowing mathematics in these multiple uncompressed forms and mapping between them is a subset of mathematical knowledge unique to teachers (Ball et al., 2008).

This study draws on a situated perspective of knowledge and practices. In this perspective, embedded within knowledge are fundamental links to the situations in which it was learned and is used (Brown, Collins, & Duguid, 1989). Teachers' "professional knowledge is developed in context, stored together with characteristic features of the classrooms and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations" (Putnam & Borko, 2000, p. 13). In this view, teacher knowledge is inextricably linked to the contexts, tasks, and practices of teaching. Therefore, knowledge and practices can vary across settings. Several examples exist in the literature documenting teachers whose mathematical knowledge differed between teaching and non-teaching contexts (Borko et al., 1992; Hodgen, 2011; Ma, 1999; Thompson & Thompson, 1994).

Methods

This paper presents a qualitative study using tasks and responses from three different studies designed to investigate MKT using cognitive interviews (Desimone & Le Floch, 2004). In the first study, nine experienced geometry teachers were interviewed while solving nine multiple-choice MKT-Geometry tasks, three problems in each domain: knowledge of content and teaching, knowledge of content and students, and specialized content knowledge.

The second study was designed to address a range of teaching practices and mathematics content. For each task, participants were presented with a classroom situation and asked to take on the role of the teacher. This included providing feedback on student work and using it to further instruction, creating formative assessment, and designing learning goals for a lesson. Twelve experienced teachers ranging across grades K-12 participated in this study.

Ten high school Algebra II teachers participated in the third study. The tasks in this study were focused on the teaching practices of giving explanations and selecting examples and on the content of rational expressions and equations. In each task, participants were either asked to pick examples to teach a particular topic, such as simplifying rational expressions, or to give an explanation for students of a particular concept, including solving a rational equation.

While each study was focused on investigating MKT in the context of particular mathematics content and teaching practices, this paper presents a broader analysis across all three studies of the ways in which the tasks used in each study shaped what could be learned about MKT from the tasks and interviews in each study. Each task and its associated responses were analyzed around the ways in which the pedagogical context and details of the task, as well as the type of response participants were asked for, shaped the facets of MKT that were made visible.

Results

The four findings that emerged from the analysis are described below with examples.

Task Type

Different types of tasks can allow different facets of teachers' MKT to be made visible. For example, several tasks from the second study asked participants to evaluate student work. For each task participants were asked to craft their own response. Because of the open-ended nature of the task, participants were more likely to draw on knowledge they would use in doing this task in their own teaching. In contrast, one of the tasks from the first study asked participants to evaluate incorrect student work and determine the student's error. This task provided four options for participants to choose from. By providing the possible student misconceptions, the knowledge that could be seen was constrained to the choices given.

Although intended to reveal participants' knowledge of student thinking, a task that asks participants to evaluate student work can surface gaps in participants' mathematical knowledge. For example, one task asked participants to evaluate a student's construction of a geometric figure. The student work was missing a description of how one component of the figure was drawn, making it incomplete, but not incorrect. A few of the participants made mathematically incorrect statements about the student's final figure, revealing gaps in their own mathematical understanding. If the participants had instead been asked to construct the figure themselves, this knowledge may not have been revealed.

Task Structure

Small changes in the tasks can reveal or obscure facets of teachers' knowledge. For example, one of the early task designs in the third study presented participants with ten problems from which to select examples for teaching simplifying rational expressions. In pilot interviews, participants spent a significant amount of time solving all of the problems, which made visible their mathematical reasoning. However, the work they engaged in was less authentic to the work of teaching and the knowledge used in doing the task was not reflective of how they might actually select examples in their own planning. The task was revised to present participants with a list of 30 problems. The list was similar to one teachers might find in their textbooks or online, and also included a list of answers. Participants who were presented with the later version of the task began by evaluating the features of the problem, including the final answer. While many of the participants worked through a few of the problems after they had selected them, none of the participants solved all of the problems. By changing the task, the work participants did in solving it more closely resembled the work of teaching, providing a better window for viewing MKT.

Another example comes from the second study. One of the tasks presented participants with student work on a permutation problem. Participants were first asked in writing what they noticed about the student's work. They were also asked to provide feedback to the student and how they might wrap up the problem with the whole class. By asking participants in writing what they noticed before they provided written feedback to the student, the task was unknowingly focusing participants on aspects of the student work they might have otherwise overlooked and distorted the ways in which they would actually engage this work in practice. Removing this question in written form provided a better understanding of how participants evaluate student work and provide feedback on it.

Mathematical Detail

One task from the first study presented a problem that involved finding the measure of an inscribed angle given other known angle measures. Four student solutions were given, all of which ended with the correct value, but provided different levels of detail. Participants were asked to determine which solution was incorrect. One solution made a mathematical jump, associating the value of two unrelated angles. The other answers used correct mathematical reasoning to reach a correct answer, including dividing the measure of the central angle in half to find the measure of the inscribed angle that intercepts the same arc. However, none of the students mentioned that the measure of the inscribed angle was half the measure of the arc it intercepts. Because the problem focused on inscribed angles, several participants interpreted these responses to be incorrect, even though they were designed by researchers to be correct.

Context Unrelated to MKT

Other details in the problem context can cause teachers to respond to a task without drawing on the MKT the task intended to elicit. On one task, participants attributed an incorrect response to the verbal mode of communication instead of the student's underlying misconception. In another case, participants chose an activity based not on the underlying mathematics, but on the tools students would use in the activity.

Conclusion

Creating tasks to study MKT is not straightforward. The types and structures of cognitive interview tasks, as well as the pedagogical contexts in those tasks can affect the facets of MKT that are made visible. Other task features also matter. Tasks that most closely resemble the work of teaching have the potential to better probe MKT. When participants respond in unexpected ways to a task, small changes to better approximate the work of teaching may enable participants to draw on their MKT as they do in teaching. Further, teachers' content knowledge may be better accessed indirectly through a teaching situation than through mathematics problems. While there is much left to consider around methods for studying MKT, the methodological considerations discussed in this paper provide a starting point for understanding how best to investigate MKT.

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EXAMINING PROSPECTIVE TEACHERS' ABILITY TO NOTICE AND ANALYZE EVIDENCE OF STUDENTS' MATHEMATICS LEARNING

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One way to strengthen the impact of teacher education, and thus address an enduring challenge in improving education, may be to teach prospective teachers (PTs) to systematically study and improve their teaching. In particular, a key skill for teachers is to appropriately notice and evaluate evidence of student understanding. In this study, we examine PTs' analyses of student learning in a lesson transcript, and compare these analyses across populations. Results indicate that prospective secondary teachers conducted higher quality analyses than prospective elementary teachers, but both populations of PTs overestimated students' understanding within the lesson transcript. Implications of this study suggest a link between mathematical knowledge and analysis skills, and indicate the need to explicitly teach analysis skills to PTs.

Keywords: Teacher Education-Preservice; Teacher Knowledge

One of the most persistent and enduring challenges for teacher educators and educational researchers is the difficulty of designing teacher education programs which produce strong and long-lasting positive effects on their graduates. Multiple researchers have suggested that one way teacher education can have long-lasting effects is by preparing prospective teachers (PTs) to be lifelong learners who become better teachers over time (Fieman-Nemser, 2001; Hiebert, Morris, & Glass, 2003). However, more research is needed on how to best achieve this goal.

One model for lifelong learning from teaching is composed of a cycle of four skills meant to be repeated systematically on a lesson or unit. This cycle, which is theoretically informed by the work of Hiebert, Morris, Berk, and Jansen (2007), is known as a *lesson experiment*. The skills of the cycle are intended to help teachers structure a personal or collaborative analysis of their own teaching in terms of its effectiveness in helping students meet specified learning goals. These skills are: (1) setting mathematical learning goals for students; (2) collecting and analyzing evidence of students' achievement of those mathematical learning goals; (3) developing cause-effect hypotheses that connect teaching to students' achievement; and (4) revising teaching based on the hypotheses to better improve students' opportunity to learn (Hiebert, Morris, Berk, & Jansen, 2007). Using these skills may help teachers both improve a lesson or unit and develop a mindset for thinking about teaching in terms of student learning (Phelps & Spitzer, 2012). Lesson experiments allow teachers to analyze and learn from their own teaching, systematically improving over the course of their career. Thus, helping PTs learn to conduct lesson experiments might be one way to strengthen the impact of teacher education programs. Learning more about how PTs might acquire and use the lesson experiment skills is an essential step in addressing the enduring challenge of designing effective teacher education programs.

Objectives of the Study

Research suggests that without explicit instruction, PTs hold a variety of misconceptions about analyzing teaching, and while they can enact some skills of the lesson experiment model, these skills are fragile and context-dependent (e.g. Morris, 2006; Spitzer, Phelps, Beyers, Johnson, & Sieminski, 2011). However, this research has mostly been conducted with prospective elementary teachers. Little work has been conducted to examine these skills in practicing teachers or in prospective secondary teachers, and no existing research compares these skills across different populations of teachers. This lack of knowledge limits our ability to successfully prepare teachers at all levels who will be lifelong learners.

In an ongoing research project, our goals are to begin to address these research gaps by investigating teachers' ability to analyze and evaluate evidence of student achievement of mathematical learning goals, and compare them across several populations. In particular, in this paper we will present the results of a quantitative analysis of prospective elementary and secondary teachers' evaluations of a sample lesson transcript in terms of student thinking and learning. The research questions for this analysis are: *How do PTs evaluate evidence of student understanding in a lesson transcript? What differences exist in the evaluations of evidence among elementary and secondary PTs?*

Methods

This project employed a mixed-methods approach in which participants completed an online survey including both open-ended and multiple-choice type questions eliciting their analysis of a sample transcript of a 5th grade mathematics lesson. This paper focuses on the quantitative portion of the study, while future analyses will investigate the open-ended responses.

Participants

Participants included PTs enrolled at either of two large comprehensive universities, one in the Mid-Atlantic region and the other in the Midwest. Participants were recruited through email, word of mouth, and announcements in classrooms, and received a token gift card as a thank you for participating. Fifty-four PTs responded to the invitation and completed the survey. This sample includes 36 prospective early childhood or elementary teachers (including 4 prospective elementary teachers with a specialization in mathematics), 5 prospective middle grades teachers with a specialization in mathematics, and 13 prospective secondary teachers with a specialization in mathematics. 87% of participants were female; 22% had completed at most one year of college, 58% had completed 2 or 3 years, and 20% had completed 4 or more years.

Measures and Analysis

To assess participants' evidence analysis abilities, we designed a task asking participants to read a transcript of a 5th grade lesson on comparing decimals using the conceptual ideas of decimal place value. The content was chosen specifically to be accessible to all participants (elementary and secondary) and because place value underlies so many important ideas of school mathematics. The learning goal of the lesson, which was prominently displayed at the beginning of the transcript, was "*Compare two decimals to the thousandths place based on meanings of the digits in each place* (National Governors Association, 2010, CCSSM 5.NBT.3.b)."

The sample lesson transcript featured teacher direct instruction, students working on problems individually, and whole-class discussions and was divided into six sections, each featuring a different student response or teacher action, and a lesson conclusion. Of the six transcript sections, two were designed to provide some evidence of student understanding (one of a misconception and one of some conceptual understanding) and the other four provided no evidence of student understanding of the learning goal. These included a student who used a procedure to compare decimals (appending zeros) but without evidence of understanding that procedure (Section A), a conceptual explanation by the teacher accompanied by student thumbs-up (Section B), a student who provides only the correct answer with no explanation (Section C), and a student who used an alternative method (converting to fractions and using a calculator) to compare decimals without reference to place value ideas (Section D).

Participants were directed to read the entire transcript first and then were presented with the sections individually. After participants read each section, they were asked "What can you tell from this section about how well the students understand the learning goal?" and chose from three options:

0: I can't tell from this section if the students understand the learning goal or not.
1: I can tell a little from this section about the students' understanding (or lack of understanding) of the learning goal.
2: I can tell a lot from this section about the students' understanding (or lack of understanding) of the learning goal.

Participants were then asked open-ended follow-up questions asking what mathematical ideas students understood (or did not understand) and what evidence from the transcript convinced them. Last, participants chose which part of the transcript they thought told them the most about student understanding. For this question, they could choose any of the six sections or the lesson conclusion, which featured mostly teacher talk with students nodding.

For analysis here, we focus only on participants' quantitative ratings of each of the four transcript sections that provided no evidence. A rating of "can't tell" was coded as a 0, "tells a little" as a 1, and "tells a lot" as a 2. Because each section provided no evidence of student understanding, lower ratings indicate a higher-quality analysis of the evidence (i.e., "can't tell" is the best response for each of these sections, and "tells a little" is better than "tells a lot").

Results

The average rating across all four sections was 1.36 (SD = 0.41), indicating that most PTs felt that these four transcript sections consistently told them something about students' understanding. Because these four sections were all specifically designed to provide no evidence of student understanding, this indicates that PTs are overly generous in their estimations of evidence. We first consider the kinds of evidence that PTs found most compelling. Table 1 shows participants' ratings for each of the four different sections.

Table 1: Participant's Ratings of Transcript Sections

	Percent of respondents choosing each rating			
Transcript Section	"Can't tell"	"Can tell a lot"	"Can tell a little"	Mean Numerical Score (*SD*)
A: Procedural Explanation	2%	74%	24%	1.72 (*.49*)
B: Teacher Instruction	35%	28%	37%	0.93 (*.80*)
C: Correct Answer	23%	36%	41%	1.13 (*.76*)
D: Alternative Method	6%	72%	22%	1.67 (*.58*)

From this table, it is clear that participants most highly rated those sections which contained a student explanation of some kind (either of a procedure or alternative method), even though neither of those explanations related to the mathematics of the learning goal (place value). In particular, only one participant recognized that a procedural explanation does not provide evidence of conceptual understanding. Participants expressed the most skepticism about a section containing only teacher instruction, but even for this section, 65% of PTs believed that the transcript provided at least a little evidence of student understanding.

Interestingly, when directed at the end of the task to choose the section which told them the most about student understanding, participants looked more favorably on teacher instruction and non-verbal student responses as evidence of learning. 20% of participants chose Section B as telling the most, and another 17% chose the lesson conclusion, which also included mostly teacher talk and students nodding. Only 20% of PTs chose one of the two sections which did provide some evidence of student thinking.

We were also interested in the differences that might exist in evaluations between different populations of PTs. The data suggests that prospective secondary (middle and high school) teachers demonstrated significantly higher quality analyses of this transcript than prospective elementary (and early childhood) teachers (Primary $M = 1.49$ ($SD = 0.32$), Secondary $M = 1.10$ ($SD = 0.45$), $p < .01$). It is likely that this difference is related to higher levels of mathematics knowledge among prospective secondary teachers; future analyses of this data will investigate this relationship more closely. We also hypothesized that PTs who had completed at least one methods course would conduct higher-quality analyses than PTs who had not. However, this relationship did not prove to be the case, as there was no significant difference found (Methods $M = 1.37$ (SD = .34), No methods $M = 1.35$ ($SD = 0.49$), $p = .88$).

Discussion

The PTs in this sample generally overestimated the evidence of student learning, particularly in cases where the student gave some kind of explanation. Those PTs who had completed a methods course performed no better than other PTs, suggesting that general knowledge of pedagogy does not automatically improve evidence analysis skills. However, secondary PTs did better than elementary PTs, suggesting mathematics knowledge may matter in analyzing evidence. Overall, these results indicate that more explicit instruction in evaluating evidence of student learning in mathematics may be needed. Future research is also needed in order to better investigate the link between mathematics knowledge, pedagogical knowledge, and the ability to analyze evidence. The results of this study provide a first look at one method for responding to the enduring challenge of effective teacher preparation. This research, and future research, will improve our ability to give all teachers the skills needed to become lifelong learners of teaching.

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MATHEMATICAL KNOWLEDGE FOR TEACHERS: OPPORTUNITIES TO LEARN TO TEACH ALGEBRA IN TEACHER EDUCATION PROGRAMS

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Preparing to Teach Algebra (PTA) Project gathered data about opportunities to learn to teach algebra provided by secondary teacher education programs. In sharing our findings, we use structures from the Mathematical Knowledge for Teaching (MKT) framework (Ball & Bass, 2003). The MKT framework is one way of situating the ways in which teacher education programs support preservice mathematics teachers in developing necessary knowledge types for teaching algebra. Using the MKT framework as an analytical framework is not straightforward, however. The framework must be interpreted and individualized to the study of algebra and to the opportunities to learn to teach algebra that we found. We present our struggles, the evolved framework, and our results in this session.

Keywords: Teacher Education-Preservice; Mathematical Knowledge for Teaching; Algebra and Algebraic Thinking; Instructional Activities and Practices

Introduction

Teacher preparation programs should provide opportunities which support the development of specific types of knowledge needed for teaching mathematics, especially at the secondary level (e.g., McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). This study is part of the *Preparing to Teach Algebra (PTA)* project which explored opportunities provided by secondary mathematics teacher education programs to learn about algebra, teaching algebra, issues of equity in algebra context, and algebra, functions, and modeling standards described in the Common Core State Standards for Mathematics. For this presentation, we address the question: *What types of knowledge do secondary mathematics teacher preparation programs support when they provide preservice teachers (PSTs) with opportunities to learn to teach algebra?*

Theoretical Framework

Ball and Forzani (2009) suggested that teacher preparation programs, including those preparing mathematics teachers, should change their curricular focus from what teachers know and believe to "the actual tasks and activities involved in the work" (p. 503). In exploring opportunities provided to preservice algebra teachers, we face a challenge to situate opportunities within a particular theoretical framework. The MKT framework offers a categorical system of knowledge types needed for successful mathematics teaching and focuses on knowledge in practice (Ball & Bass, 2003). Several researchers have made efforts to measure MKT outside of practice, suggesting that knowledge development can be supported during teacher preparation (e.g., Herbst & Kosko, 2014). In this presentation, we share our use of the MKT framework (e.g., Ball, Thames & Phelps, 2008) to better understand how intended opportunities to learn to teach algebra might support knowledge for teaching algebra.

Method

PTA project collected data from three secondary mathematics teacher education programs in the form of interviews with instructors of required mathematics and mathematics education courses at each institution, along with relevant course materials. We focused our current analysis on opportunities to learn to teach algebra, coding data in pairs of researchers. In this presentation, we analyze opportunities reported by instructors and found in their instructional materials to investigate aspects of MKT included in courses at each institution and across the three programs.

Results

In the following sections, we discuss our operating definitions of different aspects of the MKT framework. We present some preliminary results from our use of the MKT framework. Here, we focus on five aspects of MKT: *specialized content knowledge, horizon content knowledge, knowledge of content and students, knowledge of content and teaching,* and *knowledge of content and curriculum* (each defined below).

Specialized Content Knowledge (SCK)

Specialized content knowledge (SCK) can be described as knowledge that teachers need to unpack mathematical concepts and to interpret student work, especially in terms of unique approaches or solutions, that similarly qualified persons outside of mathematics teaching may not need (e.g., unpacking and relating the area and volume formulae for different shapes) (Ball et al., 2008). SCK enables teachers to delve into content from multiple perspectives, allowing them to explain concepts in multiple ways or follow students' reasoning (Ball et al., 2008).

For the purposes of this study, we refer to SCK learning opportunities as including: (a) unpacking particular aspects of mathematical concepts (including derivations, history, and proof), (b) evaluating and interpreting student arguments and solutions, (c) recognizing and valuing alternative solutions from instructors and colleagues, and (d) exploring teaching-related mathematical ideas. For example, a History of Mathematics course instructor at Kappa University expressed her desire for PSTs to appreciate the complexities present in many of the algebraic ideas they take for granted, and how historical struggles can be predictors of student struggles. She gave them opportunities to investigate these struggles by studying the process and timeline taken to develop and standardize concepts such as function notation. As the instructor noted, “Some of these early algebraic things are ideas that are really quite sophisticated and it makes sense for students to stumble.” We classified these opportunities as SCK because of the opportunities for PSTs to learn about the history and complexity of algebraic concepts and notations which could impact student understanding.

Horizon Content Knowledge (HCK)

According to Ball et al. (2008), horizon content knowledge (HCK) refers to teachers’ understanding of how particular mathematical content is connected throughout the K-12 curriculum and to college mathematics. For our analysis, we focused on opportunities PSTs had to learn about how middle school mathematical topics (e.g., variables) can be used in high school topics (e.g., exponential functions) or how college mathematics relates to secondary content.

For example, a Discrete Mathematics instructor at Kappa University wanted to provide mathematical knowledge for PSTs to see where the secondary content fit into the larger mathematical landscape. For instance, the instructor mentioned that PSTs learned when it was appropriate to allow complex number solutions to quadratic equations. We coded this example as HCK because the instructor described how teaching algebraic content varies at different levels.

Knowledge of Content and Students (KCS)

In Ball et al. (2008), Knowledge of Content and Students (KCS) is described as "knowledge that combines knowing about students and knowing about mathematics" and specifically supports teachers in tasks that require "an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking" (p. 402). Ball et al. described elements of KCS as including knowledge about students' thinking, interests or motivations, as well as common student conceptions and misconceptions.

In looking for examples of KCS from our data, we considered activities that emphasized anticipating or predicting with respect to student thinking, mathematical understanding, and motivation. For example, the Secondary Mathematics Methods instructor at Beta University emphasized gathering and interpreting data on student thinking: "Okay, you can't just gather information about right and wrong answers. So … in this system of linear equations, they were able to find the intercept. What does that mean? Do they understand what that means?" At Gamma University, the Algebra in the Curriculum instructor mentioned asking PSTs to look at written student work and anticipate what support the students would need by answering, "[H]ow can you support, from the written work what can you see as far as what are the [algebraic] habits of mind that those students need help with?"

Knowledge of Content and Teaching (KCT)

Ball et al. (2008) described Knowledge of Content and Teaching (KCT) as an important component of PCK; they proposed that this type of knowledge "combines knowing about teaching and knowing about mathematics" (p. 401). PSTs need opportunities to learn KCT in their secondary mathematics programs; we found evidence of such opportunities in our case study programs. For example, the Modeling in the Curriculum instructor at Gamma University included a textbook analysis as a course assignment. He asked PSTs to choose a textbook and find where matrix multiplication was addressed in the book. PSTs then discussed, "How are the different books doing it? What are the advantages, disadvantages?" This activity required students to grapple with both mathematical and pedagogical ideas in order to make decisions about both how to introduce particular mathematical topics and how to sequence concepts or activities associated with the topic. The Probability and Statistics instructor at Beta University emphasized that she provided her students with multiple opportunities to visualize and interpret data. She modeled this as a pedagogical strategy for those students who will be teachers as their future students will likely benefit from seeing data in various ways as well. Again, this strategy engages PSTs in what Ball et al. (2008) refer to as "an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (p. 401).

Knowledge of Content and Curriculum (KCC)

From Ball et al. (2008), KCC is the category of PCK focused on issues related to mathematics curriculum, specifically knowledge of the available programs and instructional materials designed to teach a certain topic at a particular level. KCC also includes considerations related to the affordances and complexities of selecting curriculum within a determined context. According to Shulman (1986), there are two additional types of curricular knowledge: lateral curricular knowledge and vertical curricular knowledge. The former is about the knowledge of the curriculum taught in a different subject area. The latter is related to the knowledge about the curriculum in the same subject area that has been used during previous years, as well with that that will be used in subsequent years.

For example, the Modeling in the Curriculum instructor at Gamma University reported that "[o]ne of the things that we do, actually when we do matrices. I have them each grab an Algebra II or secondary math textbook, I don't know if you have seen the math ed room, but there's tons of books

(...), so they kinda grab books and we discuss how are the different books doing it, what are the advantage, disadvantages."

Discussion

We are investigating the use of the MKT framework to analyze our OTL data from courses at three secondary mathematics teacher preparation programs. Although the MKT framework is appropriate to analyze future teachers' OTL about different types of knowledge for teaching, utilizing the MKT lens to make sense of our data has not been a straightforward process. These areas of knowledge are not discrete (as acknowledged by its developers) and many instructors design OTL to address multiple areas in connection with one another. For example, we were unclear about the categorization of an OTL in Abstract Algebra at Kappa University. The instructor reported that he had students in the course teach each other the abstract content by going to the board to explain homework questions to each other, presenting alternative solutions to problems, and working in groups in class. An argument for including this example in KCS is that by teaching each other the content, PSTs may be developing their abilities to anticipate what their classmates know or will struggle with and that may translate to their future students. On the other hand, the PSTs are not explicitly supported in anticipating their classmates' (or future students') thinking, motivation, or struggles. Therefore, we tentatively decided to categorize this OTL as SCK rather than KCS. Textbook analysis activities also proved challenging to categorize. They might seem to be obvious cases of KCC given their direct relationship to curriculum. However, we found instances in which this categorization was less clear. The example described in the KCT section above serves to illustrate this difficulty, as the students analyzed textbooks in order to investigate treatment of matrix multiplication and to consider advantages and disadvantages of different approaches; this analysis seemed more closely related to the descriptions provided for KCT than KCC.

In this session, we will describe analytic challenges inherent in using the MKT framework, present our evolving conceptions of the MKT categories, and share preliminary analysis results. We will engage our audience in discussions about our definitions, analytic decisions, and next steps in our analysis. We will share preliminary findings from three teacher preparation programs, which will help other teacher educators strengthen their programs. This presentation will help researchers who plan to use the MKT framework as an analytical framework for types of knowledge of algebra teachers.

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CAN I TEACH MATHEMATICS? A STUDY OF PRESERVICE TEACHERS' SELF–EFFICACY AND MATHEMATICS ANXIETY

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This paper presents two studies (qualitative and quantitative) with the shared goal of exploring preservice teachers' (PSTs') experiences of mathematics anxiety and self-efficacy for mathematics teaching. Findings indicate that PSTs experience high levels of mathematics anxiety, impacting current learning and preference for teaching the content, as well as the development of self-efficacy for teaching mathematics and conceptions of ideal teaching. Findings regarding anxiety (fear) of evaluation and concern about being able to inspire students in their future classrooms converged across studies.

Keywords: Mathematical Knowledge for Teaching; Teacher Education–Preservice

Mathematics education researchers agree that a confident and competent mathematics teacher is a vital necessity in the classroom (Oswald, 2008). Yet many elementary preservice teachers (PSTs) and in-service teachers experience relatively high levels of mathematics anxiety (e.g., Beilock Gunderson, Ramirez, & Levine, 2010; Swars, Smith, Smith, & Hart, 2007), and have histories of confronting anxiety and feelings of failure in mathematics during their own K-12 experiences (McGlynn-Stewart, 2010; Sloan, 2010). Further, definitions of mathematics anxiety suggest a relationship between anxiety of failure and self-beliefs such as self-efficacy (Trujillo & Hadfield, 1999). In efficacy theory, mathematics anxiety may be a source of physiological arousal that impedes the development of positive self-efficacy. Importantly, Lee (2009) found in a study of PISA results that mathematics anxiety and self-efficacy for mathematics were independent predictors of mathematics performance, indicating the need to not only measure them separately, but to understand how they differentially impact mathematics learning and achievement.

Thus, in this study, mathematics anxiety is investigated as a potential source of negative physiological arousal that impacts, but is not synonymous with, self-efficacy for teaching and learning mathematics. To the extent that PSTs feel anxiety when contemplating teaching mathematics (or learning mathematics content and pedagogy deeply enough to be effective in teaching mathematics), their experience of self-efficacy for teaching mathematics is negatively impacted (Bursal & Paznokas, 2006; Swars, Daane, & Giesen, 2006). However, there are some indications that experiences gained in mathematics education methods courses can help decrease mathematics anxiety (Gresham, 2007; Vinson, 2001). Yet little is known about what aspects of courses are efficacious in reducing mathematics anxiety and how these effects can be maintained. Thus, we undertook both a quantitative and qualitative exploration of mathematics anxiety and self-efficacy experienced by PSTs with the goal of further understanding how these perceptions impact PST education.

Methods and Data Sources

This paper reports on both a qualitative and quantitative study to explore different aspects of PSTs' experiences of mathematics anxiety and self-efficacy. PSTs were recruited while enrolled in their teacher preparation programs at a southwestern university. The PSTs who participated in the qualitative study (n = 3*)* were recruited during the 2010-2012 academic yearswhile participating in a larger ongoing research project, [TEACH MATH] that follows PSTs from their preparation programs and into early career classrooms. PSTs in the quantitative study (n = 53) were recruited during the

2013-2014 academic year. Participants in both studies were primarily interested in teaching early elementary (39.6%) or upper elementary (45.3%) grade levels and thus planned to teach all core subjects, including mathematics.

For the qualitative study, the participants (Estelle, Phoebe, and Roxanne) were selected because their mathematics autobiographies (an assignment written for their mathematics methods course) spoke clearly and powerfully about feelings of mathematics anxiety. Additional data such as individual and group interviews as well as semi-structured prompts were collected over the course of three semesters during their methods courses and student teaching. An iterative analysis (Bogdan & Biklen, 2006) was used to demarcate the narratives that pertained specifically to mathematics anxiety and self-efficacy. For each participant, narratives were identified within text passages that included key words specific to anxiety and self-efficacy. An emergent coding scheme (Marshall & Rossman, 2006) was utilized to organize and sort each participant's narratives.

For the quantitative study, PSTs were recruited from teacher education courses to participate in an evaluation of online teacher professional development for elementary mathematics. Quantitative anxiety measures [based on the Hopko (2003) and Hadley & Dorward (2011) revisions of the *Mathematics Anxiety Rating Scale - Revised*], self-efficacy measures [based on the *Mathematics Teaching Efficacy Beliefs Instrument* by Enochs, Smith, & Huinker (2000)], subject area preference measures, and amount of vicarious experience PSTs had in classrooms were collected at pre-post. This study reports on baseline measures (prior to exposure to the professional development content).

Results

Qualitative Findings

All three of the women viewed mathematics as a content area to learn as students and to know as teachers, but one that held little appeal for them. Estelle saw mathematics as something that she could not escape (i.e., that mathematics was "everywhere"). Phoebe found mathematics to be a subject area that she *strongly* disliked and believed she was just not a "math person." As a result, she often reached for the comfort of her non-math status to diminish others' high expectations of her in this content area. Roxanne viewed mathematics as "a sore subject." Having spent her lifetime trying to pass mathematics classes despite not understanding the content, Roxanne had few ideas about mathematics other than mathematics was a subject to be endured. In other words, all three women spoke of mathematics as a requirement in their academic and professional lives, but not as a rich discipline they looked forward to learning or teaching. Based on the patterns of findings over three semesters of data collection, PSTs who experience high levels of mathematics anxiety and low perceptions of self-efficacy can learn to gain an understanding of the content, but they often limit (self-handicap) their opportunities to expand their mathematics understanding.

Further, the PSTs' experiences with mathematics anxiety and self-efficacy shaped their views of the ideal mathematics teacher. For Estelle, the ideal mathematics teacher was one who would never embarrass students in front of the class, who would be truly interested in all students' mathematical thinking (not just the smart students), and who would hold high expectations for all students. In other words, the best mathematics teacher would not let students hide behind a "wall" of feigned understanding; nor would she make them wish they could. For Phoebe, the ideal mathematics teacher was one who would explain mathematics concepts and problem-solving methods in multiple ways and who would thus create a learning environment in which there were multiple ways to be a "math person." For Roxanne, the ideal mathematics teacher was one who would provide opportunities for students to actively engage in mathematics, regardless of their mathematics ability so their focus would not be on just "getting through" the content. Thus the findings indicate that experiences in learning mathematics and learning to teach mathematics influence how future teachers imagine the qualities of the ideal mathematics teacher.

Quantitative Findings

In general, PSTs rated their mathematics learning anxiety and mathematics teaching anxiety similarly ($h = .04$), with the average scores on both scales falling between "a little" and "a fair amount" of anxiety in response to items. On both scales, anxiety was highest for items set in an evaluative context. In contrast, efficacy beliefs were quite high for both learning and teaching, although PSTs felt higher efficacy for learning ($h = .16$). Item averages indicate that students tended to fall between "agree" and "strongly agree" with statements about their efficacy to learn (with the lowest efficacy for learning to inspire students in mathematics), while they fell somewhere between "uncertain" and "agree" on statements about their efficacy to teach.

Based on efficacy theory, it was hypothesized that mathematics anxiety and vicarious experience in the program would predict PST self-efficacy, which would in turn predict preference for teaching mathematics. A mediation model(using PROCESS 2.11; Hayes, 2013) was found to be significant; $R = .55, F(3, 48) = 7.33, p < .001$, with $R^2 = .31$, and CI 95% [.12, .50]. Results indicated that the mathematics anxiety ($b = -.33, p = .001$) and vicarious experience ($b = .12, p = .017$) significantly predict self-efficacy, self-efficacy significantly predicts subject area preference ($b = .83, p = .013$), but anxiety ($b = -.44, p = 0.059$) and vicarious experience ($b = .05, p = 0.658$) do not significantly predict the outcome with the mediator in the model. Thus, self-efficacy appears to mediate the relationship between anxiety, vicarious experience, and subject area preference. The indirect effect of mathematics anxiety on subject area preference is -.27 (Boot 95% CI [-.64, -.08]), and the indirect effect of vicarious experience on subject area preference through self-efficacy is .10 (Boot 95% CI [.02, .25]).

Discussion

Based on the literature, it was hypothesized that PSTs would respond with relatively high levels of anxiety and low levels of self-efficacy for teaching mathematics as has been demonstrated in other studies. While anxiety was high across the studies (especially around contexts of evaluation and failure), efficacy was higher in the quantitative study than expected and inconsistent with the ways PSTs talked about their future teaching in the qualitative study. However, the PSTs in the quantitative study felt most efficacious about their ability to learn, rather than teach, the content. Given the relatively low level of real classroom experience these PSTs had, uncertainty is probably a healthy response and is certainly consistent with self-efficacy theory, which asserts that the efficacy of novices remains fluid until such time as individuals have had successful (or unsuccessful) mastery experiences upon which to base their beliefs. It is further consistent with the qualitative findings that suggested the PSTs were greatly concerned about their abilities to become good mathematics teachers.

An interesting finding across both studies was that PSTs were concerned about their ability to inspire students in mathematics. In the quantitative work, PSTs reported lower self-efficacy for helping their students become interested in mathematics, motivating them when they lost interest, and helping build connections with families to improve student achievement in mathematics. In the qualitative work, the "ideal" teacher is clearly represented as one who can help even anxious students feel comfortable in learning mathematics. Clearly, PSTs need more support to develop skills and confidence in motivating and inspiring students to be comfortable and engaged in mathematics.

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CONNECTING THEORY AND PRACTICE IN MATHEMATICS TEACHER EDUCATION: AFTER SCHOOL PROGRAMS AS PROFESSIONAL DEVELOPMENT CENTERS

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Meeting the goals of rigorous curriculum requires providing professional development (PD) that challenges existing notions of teaching mathematics. It requires nurturing shifts from teacher-centered norms to classroom norms centered on student thinking. This paper reports a PD design that used after school programs, and supported student mathematical thinking and a shift towards student centered practices. Effects on student learning, as well as on teacher practices were based on analysis of classroom videos, written artifacts, curriculum materials, and field notes.

Keywords: Teacher Education-Inservice; Elementary School Education; Design Experiments

Objectives

Good's (2010) report of normative classroom practices from 1968 to 2008 revealed that despite changes in the curriculum standards to emphasize discourse, critical thinking, and problem solving, the normative teaching practices continue to be teacher centered. Lack of significant changes in teacher practice show that PD opportunities have not been very effective. Students continue to underperform in critical thinking and problem solving areas. For example, the National Assessment of Educational Progress (2013) showed that from 1978 to 2012, number of 17-year-old students in US that demonstrated understanding of algebra and multistep problem solving skills has never been above 8%. Meeting curriculum goals is constrained by teachers' weak knowledge and the school environment (Ball, Thames, & Phelps, 2008), placing great demands for improvements of teacher education and support. As Guskey (2009) explained, "no improvement effort in the history of education has ever succeeded without thoughtfully planned and well-implemented PD activities designed to enhance educators' knowledge and skills" (p. 226). Furthermore, Marrongelle, Sztajn, and Smith (2013) insisted that "we need studies that open the black box of PD and provide rich descriptions of the nature of the work in which teachers engage that does or does not lead to improved knowledge, beliefs, or habits of practice" (p.209). The objective of this paper therefore, is to contribute to this area of high need by reporting a mathematics PD design.

Perspectives and Methods

Sztajn (2011) discussed the need for standards for reporting mathematics PD research. These standards are necessary because of the uniqueness of mathematics PD research and for clarity when reporting features that affect the effectiveness of the PD. Sztajn recommended reporting theoretical perspectives, goals, context, and its structure.

Theoretical Perspective

This study draws from Greeno's (2003) situative perspective of knowing. Development is a product of interaction of the individual, interpersonal, and the social systems. Acquired knowledge facilitates one's participation in a community, and because "individual mental structures certainly change as part of this learning" (Sawyer & Greeno, 2009, p.364), such knowledge can be transferred from one community to another. Development and practices are influenced by the norms of the social systems; normative teacher practices are therefore a reflection of the affordances and constraints of school systems. Implications of this include a focus on content, tools (e.g., instructional materials), discourse communities, and viewing teaching as learning in practice. Based on this framework, the situative research question for the current study is: What are the affordances and constraints of this PD design?

Program Goals and Context

The current mathematics PD had two interrelated primary goals. The first goal was to support elementary school students' mathematical understanding through engagement in generalizations and justifications about mathematical patterns. The second goal was to develop pedagogical and content knowledge through theory and practice.

Six schools and 12 teachers with 6-25 years of teaching participated. Three of the school were title 1 schools with 76.67% to 99.55% students classified as poor and about 30% students were below their grade level in mathematics. For the nontitle 1 schools, about 58% of the students were poor and 14% were below grade level in mathematics.

Program Structure

This program was structured into different recursive phases. Teachers worked on pattern finding activities like in figure 1 during PD. They discussed different student strategies, watched videos of students working on the tasks, analyzed student work, and discussed best practices to support students. The teachers met with the professional developers three times for 90 minutes each meeting during this phase. In the second phase, the teachers taught classes of about 15 students who were in grades 3-5 who voluntarily enrolled in this program. They taught twice a week for 90 minutes each time for five weeks. These lessons started with an outdoor physical activity and were followed by pattern finding activities. The teachers were provided with curriculum materials including lesson plans. Phase 3 was similar to phase 1. However, during this phase, teachers reflected on their classroom experiences, and discussed the affordances and constraints in their classrooms. Teachers also made recommendations to the professional developers on the structure and content of the activities for their students.

If one person sits on each side of a square in this pattern, how many people would sit around a train of 100 squares? How many people would sit around a train of *any* number of tables. Write your rule. How do you know your rule will always work?

1 2 3

Figure 1: An example of pattern finding tasks for On Track program.

This pattern of phases continued until the teachers had three teaching (five-week) sessions distributed over a year. Two interviews, at the end and during one of the teaching sessions, were conducted with each teacher. Recordings of the interviews, classroom activities and materials, were analyzed qualitatively and quantitatively to identify the constraints and affordances.

Results and Substantiated Conclusions

Student Learning

74% of the generalizations that students wrote were explicit (i.e. students wrote correct rules for finding the nth term of the pattern-finding activities) which in this study was considered the highest possible reasoning level. The percentage of explicit generalizations, however, differed from task to task. Multiple regression analysis showed a significant positive relationship between the student growth in the End of Grade state math exams and students' explicit generalizations from this study at a p value of less than .0001 (Maher and Berenson, 2013). That is, this program supported students learning within it, and highly correlated with performance in mathematics end of year exams. Moreover, most teachers reported a positive change in students' persistence and development of habits of using different strategies.

Teacher Learning

Teachers reported that doing the math activities in the PD and teaching with the same task contributed to their knowledge more. Teacher Jane said:

> I am (now) able to see and develop what kids might do, what problems kids might run into, and how they might think. It is also interesting to me because I am like maybe they will come up with something different from the way I approached it.

Teachers reported a shift from teacher-centered practices and notions in which activities are very structured and questions funnel student responses. Teacher Mary reflected: " *I have learned doing this program that too often we tell the kids instead of asking them and giving them time to arrive at the answer*." Further, more, Liz felt that

> The program has been an interesting experience, just teaching math in a way that we do not teach in schools where you are not rushed to make them come up with an answer. It is like just do what you are capable of doing and just pushing kids but not like leading them, but just letting them do it for themselves.

The shift towards student-centered teaching was also attributed to the affordances of the after school settings especially the flexibility of how much time teachers could allocate to activities. On reflecting on her practice, teacher Deb said:

> I think I have changed from the beginning of the program to this last session because at first I was so used to having a long list of objectives. You got to cover this, you have got to do this real quick, so I think I was pushing a little bit with the kids at first. But I have come around to a little bit of –here it is, see what you can do with it and just asking them questions and not trying to lead them.

Observations aligned with reports of changes in teaching practice. Teachers became more at ease with practices such as encouraging students to question each other, requiring students to make their reasoning accessible to their peers, and encouraging finding relationships between different strategies. However, teachers did not develop some critical habits easily. One notable difficulty in the change of teaching practices which the teachers reported and the research team observed was the development of habits of creating or taking up opportunities to ask for powerful justifications. The teachers discussed the likely student justifications, appreciated the justifications with more explanatory power for all cases, and identified critical questions that will stimulate students to think of powerful justifications. For example, for the task in in Figure 1, the rule for n number of tables is $2n + 2 = s$ where s is the number of seats. Asking how the variable $2n$ and constant $+2$ connect to the question was one of the critical questions. However, teachers tended not to ask the critical questions but instead asked, "how did you get your answer" as the main way of asking for justification. As a result, describing the steps for the strategies was the most socially accepted justification scheme in these classrooms.

Significance

PD remains at the center of reform efforts, especially now as we transition to the Common Core State Standards. There is a critical need to learn what and how teachers learn in PD settings, and the impact on student learning. This study contributes to this need by reporting the PD design and it's impact. Classrooms already have ways of teaching and doing mathematics that may not support PD goals or reform efforts. This PD design removes teachers from these settings and support them in afterschool settings in which teachers can flexibly focus on student thinking, and re-establish norms and notions to develop teaching habits that align with reform goals without some constraints of schools and regular classrooms. Of course, one important goal of this PD is for teachers to transfer their PD teaching practices to their regular classrooms.

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FACTORS INFLUENCING SUSTAINED TEACHER CHANGE: A COLLECTIVE CASE STUDY OF TWO FORMER PROFESSIONAL DEVELOPMENT PARTICIPANTS

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This study considers professional development (PD) effectiveness by examining sustainability of teachers' change. The instrumental collective case describes two teachers' classroom instruction three years after the completion of a PD focused on mathematics discourse. A strong alignment between the PD's goals with the teachers' school culture was one factor in the teachers' continued use of elements from the PD. Teachers also continued to emphasize core elements of the PD, including valuing student talk as a way of mediating student thinking.

Keywords: Teacher Education-Inservice; Elementary School Education; Classroom Discourse"

Although 99% of US school teachers reported attending professional development (PD) in 2011 (NCES, 2013), it is unclear how teachers engage with professional learning experiences and change practice over time. Current work studying PD effectiveness suggests effective PD is sustained over time, focuses on content, and occurs within a community of practice (Garet et al., 2001; Boston & Smith, 2009; Desimone, 2009; Louis, Marks, & Kruse, 1994). Yet many current measures of effectiveness only examine teachers while they are participating in the professional development. Those measures of effectiveness fail to examine the lasting effects of professional development over time.

This study joins an emerging field of work that addresses an often neglected measure of PD effectiveness: *sustainability* of teacher change. For the purposes of this study, I adapt definitions from Fishman et al. (2011) and Franke (2001). *Sustainability* is defined as a teacher's continued use of the intervention in ways congruent with developers' intent after resources and support for such practices have been withdrawn.

Others' work in sustainability shows what features of PD result in sustained teacher change. Preliminary results suggest these features include teacher group cohesion, communities of practice, teachers' perception of the usefulness of the intervention, alignment of the PD with current practice, and alignment with other stakeholders' expectations of instruction (Hodges & Cady 2013, Fishman et al., 2011; Franke et al., 2001; Coburn, 2003; Gutierrez & Penuel, 2014).

This instrumental collective case study (cf. Stake, 1995) describes two teachers' mathematics instruction three years after the completion of a professional development program. It seeks to answer the overarching questions: In which ways did teachers sustain PD ideas into their practices or not? From the teachers' perspectives, what factors affected sustainability of the professional development ideas? Specifically, what classroom aspects of Project AIM have been sustained? What grade level team aspects of Project AIM have been sustained? What aspects of Project AIM beyond the second grade team have been sustained?

Theoretical Framework

Professional development intends to facilitate teacher change over time. Researchers have examined teacher change within the context of creating and adapting new knowledge (Franke 2001). I adapt a Vygotskian framework, like that described by Gallucci, Van Lare, Yoon, & Boatright (2007). It includes the following steps:

- Individual *appropriation* of particular ways of thinking through interaction with others
- Individual *transformation* and ownership of that thinking in the context of one's own work
- *Publication* of new learning through talk or action
- The process whereby those public acts become *conventionalized* in the practice of that individual and/or in the work of others (p. 296)

Methodology

As this is a sustainability study, I provide a brief description of Project AIM, the initial professional development program in which the participating teachers engaged. During the 2011-2012 school year, 26 teachers participated in a year-long, forty-hour professional development. The professional development was offered in a large Southeastern U.S. school district. Conditions of participation in the PD included teaching second grade and attending in school teams. The professional development had a focus on "math talk" or mathematics discourse.

Study participants were selected from the pool of 26 teachers who formerly participated in Project AIM in the 2011-2012 school year. By design, the PD required teachers to attend with colleagues from their grade-level school teams. In order to examine teacher group cohesion, this study follows two teachers who work together. These two teachers from Barnes Elementary were members of a group of three who attended PD together and continued to teach together in second grade after the completion of the Project AIM PD. These teachers represent two individual case studies.

Data sources for the study consist of a set of baseline data from the 2011-2012 PD year and follow-up data collected during the 2014-2015 school year. The PD year data include written pre- and post- beliefs questions and a math lesson artifact package for each teacher. The follow-up data include a follow-up math lesson artifact package, a semi-structured interview, a team focus group, and two lesson cycle observations for each teacher. Data analysis involved an iterative approach outlined in Decuir-Gunby et al. (2011), where theory, data (the participants' own words), and research goals (interest in sustainable elements of professional development) inform the code development.

Results

Barnes Elementary follow-up

Nora and June teach together at Barnes Elementary, a school with strong emphasis on Paideia seminars. Teachers at the school regularly engage their students in group discussions across subjects.

Nora. At the time of the professional development, Nora, a white female, was in her second year of teaching second grade. She earned her Bachelor of Science degree in Elementary Education. Nora valued students' comfort level when describing important attributes of math talk. By the end of the PD, she had some specific strategies to establish that comfort: building up strategies throughout the year and using question stems.

Since the completion of the PD, Nora has engaged in many additional professional learning activities. She has completed a Masters degree in Education. She is active in the National Paideia Center by attending conferences and presenting seminars. She has also taken on a number of leadership roles in her county and school; she writes and reviews math lessons for the county, while her school leadership includes serving on her school's magnet committee and serving as grade chair.

June. At the time of the professional development, June was in her eleventh year teaching at Barnes. Of her ten previous years teaching, three were in third grade and seven were in second grade. She earned her undergraduate degrees in Elementary and Special Education. At the beginning of the PD June valued student comfort as the most important aspect of math talk. By the completion of the

PD, she expressed other aspects of math talk as more important: questioning, high-level tasks, and active listening.

Since the completion of the PD, June has continued to actively work in her school mentoring newer teachers. She is passionate as she discusses her continued love for second grade. She has attended multi-disciplinary conferences that align with her school's Paideia scholarship.

Cross-case Analysis

Initial results suggest that there are multiple aspects of the professional development still being implemented and adapted three years after the completion of the PD. Some of these are found in the following table:

What aspects of Project AIM PD have been sustained in participating teachers' classrooms?

Sustained Aspect of the PD	Example
The PD introduced some specific strategies to begin to scaffold student talk in the classroom	Teachers learned a strategy called "Bet Lines" during the PD and have continuously used that strategy in much the same way as it was presented in the PD.
The PD emphasized careful planning for instruction, including selection of high-demand tasks.	One teacher now critically examines "challenge" tasks in curriculum materials, selecting and modifying tasks so that they will be of high cognitive demand.
The PD emphasized that small group work is a time when more students have the opportunity to participate.	One teacher reflected that her students with limited language skills were more engaged in pair work than with whole group work. Rather than "checking out" during whole group discussion, the opportunity to speak with a partner promoted talk.

This presentation focuses on how teachers continue to value student talk as a way to reveal thinking. June and Nora both discussed the importance of student talk. June reflected on the PD's alignment with her school's philosophy: "[W]e're not necessarily a STEM school, but we do a lot of STEM thinking. So a lot of it is just constantly using the discourse and explaining your thinking and the kids questioning and so it's constantly kind of tied together in some ways."

There was evidence of this in June's classroom, as she encouraged discussion and critical thinking in all subjects, including math. During her classroom instruction, June pushed her students by using probing questions. She also scaffolded students' interactions by encouraging them to disagree respectfully or ask one another clarifying questions.

Nora also reflected on the importance of student discussion as a way to mediate understanding: "We are starting number sense now. You know, that's so important in second grade, the major work that we do, and I wanna make sure that every kid understands it, but sometimes you have to kinda let that go and hope that maybe they will understand it better when they're hearing conversation and hearing their peers share. You have to, kind of, take away some of the control from me and give it to the kids."

There was evidence of Nora putting these ideas into action in her classroom. During one classroom observation, students were assigned rotating roles for their small group work. (This was one strategy for talk presented in the PD.) One student's role in each group was the questioner. Nora provided the questioner with some possible questions to ask relevant to the activity. As she and her class discussed the lesson at the end of the class, Nora asked her class which of the three roles had helped the students best understand the math activity. Many expressed that being the questioner had helped them to understand the task better.

Conclusion

This study shows that a strong alignment of PD goals with teachers' administrative goals allows teachers to engage with and sustain professional development ideas long after the professional development has completed. In fact, teachers expressed that Project AIM fit naturally in to the work they were already, not feeling like doing the work of the PD was "just one more thing." Future professional development should consider alignment with current administrative goals in order to make their work relevant to teachers for years to come.

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TEACHER BUY-IN FOR PROFESSIONAL DEVELOPMENT: 4 DISTINCT PROFILES

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Professional development (PD) provides an opportunity for teachers to grow in their practice, but this growth is mitigated by teachers' buy-in for the PD. We examine factors that affect teacher buy-in for PD, creating profiles for 4 different levels of buy-in. Thematic analysis based on field notes from the first year of a sustained PD focused on improving mathematics instruction showed that individual teacher buy-in was affected by (a) the teacher's view of mathematics, (b) the teacher's view of their students, and (c) the teacher's perceived need for growth in their own practices.

Keywords: Teacher Education-Inservice; Teacher Beliefs

In this report, we explore different ways in which elementary school teachers interact with a high-quality, sustained, district-wide professional development (PD) for mathematics teaching, *MathematicsStudio PD* (Foreman, 2010). How teachers interact with a PD depends on various factors, including (a) their beliefs about mathematics, (b) their views of their students' mathematical capabilities, and (c) their beliefs about their own need to grow in their teaching practices.

Perspectives / Theoretical Framework

Darling-Hammond et al. (2009) synthesized research on PD in education, noting that "effective professional development is intensive, ongoing, and connected to practice; focuses on the teaching and learning of specific academic content; is connected to other school initiatives; and builds strong working relationships among teachers" (2009, p. 5). Likewise, in her summary of successful PD, Klingner (2004) advocates for long-term support for teachers rather than one-shot PD. Klingner notes the importance of a community of practice, of administrative and mentor support, and opportunities to observe demonstrations of the target practices. She lists the following factors as contributing to teacher buy-in for PD: (1) grassroots support from teachers (as well as support at administrative levels), (2) teacher involvement in planning of the PD, and (3) transfer of ownership from the PD providers to the teachers.

The PD in this study was designed to align with Darling-Hammond et al.'s (2009) characteristics of effective PD. However, the PD was district-wide and opt-in to participate in the PD happened at the school level rather than the individual teacher level, thus support from teachers was not grassroots in all cases as advocated by Klinger (2004). Given that school- and district-wide PD is a common feature of in-service teacher education, it is worthwhile to investigate teacher buy-in for PD that they did not select for themselves. Because buy-in for PD is linked to teacher implementation of research-based practices, understanding what affects buy-in may allow PD providers to have a stronger impact on the teachers they serve. For the purpose of this paper, we define *buy-in* for PDas active engagement with and support of the PD.

Underlying Principles of the *Studio PD*

The *Studio* PD advocates for student-centered classrooms where activity centers on mathematical sense making, reasoning, and understanding by all students. All students are to engage in discourse that focuses on sense making, justifying, and generalizing mathematical ideas. In this way, mathematics is not treated as a set of rules, but rather as an interconnected and logical structure (Hiebert, 1986). However, teachers have been shown to frequently view mathematics as a "fixed set of factors and procedures" (Smith III, 1996, p. 392). As noted by Smith, this view of mathematics often occurs in conjunction with teaching that is primarily prescriptive in nature where teachers tell the mathematics rather than have students explore and engage in mathematics themselves.

Beyond just seeing mathematics as more than a set of procedures, teachers must also see students as being capable of engaging in mathematical habits and practice. The professional development advocates for a growth rather than fixed mindset (Dweck, 2007). A teacher with a fixed mindset might feel that only their “strong” students were capable of engaging in mathematics. One has a fixed amount of intelligence and no amount of work will change that. A growth mindset approach changes focus from intelligence to effort where anyone is capable of learning and improving.

Underlying the PD is the assumption that teachers are reflective practitioners. That is, the teachers are asked to reflect on their teaching and consider how they might grow in their practice. However, this presupposes that teachers have a disposition for changing their teaching. We posit that teachers must feel there is a need to grow to engage fully in the PD.

Methods

Data for this project comes from a large-scale study looking at the efficacy of a 3-year PD in an urban school district. For this study, we focus on the first year of the PD, which began with a 3-day summer workshop on best practices for teaching mathematics and then included 5 two-day PD sessions situated within each school spread across the school year. Each of these two-day sessions included a morning of leadership coaching with the principal, an afternoon of PD with the studio teacher (a resident teacher who opened her classroom for facilitated observations by the other resident teachers), and a full day of PD with the studio teacher and other resident 3-5th grade teachers. The first year of the PD focused on the *Mathematical Habits of Mind and Interaction*(Foreman, 2010), a set of metacognitive skills for engaging with mathematical ideas and problems, in conjunction with the *Mathematically Productive Teaching Routines*(Foreman, 2010) which support student engagement in the Habits. Posters of the Habits were prominently displayed in each classroom.

Researchers observed each PD session and took detailed field notes at two case study schools. The PD sessions were also video-recorded and relevant sections transcribed for analysis. The field note data was analyzed using thematic analysis (Braun & Clarke, 2006). To examine the varying levels of buy-in and engagement among the teachers, the data was analyzed to look for trends across teachers.

Results & Discussion

Variations in teacher buy-in for the PD were linked with two primary characteristics: alignment of the underlying principles of the PD and the teacher beliefs about students and mathematics, and teachers’ own perceived need to grow. Table 3 shows four different types of buy-in based on these primary characteristics. While the profiles for high and low buy-in had clear-cut relationships with belief alignment, mid-level buy-in took a number of forms. We present two teachers with mid-level buy-in: Kim and Nina. Kim’s beliefs about mathematics and children aligned with the principles of the PD, however, her buy-in was mitigated by an inconsistent perceived need to grow in her practice. Nina’s beliefs and perceived need to grow fluctuate leading to varying levels of engagement and support of the PD. In this report we focus on the teachers’ perceived need to grow due to space limitations.

Table 3 Four profiles of teacher buy-in for PD, focusing on two attributes of teacher beliefs

Teachers	Cora	John	Kim	Nina
View of mathematics and their students	Aligned	Not aligned	Aligned	Conflicted
Perceived need to grow in their own practice.	Yes	No	Conflicted	Conflicted

Buy-In	High	Low	Mid	Mid

Kim can be described as seeing mathematics as a sense-making subject and a context for rich student discourse, viewing her students as capable of engaging in the Mathematical Habits advocated by the PD, and generally being conflicted about her perceived need to grow.

In terms of her perceived need to grow Kim often felt she was already engaging in the practices advocated by the PD (demonstrating low buy-in). By advocating for reflection and alterations in teaching, she felt the PD providers were challenging her status as a professional.

> I think most of things we are [already doing]. I feel like, … like we feel like [we are treated as if we] were incompetent or something, you know. [The PD should recognize] that we have to feel that some of these things we do! … it's not like right now we are all the sudden changing our teaching skills or our teaching strategies, you know.

However, Kim also recognized the PD as useful (demonstrating high buy-in) when she found that it provided her with tools that she saw as immediately applicable in her classroom to solve one of her problems of practice. For example, consider Kim's reflection on a discussion protocol discussed in the PD:

> Well, I think with this particular strategy [having students interpret and compare their responses with each other], it's great because ... you are made to look at someone else's problem and explain to the other person what it is. So, it makes them [the children] become truly the listener and then maybe by doing that you are clarifying … your idea or seeing things, you know, so they are processing.

In contrast to Kim, Nina waivers in her fundamental beliefs. She fluctuates between seeing mathematics as a set of rules or a sense-making subject and whether she sees her students as capable of engaging in the mathematical habits advocated by the PD or not. Her perceived need to grow is mitigated by her doubts around her mathematical ability and her doubts of her own efficacy when transitioning away from a *teaching as telling* model (Smith III, 1996) towards a student-centered classroom.

In terms of her perceived need to grow, Nina recognized some positive effects of the PD and she wavered between seeing teaching as something to be refined/improved and reverting to her "traditional" approach. She explained:

> I'm still old-school, I am learning it with the students. I think I need as much practice as they do. ... I've got to just make it work better next year. I'm practicing [to engage students in sense-making] and I see how it is working for everyone, especially the slower student who thinks math a little less quickly as others.

In this statement, Nina's conflicted beliefs are evident by her self-label as "old school" but are in conflict with Nina seeing that the changes advocated by the PD are working for students (demonstrating a high level of buy-in). While Nina appears to be positively approaching change, she also explains she reverts back to her prior teaching style when students are not meeting her expectations (low buy-in).

> Once they can't reach it [sense-making], I'm like, 'Okay, this is how we did it in 5th grade' because back in the day we didn't get to talk. We had to do, do, do – … So I show them how I did old-fashioned math that got us to Einstein, to build a rocket and everything else. Now, we're talking math and talking math and talking math and sometimes they don't get it through the language.

Nina engaged in the PD and attempted to make changes to her classroom. She responded positively to the potential effects of the new teaching routines and student mathematical habits when she saw evidence that her struggling students found success. However, when facing doubts of efficacy, she often abandoned the practices advocated by the PD.

Conclusions/Take-Away

In order for high quality professional development to be useful, the teachers receiving the PD must buy-into the PD and take on the work in their own teaching practice. This is particularly true in the case of district-wide PD which is not self-selected by each teacher. The teacher profiles provided insight into what teacher buy-in may look like and how individual teacher differences may lead to differing levels of buy-in within the same PD. It may be difficult to generate teacher buy-in if their beliefs about mathematics and related pedagogy do not align with those advocated by the PD. When aligned, teachers can look to the PD to support their teaching practice. When unaligned, teachers may choose not to incorporate the principles of the PD into their practice. Additionally, we found that a perceived need to grow in teaching practice was a mitigating factor for buy-in. If a teacher is willing to acknowledge problems of practice in her own classroom and she is willing to try new teaching techniques, then a district-wide PD can provide welcome support for growth. If teachers believe they are already doing well enough, they may have little motivation to engage with "yet another" district-wide PD. Therefore, we advocate that PD providers consider how to actively leverage teachers' own perceived needs when designing and enacting PD.

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PROSPECTIVE ELEMENTARY TEACHERS' PROFESSIONAL NOTICING OF CHILDRENS' FRACTION STRATEGIES

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60 PSTs engaged in the practice of professional noticing of children's mathematical thinking of one student's work in finding ¼ of a set of 8 baseball cards. While PSTs were mostly successful (> 65%) when examining the facets of professional noticing as individual skills, < 50% of PSTs were successful in mobilizing all three in a coordinated manner.

Keywords: Teacher Education-Preservice; Elementary School Education; Instructional Activities and Practices

Educational approaches have suggested effective instruction should build upon students' ways of understandings mathematics (NGA & CCSSO, 2010; NCTM, 2014). Professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010) is a construct that has been introduced and used frequently as a framework to understand the ways in which teachers attend to, interpret, and respond to children's mathematical thinking (e.g., Dietiker et al, 2014). Professional noticing has been described as a difficult practice to develop, but also as one that can be learned. In this paper, we present our analysis of PSTs' responses to a pedagogical activity designed to engage them in professional noticing within a fraction context.

Theoretical Frame

Our "Baseball Card Problem" activity design drew on the work of Jacobs and her colleagues (2010). Professional noticing is comprised of three related skills: attending to children's strategies, interpreting children's understandings, and responding to children. Our activity provided a sample of student work along with specific questions designed to examine PSTs' processes of professional noticing within the context of a fraction multiplication problem. Fraction operations have long been considered to be the most difficult topic for elementary teachers to both understand and teach effectively (e.g. Ball, 1990); similarly often demonstrate "limited understanding of the meaning of multiplication and division of fractions" (Armstrong & Bezuk, 1995, p. 87).

Methods

Data came from 72 PSTs from two different university sites during the 2011-2012 academic year. Each was enrolled in an elementary mathematics methods course jointly designed and delivered by a sub-set of the authors. Within the section of our course addressing making sense of and responding to student work, we have designed activities to support the development of PSTs' professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010); specifically in the area of responding by posing a subsequent problem (Tyminski et al., 2014). The Baseball Card Problem was set within Tracey's 4th and 5th grade, multi-age classroom, in which she posed the following problem:

> Dustin has ______ baseball cards. He gives ¼ to his friend. How many baseball cards did he give to his friend?
>
> 8 24 44 60 100 144

Part I of the activity addressed Liev's (pseudonym) work finding ¼ of 8 baseball cards (Fig. 1).

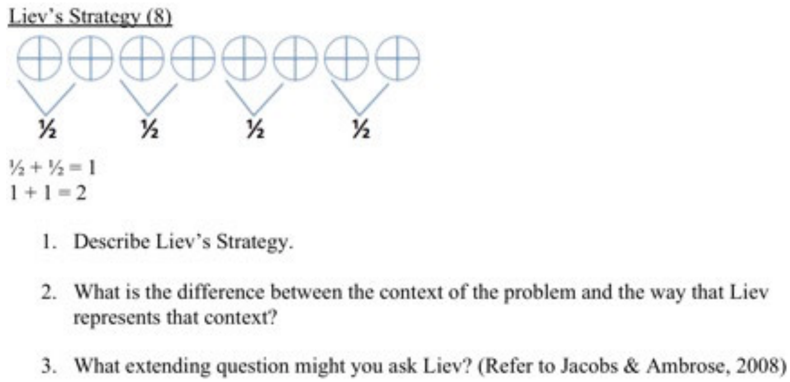

Figure 1: Liev's Written Response to the Baseball Card Problem

We associated question 1 with the skill of attending, question 2 with interpreting, and question 3 with responding. We specifically asked PSTs to respond by using an extending question, as outlined in a course reading, *Making the Most of Story Problems* (Jacobs & Ambrose, 2008).

Analysis

From the 72 PST responses, we selected a random sample of 12 from across the three sections of classes and included responses from both university sites. The authors utilized these 12 responses using open and emergent coding techniques (Strauss & Corbin, 1998) in order to develop and refine a coding scheme for the remaining 60 responses. The authors began by individually and independently reading through the 12 sample responses and creating a coding system for the responses to questions 1 and 2, which were negotiated and agreed upon. Question 3 was to be coded using *a priori* codes from Jacobs & Ambrose (2008). Once codes were established the second and third authors independently coded the responses for the remaining 60 PSTs. Reliability rates for each question were greater than 81%.

Results

We present results of PSTs' responses to the questions, including representative samples as well as a holistic view of PSTs' practice within this example of professional noticing.

Question 1

Question 1 asked PSTs to describe Liev's strategy. We coded PSTs' responses on a scale of 0-3. We next present descriptions of each code along with a representative example for each. All PST names are pseudonyms. Leah's response was coded as 0 as it was an incomplete description of Liev's work., "Liev has divided each of his baseball cards (which are round) into fourths and is giving one-fourth of each card to his friend". Erica's response was coded as 1 as it contained both correct and incorrect information.

> Liev is dividing each card into four equal parts. He then combines¼ of a circle to ¼ of another circle to show 2/4 of 2 circles is the same as ½ of 2 circles. He shows that ¼ of 8 circles will give him ½ of 4. He then joins the 4/2 to get 2.

Joan's response was correct, but omitted a key piece of information; it was coded as 2, while Angela's response was coded as 3 as it was correct, complete and precise:

> Liev draws eight circles to represent the eight baseball cards. He then divides each circle into fourths. Once he has divided the circles into fourths, he adds ¼ of the first circle to ¼ of the second circle and gets one half. Once he has done this for all of the circles, he adds ½ and ½ to

> get a sum of one. He then adds 1 and 1 to get a final answer of two. However, he does not take into account that Dustin gives away two WHOLE cards. By the work provided here, Liev makes it seem that Dustin is giving ¼ of each one of the eight cards to his friend.

Across the 60 PSTs, 25% of responses were coded as 0, 10% were coded as 1, 46.7% were coded as 2, and 18.3% were coded as 3. In all, 65% of PSTs' responses were scored 2 or above, indicating they had a complete or mostly complete and correct description of the student work.

Question 2

Question 2 was asked to support PSTs' interpretation of Liev's thinking beyond the fact he arrived at the correct answer. The context of the problem involves sharing baseball cards, but Liev's solution strategy involved cutting each card up into four equal pieces and giving his friend ¼ of each of the 8 cards. Although his answer is correct, his solution path does not make sense within the context of the problem. We coded PSTs' interpretation of this situation on a scale of 0 – 2, where 0 represented an incorrect interpretation, 1 indicated a correct interpretation with no detail concerning area models versus set models, and 2 indicated a correct interpretation with details. Michelle's is an example of one coded as 0:

> He is trying to find ¼ of 8 baseball cards, but when solving the problem it looks as if he took ½ of the baseball card to solve the problem but still looks as if he divided the 8 cards into four groups. The context of the problem was figuring out ¼ of 8 baseball cards. Therefore, it would make more sense if he would have divided the 8 circles into 4 groups and just took two from that instead of adding the ½ and ½.

Hazel's response was a correct interpretation, but contained little detail as to the mathematics of the situation, "He cuts up the baseball cards into ¼, but in real life you would not do that because then the baseball cards would have no value or make sense". We considered Norah's response an exemplary representation of a 2:

> The context of the problem is that Dustin is giving ¼ of his baseball cards to his friend. He is giving his friend a part of the set, not a part of each piece of his set. Liev does not divide the set into equal whole parts. He divides each part in the set into fourths. In other words, when giving baseball cards to a friend one would not give a piece of each card away. They would give whole cards. Liev does not make that connection to the context of the problem.

Of the 60 responses, 28.3% were coded 0; 40.0% were coded 1; and 31.7% were coded 2. Roughly ⅔ of PSTs correctly interpreted the situation

Question 3

Question 3 asked PSTs to respond to Liev by posing an extending question, as he was able to arrive at a correct answer. We coded only the first question PSTs included in their response according to the four categories from Jacobs & Ambrose (2008, p. 263). Considering the disconnect between the task context and Liev's approach, PSTs who asked a question to promote reflection on the child's current strategy (n = 32) seemed appropriate. We also viewed extending questions encouraging Liev to consider solving the problem another way (n = 19) as an appropriate response for the situation. There were no responses coded as 3, since Liev wrote number sentences, and the concerns most PSTs had were with his approach, not his symbolic representations. Nine PSTs responded with follow up problems for Liev. Of these, 3 PSTs suggested trying the same number choice, but with an easier fraction (½), suggesting that asking for one-half of the cards would encourage Liev to think about splitting the set into two equal parts, a response that seems appropriate. The remaining 6 PST did not suggest follow up problems that we considered appropriate based on Liev's thinking. In all, 90% PSTs responded with an appropriate extending question.

A Holistic View

We believe that the three skills must also be viewed holistically if we wish to understand PSTs' abilities to engage in this practice. We conducted a process of filtering PSTs' responses to determine how many engaged in each of the three skills in an appropriate and connected manner. We considered responses to Question 1 coded as either 2 or 3 as being successful in attending to Liev's work (n = 39). We considered responses for Question 2, coded 1 or 2 to be successful in interpreting Liev's work (n = 30). For Question 3, responses coded 1 or 2 were deemed as appropriate, along with the 3 PSTs who posed new questions using ½. In all 45% of PSTs successfully engaged in the professional noticing of Liev's work in a connected manner.

Discussion and Implications

The examination of PSTs' responses to individual skills within the construct of professional noticing reveal this group of PSTs was largely successful as all percentages were greater that 65%. However, when we took a holistic perspective of PSTs' responses, only 45% of PSTs were successful engaging in the three skills appropriately and in concert. The differences within the individual skills and the holistic view raise questions for future study.

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THE DEVELOPMENT OF MATHEMATICS INSTRUCTIONAL VISIONS: AN EXAMINATION OF ELEMENTARY PRESERVICE TEACHERS

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The purpose of this study was to examine the development of preservice elementary teachers' visions of high-quality mathematics instruction as they progress through their teacher preparation program. Four case study participants were purposefully selected based upon whether their experiences with school mathematics as a child were predominantly positive or negative. Interviews were conducted at three time points, and the resulting data were analyzed for similarities and differences across participants. Findings indicated similar vision trajectories for those with the same type (positive or negative) past experiences in mathematics. Implications for elementary teacher preparation programs are discussed.

Keywords: Elementary School Education; Teacher Education-Preservice; Affect and Beliefs; Teacher Beliefs

Over the past five years, there has been a call for rigorous, well-designed, longitudinal studies of teacher preparation programs. (e.g., American Association of Colleges for Teacher Education, 2010; Cochran-Smith & Zeichner, 2010). Teacher preparation programs have come under fire, as evident in the recent suggestion by the U.S. Secretary of Education, to tie the "effectiveness" of teacher preparation programs to the assessment outcomes of graduates' K-12 students. Potential policies such as this speak to the importance of a carefully studying how preservice teachers develop on various constructs. Results of such studies can inform modifications to existing programs, and simultaneously, the findings can likely demonstrate impact of programs on other constructs beyond K-12 student assessment outcomes. The current study aims to begin to fill the void in the literature by examining how elementary preservice teachers develop in their visions of high-quality mathematics instruction over the course of their undergraduate preparation program. While there has been work on practicing teachers' development of visions (e.g. Munter, 2014), the current study focuses on preservice teachers and is particularly fitting for this population since "visions" implies a view of their future work as elementary teachers.

Teachers' visions, as outlined by Hammerness (2001) are "a set of images of ideal classroom practice" (p. 143); the development of instructional visions cannot be separated from past experiences (Hammerness, 2001). Typically, elementary teachers do not choose their occupation due to a love of mathematics, unlike secondary teachers of mathematics. Elementary teachers may recall their childhood experiences with school mathematics as positive, or they may describe negative experiences. Since past experiences matter for the development of instructional visions, the work of the current study is warranted. The specific research questions guiding this study were: (1) How do elementary preservice teachers with positive childhood experiences in school mathematics develop in their vision of high-quality mathematics instruction over the course of their teacher preparation program?; (2) How do elementary preservice teachers' with negative childhood experiences develop in their vision of high-quality mathematics instruction?; and (3) What are the similarities and differences in the development of instructional visions across all teachers?

Theoretical Framework and Related Literature

The current study draws on the framework proposed by Hammerness (2001) on teachers' visions and on more recent work by Munter (2014). Hammerness (2001) noted that teachers' visions tend to vary across focus (what are the areas of focus as they describe their vision), range (how narrow or broad is the articulated vision), and distance (how far or close the vision is to their actual practice). The current study examines the first of these ideas – on what the preservice teachers ***focus*** as they describe their visions of high-quality mathematics instruction. Drawing on standards-based practices in the mathematics education literature, Munter (2014) developed a framework for examining the role of the teacher on five progressively sophisticated levels: motivator, deliverer of knowledge, monitor, facilitator, and more knowledgeable other. As one moves up the levels in the framework, the descriptors are more aligned with a vision of standards-based mathematics teaching practices.

Methods

Participants

Participants were purposefully selected from a larger sample of 19 preservice elementary teachers who were participating in a larger grant-funded study (Project ATOMS) focused on program

Table 1: Participants' Childhood Experiences in School Mathematics

Participant	*Experiences*	*Quote*
Felicia*	Positive	"I've always loved math, it was, I think it was my always my strong point in elementary school. I was really good, but it was the way it was taught to me, it was purely memorization, but I was so good at that. So, I I didn't understand why other people were getting frustrated and I just completely understood, you have a formula plug and chug. So, that was always my favorite homework to do."
Adrienne*	Positive	"Math has always been my favorite subject. It's always been a thing that kind of comes easy to me. I like working with numbers and filling out equations and puzzles and stuff like that. It's something that's always interests me, so I've always enjoyed math."
Paul*	Negative	"Math is a, oh God, and Dr. Smith* would probably just cringe but it's, that's just, it's no, I never had that one teacher that made me be like oh, math is the answer to everything, it was, like I said, I just kind of flew under the radar. I could do my multiplication table, I mean, I could, I mean, I wasn't amazing at them, but AIG I had a teacher named Ms. Rutherford* and she was the scariest woman on earth and I think that made me also shy away from it cause she was so mean, she would push us and push us but there was no reciprocation, no caring…So, I've always been the one that was kind of on the lower end of math and I never took calc in high school cause I didn't, I didn't want to push myself so that was kind of me."
Terri*	Negative	"Math was a huge source of frustration for me, mostly because I, I think it was because I was really good at everything else, I was a really good student but I wasn't really quick with math and math took a little bit more effort and I attributed to me not being good at it and me failing so the timed multiplication quizzes I would get really stressed out about those, those were always just really hard, I hated them and once we got into long division I had a teacher who was actually a teacher who influenced me a lot in other ways but she only explained it to me one way and she would not explain it to me any other ways and I just didn't get it and that just made me hate math and from then on I just hated math, like I would go home and cry doing my homework."

*Names are pseudonyms.

evaluation. The four participants were selected based upon their articulation of their own experiences with school mathematics as a child. Two of the participants (Felicia and Adrienne) had positive experiences with school mathematics. In contrast, the other two participants (Paul and Terri) described negative experiences with school mathematics from their childhood. Table 1 presents each participant along with quotes describing their childhood experiences in mathematics.

Data Sources

The data for this study comes from three interviews among a larger set of interviews completed with each of the participants. The three interviews included an introductory interview at the beginning of their full-time enrollment in professional studies courses (beginning of junior year), a midpoint interview completed halfway through their professional studies courses (end of junior year), and a final interview completed just before graduation from the program (end of senior year). The analysis was not limited to, but primarily focused on participants' responses to two questions: (1) Describe a math lesson in an elementary school classroom that you would consider to be effective and explain why you consider it to be effective; and (2) What the teacher does and what the students are doing during mathematics instruction are really important. Describe what you think the teacher should be doing most of the time, and describe what you think the students should be doing most of the time.

Analysis

This study used a qualitative, case study research design. The pair of preservice teachers with positive experiences in school mathematics constituted one case; the other pair made up the second case. There were two levels of analysis: (a) an analysis of each case (or pair of teachers), and (b) a cross case analysis (Yin, 2009). The framework presented by Munter (2014) served as the foundation for coding and for analyzing patterns within a case and across cases.

Findings

RQ #1: Case #1 → Positive Childhood Experiences with School Mathematics

Both Felicia and Adrienne ended their preparation programs with descriptions of their instructional visions as the teacher as "facilitator." In other words, in their descriptions, they tended to focus on the teacher as promoting student-student discourse and probing students as needed as they work in small groups. For example, Felicia said the ideal lesson would be:

> Just the students having really good discourse and not just working through a ton of like procedural problems, but having two or three where they are really able to have all those conversations and make sure that they are really like working as mathematicians and explaining their thinking, talking it out, showing multiple representations and then the teacher is there as a facilitator walking around in between groups listening to what's being said and interjecting when necessary.

At the midpoint of the program, both participants also viewed the teacher as "facilitator." For example, Adrienne said,

> I feel like the teacher needs to possibly pose a problem or a question and then the kids need to discuss it and they need to not only think about it themselves, so maybe kind of do like a think pair/share thing where they're thinking about it and then they're sharing with their partner to kind of and when they're sharing that you could always kind of purposely pair them.

The participants differed at the beginning of the program. Felicia started the program describing the teacher as a monitor in which she described the teacher sharing some knowledge but giving students opportunities to apply the knowledge while the teacher monitors the small group work. In contrast, Adrienne started the program with the teacher having the role as motivator and deliverer of knowledge. She said,

> I would say the teacher would need to first of all be excited about what they were teaching about. Second of all, actually explaining what they are doing not just presenting them with a set of procedures and going off of it but actually explaining why they have to do that and why they have to, why they get the answer.

RQ #2: Case #2 → Negative Childhood Experiences with School Mathematics

Paul and Terri both ended the program describing the teacher as a monitor in which the teacher still disseminates knowledge to the students but gives the students the chance to practice or apply that knowledge in small groups while the teacher monitors. For example, Terri said,

> There should be some direct instruction… I don't… not… not a lot of up front full class instruction. Um, the teacher should be explaining things uh, not from a procedural point of view from the beginning obviously, it should be um, teaching the foundational skills and teaching why things are in questioning and encouraging a lot of discourse throughout the classroom. Asking kids to explain and justify their thinking. The kids should be working in groups. Um I think there should be a good mix of independent work and working groups where there should be a lot of collaboration.

Terri remained at the "monitor" level throughout her teacher preparation program in the description of her vision, but Paul started as describing the teacher as a motivator and deliverer of knowledge when he says, "enthusiasm, enthusiasm, you got to, you got to show the kids this is fun" and moved to describing the teacher as monitor at the midpoint of the program.

RQ #3: Cross-Case Analysis

With the exception of Terri, all participants progressed in their visions of high-quality mathematics teaching. Both participants who had negative experiences during childhood mathematics ended the program describing the teacher as a monitor, whereas the two participants with positive experiences ended the program describing the teacher as a facilitator, which is more aligned with standards-based instruction.

Discussion

It seems understanding our students' backgrounds and experiences in school mathematics is important for our work as mathematics teacher educators. We are not attempting to generalize to the larger population; however, the themes among these four participants in their articulation of their instructional visions suggest backgrounds do matter, particularly in how they progress along a standards-based vision of high-quality mathematics instruction.

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DEVELOPMENT OF PRESERVICE TEACHER NOTICING IN A CONTENT COURSE

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This study addresses the continuation of a teacher noticing project as we ask: How does preservice teachers' noticing develop in a mathematics content course for teachers that incorporated unique opportunities for the PSTs to analyze middle grade students' work? In a previous study (Warshauer, Strickland, Hickman,& Namakshi, 2014), we learned that preservice teachers enrolled in a mathematics content course improved their knowledge of content and students as well as their content knowledge as measured by the Learning for Mathematics Teaching instruments (Hill, Schilling, & Ball, 2004). Our aim is to now investigate the relationship between that growth and a specific writing intervention targeting the preservice teachers' noticing of middle school student thinking.

Keywords: Teacher Education-Preservice; Mathematical Knowledge for Teaching; Teacher Knowledge

Purpose and Introduction

Recognizing teacher quality as critical to student learning, the National Research Council (2010) underscores the importance of strengthening teacher preparation programs. University preparation of future teachers typically includes coursework in school subject areas (mathematics in our case), in education and methods, as well as classroom observation, student teaching, and other field experiences (NRC, 2010). Inspired by research that suggests *teacher noticing* can be developed (Miller, 2011), we designed our study to examine how such development might take place in a mathematics content courses for teachers at a university.

In a previous study (Warshauer et al., 2014), we learned that PSTs enrolled in a mathematics content course improved their knowledge of content and students (KSC) as well as their content knowledge (CK) as measured by the Learning for Mathematics Teaching (LMT) instruments (Hill, Schilling, & Ball, 2004). We began the investigation of the relationship between that growth and a specific intervention in the form of writing assignments (WAs) targeting the PSTs' noticing of middle school student thinking. We purposefully chose a WA-based noticing study rather than the more common video-based studies (Sherin & van Es, 2009) because we wanted to reduce the complexities that occur in most classrooms (e.g., teaching moves) and provide a static object that the PSTs would examine for their noticing assignment. Furthermore, assessing homework and paper-based work by students is a common aspect of instructional practice and therefore a natural context for focusing on student thinking. For each WA, the PSTs engaged in a mathematical task during class in groups, participated in whole class discussion of their strategies, and were then given a packet of middle school students' work on that same task and asked to analyze the students' thinking; this was enacted for three rounds with new tasks each round. This proposed talk addresses the continuation of the project as we ask: *How does PSTs' teacher noticing develop in a mathematics content course that incorporated unique opportunities for the PSTs to analyze middle grades student work?*

Theoretical Framework

Research shows that *teacher noticing* is a critical element for effective teaching (Sherin, Jacobs, & Philipp, 2011). We use the definition of *teacher noticing* proposed by Jacobs, Lamb, and Philipps (2010) as three interrelated skills consisting of attending to student's strategies, interpreting student thinking, and deciding on how to respond based on student understanding. *Principles to Action* (NCTM, 2014) includes the mathematics teaching practice, "elicit and use evidence of student thinking" (p.11), and emphasizes the use of evidence of student thinking as an effective practice that informs teaching and supports learning. In particular, what teachers attend to and how they interpret students' mathematical thinking is consequential to the decisions they make in their teaching of mathematics (Jacobs, Lamb, & Philipp, 2010).

A role of mathematics teacher education is to develop teachers' mathematical content knowledge (Hill, Sleep, Lewis, & Ball, 2007) by providing experiences for PSTs to deepen their conceptual understanding of mathematics (Ball, Hill, & Bass, 2005) and to assess students' mathematical work (Hiebert, Morris, Berk, & Jensen, 2007). Philipp (2008) contends PSTs benefit from learning "…about children's mathematical thinking concurrently while learning mathematics." (p. 8). In doing so, Philipp and his colleagues (2007) found that the PSTs were more motivated to learn mathematical concepts beyond just the procedures in order to teach students mathematics for understanding. While studies suggest that *teacher noticing* can be developed (Miller, 2011), findings from Jacobs et. al. (2010) indicate that *teacher noticing* develops with deliberate practice involving particular experiences. Previous studies have explored PST *noticing* in methods courses and student teaching. An important question to investigate, therefore, is how a content course, taken earlier in their program, could be a fruitful environment for starting a trajectory of *noticing* that can develop as PSTs progress through their certification courses and experiences.

Methods

The study reported here was nested within a larger mixed methods project conducted in 2014 over a 14-week semester at a state university in the southern United States. Participants included 128 elementary and middle school PSTs enrolled in six sections of a mathematics content course focused on number and operations. The only selection criteria for the participants were their enrollment in the course and that they gave consent. We report on the qualitative analysis of data related to only eight PSTs who were selected because their first WA seemed to place them in a lower or higher category of noticing based on our rubric.

We used three WAs spaced throughout the semester and interviews to investigate our research question: *How does PSTs' teacher noticing develop in a mathematics content course that incorporated unique opportunities for the PSTs to analyze middle grades student work?* The *writing* intervention was unique in that it was implemented in a content rather than a methods course or later in the PSTs' teacher preparation experience such as student teaching. Each WA cycle began with class time for the PSTs to collaboratively solve a problem-based task intended for the middle grades curriculum (Schoenfeld, 1999) and then discuss their solutions as a whole class. Afterwards, PSTs were provided packets of middle grade students' work (ranging from four to six students per WA) on the same task. The PSTs were instructed to analyze the student work at home and write a paper addressing each student's understanding (or lack of) while providing evidence from the student work to support their claims. Guided by the coding described by Sherin, Jacobs, and Philipp, (2011), we developed a scoring rubric for assessing the PSTs attention to the *noticing* components of *attending* to and *interpreting* student thinking.

Task-based interviews with the eight focus PSTs were conducted at the end of the semester in which they were asked to analyze a single student's work from WA1 that had not been previously assigned. The interviews concluded by prompting PSTs to reflect on what they had learned via the

WAs. To analyze the interviews we used open coding (Strauss & Corbin, 1990) to determine themes that arose from the transcripts of the eight PSTs' interviews with a focus on how they interpreted the WAs in relation to their teacher preparation development.

Results

We report our findings from two main sources to shed light on our research question: *How does PSTs' teacher noticing develop in a mathematics content course for teachers that incorporated unique opportunities for the PSTs to analyze middle grades student work?*

Using data from all 128 PSTs in a multiple-regression model, we examined the effect of their WA1, WA2, Pre-CK and Pre-KSC scores on their Post-CK and Post-KSC scores, respectively. In both models, PSTs' scores on Pre-CK and WA2 were significant. This suggests that by the time the PSTs enacted WA2, they had improved their noticing and this was correlated with their CK and KSC post-scores. Also, in both models, pre-CK was the strongest predictor of their post-LMT scores.

Our findings from the eight focus PSTs' WA data suggest: (1) Among the PSTs with Low levels of attending and interpreting there was a lack of content-specific insights; (2) The growth among PSTs with High levels of noticing varied little across the three WAs possibly due to a ceiling effect; (3) PSTs frequently used subjective accounting **for** and not accounting **of** student work (e.g. attributing errors to "laziness") (Mason, 2002); and (4) Slight growth detected in noticing skills with enriched descriptions of student work from WA1 to WA2 or from WA2 to WA3 in five out of eight focus PSTs.

Upon analyzing the PSTs' interview data, we found four main themes. (1) PSTs found it helpful to hear a variety of their peers' strategies during the in-class problem solving of WA tasks when later attending to and interpreting the student work; (2) WAs were a challenge, as the PSTs had never done anything like this before; (3) PSTs paid closer attention to students' explanations and did not just evaluate students' answers in order to determine what students understood; and (4) Instructor feedback guided PSTs on what to look for in subsequent WAs.

Conclusion and Implications

Analysis of the interview data suggests that PSTs have had little experience with examining student work prior to this course. The WAs therefore created opportunities for PSTs to engage in the important instructional practice of examining student work to understand what the students were thinking and understanding in addition to learning the mathematical content related to the students' work. Through their own enactments of the WA task problems in class, the PSTs began to think of and appreciate the multiple approaches to problems. This seems to have informed the PSTs that students, too, might approach problems differently. The WAs appear to support the development of *teacher noticing* with a focus on student thinking. The challenges faced by the PSTs appear to stem from lack of experiences in writing about students' mathematical thinking, particularly for students who showed little work. A WA format rather than a video, for example, gave PSTs an opportunity to review the student work at their own pace and refer to class notes and other resources to reflect upon and write about their noticing. However, PSTs initial interpretations relied more on their own experiences than possible mathematical reasoning based on students' work. In other instances, they used non-mathematical bases such as students' laziness, sloppiness, or being in a rush as reasons for the work or lack of work on the students' papers. Instructor feedback of the WAs and PSTs' reflections on their previous ones hold promise as supports for PSTs' development of *teacher noticing*. In a continuing project, we plan to follow a cohort of PSTs from this study to examine the development, if any, of their *teacher noticing* as they progress beyond the content courses into subsequent teacher preparation courses as well as conduct a quasi-experimental study controlling for WAs.

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DESIGNING SIMULATED STUDENT EXPERIENCES TO IMPROVE TEACHER QUESTIONING

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In this paper we share preliminary outcomes from the first enactment of a set of "simulated student" experiences designed to improve preservice elementary teachers' ability to recognize and pose "high leverage" questions in response to students' mathematical thinking. Implications for future iterations are discussed.

Keywords: Teacher Education-Preservice; Instructional Activities and Practices; Teacher Knowledge

Purpose

The purpose of the study is to develop and study an instructional intervention aimed at helping preservice elementary teachers (PSTs) develop their skill in specific teaching practices. In this paper we describe how a "simulated student" tool can support PSTs in developing the skill of asking questions that extend or build on students' mathematical thinking. This project responds to recent calls in mathematics education for better defining the specific practices involved in teaching mathematics and shifting emphasis in teacher education from simply developing knowledge about teaching to providing structured opportunities to practice carrying out this work (Ball & Forzani, 2009).

Theoretical Framework

We see teaching as highly situated, which means that "how a person learns a particular set of knowledge and skills, and the situation in which a person learns, become a fundamental part of what is learned" (Borko et al., 2000, p. 195). One implication of this view is that if preservice teachers are going to draw on the knowledge and skills that they gain in their education courses, their learning experiences need to simulate real teaching. This means creating "approximations of practice" (Grossman, Hammerness, & McDonald, 2009) that provide low-risk situations for novices to try, fail, and learn from their practice. A simulated student provides such opportunities as well as a standardized experience that can be studied and refined over multiple iterations.

Jacobs, Lamb, and Philipp (2010) described the teaching practice of *noticing student thinking*, which includes making decisions about how to respond to students when they exhibit different kinds of mathematical thinking. One way that teachers respond to student thinking is by asking questions, although little is known about how teachers develop the ability to ask effective questions. One of the central goals of this project is to understand how the development of this skill can be supported through a simulation that approximates a teaching situation.

Methods

Design-based research

Design-based research assumes that instructional contexts consist of complex, interwoven parts that are not isolatable; thus design approaches do not attempt to control confounding variables. Instead, they seek to characterize with a high degree of specificity the situation in which an instructional design is implemented, as well as capture multiple measures of different outcomes and a detailed record of alterations to the design over multiple cycles (Collins, Joseph, & Bielaczyc, 2004). Thus far, we have completed one complete implementation cycle, which includes three distinct simulated student experiences, a reflection, and a pre-post assessment.

Context

The participants were recruited from five sections of an undergraduate mathematics course for elementary teachers offered at a large university in the Midwest region of the United States. Much of the mathematical attention of the course is focused on fractions, including operating on quantities and making sense of the unit (Chval, Lannin, & Jones, 2013). The intervention was included in the required assignments for the course and consisted of a set of three "experiences," a reflection assignment, and a pre- and post-test. Fifty-four PSTs participated in all parts of the intervention and gave consent for their work to be collected and used in the research project.

Design

The experiences were designed to provide opportunities for the PSTs to interact in a limited way with a virtual elementary student. Using the online Lessonsketch platform, the PSTs viewed a storyboard representation of a portion of a mathematics lesson withthe roles of teacher and studentsplayed by nondescript cartoon characters. The simulated students convey information to the PST through speech bubbles and facial expressions and follow pre-established paths determined by choices made by PSTs.

An "experience" consisted of a series of tasks and choices. First, PSTs were asked to solve a mathematical task and describe what main mathematical ideas were embedded in the task. Then they viewed the storyboard lesson, which ended with a representation of student thinking containing a critical conceptual "error" closely related to the content of the course (Chval, Lannin, & Jones, 2013) For example, in one experience the cartoon teacher highlighted a students' response to the task in Figure 1, saying "Susan said there were three fourths of the two brownies shaded. I'm curious about what you think. Is Susan correct?" After a time of partner discussion, Matthew was called to the front of the class to share his thinking. He responded, "I don't think Susan can be right, because they are cut in half, not in four squares."

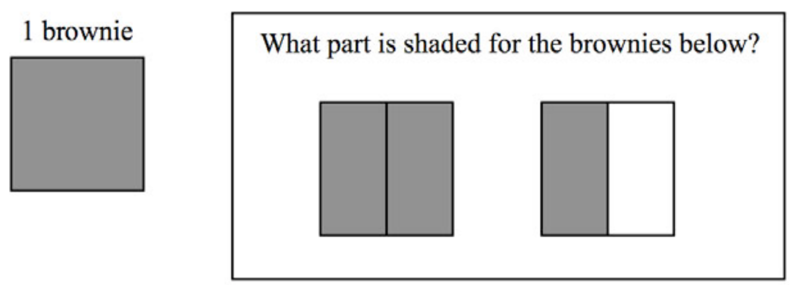

Figure 1: The Brownies Task (from Chval, Lannin, & Jones, 2014)

After viewing an episode, PSTs were asked to interpret the student's thinking and write a teacher question in response. Next they were asked to select from a list of between two and six questions, explain why they believed the question they chose would be effective, and describe how they expected the student to respond. Then they viewed the student's response.

We designed teacher questions to fall into one of two categories, *high leverage* and *low leverage*, drawing on characteristics described in NCTM's *Principle to Actions* (2014). High leverage questions are those that "build on, but do not take over or funnel, student thinking," "make mathematical thinking visible" (p. 41), and target specific ideas rather than general. See Table 1 for examples from the Matthew experience.

Table 1: Examples of question types and student response

	Teacher Question	Matthew's Response
High Leverage	What do you mean about the four squares? Can you show me how that would look?	[draws picture of the two brownies, each split into four squares]. "These are fourths. Susan is wrong because she said THIS [pointing to a half-brownie in the original diagram] is a fourth, when it is really a half."
Low Leverage	If you put the two brownies together, how many pieces would it be cut into?	"Each one is cut into two pieces."

In each experience, the PST was given an opportunity to see a response from the student to their question, and then they would evaluate the question with one of three choices:

1. It was a good question; it accomplished what I wanted it to accomplish
2. It was a good question, but [the student] didn't respond in the way I expected
3. It was maybe not the best question; I should have asked something different

Finally, the PST was given an opportunity to "go back in time" and ask a different question. In the Matthew experience, PSTs wererequired to contrast responses to a high leverage question with responses to a low leverage question, and then say which question they believed was better.

Outcomes

The chart in Figure 2 shows how *low leverage* questions were evaluated after PSTs viewed the students' response. In general, PSTs tended to claim that the low leverage question they chose was good, even though they were often unsatisfied with the student's response. Often, they expected that their question would lead to a resolution of the student confusion, or at least a correct solution to the task. For example, PSTs made statements such as, "I think the question got across the point I was trying to make but Matthew seemed to not process what I was asking." These results seem to suggest that, by themselves, our designed student responses did not generally help PSTs move away from low leverage questions.

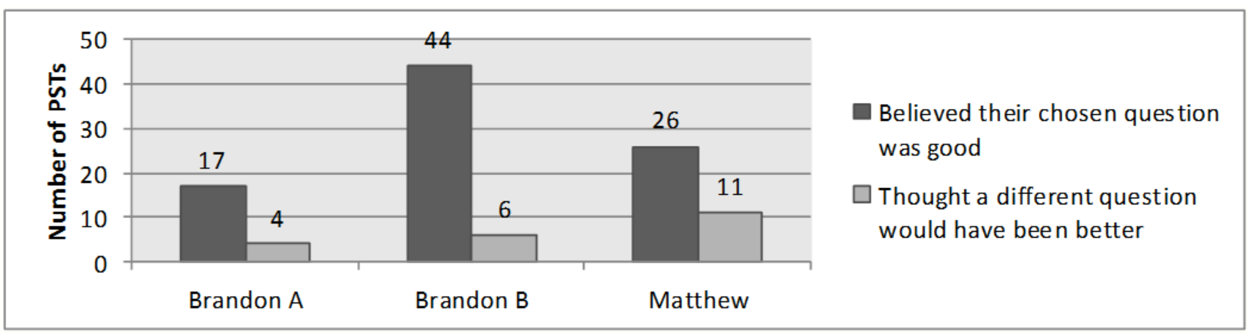

Figure 2: PSTs' evaluations of low leverage questions after viewing the student response

Recall, however, that in the Matthew experience PSTs were given the additional opportunity to see Matthew's response to a high leverage question. Of the 37 PSTs who initially chose the low leverage question, 16 preferred the high leverage question after viewing both responses, while only 3 preferred their initial question (18 indicated no preference). Of the 16 PSTs who preferred the high leverage question, 10 had reported that they still liked their low leverage question *until* they saw the second response.

We conjecture that the opportunity to directly contrast the effect of the two questions was an important design component of the Matthew experience. This is supported by some of the PSTs' evaluations of their questions. After seeing Matthew's response to the initial question, one PST said, "It was a good question and that's where his thinking is wrong but he still isn't thinking of the two brownies as a whole. He is still just looking at a single brownie and seeing one half." But after viewing Matthew's response to the second question, this PST decided that the high leverage question was better: "I didn't realize that if you split up each brownie into 4ths and color in three fourths of each that it would still be three fourths. So technically Matthew is correct too I guess which I never thought about it but still Susan is correct also and he needs to understand that concept too." This example shows a PST not only learning from Matthew's response to the high leverage question, but also recognizing and appreciating that the response helped the PST better understand the relationship between Matthew's and Susan's ideas. This change seemed to be facilitated by the opportunity to see Matthew's response to both questions.

Discussion

Our analysis of our initial outcomes has suggested that an important design feature of our simulated student intervention was the opportunity to directly contrast student responses to high and low leverage questions. That is, simply showing a students' response was not enough to help PSTs recognize the limitations of their initial selected question. This seemed to be because PSTs' default criterionfor an effective question was that it should resolve student confusion. It was only when they saw two questions thatdid not result in resolved confusion that they could look past this criterion and search for other value in the student's response.

This observation suggests that more direct contrasts would continue to help PSTs move towards high leverage questions. That is, in order to help them attend to characteristics of high and low leverage questions, we need to design student responses that are invariant in terms of characteristics PSTs tend to value (e.g., resolving student confusion), but variant in terms of the leverage characteristics (e.g., funneling student thinking versus building on student thinking). This is consistent with the idea that variation needs to be systematically coordinated to help learners differentiate between the ideas they are intended to learn (Marton & Pang, 2006). In the future, we plan to continue to refine our experiences to provide such patterns of variation and invariance to continuedeveloping the skill of asking high leverage questions.

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ANALYZING COHERENCE OF TEACHER'S KNOWLEDGE RELATING FRACTIONS AND RATIOS

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In this study we used Epistemic Network Analysis (ENA) to highlight connections among knowledge resources that middle school mathematics teachers evoked while relating fractions and ratios. The findings discuss how these groups were formed using ENA and discuss the differences in the connections of the participant's knowledge resources in each group. Implications for using ENA to understand teacher knowledge are included.

Keywords: Teacher Knowledge; Middle School Education; Rational Numbers

Introduction

Over the past decade, interest in teachers' mathematical knowledge has been a focus for many researchers (e.g. Hill, Ball, & Schilling, 2008; Thompson, Carlson, & Silverman, 2007). Despite the complexity of researching this domain, researchers have made headway in understanding critical elements of teacher knowledge. For example, researchers have found that there is an important link between the amount of knowledge a teacher has and its organization (Ma, 1999; Orrill & Shaffer, 2012). This aligns with research on expertise that suggests expertise in a domain requires both a certain amount of knowledge and particular organizations of that knowledge (Bédard & Chi, 1992).

Despite the compelling argument that knowledge organization matters, very little research has focused on such organization. In this study, we use Epistemic Network Analysis (ENA; Shaffer et al., 2009), to explore connections between knowledge resources that teachers evoke. For this study, we were interested in investigating how ENA can be used to investigate the connections of teachers' knowledge resources for differentiating between fractions and ratios.

Theoretical Framework

We rely on Knowledge in Pieces (KiP; diSessa, 2006) to explore teachers' knowledge. KiP asserts that the learner holds understandings of various grain sizes that are used as knowledge resources in a given situation (Orrill & Burke, 2013). We posit that as teachers develop their expertise, more connections among resources allow more knowledge resources to be invoked in appropriate situations. Thus, coherence refers to multiple knowledge resources that are connected in robust ways allowing for *in situ* access to the resources. Coherence, combined with a robust set of knowledge resources, allows teachers to deal with complex situations efficiently. This is consistent with previous research on expertise (e.g., Bédard & Chi, 1992).

We chose to focus our study on the relationship between fractions and ratios, which are key concepts in the domain of proportional reasoning (Lamon, 2007). Despite the importance of the relationship between fractions and ratios in proportional reasoning, there has been little research on how teachers differentiate between the two (Clark, Berenson, & Cavey, 2003).

Methods & Data Sources

This study is part of a larger study of middle school teachers' knowledge of proportional reasoning. The participants included 8 licensed middle school teachers (5 female) ranging from 1 to 18 years of experience (med=11.5) from multiple schools. Data were collected from two interviews. The interviews were recorded and transcribed verbatim for each participant.

The four tasks analyzed for this study asked participants about the relationship between fractions and ratios. The first task addressed the multiplicative relationship between the two sides of similar triangles. The second task asked participants to interpret and respond to four teachers' explanations about a 2:5 ratio of vinegar to oil in a salad dressing. The third task had participants consider the concept of equality in proportional situations. The fourth had participants consider the difference between ratios and fractions by presenting a non-standard ratio-combining situation.

Data Analysis

Analysis for this study relied on ENA (Orrill & Shaffer, 2012; Shaffer et al., 2009). For each utterance (typically a response to a single question posed by the interviewer), binary coding was used: a 1 indicates that a particular knowledge resource (see Table 1) was used in the utterance and 0 indicates that it was not. In terms of our theoretical frame these codes represent the pieces of knowledge resources observed.

Table 1: Selected Qualitative Codes of Knowledge Resources That Emerged From Data

Fraction Ratio Concepts (FRC)		
Fraction is part/whole	FRC 1	Refers to fractions as representing a part-whole relationship.
Ratio is part/part	FRC 2	Talks about ratios as being a part-part relationship.
Ratio is part/whole	FRC 3	Talks about ratios as being a part-whole relationship.
Fraction Ratio Differentiation (FRD)		
Context / Units	FRD 1	Uses context or units as a distinguishing feature of fraction or ratio.
Symbolic Representation Use (SRU)		
Mathematical Reasoning with Symbols	SRU 2	Uses a symbolic representation to reason about the relationships within the problem as part of the problem-solving process.
Uncategorized (UC)		
Task Interpretation	UC 1	Talks about what the task description means. Makes sense of what he or she is being asked to do not just answering the question. This includes making sense of symbols like in the case of the free throw problem.
Attending to Context	UC 2	Uses numbers and quantities that are accompanied by labels, directly related to context. In contrast to the use of decontextualized numbers.

ENA took the coded data and placed it into a matrix, broken into vectors by item. This allowed every co-occurrence of the code within a task to be captured. These co-occurrences were then translated into three-dimensional graphs in which the codes are placed relative to their frequency across the dataset, thus allowing variance to be seen. Equiload graphs (Figures 1 & 2) were generated for each participant that showed the co-occurrence of the codes for that participant. The thickness of the lines connecting the vertices indicated the relative frequency of the co-occurrences of those codes.

From a KiP perspective, ENA offers a tool for looking at coherence when it is defined as connections between knowledge resources within the same task. We would expect teachers with more coherent understandings to have one or more very strong pairings of codes in their equiloads, whereas teachers with less coherence would have weaker connections between resources drawing on a wide array of knowledge resources.

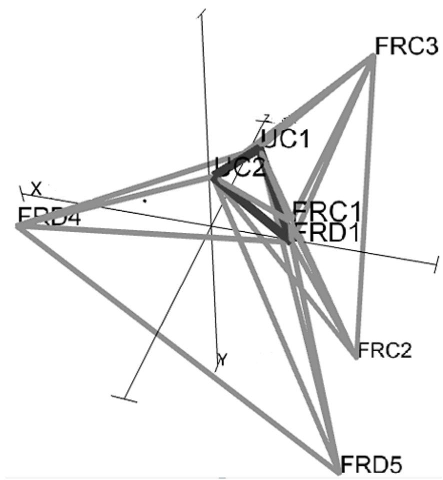

Figure 1: Allison's Equiload Graph

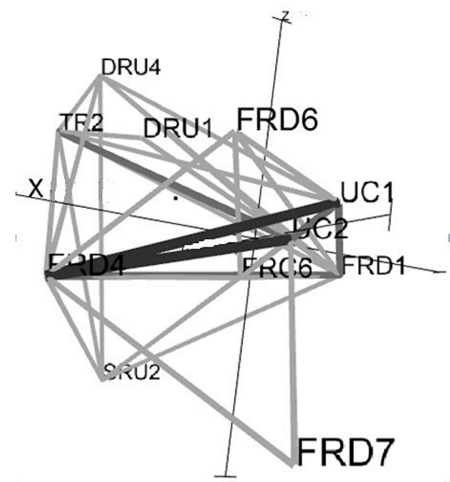

Figure 2:Autumn's Equiload Graph.

Results

Our analysis of the ENA equiloads suggested possible differences in the participants' use of resources. Because our dataset is small and this is exploratory work, we initially separated our participants into groups (Table 2), based on their response to a hypothetical teacher's statement that "fractions and ratios are the same".

Table 2: Participant Groupings

	Participants	Grouping Characteristic
Group 1	Ella, Bridgette, Allison, Mike	Participants rely on fractions as part whole relationships and on ratios as part to part relationships
Group 2	Alan, Larissa	Participants rely on ratios as part to part relationships
Group 3	Autumn, David	Participants do not use either of the knowledge resources above in connection with other knowledge resources

Differences between Group 1 and Groups 2 and 3

Group 1 attended to thinking about part-part and part-whole relationships in discussions of ratios and fractions. For instance, Allison's graph (Figure 1) shows *ratios are part-part*(FRC2), *ratios are part-whole* (FRC3), and *fractions are part-whole* (FRC1)relationships have many connections to one another as well as other resources such as *task interpretation*(UC1), *context*(UC2), and *context/units*(FRD1). In contrast, Alan, Larissa, Autumn (Figure 2)and David's graphs showed no connections to any part-part or part-whole knowledge resources.

We also found differences in the way teachers in group 1 used knowledge resources compared to those in groups 2 and 3. For example, Allison (Figure 2) relied on fewer connections than Autumn (Figure 3). However, Allison's connections drew more on resources related to fraction and ratio concepts (FRC) and fraction and ratio differentiation (FRD). Autumn drew from many different resources useful for problem solving, rather than relying on content knowledge resources. We interpreted Allison's repeated reliance on the same knowledge resources to be a sign of coherence.

Thus, Autumn could be considered less coherent than Allison in terms of the knowledge resources she used.

Resources that differentiate group 2 from group 3

The difference between groups 2 and 3 was not as clear. However, we noted some important differences. The main difference between participants in groups 2 and 3 was their use of the idea that *fractions are part-whole* (FRC1)and *ratios are part-part* (FRC2)relationships. Both Alan and Larissa in group 2 relied on these resources in their responses. In group 3, Autumn never used either idea to respond to any of the tasks and David used these ideas in a single utterance.

Scholarly Significance

Teacher knowledge is critical for supporting student learning, however it is important to consider not just the amount of knowledge, but coherence (Orrill & Shaffer, 2012). ENA has proven to be a useful approach in analyzing the connections of teachers' knowledge resources in conjunction with the KiP framework. This research is a first step in using an analysis method new to the field of mathematics education to contribute to our understanding teachers' knowledge for differentiating between fractions and ratios and to the domain of proportional reasoning.

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MODIFYING CHILDREN'S TASKS INTO COGNITIVELY DEMANDING TASKS FOR PRESERVICE ELEMENTARY TEACHERS

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This paper expands upon previous work on task development by Tobias et al. (2014), which identifies an eight-step task design process for selecting, modifying, and reflecting upon the implementation of children's mathematical tasks with preservice elementary teachers. Here we explicate the modification step, which includes altering children's tasks to be appropriately challenging and offer opportunities for teachers to move beyond their incoming conceptions. We provide examples of modifications we made to a children's fraction comparison task and discuss the goals we targeted to increase the task's level of cognitive demand. We share results from our analysis of 61 preservice elementary teachers' written work on this task and reflect upon the effectiveness of our modifications.

Keywords: Teacher Education-Preservice; Elementary School Education; Rational Numbers

"What students learn is largely defined by the tasks they are given," (Hiebert& Wearne, 1993, p. 395). Thus, mathematics teacher educators (MTEs) are faced with the challenge of designing and implementing high-quality tasks that can help preservice elementary teachers (PSTs) develop deep and flexible knowledge for teaching mathematics. Our task design research team is comprised of six MTEs from six different universities in the U.S., a collaboration that resulted from a PME-NA working group on PSTs' knowledge for teaching.

Background

Our work is based on the task design cycle developed by Liljedahl, Chernoff, and Zazkis (2007). We have modified their framework to integrate the suggestions made by Yackel, Underwood, and Elias (2007) for drawing tasks from K-12 curricula and modifying them for use with PSTs. This process for task design is meant to provide PSTs with opportunities to view the development of mathematical concepts in the same way their future students will. Not only can this practice support PSTs in developing their knowledge of curriculum, but also by connecting the work PSTs do in teacher education courses to their future work in elementary classrooms we can increase task *authenticity*. Students who experience more *authentic* tasks (i.e., those they consider to be valuable, worthy of their effort, and relevant to their future profession) demonstrate higher levels of achievement (Newmann, King, & Carmichael, 2007).

As MTEs select particular children's tasks to use with PSTs, it is important that they consider the intellectual demands of the tasks and whether (and how) PSTs will have opportunities to engage in high-level thinking and reasoning. Tasks that would pose a high-level of *cognitive demand* (Stein, Smith, Henningsen, & Silver, 2009) for children will not necessarily be high-level for PSTs. Therefore, children's tasks may need to be modified to provide PSTs with an appropriate level of challenge.

Task Development

Our team has collectively developed a fraction comparison task for PSTs by selecting and modifying a children's task from a common U.S. elementary curriculum (Russell et al., 2008; for further discussion see Tobias et al., 2014). The original fifth-grade task provides children with four pairs of fractions and asks, "Which is greater? Solve the problems below and explain or show how you determined the answer" (see Table 2 for original problems). Based on prior experience and research regarding PSTs' use of fraction comparison strategies (e.g., Yang, Reys, & Reys, 2009), PSTs tend to be most familiar with algorithmic procedures (such as converting to decimals or percents or finding common denominators); hence, we anticipated that the PSTs in our classes would tend to use such procedures, as opposed to more reasoning-based methods for comparing fractions. Thus, to increase the cognitive demand of this task we developed learning goals to help PSTs develop conceptual knowledge of fractions as numbers and compare fractions using multiple reasoning and sense-making strategies (see Table 1).Our modifications were led by the following three goals: (a) *discourage familiar (algorithmic) procedures*, (b) *develop multiple fraction comparison strategies based on reasoning*, and (c) *create opportunities for PSTs to reason about fractions greater than one and with benchmark values other than one*. At least two goals were targeted in the modification of each problem (see Table 2).

Table 1:Targeted Fraction-Comparison Reasoning Strategies

Strategy Name	Abbreviation	Example Application
Same Number of Pieces	SNP	4/7 > 4/9, since the fractions have the same number of pieces (four) and sevenths are larger than ninths
Benchmark Value Between	BVB [#]	7/5 > 5/7, because 7/5 > 1 > 5/7
Distance from a Benchmark Value	BVD [#]	4/5 > 3/4, because 4/5 is 1/5 less than one and 3/4 is 1/4 less than one; since 1/5 < 1/4, 4/5 is closer to one than 3/4
Greater Number of Larger Pieces	GLP	8/11 > 7/13, because 8/11 has more pieces (8 > 7) and its pieces are larger (elevenths are larger than thirteenths)

Table 2: Modifications to Problems from Original Children's Task

#	Original Problem	Modified Problem	Intended Strategy
1	7/10 or 3/5	7/10 vs. 8/9	GLP
2	7/8 or 9/10	8/9 vs. 12/13	BVD [1]
3	4/3 or 3/4	24/7 vs. 34/15	BVB [3]
4	3/8 or 1/3	3/7 vs. 6/11	SNP or BVB [½]

For problem #1, we replaced 3/5 with 8/9 to discourage the use of common denominators (by renaming 3/5 as 6/10) and elicit the GLP reasoning strategy. When paired with 7/10, the fraction 8/9 creates an opportunity for PSTs to reason about the fractions' numerators and denominators simultaneously to see that 8/9 is greater than 7/10 because it has more pieces (8 > 7) and its pieces are larger (ninths are larger than tenths). In problem #2, we kept the format of the original problem, which compares fractions that are each one piece less than one whole. This problem type can help PSTs reason about fractions by comparing their distances to a common benchmark value. However, to discourage the use of common denominators and percents, we replaced both fractions with a pair that has odd and relatively prime denominators. Since these fractions are 1/9 and 1/13 less than one, and 1/9 is larger than 1/13, the distance from 12/13 to one must be shorter than that from 8/9 to one, making 12/13 the greater fraction. Although BVD [1] could have been used to solve the original problem, we anticipated that our PSTs' familiarity with these fractions would lead to their use of percents or decimals or a reliance on drawing pictures. For problem #3, we maintained the original goal of comparing fractions to a benchmark value lying in between them, but by selecting larger fractions with relatively prime denominators (and ones that are not reciprocals of one another) we intended to discourage familiar procedures and create opportunities for PSTs to reason about fractions greater than one and with benchmark values other than one. As for problem #4, our modified version serves to develop multiple fraction comparison strategies: SNP (3/7 = 6/14, which is less than 6/11) and BVB [1/2] (3/7 < 1/2 < 6/11). In all, our task included ten problems, these four modified from the original task and six others we created. In this paper we will only discuss the four problems above (for further discussion of all ten problems see Tobias et al., 2014).

Methodology

Our task was enacted by three of the research team's MTEs in their respective mathematics content courses for elementary PSTs across four classrooms (n=61). PSTs were explicitly instructed not to use common denominators or calculators on the task. PSTs worked on the task in groups with their peers during class time while the MTEs facilitated individual conversations with students and groups as they monitored the class. The PSTs' written work on the task was collected after group work, but prior to whole class discussions during which the four reasoning strategies (SNP, BVB, BVD, and GLP) were explicated and named. PSTs' responses to each problem were coded by at least two MTEs with respect to: (a) the strategy used (or attempted to use) to compare the given fractions,(b) whether the strategy used (or attempted to use) was one we intended to elicit, and (c) whether the problem was answered correctly (for further discussion see Thanheiser et al., accepted).

Results

Results indicate that PSTs used a variety of strategies in solving the four modified problems and were mostly successful (for each problem at least 85% of responses received were correct). At the same time, not all PSTs solved the problems using the strategies we intended(see Table 3). Some used more familiar algorithmic strategies, such as finding common denominators and converting to percents, while a small amount applied additional valid strategies.

Table 3: PSTs' Responses to our Modified Problems

#	Modified problem	Intended strategy	Number of responses received (% out of possible n=61)	Responses with correct answer (%)	Responses using or attempting to use target strategy (%)	Responses using common denominators (%)	Responses using conversions to decimals/percents (%)
1	7/10 vs. 8/9	GLP	52 (85%)	98%	6%	10%	10%
2	8/9 vs. 12/13	BVD [1]	53 (87%)	85%	68%	0%	8%
3	24/7 vs. 34/15	BVB [3]	43 (70%)	95%	77%	2%	5%
4	3/7 vs. 6/11	SNP; BVB [½]	59 (97%)	98%	8%; 58%	3%	2%

Discussion and Conclusion

Our modifications to problems #2 and #3 proved to be successful at addressing our first goal, to discourage familiar (algorithmic) procedures, as only 8% and 7% of PSTs who answered these problems, respectively, did so by finding common denominators or converting to percents or decimals. However, 10% of all PSTs who solved problem #1 used common denominators, while an additional 10% converted to percents or decimals. We note that our selection of denominators in problem #1 (10 and 9) did not impede the finding of a common denominator and 7/10 lent itself to easy conversion to percent and decimal form; thus, this problem will require readjustment. We also note that cross multiplication may be another familiar algorithmic procedure, yet we only received one application of this strategy in a response to problem #4.

Our second goal was to help PSTs develop multiple fraction comparison strategies based on reasoning (i.e., SNP, BVB, BVD, and GLP). Problems #2, #3, and #4 were successful in eliciting BVB and BVD strategies from a majority of PSTs. Problem #4 was much more successful at eliciting BVB [1/2] than SNP (58% and 8% of responses, respectively), while two PSTs solved this problem using BVD [1]. The strategy that was least successfully elicited was GLP, with only 6% of PSTs who responded to problem #1 having identified 8/9 as having a greater number of larger pieces than 7/10. Although 98% of responses were correct, we found frequent use of procedural strategies (20%), one PST who used BVB [3/4], and several who claimed that 8/9 was "obviously" larger than 7/10, without providing much detail about their thinking.

Our third goal, to create opportunities for PSTs to reason about fractions greater than one and with benchmark values other than one, was met by our careful selection of fractions in problems #3 and #4. As mentioned above, the majority of responses to problem #4 utilized a benchmark value of 1/2.For problem #3, 95% of the responses received correctly stated that 24/7 > 34/15. We note that this problem was only answered by 70% of the PSTs, but due to its position as last of the ten problems, we believe the low completion rate was an issue of time and not content.

In conclusion, mathematics tasks designed for children can provide an authentic foundation for MTEs when developing tasks for PSTs. This paper does not exhaust all modifications required when using children's tasks with PSTs, but offers insight into ways to increase cognitive demand to provide PSTs with an appropriate level of challenge. By reflecting on PSTs' work on our task, we found that many of our modifications were successful at increasing cognitive demand by developing PSTs' conceptual understanding of fractions, while encouraging them to justify their reasoning and use multiple comparison strategies. Other modifications will require readjustment and further evaluation as we continue to develop this and future tasks for PSTs.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' APPLICATION OF THEOREMS AND ABILITY OF WRITING GEOMETRY PROOFS

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Keywords: Reasoning and Proof; Geometry; Teacher Education-Preservice

It is widely accepted by the field of mathematics education that proof and reasoning should be integrated into students' mathematical experiences across all grades and across a breadth of content areas (National Council of Teachers of Mathematics, 2000; Schoenfeld, 1994; Yackel& Hanna, 2003). In spite of the important role of proof in mathematics curriculum, a substantial number of secondary learners encounter difficulties with proof in school mathematics (Chazan, 1993; Senk, 1985). Mathematics educators have pointed out that students' low achievement and misconceptions in proofs may actually be caused by teachers' ineffective instructions (Schoenfeld, 1994; Senk, 1985).In order to ensure secondary students' learning and understanding of proof, secondary teachers' knowledge of proof needs to reach to a certain level. Teachers should be able to "Develop and evaluate mathematical arguments and proofs" and "Select and use various types of reasoning and methods of proof" (National Council of Teachers of Mathematics, 2003, p. 1).

The goal of the study is to look in-depth at the nature of preservice secondary mathematics teachers' (PSMTs) learning of geometry proofs in order to help them develop their proof writing ability more effectively. To reach this goal, the following research question will be addressed by using a collective case study design: How do tasks designed specifically to target the application of theorems in geometry build PSMTs' ability to use these applications in proof writing?

A pre and posttest design will be used to collect evidence of the impact of the theorem application tasks on six PSMTs' proof writing ability. The task-based learning sessions (TLSs) are developed for helping the teachers gain familiarity with the application of geometry theorems. The entire data collection for each participant consists of three phases: a pre proof writing test, three TLSs, and a post proof writing test. The theoretical tool, Principles of the Application of Theorems, is comprised of three aspects of applications: using diagrams, words and symbols, analyzing given proofs, and analyzing given conditions. The principles are used for developing and analyzing the tasks for TLSs. A small-scale pilot study will be conducted to test all the data collecting instruments before the official data collection. Results of the study might be able to provide suggestions for the design of undergraduate level geometry courses and other in-service proof training programs.

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USING A VIRTUAL ENVIRONMENT TO PREPARE PROSPECTIVE TEACHERS TO TEACH EQUITABLY

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As the population in the United States becomes more diverse, we see a pressing need to build teacher education programs to better prepare elementary mathematics and science teachers to teach students from diverse backgrounds in equitable and culturally responsive ways. According to the InTASC standards, "Teachers must have a deeper understanding of their own frames of reference (e.g., culture, gender, language, abilities, ways of knowing), the potential biases in these frames, and their impact on expectations for and relationships with learners and their families" (p. 3). At the same time, recent research has shown practice-based strategies, such as rehearsals, can support change in teachers' pedagogies (Lampert, Beasley, Ghousseini, Kazemi & Franke, 2010). To address this need and the research on learning, we combined virtual and traditional environments in an innovative, new, rehearsal-based course for elementary education majors that will "authentically" engage prospective teachers in practice-based rehearsals with diverse learners in ways that are typically not possible in traditional courses.

The purpose of this poster is to describe the steps underlying the development of a course that seeks to provide opportunities for prospective teachers to 1) develop an awareness of their hidden biases and stereotypes of who can and cannot do mathematics and science and 2) rehearse alternative responses and practices that serve students of diverse social identities more equitably.

The course centers on a virtual school where teacher candidates experience teaching students from diverse backgrounds and rehearse culturally responsive, practice-based instruction within the safe space of the university classroom. The virtual school environment was constructed using OpenSim, a free, open source, multi-platform 3D virtual world. For the virtual school, the prospective teacher designs her/his own personal "teacher" avatar (a visual representation of a person). In addition, each prospective teacher assumes the "identity" of a student avatar. We developed the elementary student avatars to represent a diverse set of students' identities and created detailed background profiles of each student avatar. By participating in this course and the virtual school, teachers will develop deeper understandings of and be better prepared to meet the needs of all students in their classrooms.

Prospective teachers will have the opportunity to rehearse interactions with students of color, identify personal biases, and enact equitable teaching practices. The use of a virtual platform in this way has the potential to significantly reshape approaches to culturally responsive teacher preparation at our University and more widely.

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CREATING CONDITIONS FOR EQUITY: A CASE OF TEACHER-LED CULTURAL CHANGE ENABLING DISTRICT-WIDE DETRACKING IN MATHEMATIC

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Keywords: Equity and Diversity; Policy Matters; Teacher Education-Inservice

Research has established that tracking in secondary schools is a pernicious barrier to equity (Lucas, 1999; Oakes, 2005; Oakes, Wells, Jones, & Datnow, 1997; Wells and Serna, 1996). This poster tells the story of a district-wide, ambitious detracking effort in mathematics, which is taking place in a large, urban school district on the West Coast of the United States. It is a story of carefully crafted, teacher-led change meeting with national reforms to set the stage for district leadership to roll out a plan for equity and excellence that eliminates tracking in mathematics from Kindergarten through the 10th grade.

In this poster, we share the design of the ambitious, multi-component, six-year professional development (PD) effort, as well as evidence that it has supported a shift in discourse in mathematics departments around the district and in the district-wide mathematics community to focus on the varied, often hidden ways in which the district's diverse students are all mathematically "smart." We describe the ways in which this cultural shift met with ideas from the Common Core State Standards in Mathematics to provide the context within which detracking mathematics was both sensible and possible, from the perspective of district leadership and the school board.

As we write this paper, the district is engaged in a heated and public argument about what mathematics instruction should be available to different groups of students. As the district leadership works to defend its detracking efforts against stakeholders who stand to lose positions of privilege (e.g. through elimination of the "honors" track), it continues to lean on the community of mathematics educators committed to excellence and equity that has been fostered by this professional development effort.

The poster describes the evolution of the PD from its inception in the 2009-2010 school year until now, including ways in which it has attended to the development of (a) classroom practice, (b) departmental communities organized to collaborate around a shared vision, (c) district-wide community of mathematics educators committed to equity, (d) district capacity to sustain the effort without participation from outside supporters. The effort began modestly–a week-long workshop for 24 teachers in Complex Instruction–and has evolved into a multi-level PD effort involving 17 schools and more than 120 teachers.

Our hope is that this poster will foster discussion across our community about how systemic, research-based change toward equity can be strategically supported to overcome immense institutional, cultural, and political barriers.

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PRE-SERVICE TEACHER NOTICING OF STUDENT PROBLEM SOLVING STRATEGIES

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There is a need for high quality, effective mathematics teachers recognize and use a variety of problem solving strategies. Professional noticing has been identified as a critical component of teacher preparation, and this skill can be developed through practice (Miller, 2011). To examine this development, I ask "What do Pre-service teachers (PSTs) notice about specific problem solving strategies as they examine student work on a rich problem-solving task?"

Jacobs, et. al have found that professional teacher noticing, although nonroutine, can be learned and supported by professional development (Jacobs, Lamb, & Phillip, 2010). Allowing pre-service teachers to work with multiple representations of word problems can diversify the approaches taken to the problem (Ozdemir & Reis, 2013). This study examines a small piece of this in looking at what PSTs notice about student problem solving strategies.

The participants are PSTs in a mathematics content course as part of a teacher preparatory program at a large university. The participants completed a task involving finding the number of matches involved in a round-robin tennis tournament, and were given student work on this task to analyze in writing, focusing on student thinking and instructional strategies. What problem solving strategies PSTs are noticing, and how they are interpreting these strategies is the focus of the analysis. The writing assignment was given as part of a larger study on teacher noticing.

Initial results reveal that the PSTs focused on, described, and interpreted student problem solving strategies. Other PSTs incorrectly interpreted strategies or made unfounded suggestions, such as rereading the problem. The implications are that PSTs need practice and tools to understand student problem solving strategies, and are willing to do so. Further research is needed to determine the effect of these types of exercises on PSTs growth in terms of their own problem solving toolkit, and their interpretations of students' strategies.

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MATHEMATICS TEACHERS USING DATA IN PRACTICE: EXAMINING ACCOUNTABILITY AND ACTION RESEARCH CONTEXTS

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Recently, there has been a press for the use of data and evidence in schools. There is, however little research that provides insight into the specific practices used by teachers or into the influences of school and policy contexts on teachers' daily uses of data (Little, 2012). Current accountability policies require teachers to use particular data to answer questions posed by administrators and school boards outside the classroom. In contrast, when conducting action research, teachers pose their own questions and use data to seek their own answers to problems in their classroom. Examining teachers' work within the contexts of: (a) their school and its accountability policies, and (b) their work as teacher researchers, offers insights into potential conflicts for teachers using data and evidence in practice. This poster presents preliminary findings from a dissertation study about the ways mathematics teachers use and make sense of data and evidence, addressing the questions: (a) *What data do teachers select or identify as valid evidence?* and (b) *How are teachers making sense of data in relation to school accountability policies and action research efforts?*

This presentation builds from relevant literature related to teacher data use, teacher action research, and teacher agency. I use the theoretical framework developed by Coburn and Turner (2011) to support the understanding of how teachers use data in practice. The four interrelated categories of this framework, include: (1) interventions to promote data use, (2) organizational and political context, (3) processes of data use, and (4) potential outcomes. Coburn and Turner build primarily from Sensemaking Theory and interpretivist traditions, to which I add critical considerations of teachers' agency and power.

The participants of this study include four mathematics teachers at a diverse suburban middle school in the Midwest, US, along with their mathematics coach and administrators. These teachers engaged in a yearlong action research study group focused on improving their practices related to mathematics classroom discourse. The group met 1-2 times monthly to discuss progress and analyze findings related to their collected data. My data sources include: (1) semi-structured interviews with participants across the year; (2) video and audio recordings of all action research study group sessions; and (3) copies of student work identified by participants. I use techniques of grounded theory (Strauss & Corbin, 1990) to generate themes across *what* data teachers identify as valid evidence. I also adopt a dialogic perspective of language use (i.e., Wortham, 2001) and I use narrative discourse analysis techniques to determine *how* teachers discuss and make sense data and evidence. Early analyses suggest that teachers' uses of data vary between action research and accountability contexts and that administrators' and teachers' implicit understandings of quality data use often conflict and may impact teachers' agency.

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PAVING THE WAY FOR SOCIO-MATHEMATICAL NORMS

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Over 20 years ago, the *Professional Standards for Teaching Mathematics* (PSTM) (NCTM 1991) identified critical elements that support meaningful learning. In particular they detailed the centrality of 'classroom discourse,' the use of full class discussions to make mathematical meanings, reasoning, and justifications public and shared. Hiebert et al. (1997) continued to clarify the PSTM elements as well as giving more detail to the social culture of the classroom and its relation to equity and accessibility. Yackel and Cobb (1996) provided nuanced analyses of classroom discussions that distinguished *social* norms for general participation from expectations of mathematical thinking which they called *socio-mathematical* (SM) norms. SM norms focus on mathematical reasoning as the clear intention of classroom sharing, questioning, explaining, and justifying. The Standards for Mathematical Practice of the *Common Core State Standards* (CCSSI, 2010) continued to emphasize these same norms by expecting students to produce mathematical arguments and to understand and critique those of others. Recommendations for teachers concur that teachers themselves need an orientation to these norms (CBMS, 2012).

Despite the research and recommendations on SM norms, we have found it challenging to provide future teachers opportunities to experience these norms first-hand. Further, mathematics educator colleagues have asked for more details on how to make these norms routine in their content courses for pre-service teachers (PSTs). To address these concerns, our team investigated the specific classroom steps an experienced instructor took to initiate elementary PSTs into SM norms. Fall, 2014 we used a mathematics course for PSTs as a case study to examine how SM norms were introduced. We adapted and enhanced a framework from Elliott et al. (2009) to analyze video recordings, field notes, instructor interviews, student surveys, and student mathematical work from the course. We found that orienting PSTs to one another's reasoning was central in beginning to establish SM norms. In this poster we share some of the specific initial practices. For example, an instructor may invite PSTs to analyze and build on a peer's incomplete strategy or representation and reach a solution as a whole group. Preliminary results indicate that, as SM norms become more routine, PSTs begin to challenge each others' mathematical ideas, openly share their confusions, and ultimately use meanings rather than rules to justify their solutions.

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EMPOWERING PRE-SERVICE TEACHERS TO ENACT EQUITY PEDAGOGY

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Keywords: Equity and Diversity; Teacher Education-Preservice; Elementary School Education; High School Education

Teachers must develop dispositions of advocacy if they are to teach mathematics for equity and social justice (Gutiérrez, 2013). We discuss how to use the research on mathematics teaching for equity and social justice to facilitate pre-service teachers' development of advocacy dispositions. We hope to facilitate discussion on how to develop equity and social-justice dispositions among PSTs within single-semester college methods, content, or connections courses.

Our framework starts with three pillars outlined in NCTM's recent research brief on equity: *Reflecting*, *Noticing*, and *Engaging in Community* (Chao, Murray, & Gutiérrez, 2104). For reflecting, PSTs engaged with the Multicultural Mathematics Dispositions Framework by White, Murray, and Brunaud-Vega (2012) to self-reflect on their own dispositions. For noticing, PSTs analyzed video case studies of students using Jacobs, Lamb, and Phillipp's (2010) noticing framework to attend to, interpret, and respond to students' mathematical thinking. Finally, for engaging in community, students used Agruirre, Mayfield-Ingram, and Martin's (2013) work to learn about students' community knowledge and then build mathematical lessons that invited and honored their students' knowledge. Finally, PSTs read Gutíerrez's (2013) work to explore counter narratives and practices of creative insubordination to connect to the oppression of marginalized peoples.

We piloted this work at two sites. The first author worked with over 60 elementary PSTs in a methods for mathematics teaching course. The second author worked with over 60 secondary PSTs in a mathematics content class connecting traditional content and methods courses for PSTs (2013).

Our preliminary analysis of how PSTs developed dispositions for equity and social justice through these courses showed that, while teachers still had difficulty connecting mathematical thinking with their students' community knowledge, they did end up creating mathematical tasks that showcased societal inequity. For instance, one lesson had children analyze traffic stops data in Ferguson by race, which allowed children to unpack the goals of the #BlackLivesMatter movement. We hope our poster invites discussion in how other educators and researchers explore ideas of equity and social justice within math teacher education.

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FOCUSING TEACHER SELF-CAPTURE VIDEO TASKS USING SPECIFIC PROMPTS TO SUPPORT TEACHERS' ENGAGEMENT WITH STUDENT THINKING

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Keywords: Teacher Education-Inservice; High School Education

Video-based professional development (PD) is an ever more popular way to support teachers' adoption of reform-oriented practices, particularly in engaging with student mathematical thinking (Borko, Koellner, Jacobs, & Seago, 2011; Tripp & Rich, 2012). Much of the video-based PD studied uses researcher-captured video, but relatively little work has examined teacher-captured video (Dyer, 2013; van Es, Stockero, Sherin, van Zoest, & Dyer, in press). This poster examines two types of self-captured video tasks designed to engage teachers in identifying student mathematical reasoning: noticing tasks and experimental tasks.

The noticing tasks were used to prompt teachers to notice and reflect on particular events in the classroom in the midst of teaching. The video equipment used by teachers allowed them save video of an event after it happened. Some of the prompts given to teachers included capturing moments when students shared thinking that the teacher could not make sense of in the moment, and moments when students made productive connections that the teacher did not anticipate.

While the noticing tasks focused on spontaneous moments during teaching, the experimental tasks are based teachers experimenting with an aspect of their practice. Each experimental task asked teachers to plan and try something new in their teaching and capture it on video. Some examples of experimental tasks asked teachers to develop questions to uncover the details of student thinking and modifying a task to incorporate students' real-world experiences.

Data from 11 secondary mathematics teachers participating in PD designed around these two types of tasks were examined using an open coding focusing on teachers' reactions to the two types of tasks. The findings suggest that the teachers preferred the experimental tasks over the noticing tasks. However, across the debriefing discussions in the group sessions it was clear that the noticing tasks provided teachers with a new frame to guide their noticing in the classroom related to the specific prompts of the task. Additionally, teachers commented that these tasks were particularly useful to examining moments that were too complex to unpack while teaching. Finally, from a teacher educator's standpoint, both tasks often provided a unique window into teachers' thinking about the specific prompts (e.g. what moments teachers believed showcased student reasoning), as well as what those moments looked like in different teachers' classrooms. These findings suggest that while noticing tasks may be productive tasks for teacher learning, they may require more effort and be more stressful for teachers than the experimental tasks.

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DRAWING CONNECTIONS BETWEEN STUDENTS' MISCONCEPTIONS AND TEACHERS' INSTRUCTIONAL PRACTICES

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This study investigated the connections between students' misconceptions and teachers' instructional practice. We examined four elementary teachers' experiences in implementing a fractions unit they designed collaboratively as part of a 40-hour summer professional development (PD) program. These teachers were participants in a three-year project focused on improving teachers' mathematical knowledge for teaching (MKT) through a series of monthly workshops, coaching and co-teaching experiences. In particular our goal was to examine the ways in which teachers' instructional practices serve to strengthen or resolve students' misconceptions.

The teachers created a fraction unit to address students' misconceptions in summer professional development sessions and implemented it in their classroom. The main data sources such as videos from the unit development PD, teachers' implementation of the unit, the debriefing sessions following the implementation, and students' task-based interview videos were analyzed using grounded theory (Corbin & Strauss, 2008).

Our findings suggest that teachers were able to identify students' misconceptions about fraction concepts from interview videos and also to design a unit to address them. However, in the process of implementing the designed unit, the teachers demonstrated some difficulties in engaging students' appropriately in reasoning about fractions and addressing students' misconceptions about the whole concept and the benchmark strategy. We attributed the possible reasons to teachers' weaknesses in using precise mathematical language to indicate fraction quantities, in probing students' thinking behind responses, and in making appropriate instructional decisions to address students' misconceptions in the moment of teaching, which all related to core components of MKT, specifically knowledge of content and teaching (KCT) (Ball, Thames, & Phelps, 2008). This study also highlights possible connections between students' misconceptions, the use of inappropriate mathematical language in teaching and teachers' struggles to focus classroom discussions to emphasize key mathematical ideas (Sleep, 2012).

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USING ARGUMENTATION TO INVESTIGATE THE IDENTITY AS TEACHER OF A PROSPECTIVE TEACHER

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Keywords: Teacher Education-Preservice; Research Methods; Affect and Beliefs

The development of an identity as a teacher-of-mathematics requires more than internal reflective work. The "rational other" (Gee, 2001) is necessary to provide feedback as to whether the actions, activities, or discourse of the individual are being interpreted as desired. Snow and Anderson (1987) call identity work the acts or activities one partakes in to define oneself in a context. Identity work can take many different forms whether it is through appearance and manner (Goffman, 1959), or embracing and distancing oneself from roles (Snow & Anderson, 1987). Acts of identity work are discursive moves used in an attempt to convince the rational other of the ownership of certain traits, beliefs, or knowledge that make up one's identity.

Specifically for prospective teachers of mathematics, identity work refers to the actions taken to convince the other that they are teachers-of-mathematics.

For this poster, I focus on the participant's discursive identity or D-Identity (Gee, 2001). D-Identities include the ways that one may characterize her or himself and how others portray the person. Recognition then becomes crucial in the development of an identity. "If an attribute is not recognized as defining someone as a particular 'kind of person,' then, of course, it cannot serve as an identity of any sort" (Gee, 2001, p. 109). Identity work will be different depending on the community he or she is looking to be recognized as belonging to. Prospective teachers must conduct identity work to argue the ownership of a teacher-of-mathematics D-identity. In general, arguments are the way one attempts to convince another that something is true. Argument in its basic form is made of three components: (a) claims, (b) warrants, and (c) data. In this poster, I explore the identity work of a prospective elementary teacher as she discursively enacts her D- Identity in an interview setting. I explore how she argues the ownership of particular traits.

Additionally, I describe the use of an argumentation lens on social identity of a prospective elementary teacher.

The framework breaks down the participant's claims, evidence provided, and her anticipatory statements used to argue her social identity as a teacher-of-mathematics. Eyre's (pseudonym) claims were categorized by the level of ownership or the discursive strength behind the claim.

The criteria for ownership were determined by the use of I-statements, hedges, and intensifiers. Evidence was separated into either anecdotal or belief-based, and anticipatory statements were categorized as being used to provide more details, recognize limitations of teaching, or defensive/protective moves (Goffman, 1959). In this poster, I share the social identity of Eyre I was able to construct using this framework. Her social identity included the centralization of an equity position and the strong desire for student engagement. Implications for teacher education programs are also discussed.

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TEACHER CANDIDATES' RESPONSES TO A NON-STANDARD STUDENT SOLUTION ON AN ALGEBRAIC PATTERN TASK

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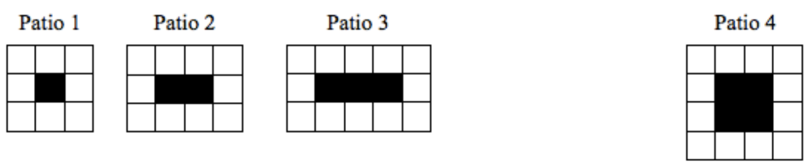
Keywords: Algebra and Algebraic Thinking, Teacher Education-Preservice

An enduring challenge facing teacher educators is how to prepare teacher candidates for more ambitious teaching practices. One response has been to help teachers focus on student thinking, which requires teachers to elicit and respond to student thinking (Franke, Webb, Chan, Ing, Freund, & Battey, 2009). However, this can be especially challenging when student thinking does not align with teacher'smathematical thinking.

This poster presents 15 teacher candidates' responses to a non-standard student solution to a pattern task in a secondary mathematics methods course. Teacher candidates were asked to explore the *Tiling A Patio* problem by extending the pattern shown below (Smith & Stein, 2011, p. 19). They were then provided with a non-standard student solution (see Patio 4) and asked to create a fictional dialogue between themselves and the student who created that pattern.



Initial results show teacher candidates were responding to student thinking by posing various types of questions, which were coded using Driscoll's (1996) question type framework. Fourteen of the 15 teacher candidates began their fictional dialogue by *prompting mathematical reflection* and asking students how they arrived at their pattern.However, none of the teacher candidates followed their initial prompt with a question that would *elicit algebraic thinking*. A closer examination of these questions shows that 11 of the teacher candidates were in fact *orienting* and more specifically *funneling* (Herbel-Eisenmann & Breyfogle, 2005) students toward a pattern that matched the candidate's own thinking. A whole-class discussion following the creation of the fictional dialogues showed that many candidates believed there was a single correct answer to the pattern task and did not try to unpack the student solution to see potential value in the response (e.g., cheaper patio). These results suggest that providing teacher candidates with pattern tasks that allow for multiple solutions may help teacher candidates focus on student thinking, while also challenging their conceptions about mathematics.

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DESIGNING PROFESSIONAL LEARNING TASKS FOR LEARNING TO POSE PROBING QUESTIONS

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Teachers' questioning practices have been shown to support students' explanations, clarify teachers' understanding of students' mathematical thinking, and increase student learning (Webb et al., 2009). Research suggests that all students in the classroom benefit when teachers pose sequences of probing questions after students' initial explanations; however, this practice is often difficult for teachers (Franke et al., 2009). Following recommendations of Grossman et al. (2009) that urge teacher educators to work with teachers on problematic aspects of practice, researchers are exploring *rehearsal* as a pedagogical strategy to work with prospective teachers in university-based methods courses (Lampert et al., 2013). Rehearsal as an approximation of practice supports novice teachers by providing opportunities to learning about, practice, and reflect upon important aspects of practice while receiving in-the-moment feedback from teacher educators. While still emerging, this research on the use of rehearsal to support prospective teachers in enacting the complex work of teaching is promising.

Our research explores rehearsal as a teacher educator pedagogy for working with practicing teachers and shares findings from a pilot design experiment seeking to create and refine rehearsals that support teachers' learning to pose probing questions. We explore the research: *What factors influence teachers' learning to focus on responding to students' mathematical thinking with probing questions in rehearsal tasks?* Working with three middle/secondary teachers in three 1.5-hour professional development (PD) sessions highlighting recent research on teacher questioning, teachers engaged in rehearsals focused on questioning related to students' understanding of functions. Our initial conjecture was that teachers' familiarity with the instructional task affects teachers probing questions. Data consisted of video recordings of sessions, written teacher reflections, post-rehearsal interviews, and our conjecture log. We used constant comparative methods (Strauss & Corbin, 1998) to analyze our data and concluded with a revised conjecture: *Familiarity with the instructional task and its learning goal, mathematics knowledge for teaching, knowledge of students, the degree to which teachers had time to anticipate students' conceptions and errors, and the supporting activities of the PD affects teachers probing questions.* Our findings suggest that mathematics teacher educators should attend to these factors as they explore rehearsals in PD.

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UNDERSTANDING ELEMENTARY SCHOOL TEACHERS' PERSPECTIVES ON CHILDREN'S STRATEGIES FOR EQUAL SHARING PROBLEMS

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Keywords: Teacher Education-Inservice; Teacher Knowledge; Rational Numbers; Elementary School Education

Over the past several decades, the growing research base on children's mathematical thinking has provided a way for teachers to structure what they see and hear so that they can better respond to children's ideas in the midst of instruction. Research has also shown that learning is enhanced when teaching is focused on children's thinking (Wilson & Berne, 1999). In this study, we worked with teachers who were learning about children's fraction thinking, and our goal was to investigate how teachers thought about children's thinking. In much the same way that understanding children's thinking has helped teachers honor and build on children's thinking, understanding teachers' perspectives can help those trying to support teachers' development.

We explored teachers' initial perspectives on children's strategies for equal sharing problems in which the answer is a fractional amount. Prior to professional development on fractions, teachers in grades 3–5 analyzed written work for 12 student strategies for the following problem: *Six children are sharing 16 brownies so that everyone gets the same amount. How much brownie can each child have?* Coding involved an iterative process of analyzing video- or audio-recorded discussions of groups of teachers who were asked to order strategies in terms of sophistication.

We identified five perspectives teachers used in deciding levels of sophistication of children's strategies: (a) whether leftover items were partitioned; (b) whether whole items were distributed prior to any partitioning; (c) whether the largest possible partitions were used; (d) whether fraction notation was predominantly used (vs. drawing); and (e) whether the answer was in the form of a mixed number (vs. an improper fraction or informal notation). Some of these perspectives were consistent with research on children's fraction thinking (Empson & Levi, 2011) such as when teachers viewed strategies involving partitioning of leftovers as more sophisticated than those which ignored leftovers. Other perspectives, while reflective of traditional curriculum sequencing, were inconsistent with research on children's fraction thinking such as when teachers viewed distributing whole brownies prior to any partitioning as more sophisticated regardless of the nature of the partitions. For example, partitioning all items strategically (e.g., linked to the number of sharers) was viewed as less sophisticated than distributing wholes and then using a more basic partitioning (e.g., repeated halving). Implications for researchers and professional developers include understanding teachers' starting points and potential confusions as well as appreciating the need to honor and build on these perspectives.

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MATHEMATICS TEACHER LEARNING THROUGH INSTRUCTIONAL PRACTICES

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This study investigates how mathematics teachers' instructional moves create opportunities for them to learn about student thinking and mathematics content. In particular, it addresses the *role of assessment in teaching and learning*, by focusing on how specially-designed *formative assessment lessons (FALs)* can be used as a mechanism for teacher professional development.

Learning and development in the workplace is a central ongoing experience for any professional, including teachers (Cohen & Ball, 1990; c.f., Schön, 1983). However, there is little research on teacher learning and development *through* practice; most research on teacher learning and development focuses on teachers' cognitive processes as viewing learning *from* formally organized contexts (e.g., teacher education programs or PD workshops) or focuses only on the social relationships and/or environment *within* a community (e.g., shift in participation within communities of practices). This study aims to fill this gap, and investigates how teachers learn and develop through practice by adapting a socio-cultural developmental framework (c.f., Saxe, 2012) to analyze the process of how mathematics teachers develop new teaching practices to accomplish their own pedagogical goals.

To explore teacher learning and development through practice, this study focuses on two 7th grade mathematics teachers (Ms. Lee and Ms. Janet) and their classrooms as the teachers attempted to implement formative assessment practices by using innovative curricular support materials (e.g., *FALs*) and participating in professional development sessions over the course of one school year. The analysis compares the two teachers' classroom practices, focusing on how Ms. Lee's instructional moves afford the means of teachers' making sense of the content—proportional reasoning—and student thinking of the content, while Ms. Janet's instructional moves do not afford the same kinds of learning opportunities. Preliminary findings illustrate that particular *forms* of instructional practices that are supported in professional development sessions serve different *functions* in two teachers' classrooms (see the "form-function" relationship in Saxe, 2012) and it has different impacts on longitudinal shifts in practices. In the poster, I will provide more information and examples of what forms of instructional practices were supported and how their implementations served different types of functions, and how it shifts or not.

This study has critical implications for mathematics teaching and learning. First, the analytical categories identify and suggest particular practices that support teachers in incorporating formative assessment in their instruction, such as making effective use of student mathematical thinking during instruction. Second, unpacking the process of teachers' developing new instructional practices provides insights on how to better support teachers' professional development and teacher learning.

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RELEARNING MATHEMATICS FOR CONCEPTUAL UNDERSTANDING: PRE-SERVICE TEACHERS' PERCEPTIONS OF THEIR COURSEWORK EXPERIENCE

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Keywords: Teacher Education-Preservice; Affect and Beliefs

An important role of mathematics content courses [MCC] during elementary teacher preparation is to help prospective elementary teachers [PSTs] relearn elementary mathematics more deeply while experiencing the type of teaching mathematics educators hope they will practice in their future classrooms. However, research suggests PSTs bring their own sets of beliefs and attitudes that affect what they learn from such programs. There are times when such entering beliefs "are not quite contextualizing, illuminating, and helpful, so much as they are powerful, potentially misleading, and unproductive as resources for learning the principles we hope to teach" (Holt-Reynolds,1992, p. 327). For example, when attending a thoughtfully planned course designed to engage them in rich mathematical thinking, some PSTs are resistant to explore, create, or prove mathematical ideas (Boaler, 2002). Experiencing dissonance, tension, and frustration, many PSTs "react to the course in a perfunctory manner" (Phillip, et al., 2007, p. 439). If PSTs' prior beliefs about learning are at odds with the goals and demands of such coursework, they can create barriers to their personal growth during teacher preparation.

The purpose of this study was to better understand how PSTs perceive opportunities intended to develop their mathematical knowledge for teaching. One hundred fifteen participants who had completed one, two, or three MCC for PSTs completed an electronic survey that measured their perceptions of the relevance of the coursework as well as the appropriateness of specific types of tasks or pedagogical strategies they encountered throughout the coursework.

Findings revealed PSTs' perceptions of the relevance of the MCC and the course learning goals appeared to be associated with their views about the pedagogical strategies typically employed in the MCC. PSTs who viewed one or more pedagogical strategies, and in particular, productive struggle, as *onlysometimes appropriate* [*SA*] consistently reported more unproductive MCC views in terms of course relevance and perceptions of course learning goals than their peers who viewed such strategies as *always appropriate*[AA]. Furthermore, PSTs who indicated *productive struggle* is *only SA* for MCC were twice as likely to view additional pedagogical strategies in the same way compared to those PSTs who found a pedagogical strategy, but not productive struggle, as only *SA* for the MCC. Analysis of open-ended responses from PSTs who perceived one or more pedagogical strategies as *only SA* suggested PSTs desire greater guidance during sense making activities, as well as instructor verification before moving on from a task. Rather than viewing cognitively demanding activities that provide opportunities to make sense of mathematics as potential learning experiences, responses revealed PSTs' overall concern for the unnerving feelings of confusion and frustration prompted by such episodes of struggle.

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UNDERSTANDING THE PRACTICE OF TEACHING FOR EQUITY

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While mathematics teacher educators (MTEs) have described their approaches to preparing teachers to incorporate equitable instructional strategies in mathematics (e.g. Aguirre, 2009), there has not been a systematic examination of how MTEs prepare teachers to effectively teach mathematics to diverse student populations (McLeman & Vomvoridi-Ivanović, 2012). This type of examination will help support MTEs to develop practices that integrate issues of race, class, gender, language, culture, or power in their mathematics teacher preparation courses.

Our goal is to obtain an initial understanding of how MTEs who make issues of equity a priority in their practice facilitate this kind of work. Our research questions are:

- How do MTEs who make equity a priority in their instructional practice conceptualize equity in their teacher preparation courses and how is this reflected in their course syllabi?
- What challenges do MTEs who make equity a priority in their instructional practice face when teaching mathematics methods courses and how do they work towards resolving these challenges?

We analyzed 23 MTEs' responses to a brief survey and their course syllabi. Utilizing components of grounded theory (Glaser & Strauss, 1967), our findings suggested a mismatch exists in the way MTEs conceptualize equity and the way it is addressed in their course syllabi. Further, our findings also suggested there is a mismatch between the nature of the challenges MTEs face and the nature of the resolutions they employ to work towards resolving these challenges.

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SOURCES OF INFLUENCE ON THE CURRICULAR CHOICES OF SECONDARY MATHEMATICS PRE-SERVICE TEACHERS

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An enduring challenge in mathematics education is the need to understand the types of knowledge and reasoning that teachers need (Ball, Thames, & Phelps, 2008; Breyfogle, Roth McDuffie, & Wohlhuter, 2010; Shulman, 1986). Novice teachers are being evaluated in ways that affect their own career and will soon be evaluated in ways that affect their teacher preparation program (Teacher Preparation Issues, 2014). To serve our secondary mathematics pre-service teachers (SMPSTs) we must continue to improve our understanding of SMPST knowledge and reasoning and respond critically by examining the way that we teach SMPSTs.

Research Problem and Question

This study seeks to explore the sources that influence the choices that SMPSTs make when planning lessons. Based on conversations with SMPSTs during the course of the author's responsibilities as a field placement supervisor, it has become obvious that SMPSTs at times struggle to exhibit strong curricular reasoning as defined by Breyfogle, Roth McDuffie, and Wohlhuter (2010). In an effort to understand the choices that demonstrate curricular reasoning, this research systematically examines what sources influence the curriculum choices of SMPSTs.

Description of Method and Timing

Data will be collected during the spring of 2015 using five SMPSTs who are currently student teaching using lesson plans, interviews with participants prior to implementing lessons, classroom observations, and interviews with participants after they implement the lesson. Sources of influence will be identified and grouped in ways that allow us as teacher educators to understand how we can help improve the curricular reasoning of SMPSTs.

Critical Use of Findings

By understanding the sources that influence the curriculum choices of SMPSTs we can respond in a way that improves the curricular reasoning of SMPSTs. Breyfogle, Roth McDuffie, and Wohlhuter (2010) explain that teachers need to be able to "adapt, supplement or omit portions of the [provided] curriculum materials" (p. 307). This study will shed light on the way that SMPSTs reason about curriculum in their field placement, and will allow those involved with teacher preparation, such as faculty members and field placement supervisors, to change the way they help SMPSTs learn to make decisions about curriculum.

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DEVELOPING A FRAMEWORK FOR OPPORTUNITIES TO LEARN ABOUT EQUITY IN SECONDARY MATHEMATICS TEACHER EDUCATION PROGRAMS

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In this poster presentation we will describe a framework developed to investigate the topics, themes, and types of emphases related to equity that preservice teachers (PSTs) experienced in five secondary mathematics teacher education programs. Our aim is to describe the motivation and development of this framework and its affordances and constraints.

The National Council of Teachers of Mathematics (2000) included “equity” as one of its six core principles in *Principles and Standards for School Mathematics*, but in teacher education programs PSTs often learn about equity issues independent of mathematics. Future mathematics teachers need an understanding of equity as it relates to mathematics in particular (Gutiérrez, 2009; Gutiérrez, 2012b). Thus, it is important to examine the required courses within secondary mathematics teacher education programs to investigate the topics, issues, and learning opportunities related to equity. We are also interested in the nuances of equity in teaching and learning algebra. This is motivated by descriptions of algebra as a civil right and a gatekeeper (Moses & Cobb, 2001).The framework we developed uses Gutiérrez’s (2012a) conception of equity and levels of specificity to algebra to answer the following: What is the nature of opportunities that secondary mathematics PSTs have to learn about issues of equity in their teacher education programs?

The data used in this is study is from a larger *National Science Foundation*–funded research project, *Preparing to Teach Algebra*. This project investigated how emphases on algebra were reflected in teacher preparation programs. Here we focus on the case studies of five teacher education programs. We examined all instructor interview transcripts and tagged any excerpts in which the participants mentioned issues related to equity; then we categorized tagged sections using Gutierrez’s equity constructs (access, achievement, identity, and power) and specificity to algebra (e.g., algebra, mathematics, general education). To promote reliability, we had two researchers code each transcript and discuss discrepancies to come to consensus; in the process, a coding book of examples was also developed. Preliminary coding experiences revealed successes and challenges when using the framework. In our presentation we will discuss results of analyses and the application of our two-dimensional framework.

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DEVELOPING PRESERVICE ELEMENTARY TEACHERS' KNOWLEDGE OF THE CONNECTIONS BETWEEN FRACTIONS, DECIMALS, AND WHOLE NUMBERS

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Keywords: Mathematical Knowledge for Teaching; Number Concepts and Operations; Rational Numbers; Teacher Education-Preservice

Arithmetic involving whole numbers, fractions, and decimals is a significant component of the content of elementary mathematics. Prior research has found that preservice elementary teachers (PSTs) tend to rely on memorized rules and procedures for solving arithmetic problems and struggle to explain why the procedures work (Thanheiser, Browning, Edson, Kastberg, & Lo, 2013). Ensuring that prospective elementary teachers have the opportunity to develop a deep understanding of these areas is thus an important part of teacher education.

Participants in this study were 32 undergraduate, prospective elementary teachers enrolled in the first of two required mathematics content courses at a large, Midwestern university. This study aimed to examine the PSTs' understanding of the relationships between fractions, decimals, and whole numbers before, during, and after their participation in an eight-week instructional unit related to these topics. The goals of this study were to (a) document the ways in which individual PSTs understood the *connections* between fractions, decimals, and whole numbers, and (b) the ways that models such as stories, pictures, and number lines helped and/or hindered the PSTs in making these connections. Two theoretical frameworks guided this study. First is Gravemeijer's (1994) work on the ways models (i.e. situations, drawings, empty number lines, ways of notating) may be intentionally used to support student understanding of formal mathematical concepts. Second is the notion that the development of mathematical knowledge for teaching should entail helping PSTs develop "coherent and generative understandings of the big mathematical ideas that make up the curriculum" (Silverman & Thompson, 2008, p. 5).

Data sources included pre- and post-tests, classwork, and homework from participants as well as field notes taken by the researcher (who was also a co-instructor during the instructional unit). Using qualitative techniques, data were analyzed for patterns in student responses to tasks designed to support students in making connections between the number types, and for the types of models used by students in their work. Preliminary results suggest that the majority of the students developed a more connected understanding of the three number types during the unit, and that stories and pictures were particularly supportive models for many students. Number lines were found to be both supportive and problematic as a model, particularly for the connections between decimals and fractions.

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ANALYSIS OF SINGULAR-PLURAL DIALOGUE OF MATHEMATICS TEACHERS

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Keywords: Classroom Discourse; Teacher Beliefs

Introduction

The purpose of this paper is to describe the results of an empirical comparison of singular-plural usage during lessons by specially chosen mathematics teachers (whose classrooms demonstrated healthy student dialogue) with the usage seen in "typical" American mathematics lessons. Recent reform recommendations emphasize the importance of mathematics classrooms as learning communities where students have a chance to actively discuss their nascent understandings (e.g., NCTM, 2000). I hypothesized that in classrooms with such healthy student dialogue, teacher dialogue would manifest low singular (*me*, *I*, and *my*) as opposed to plural (*us*, *we*, and *our*) usage. Language often reflects the dominant paradigm guiding classroom activity: If teachers view themselves as the primary content deliverer, using singular terms more frequently than plural is expected; if, however, teachers view themselves as members of a developing mathematical community, shifts in descriptions of self should be manifested.

Methods and Results

The teachers demonstrating classrooms with healthy student dialogue participated in *The Mathematics Class as a Complex System* (MCCS) study, described in greater detail elsewhere (Ricks, 2007). For comparison, I chose the four American lessons highlighted in the 1999 Trends in International Mathematics and Science Study (TIMSS) video study as "typical" classes (National Center for Educational Statistics, 2003). Analysis of classroom dialogue (one lesson per MCCS teacher) demonstrated that the MCCS teachers had similar singular-plural usage with each other (average: .42), which varied substantially from the TIMSS mathematics teachers (average: 1.26). As a benchmark, a singular-plural ratio of 1.0 would indicate equal singular and plural use. Thus, all the MCCS teachers had low singular-plural usage (using singulars less than half as much)—as hypothesized, while the "typical" American teachers generally had a high singular-plural usage (using singulars more than plurals, some higher than twice as much).

Discussion and Conclusions

I interpret these findings as evidence that—compared with typical U.S. teachers—the MCCS teachers (with their classrooms manifesting healthy student dialogue) were more likely to use language that signaled their identification as part of a robust classroom community. This study contributes to the growing body of research on teacher dialogue by demonstrating a correlative relationship between classrooms manifesting healthy student dialogue and low teacher singular-plural usage. More research is needed to understand the observed statistical relationship; examining student singular-plural usage may also indicate their perceived participation, as well.

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EXPLORING MENTORING CONVERSATIONS: WHAT MAKES THEM MORE OR LESS EDUCATIVE?

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One way that mentors support prospective teachers' development within field placements is through conversations about classroom teaching. However, little is known about these conversations and the extent to which they are - or can be - educative. This study bridges the literature on educative mentoring (Feiman-Nemser, 2001) and teacher noticing (van Es and Sherin, 2002) by considering *educative conversations* as a product of both professional practices. For this study, I theorized a definition of what an educative conversation would include and used the definition's criteria as indicators to identify *potentially educative* mentoring episodes in a set of pre/post mentor-intern conversations about video teaching episodes. I report on the nature and quality of these conversations, and I consider the affordances of using noticing and wondering language in making mentoring conversations more educative.

This qualitative study examines three mentor-intern pairs' pre- and post-conversations about two publicly available video teaching episodes (BTSA, 2012). They did this before and after a short orientation session which introduced them to Smith's (2008) *noticing and wondering* model. The mentor-intern conversations were coded using the educative conversation criteria, defined as mentor-intern talk in which the purpose is to provide opportunities for novice teachers to: (a) learn in and from practice, (b) generalize from particular instances of their teaching practice; and/or, (c) focus their attention on student learning. The frequency of the terms "notice" and "wonder" was considered to evaluate the usefulness of this language. Out of the 86 educative mentoring episodes, 41% were identified as *potentially more educative* mentoring episodes, because they met all of the criteria, and 59% were identified as *potentially less educative* mentoring episodes, because they met only two of the three criteria. Further comparison of individual pairs' pre- and post-conversations demonstrated increases in the number of *potentially educative* mentoring episodes, improved quality of educative conversations, and increased usage of the words "notice" and "wonder."

Overall, understanding more about educative conversations is crucial both for prospective teachers to maximize field placement learning and for teacher educators and mentors to develop more intentional mentoring practices that support the development of mathematics teaching practices. This study acts as a starting point for this work by providing language, criteria, and examples for discussing conversation quality with teacher educators, mentors, and interns.

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A CURRICULAR ACTIVITY SYSTEM USED IN AN URBAN SCHOOL DISTRICT

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The Common Core State Standards for Mathematics (CCSSM) has provided the field of mathematics education with clear and consistent mathematics content foci, as well as practices for students to engage in (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). As mathematics teachers have begun to teach to these standards, they have been challenged to consider curricula that address the content foci and mathematical practices outlined in the CCSSM as well as, investigate research-informed teaching practices that allow their students to engage in mathematical reasoning and sense making. One particularly useful framework for addressing the standards and teaching practices is to embed technology into a progression of key learning activities through a curricular activity system (Vahey, Knudsen, Rafanan, Lara-Meloy, 2013).

The preliminary results of this study provide evidence that a curricular activity system in which teacher professional development is aligned with a technology-embedded curriculum impacts teaching of rate and proportionality. By having the participating teachers explore the mathematics in the learning module as learners prior to teaching it there was "an explicit focus on subject matter" by providing experiences that "engage teachers as learners" (Borko, 2004 p. 5). This allowed the teachers to leverage their mathematical knowledge as a way to explore their teaching practices, to provide insight into students' mathematical thinking, and as way to transform their teaching. Moreover, middle school students developed a deep understanding of crucial mathematics when their teachers engage in teacher professional development that is coherently aligned to a technology-embedded learning module.

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STUDYING FRACTION CONTENT AND PEDAGOGICAL KNOWLEDGE GROWTH AFTER A VIDEO-BASED INTERVENTION

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Research has shown that there is much to be gained by teachers studying video episodes of children learning mathematics (e.g., Maher, 2008). Video provides learning opportunities for pre-service teachers to view certain teaching practices in action which may not be visible to them based on their field placement (Philipp et al., 2007) as long as the selected video clips align with the instructional goals of the teacher education context (Zhang et al., 2011).

This poster will describe a video-based intervention given to 137 pre-service teachers (89 experimental and 48 control). During the intervention, participants in the experimental group worked on open-ended problem solving tasks involving Cuisenaire rods and watched videos from a major video repository, the Video Mosaic Collaborative (http://www.videomosaic.org), of fourth grade children working on the same tasks as them. As a part of the tasks, the participants in the study and in the video were asked to construct arguments for their solutions and justify them to their classmates. Participants in the control group learned fractions in ways consistent with the NCTM standards, but did not engage with the tasks or videos.

Both groups were administered a pre- and post-test to measure growth in their fractional content knowledge after learning about fractions. Participants in the experimental and control both showed significant growth in their fractional content knowledge. Several questions on the assessment required the participants to explain how they would help students who held common misconceptions that arise during the teaching of fractions (e.g., adding 1/5 + 4/17 and getting 5/22). Pre-service teachers who underwent the video-based intervention showed significant growth in their ability to help the students who held the misconceptions by building conceptual understanding in the student (e.g., using benchmark numbers), whereas the control group members provided algorithmic explanations (e.g., teach about common denominators by multiplying) on both the pre- and post-assessment. This study underscores the value of using classroom video in teacher preparation and professional development interventions. Videos that focus on student reasoning, along with providing teachers time to work on the same problems as the students in the video, can be useful in preparing teachers to help their students understand mathematical concepts rather than teach them algorithmic approaches.

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DEVELOPING A JOINT KNOWLEDGE BASE FOR NOTICING STUDENTS' PRIOR KNOWLEDGE WITH ANIMATED CLASSROOM VIGNETTES

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The study aims to identify ways teachers related a set of animated vignettes to their classroom practice to develop a joint knowledge base amongst the participants (Hiebert, Gallimore, & Stigler, 2002). In order to investigate this question, video (and subsequent transcripts) from two teacher study group sessions were analyzed using the modified version of Toulmin's (1969) model of argumentation developed by Chazan, Sela and Herbst (2012).

The data comes from a professional development program for geometry teachers from high-needs schools to promote teacher noticing of students' prior knowledge. The research team created animated vignettes to provoke discussion of classroom practices (Herbst, Nachlieli, & Chazan, 2011). The study analyzes the discussion of vignettes illustrating possible launches for a problem-based lesson on dilations. The main question is, *what claims and justifications do the teachers provide in reaction to the vignettes, and how do the claims and justifications relate to their classroom practice and knowledge of students' prior knowledge, if at all?* This question addresses whether and how the teachers notice students' prior knowledge when viewing and discussing representations of teaching.

During the discussion of three launches, there were 33 claims, with 27 explicit justifications. In the discussion of the first launch, the teachers claimed that several aspects of the launch would resonate with their own students' prior knowledge, such as drawing 3D figures and playing video games. The teachers also provided rebuttals of others' claims when they were not representative of their own classroom experience. In the discussion of the second launch, one teacher shared her experience of using a similar problem to the one shown in the vignette, and how her own students used their prior knowledge of the buildings in their town to make sense of the problem. In the discussion of the third launch, the teachers provided several claims about their students' prior knowledge of vocabulary and experiences with video games and how that might affect their ability to relate to the launch. Findings show that the study group discussions provided the opportunity for the teachers to share their practitioner knowledge as they connected the vignettes to their classroom experiences, and their understanding of their own students' prior knowledge. This study is relevant in identifying resources for eliciting teachers' practical knowledge and promoting its transition towards professional knowledge. Understanding how professional development activities, such as discussions of representations of teaching, can provide opportunities for developing a joint knowledge base is fundamental in teacher education.

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MATHEMATICAL KNOWLEDGE FOR TEACHING AND REASONING ENTAILED IN SELECTING EXAMPLES AND GIVING EXPLANATIONS

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Recent attempts to improve mathematics teaching include two prominent lines, teachers' knowledge and teaching practices. Much of the research on mathematical knowledge for teaching has focused on understanding the knowledge entailed by the work of teaching (e.g., Ball, Thames, & Phelps, 2008; Baumert et al., 2010). This line of research has been fruitful, yet there is much left to understand about how teachers deploy their knowledge in doing the work of teaching. Likewise, research on teaching practices does not directly examine how teachers use content knowledge in the context of enacting these practices (Grossman, Hammerness, & McDonald, 2009). Despite this research disconnect, in the classroom, teachers' knowledge and teaching practices are intertwined. This poster describes a study asking: What knowledge and reasoning are entailed in the teaching practices of selecting examples and giving explanations? It draws on the theoretical framework of mathematical knowledge for teaching (Ball et al., 2008) and is part of a larger study, which also includes classroom observations.

Ten experienced Algebra II teachers participated in cognitive interviews, focused on the content of rational expressions and equations. They were presented with tasks where they were either asked to pick examples to teach a particular topic or to give an explanation they would use with students of a particular concept. For example, one task asked teachers to select example problems to teach simplifying rational expressions. Data were analyzed across two dimensions, components of each teaching practice and teachers' knowledge and reasoning, looking for patterns within each dimension and across the two practices.

The findings reveal that the knowledge and reasoning entailed by each task are unique to the task. In particular, they suggest that knowledge and reasoning cannot be separated from the particular tasks of teaching in which they are deployed. Although tied to particular tasks, teachers' knowledge and reasoning varied widely in selecting examples and in preparing and giving explanations. For example, on the item focused on selecting problems for teaching simplifying rational expressions, one teacher selected problems based on particular features of individual problems, such as a negative leading coefficient, or values that were less common for students or might evoke a common misconception. In contrast, another teacher described more broadly the types of problems he would want to use and then selected a problem that met each set of criteria. Full results will be displayed on the poster.

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USING JOURNALS TO SUPPORT LEARNING: CASE OF NUMBER THEORY AND PROOF

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This study investigated the questions: (1) How do journals support students' learning to prove in an undergraduate elementary number theory course? (2) How do journals demonstrate the development of students' thinking about proof in an undergraduate elementary number theory course? Students' performance in proof writing has been investigated in recent years (Weber, 2001; Raman, 2003); however, research is still needed to investigate innovative pedagogical approaches to teaching proof and how students' thinking about proof develops. Journaling has been shown to be a valuable tool for supporting students' learning and providing insight into students' thinking in various mathematical domains (Borasi and Rose, 1989). This study seeks to gain a better understanding of the power of journals in developing student thinking, and suggests possible ways to connect journal writing more closely to proof writing as an integral part of learning mathematics.

Methods

The 17 undergraduate students in the course wrote weekly journal entries related to the elementary number theory course and submitted them online. The instructor of the course then read and responded to each student's submission. The journal assignments consisted of both structured and unstructured prompts. Our data collection included pre-post surveys asking the students' views about mathematical proof and their perceptions about journals; students' open-ended and structured journal responses and task-based interview transcripts with five of the students. The data were coded according to the framework provided in Borasi and Rose (1989) for the journaling component, and the students' proof attempts from the task-based interviews were coded using Raman's (2003) framework for students' proof ideas.

Results

Our findings suggest the students used the unstructured journal assignments primarily as a means to reflect on their feelings about the course material and their learning. However, in the structured journals, the students wrote specifically about the process of proving and their views about mathematics and proving. Implications of this study suggest that journaling creates an added dimension of communication for students and the instructor to support students learning the course material in a more responsive manner. Future studies will investigate the role of directed and more structured questions to focus student journals on specific mathematical ideas.

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WHAT FIDELITY MEANS FROM A SOCIO-PSYCHOLOGICAL PERSPECTIVE: SITUATING KNOWLEDGE CONSTRUCTION IN MATH PROFESSIONAL DEVELOPMENT

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This study takes a socio-psychological approach (Wertsch, 1991) to examine how mathematics professional development facilitators and teachers enact a professional development program (PD) in the absence of strict fidelity expectations. In this view, how individuals make sense of their world is always mediated by their own socio-historical past knowledge as well as their own current and envisioned goals for the knowledge. Juxtaposed against this view of how adults make sense of new information is the construct in professional development of *fidelity*. Though often a major concern of educators and policy makers, ensuring fidelity of implementation can be an elusive goal if we are to understand knowledge construction as being culturally mediated and situated within local contexts. This mediation takes place on multiple levels: (a) as facilitators are trained by PD designers/developers, (b) as these facilitators then train teachers, and (c) as teachers instruct students.

If we are to understand the impact of PD, it is important to know more about these inevitable adaptations. We seek to shed light on this process by examining the way that a single PD is iterated across various regions in Texas and then how that PD is enacted across various schools and districts in each region as it is implemented in the classroom. Two research questions guided us: 1) What changes/adaptations are made each time the PD is implemented? 2) What reasons do participants give for making those changes? To investigate these questions, facilitators and teachers were administered parallel surveys after being trained and after implementing the PD in their own trainings (facilitators) or classrooms (teachers). Questions on these surveys mirrored one another so that comparisons could be made across the four surveys.

Open-ended responses were coded according to particular aspects of the original PD. Descriptive analyses of the data yielded a portrayal of the variations between sites at each level of implementation. Conditional probability analysis supplemented the description of what is implemented of the PD. Emergent themes from open-ended responses indicate distinct categories of interpreting and enacting PD that seems to be at least partially determined by contextual factors. These results have implications for the design and implementation of PD and how to support facilitators as decision-makers who are responsive to the contexts in which they teach.

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LAUNCHING PROBLEMS: EXPANDING TEACHERS' SCHEMA FOR STUDENT AND TEACHER RESPONSES

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Keywords: Teacher Education-Inservice; Teacher Knowledge

Mathematics education researchers have long called for the use of rich tasks (Stein, Grover, & Henningsen, 1996). More recent work has identified challenges involved in launching such tasks (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013).

Cognitive science researchers have shown that experts and novices alike rely on *schema* to make sense of situations and inform decision-making (Carter, 1990). The long prevalence of initiation, response, evaluation (IRE) in American classrooms (Franke, Kazemi, & Battey, 2007) implies that teachers' schema for student thinking (right or wrong) and teacher responses (praise or correction) are relatively simplistic.

We sought to answer the following research questions:

- Would engaging in a multiple-choice activity involving diagnosing and responding to students' thinking enlarge participants' schema for student thinking and teacher response?
- Would this activity lead to participants adopting common language around student thinking and teacher responses in the context of a lesson launch?

The researchers created an experience on LessonSketch, an online computer environment that allows participants to view cartoon storyboards of teaching situations. In the experience participants encountered typical student responses to a proportional reasoning task. Then they chose among several diagnosis and response categories that we had chosen, based on research of teaching and student thinking. We hypothesized that having to think about the categories several times in the context of analyzing a lesson launch would support participants in developing more complex schema for diagnosis and teacher response.

Participant data indicate that the LessonSketch experience did help teachers and PST's develop richer, and more uniform schema of student reactions and teacher moves. This approach challenges teacher educators to continue to identify helpful schema for launching rich tasks.

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PROSPECTIVE TEACHERS AND *COMMON CORE STATE STANDARDS FOR MATHEMATICS:* ACTIVITIES USED BY MATHEMATICS TEACHER EDUCATORS

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Keywords: Standards; Teacher Education-Preservice

Studies of mathematics methods courses show substantial variation in activities intended to prepare teachers to teach mathematics (e.g., Kastberg, Sanchez, Edenfield, Tyminski, & Stump, 2012; Taylor & Ronau, 2006). However, the wide-spread adoption of the Common Core State Standards for Mathematics (CCSSM) (NGA & CCSSO, 2010) offers the possibility of more consistency across courses (e.g., Heck, Weiss, & Pasley, 2011) as mathematics teacher educators (MTEs) take up a common focus on teaching about the content standards and standards for mathematical practice in CCSSM. Our study sought to consider the variations in activities MTEs use in mathematics methods and content courses. This study is part of a larger project examining CCSSM in mathematics teacher preparation. We surveyed 370 MTEs who taught elementary, middle school, or secondary methods or content courses. This poster presents findings from analysis of 269 descriptions of activities used to teach PSTs about CCSSM.

Consistent with other studies, our findings show a wide variation in activities, including examining the standards documents (6.5%); watching videos or observing teaching to identify standards (6.5%), analyzing curriculum for alignment with CCSSM (2.3%), and engaging in a task consistent with the standards (8.8%). The activity with the most responses was preparing for or creating lessons or units using CCSSM (14.1%). In addition, many MTEs provided responses that described a mathematical or pedagogical activity, but did not elaborate how the activity was connected to CCSSM (8.6%). MTEs may have felt that the connection to CCSSM was implied or obvious and so needed no elaboration. Also interesting were the small but still not negligible number of MTEs (1.4%) who do not incorporate any CCSSM-related activities in their courses. Our findings suggest that MTEs see many ways in which PSTs should interact with CCSSM. As such, while CCSSM may provide a common area of focus of MTEs, it does not seem to narrow or restrict instructional activities nor result in more consistency across courses or institutions.

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Chapter 10

Teaching and Classroom Practice

Research Reports

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Brief Research Reports

Poster Presentations

MATHEMATICS, LANGUAGE, AND DEGREES OF CERTAINTY: BILINGUAL STUDENTS' MATHEMATICAL COMMUNICATION AND PROBABILITY

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While all mathematics is mediated by language, the role of language is especially complex in bi- and multilingual mathematics classrooms, and more so in bilingual education programs in which the explicit goals of both language and mathematics learning intersect. We explore bilingual French immersion students' linguistic and mathematical repertoires as they work through a series of probability problems. Focusing on the collaborative dialogue that occurred between students and researchers, our discourse analysis was informed by sociocultural theory and systemic functional linguistics. Findings indicate that students' linguistic and mathematical repertoires are intertwined, and that collaboration can offer opportunities for supporting bilingual learners' language and mathematics development. We conclude with implications and challenges for bilingual mathematics education.

Keywords: Classroom Discourse; Instructional Activities and Practices; Probability

One of the most novel characteristics of French immersion programs is that students whose first language is not French learn this target language not only through French language classes but also through content courses such as mathematics. Both mathematics learning and language learning are explicitly stated goals of the immersion program (Swain & Johnson, 1997). Around the world, students learn mathematics through languages other than their first or home language(s) in a variety of bi- and multilingual mathematics classroom contexts. Consequently, it is important for researchers and educators to examine how mathematics and language learning simultaneously take place in the classroom.

Theoretical framework

Work by Halliday (1978) and others has brought to the forefront issues related to language in mathematics and, more specifically, the mathematics classroom. Halliday described three aspects to consider with regard to any linguistic situation, including mathematical discussion: "first, what is actually taking place; secondly, who is taking part; and thirdly, what part the language is playing" (p. 31). Drawing on these ideas, we focus on mathematics as a meaning-making activity rooted in the social interactions of the learners. The notion of the mathematics register describes this in more detail: The mathematics register involves "the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes" (p. 195).

From a second language education standpoint, we adopt a theoretical framework that supports our view of mathematics and language. We draw on the work of Vygotsky (1962, 1978) and neo-Vygotskians (Cole, 1985; Donato, 1994; Lantolf, 2000; Lantolf & Appel, 1994; Swain, 2000; Swain, Kinnear, & Steinman, 2011; Wertsch, 1985, 1993), which has underscored the social element driving all individual cognitive functions. Moreover, this work has emphasized the key role language plays in all human interactions and learning. Language is viewed as a mediational means, in other words, language mediates thought and is not strictly a conveyor of thought. This approach, rooted in the exploration of language and learning through social interactions, guides our analysis of mathematical discourse.

Selected literature

Our review of selected literature has two main parts: first, a discussion of studies of mathematics and language from the mathematics education field, with particular focus on those based in bi- or multilingual contexts; and second, an exploration of some key sociocultural concepts related to second language education.

Mathematics and multilingual classrooms

Research based in a variety of multilingual mathematics classrooms has highlighted a number of important issues with regard to mathematics and language. In particular, scholars have pointed to a need to recognize that the mathematics register is enacted in unique ways within the mathematics classroom. The specialized language of the mathematics classroom is distinct from the specialized language of mathematicians (Barwell, 2005, 2007, 2009b; Barwell, Leung, Morgan, & Street, 2005; Morgan, Craig, Schüte, & Wagner, 2014; Moschkovich, 2003, 2007, 2010; Pimm, 2007; Setati & Adler, 2000). This work has underscored the importance of classroom context, and has viewed mathematics as a social, discursive activity. From this standpoint, "mathematical discourse includes not only ways of talking, acting, interacting, thinking, believing, reading, writing but also mathematical values, beliefs, and points of view" (Moschkovich, 2003, p. 326). Far from a homogeneous set of practices and norms, some general characteristics of mathematics classroom discourse can include "being precise and explicit, searching for certainty, abstracting, and generalizing,[… and] imagining" (Moschkovich, 2003, p. 327).

With regard to bilingual learners in particular, research has called for a refocusing on the resources these learners bring to the mathematics classroom rather than on their so-called problems or deficiencies (Barwell et al., 2005; Moschkovich, 2003, 2007). In this vein, studies have suggested that allowing for ambiguity, or the acceptance of multiple meanings, during mathematical collaboration can be a resource for mathematical understanding, particularly for bilingual students (Barwell, 2005). Moreover, hearing the mathematical in students' so-called everyday talk is also key to supporting bilingual learners, who may use this everyday talk to contextualize and understand the linguistic and mathematical aspects of problems (Barwell, 2009a; Moschkovich, 1999, 2003, 2005, 2009a, 2009b). This approach does not view mathematical and language learning as separate, but rather as intertwined and co-developing in a reflexive relationship (Barwell, 2005).

A further concept for consideration in bilingual mathematics classrooms is if, when, and how multiple languages are used and valued (or not). Often referred to as codeswitching, that is, the switching of languages "within the course of a single conversation, whether at word or sentence level or at the level of blocks of speech" (Baker, 2011, p. 107), this phenomenon has been explored in both mathematics and second language education contexts. As scholars in both fields have explained, historically codeswitching has been perceived as indicative of a deficiency with regard to bilingual students' mathematics and language proficiency. Recent work, however, has challenged this view. Researchers have argued for a positive, resource-oriented view of bilingual learners that recognizes the resources they bring to the mathematics classroom and, in line with sociocultural theory, this may include learners' first or home language(s). This approach challenges the monolingual norm and views codeswitching as socially and cognitively complex. However, codeswitching remains a contentious and controversial issue and the use of multiple languages in the mathematics classroom often conflicts with political agendas and language policy goals (Adler, 1999; Barwell, 2009b, 2014; Cummins, 2007; Moschkovich, 2005; Planas & Setati-Phakeng, 2014; Swain & Lapkin, 2000; Turnbull & Dailey-O'Cain, 2009; Setati & Adler, 2000).

Second language learning through content

With regard to second language learning in mathematics, two key sociocultural concepts emerge that are pertinent to our analysis. The first is the notion that as learners interact with a more capable other, who could be their teacher or their peers, they can achieve more than would have been possible on their own. In this scenario, the learner eventually gains control over the task, internalizes the skill, and is able to perform it independently. This movement from other- to self-regulation is described as what happens in the zone of proximal development (Vygotsky, 1978). It relates to a pedagogical notion called scaffolding, in which a temporary scaffold provided by an expert other is used to help learners with a particular learning task (Cole, 1985). The scaffolding is eventually dismantled as the learner becomes more capable and the responsibility for the task is gradually transferred from expert to learner.

The second pertinent concept is the notion that language learning occurs during collaborative dialogue. According to Swain (2000), collaborative dialogue "is where language use and language learning can co-occur. It is language use mediating language learning. It is cognitive activity and it is social activity" (p. 97). In this view, when language learners engage in problem-solving tasks they are able to notice and pay attention to linguistic elements and co-construct knowledge through producing output through collaborative dialogue. These language-related episodes mediate the learners' understanding of the problems and solutions (Donato, 1994; Swain, 2000; Swain & Lapkin, 1998).

The study

The current study is framed within a larger, 3-year longitudinal study entitled "Students' language repertoires for investigating mathematics" (supported by the Social Sciences and Humanities Research Council of Canada, Principal Investigator: David Wagner). In this paper, we focus on bilingual French immersion students' linguistic and mathematical repertoires during collaboration with an interviewer-researcher on probability-related problems and activities. (We have discussed different aspects of the larger study elsewhere. See, e.g., Culligan, Dicks, Kristmanson, & Wagner, 2014; Wagner, Dicks, & Kristmanson, 2015.)

Context and participants

The participants in the current study were Grade 3 French immersion mathematics students in their first year of the program. Students first engaged in a whole-class probability-based activity (Skunk die game, described in the next section) and then worked on related problem-solving tasks in small groups of two to three. As a follow-up, students interacted with an interviewer-researcher as an extension of the whole-class activity. During these interviews, students were introduced to a second probability-related activity (Skunk card game, described in the next section) and responded to questions related to the two games. The students were asked about their strategies for playing both games, about the differences between the two games, and about different words of interest (related to probability and degrees of certainty) they had used while responding to these questions and/or engaging in the problem solving.

The probability activities

In both the Skunk die game and the Skunk card game, students were introduced to the problem with a narrative: You are picking berries in the forest and trying to collect as many berries as possible before the skunk comes. Numbers 1 to 5 represent the berries you collect on each roll of the die. The number 6 represents the skunk and the end of the turn. If the skunk comes, you lose all of the berries you collected on that turn, unless you have "gone home" to avoid the skunk when you had enough berries. You do this for seven days (Monday to Sunday). The player with the most berries at the end wins the game.

In the Skunk card game, the interviewer-researcher laid playing cards (numbered 1 to 6, with 6 being the skunk) out on the table one by one, rather than rolling a die. The cards were not picked up once laid down. Rather, the interviewer-researcher continued laying down cards one by one as long as the student wished to continue. We moved on to the next day of the week once the students had decided to stop collecting berries for the current day, or once the skunk card was played. Cards were not picked up and reshuffled until all six cards had been laid down; this could happen in the middle of the current "day." Thus, the probability of getting the skunk on any given turn differs in the card game compared to the die game. In the card game, the events are mutually exclusive, and in the die game, the events are independent.

Data collection and analysis

Students were audio and video recorded during the whole-class activity and the follow-up interviews. Data were transcribed and written transcripts were the primary source for analysis. We analyzed the data using Swain and Lapkin's (1998) approach to discourse analysis, which entails describing and interpreting language-related episodes. Furthermore, we drew on the field of systemic functional linguistics (e.g., Halliday, 1994), which enabled us to describe and interpret specific instances of language use within our particular context.

Results

To discuss our results, we present selected excerpts of transcripts from the students' interviews with an interviewer-researcher and offer our interpretations.

Excerpt 1: Linguistic and mathematical uptake of "*absolument*"

The following is an excerpt from a Grade 3 interview. This is the first year of French medium learning for these children. English translations are provided on the right. The interviewer-researchers (R1 and R2) are asking the students (S1, S2, and S3) their predictions regarding the upcoming cards and their degree of certainty regarding these predictions. One researcher (R1) leads the interview and the other (R2) is behind the camera, taking note of students' language use and then participating in the interview later.

115	*R1:*	Est-ce que c'est absolument le quatre?	*Is it absolutely the four?*
116	*S3:*	Oui.	*Yes.*
117	*R1:*	Est-ce que tu es certain que c'est le quatre?	*Are you certain it's the four?*
118	*S3:*	Oui, non.	*Yes, no.*
119	*S3:*	Ça peut être une trois aussi.	*It could be a three too.*
120	*R2:*	Quelles sont les chances que ça soit un trois?	*What are the chances that it's a three?*
121	*S2:*	Beaucoup….	*A lot…*
…			
166	*R2:*	Alors ça doit être quoi ici?	*So it has to be what here?*
167	*S2:*	Il doit être, un, deux trois, quatre, cinq.	*It has to be one, two, three, four, five.*

168	*R2:*	Ça doit être des fraises (et non pas la moufette)?	*It has to be berries (and not the skunk)?*
169	*S1, S2, S3:*	Oui.	*Yes.*
170	*R2:*	Absolument des fraises?	*Absolutely berries?*
171	*S2, S3:*	Oui.	*Yes.*
172	*S1:*	Absolument.	*Absolutely.*

In this exchange, we see linguistic uptake of *absolument* (used by R1 line 115, R2 line 170;taken up by S1 line 172). Moreover, there is mathematical uptake of *absolument*, a concept related to probability. The students go from being very certain (line 116), to questioning/hedging (line 118), to using a modal expressing a greater degree of uncertainty (line 119). Throughout the exchange, the researcher-interviewer acts as a more knowledgeable other, providing scaffolding and pushing students to go farther than they may have done on their own. Notably, however, the questioning of the student's response did not lead the student to change her answer ultimately—she worked through the task and decided she was *absolument certaine*. Here, mathematics and language work together to solidify the students' understanding of the probability concept.

Excerpt 2: Explaining the meaning of "*çadoit*"

In this Grade 3 excerpt, the interviewer-researcher (R1) is asking the students (S1, S2, and S3) the difference between "it has to be a 6" (card game) and "you have to brush your teeth" which present different senses of the modal verb "have to"—one indicating logic and the other obligation. The students relate this distinction to the English expressions "it is going to be a 6" and "you are going to brush your teeth." (In the translation at right, the underlined text is not translated because it is English in the original.)

388	*S1:*	En anglais « doit » dans la première phrase, ça doit être une moufette.	*In English "has to" in the first sentence, it has to be a skunk.*
389	*S1:*	Et, dans l'anglais, ça veut dire « it's going » and, dans l'autre phrase, ça dire « you have to. »	*And, in English, it means "*it's going*"* and*, in the other sentence, it means "*you have to*".*
390	*R1:*	« You have to », comme tu doit te brosser les dents et « it's going. »	*"You have to", like you have to brush your teeth and "*it's going*."*
391	*S2:*	Tu n'as pas une choix.	*You don't have a choice.*

Here, the students use their first language, English, to clarify their ideas. The first language seems to provide them with resources to strengthen and confirm their explanation that there is a difference between the two uses of "have (has) to". Students use English to clarify or confirm their interpretation of the French expression "*çadoit*". Comparing "it has to be a six" in the card game, in which students knew the next card "had to" be the skunk (it was the only card left to be played), to the sentence "you have to brush your teeth" was a cognitively challenging activity both mathematically and linguistically. In the last line, S2 raises the question of choice, which is inherent in the “you have to brush your teeth” example, but not in the “it has to be the skunk” example. Students use their first language as a tool for discussing the multiple meanings of “*çadoit*” and, in so doing, construct both mathematical and linguistic understanding.

Discussion

Our results highlight the mathematical and linguistic understanding that can occur during collaborative dialogue in the bilingual mathematics classroom. When viewed through a sociocultural theory lens, in the first excerpt, the learners, through the scaffolded guidance provided by the interviewer-researcher, are able to go farther, mathematically and linguistically, than they may have been able to individually. Through the interviewer-researcher’s introduction of the term “*absolument*”, students are able to pick up that language and use it to explore the mathematical concept of certainty. Similar to the reflexive relationship described by Barwell (2005), in this study students’ mathematical understanding of the probability-related concept of certainty develops in an interwoven fashion with their linguistic understanding.

In the second interaction, students engage in a phenomenon that is of particular interest to many working in bilingual mathematics classrooms—codeswitching. Despite some traditional, deficit-oriented views of codeswitching, recent research in the field of second language education has argued that in the classroom, judicious use of students’ first language can serve as a resource for second language learning (e.g., Cummins, 2007; Swain & Lapkin, 2000; Turnbull & Dailey-O’Cain, 2009). Moreover, research in mathematics education has argued that bi- and multilingual learners use their first language, home language(s), or shared language(s) as a resource for mathematical learning and that it plays an important social and political role (e.g., Adler, 1999; Barwell, 2014; Moschkovich, 2005; Planas & Setati-Phakeng, 2014). Barwell and Setati (2005), for example, have urged mathematics educators to find “ways of dealing with linguistic diversity that avoid reducing mathematics classroom interaction to a monolingual (English language) norm” (p. 23). Although the research contexts referred to here are varied and each is unique, codeswitching is a phenomenon that seems to occur throughout. A sociocultural theoretical framework that views language as a mediator of thought and as a cognitive tool, allows us to view students’ codeswitching in this study as a resource for mathematical and language learning, rather than a problem or deficit to be overcome.

Implications and challenges

The two excerpts featured here point toward implications, and corresponding challenges, for mathematics educators working in bi- and multilingual contexts. First, we suggest that providing opportunities for students to engage in collaborative dialogue with each other and with their teacher is important mathematically, linguistically, and socially. Taking the time to allow these interactions to unfold is challenging when faced with the demands associated with covering curriculum outcomes and assessment, but can result in learning that is mathematically and linguistically valuable. It will be imperative for mathematics educators to recognize, value, and build upon the mathematics present in students’ multiple meanings, and in their everyday talk. This is particularly true for bilingual learners. Viewing both mathematics and language as social, discursive activities may help foster collaborative exchanges.

Second, the ways in which multiple languages are used in any context, including the mathematics classroom, are complex. Interpretations of codeswitching practices must take into account contextual, political, and language policy factors. Nonetheless, researchers across contexts are increasingly viewing student codeswitching as a potentially resourceful way of understanding complex mathematical and linguistic content. In spite of this, local policy often dictates that one language only, the target language, be used as the language of teaching and learning in the classroom (and this is certainly the case in our study’s context, French immersion). The challenge will be for researchers and educators to continue to explore in more detail if and how students’ multiple languages can be used in the mathematics classroom, and how to do this in a way that results in effective and efficient language and content learning.

Conclusion

In sum, our results suggest that collaborative dialogue can be a meaningful activity in the bilingual mathematics classroom. In particular, interaction may provide opportunities for bi- and multilingual learners to learn not only mathematical content but also language. Learners can build on the scaffolding provided by teachers and even their peers to extend their understanding of linguistic and mathematical concepts.

We argue for a need to move beyond viewing strictly academic mathematics vocabulary as the only acceptable or valuable mathematical communication. While gaining control over mathematics terminology is without a doubt important, students also need to acquire the language necessary to talk about mathematics. Moreover, language use, language learning, and mathematics learning are largely, if not entirely, inseparable.

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THE DISCOURSE OF ATTENDING TO PRECISION IN SECONDARY CLASSROOMS

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Attending to precision (ATP) is essential in mathematics. This study examined ATP instances through the lens of univocal (functioning to convey information) and dialogic (functioning to generate new meaning) discourse. Analysis of data from five secondary mathematics classrooms focused on whole-class instances of ATP with coding based on the univocal or dialogic nature of the discourse. Although instances were predominantly univocal, there was variation in whether the teacher's or student's idea was being transmitted. We share examples of the rare dialogic instances where the co-construction of meaning through discourse involved ATP in qualitatively different ways than the univocal instances.

Keywords: Classroom Discourse; Problem Solving; Metacognition; High School Education

Within mathematics, precision is highly valued because imprecision can lead to holes in arguments or faulty conclusions and miscommunication can prevent the development of shared meaning. As a steward of the discipline, then, school mathematics must help students learn to attend to precision and recognize that standards of precision are different within mathematical communities than they are in other communities. In the United States, the importance of attending to precision (ATP) was affirmed by its inclusion within the *Common Core State Standards for Mathematics* (2010). Not only is ATP a worthwhile end in its own right but also has the potential to support student learning. For example, by paying careful attention to the precise meaning of algebraic symbols, students can successfully transition from arithmetic reasoning to algebraic reasoning and be prepared for higher levels of mathematics (Kieran, 2007). In general, students, by interacting with teachers and classmates, can express and refine mathematical ideas together, constructing shared meanings and becoming legitimate participants in mathematical discourse (Lave & Wenger, 1991) rather than passive recipients of knowledge.

This study focused on ATP in secondary mathematics classrooms. Our goal was to see a broad range of instances of ATP and analyze the nature of the discourse in those instances within the whole-class public discourse. This work adds to the literature on ATP, which is relatively thin compared to well-developed topics such as problem solving or reasoning-and-proving.

Theoretical Framework

For this study we took a sociocultural perspective wherein student learning is viewed as intertwined with social interactions (Vygotsky, 1978) and used tools from discourse analysis to examine the interactions. In particular, we drew on Herbel-Eisenmann and Otten (2011) who framed the learning of a subject as the process of coming to participate meaningfully in the discourse of that subject's community. We consider ATP to be one of the characteristic practices of the mathematics classroom community as well as the broader mathematics community.

Discourse serves two important functions—to transmit an existing meaning from one person to another and to generate new meanings through the process of interacting (Bakhtin, 1986). The term *univocal* refers to discourse that is primarily intended to fulfill the transmission function, whereas *dialogic* refers to discourse that is primarily intended to generate new meaning. Mathematics classrooms in the United States predominantly feature univocal discourse (National Council of Teachers of Mathematics, 2014), yet various scholars have provided evidence for the value of dialogic discourse (e.g., Lobato, Clarke, & Ellis, 2005; Otten & Soria, 2014) and called for more research on benefits of different forms of classroom discourse (Howe & Abedin, 2013).

Research Question

This study addresses the following question: What is the univocal or dialogic nature of the discourse within instances of ATP in secondary mathematics classrooms? Although the question draws a binary distinction between univocal and dialogic discourse, we recognize that the underlying characteristics form a continuum (Truxaw & DeFranco, 2008) and that all discourse involves both deciphering meaning (univocal) and generating meaning (dialogic). Nonetheless, like others (Peressini & Knuth, 1998), we see practical value in distinguishing between discourse that is more prevalent in one than the other. This distinction is especially appropriate with regard to a mathematical practice such as ATP because scholars (e.g., Barwell & Kaiser, 2005) have pointed out that dialogic discourse is a practice-oriented process within a community.

Method

Setting and Participants

This study is part of a larger project focused on the mathematical practices of reasoning-and-proving, generalizing, and ATP. The project has two interrelated goals: (1) to support mathematics teachers in understanding and implementing these practices, and (2) to better understand how practicing teachers and students at various grade levels engage in the practices (or not) during classroom instruction. We worked with a group of teachers during their summer break to achieve goal 1, and we conducted classroom observations throughout the following academic year to achieve goal 2. This particular study reports on an analysis of ATP within the classroom observation data of the secondary teachers.

The project took place in a rural school district in the central United States. The district is predominantly white in ethnicity but comprises substantial economic diversity and diversity in parental education levels. The district performed slightly above the state average on the secondary mathematics standardized assessment.

Eight mathematics teachers volunteered to participate in the larger project, spanning grades 5–12. The teachers' backgrounds, teaching experiences, and philosophies toward mathematics education varied. For this study, we focused on five teachers (grades 8–12) and one focal class per teacher. We observed the focal classes on typical instructional days in the fall and winter. We intended to conduct three observations per class to capture a range of mathematical topics and gain a sense of the variability of discourse per classroom. When the discourse was limited, two observations were sufficient. When the discourse involved a range of interactions, we made three or four observations. This does not threaten our analysis since we do not seek to compare classes. Instead, we focus on the ATP instances when they arise and the nature of the discourse therein.

Data and Analysis

Classroom observations involved a single video camera and four digital audio recorders. Analysis began by flagging instances of ATP in the whole-class public discourse using a coding scheme based on the Common Core State Standards (2010), Koestler and colleagues (2013), and Fennell, Kobett, and Wray (2013, January). In particular, we looked for the following indicators:

- Emphasis, clarification, questions, or discussion in regard to
    - Defining terms or using them appropriately
    - Defining symbols or using them appropriately
    - Labeling units, graphs, or diagrams
    - Precision of calculations
    - Precision of measurements
    - Rounding or estimating

- Making or refining claims
- Giving explanations or justifications
- Appropriate mathematical precision within a non-mathematical problem context

Note that a teacher or student merely *being* precise did not necessarily result in a flag for ATP. More important was whether explicit *attention* was given to the precision. Having flagged the instances of ATP, we marked their beginning and end based on when the focus of the discourse shifted to and from the issue of precision and then transcribed each instance. We compiled durations and descriptive statistics for each class and for the set of ATP instances overall.

Next, we coded each instance of ATP based on the distinction between *univocal* and *dialogic* discourse. Building on the definitions above, we operationalized these constructs using the following characteristics, adapted from Wegerif (2006) and Truxaw and DeFranco (2008):

- Univocal
  - transmission of meaning (giver and receiver(s))
  - closed discourse (answer or information concludes the exchange)
- Dialogic
  - shared development of meaning (give-and-take)
  - open discourse (information or ideas spur further discourse)

Note that having multiple speakers does not imply dialogic interaction. Multiple speakers can be involved in transmitting meaning rather than in opening up a discourse space for making new meaning. Also, a single turn, such as a teacher's question, cannot be coded on its own because the potential to spur dialogic discourse does not necessarily do so. Thus, we considered interactions overall when making coding decisions. In rare cases, there was difficulty coding but these were discussed until consensus was reached.

Our final phase of analysis was to look across the coding for patterns and themes both within and across the univocal and dialogic instances. These themes were used to structure the Findings, presented below with several examples used to illustrate the themes.

Findings

We found 140 instances of ATP in the classroom data, with an average duration of around 40 seconds per instance. Some were very brief and others lasted a few minutes. Overall, univocal instances (131) were more common than dialogic instances (9). Only two classes exhibited dialogic instances and univocal ATP was still predominant. Below we illustrate some themes that emerged from the univocal instances of ATP and give examples of the rare dialogic instances.

Univocal Instances of Attending to Precision

The 131 instances of univocal ATP comprised 93.6% of the total instances and ranged in duration from 3 seconds to 2 minutes, 48 seconds. The nature of the ATP in these instances varied. In some cases, teachers briefly pointed out a lack of precision, a need to be precise, or prompted students to use precise terminology. In other cases, teachers and students attended to precision of their communication to assure that the transmission of meaning was successful.

Many instances of univocal ATP were relatively brief, involving a teacher making an explicit remark about precision. For example, when working with graphs, Mr. Forrest said, "[The graph] doesn't have to be perfect, but I'm going to make sure I put the x-intercepts about where they are on this graph and the y-intercept about where it is on that graph." These instances were univocal because they simply involved a teacher transmitting an idea or warning with regard to ATP.

Other brief instances of univocal ATP involved teachers pressing for precise terminology. For example, Ms. Finley was working with students to prove that two triangles were congruent. One of the steps in the proof involved side NG being shared by both triangles.

Finley: OK, GN is equal to NG, which is true. My question to you is, why?
Students: Same line?
Finley: It is the same line, but we have a name for that… It's a property that we did back in Chapter 2, that seemed really silly at the time, but…
Students: Reflexive!
Finley: Reflexive. OK?
Male Student: I said it. I knew it.

Ms. Finley prompted the class to use the precise term in the written proof rather than simply saying that GN and NG are the "same line" segment. This interaction was univocal because the teacher knew the property and terminology she wanted and the interaction ended as soon as the term was supplied. It was not opened up into a discussion of the reflexive property itself or to other imprecisions such as the conflation of line and line segment. The student's final turn confirms that this interaction was univocal, directed toward supplying known information.

Longer instances of univocal ATP most often involved the teacher and students engaging in discourse to clarify or correct the communication of a mathematical idea. These instances were univocal because they were, in essence, a transmission of someone's idea and ATP played a role in assuring that the meaning was transmitted and received as intended. For instance, Ms. Finley drew a pair of triangles on the board (see Figure 1) and claimed they were congruent, but she purposefully wrote the congruence statement with the vertices listed in an incorrect order.

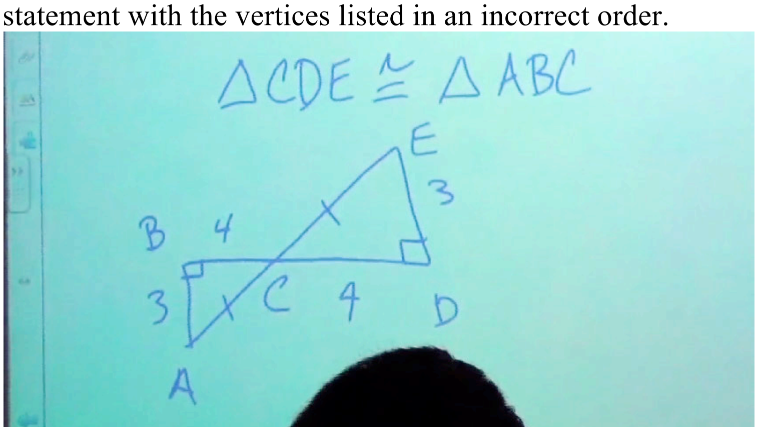

Figure 1: An incorrect statement about congruent triangles

Finley: OK, so, I'm telling you I'm wrong. Emily thinks I'm wrong because I can't draw. I get that. The question is, why am I really wrong?
Female Student: Does it have something to do with the right angle?
Male Student 1: You don't have congruent angles.
Students: Yeah.
Ms. Finley guided the students through a quick verification that, in fact, all the corresponding sides and angles were congruent to one another, so the triangles were truly congruent.
Finley: So there's something else that must be wrong with my thinking. Melissa?
Melissa: The numbers are wrong, or the letters.
Finley: The letters are in the wrong order. What did I not match up right?
Male Student 2: Ohhh, I see.
Finley: C should match with…
Students: C.

Finley: C [instead of A]. So, this [congruence statement] is not true, and don't make this mistake on your homework because if you do, we'll have a problem.

Note that there was not only ATP in this interaction but also a suggestion from Ms. Finley to be precise in the future. This interaction was univocal because Ms. Finley had a particular error she wanted to discuss and the exchange concluded when it was identified. Students made bids to open up a dialogic exchange when they questioned the angles instead of the labeling, but this did not lead to dialogic discourse as Ms. Finley led students through verifying the corresponding parts of the triangles were indeed congruent and then proceeded with the ATP of the labeling.

Less often, the long instances of univocal ATP focused on students' ideas. The following instance is from Mr. Forrest's lesson on graphs of polynomials. Mr. Forrest asked the class about "tangent to" and the interaction then focused on clarifying a student's idea about the phrase.

Forrest: Is the graph tangent to the x-axis, or does the graph continue through the x-axis at each x-intercept? … Does anybody know what "tangent to" is?
Dustin: Stops at it.
Forrest: How do you mean, "Stops at it"?
Dustin: Like, whenever it goes down (drawing in the air) and touches it [the x-axis], it immediately goes right back up and that's either the furthest point up or the furthest point down in that part.
Forrest: Alright, I like your explanation Dustin. Could you draw a picture on the board? [Dustin draws a curve on the board; see Figure 2]

Figure 2: Dustin's example of a curve tangent to the x-axis.

ATP occurred as Mr. Forrest asked Dustin to clarify his words, "Stops at it", and to represent his idea graphically. Although Mr. Forrest prompted the ATP, Dustin's idea was the focus. The interaction was univocal because it involved clarifying Dustin's transmission to the class.

Dialogic Instances of Attending to Precision

Although dialogic instances of ATP were rare, they are important to consider because they illustrate a different form of interaction and engagement with ATP. All 9 dialogic instances involved precision of language or communication, whether it be constructing definitions, refining student conjectures, or building upon (rather than merely transmitting) a student's idea.

The following excerpt is from Mr. Forrest's class as they discussed the graph of $y = \sqrt{-x}$. Mr. Forrest asked a small group how the calculator was able to display a graph even though the square root of a negative number is imaginary. He then engaged the whole class in this idea.

Forrest: Can anybody explain why you still got a graph there? What do you think, Matt?
Matt: Because all the x-values are negative, it's gonna make it positive.

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Forrest: So, "Because all the x-values are negative, it's gonna make them…"
Matt: Like, the x-values in the graph, whenever you plug them in it's gonna invert them into a positive number.
Forrest: Alright, so can you give me an example? Because we got [writes on board] y equals the square-root of negative x, right?
Matt: So x is negative three.
Forrest: So if we made an xy-table, plugged in negative three?
Matt: Yeah.
Forrest: We get the square root of…
Matt: Three.
Forrest: Three, right? Because, if you think of it another way, wouldn't that be the square root of negative negative three?
Female Student: Uh huh.
Forrest: Which is the square root of negative negative 3. This is why it's possible. Because even though it's the square root of negative x, what kind of values can we plug in for x? We can plug in negative values for x to give the square root of x. Does that make sense?

Matt offered an idea and the subsequent discourse involved attending to the precision of his language to facilitate communication while also building new meaning. For example, Mr. Forrest repeated Matt's original statement, which contained two vague referents ("it's" and "it"), then Matt added to his explanation. Mr. Forrest also requested an example and said, "So if we made an xy-table, plugged in negative three?" to push Matt to express his idea more clearly. Matt was able to give a specific example and Mr. Forrest then built on that idea to move toward a more general idea based on that shared meaning. It is important to note that Mr. Forrest was not the one doing all the reasoning. He did contribute, but he also pushed Matt to illustrate the reasoning behind his idea, which allowed Mr. Forrest to build upon and extend Matt's original reasoning.

This second example is also from Mr. Forrest's class because his class exhibited nearly all the dialogic instances of ATP. Here, Mr. Forrest was reviewing the definition of a polynomial in a univocal manner when dialogic discourse took place around the definition of whole number.

Forrest: Polynomial functions have only whole number exponents on the variables. What are whole numbers?
MS: Like, number one would be a whole number.
Forrest: That's true. Specifically, what is the set of whole numbers? … [pause] Because when we say whole numbers, it's actually a very specific mathematical set, right?
Zack: One, two, three, four.
Forrest: Say it again.
Zack: I just counted them.
Forrest: OK.
Ben: Like zero to infinity, positive.
Forrest: Zero to positive infinity?
Ben: Mm hmm.
Male Student: Wait, is there other kinds of infinity?
Henry: It could be fractions and decimals, and those aren't.
Forrest: Alright, so if we say zero to positive infinity, we're including too much? Is that what you're saying, Henry? [Henry nods] OK. So be specific.
Dustin: Starting at one and counting up by ones.
Forrest: Starting at one and counting up by ones. Is that the whole numbers?
Female Student: Sure.
Forrest: Sure.

Amy: It doesn't have zero.
Male Student: No, zero isn't.
Forrest: I don't really like pestering anyone, but I see lots of college algebra books right in front of people. You could find out, right?

The precision of language is apparent as Mr. Forrest repeatedly called on students to clarify how they describe whole numbers. In trying to precisely describe the set of whole numbers, several students contributed ideas in a dialogic manner. One student gave an example of a whole number ("one"), Zack then gave a pattern of whole numbers ("one, two, three, four"), and Ben attempted a definition ("zero to infinity, positive"). Then, Henry pointed out that Ben's definition included too many numbers. Mr. Forrest clarified this idea but then did not refine the definition. Instead, he left the discourse open and Dustin then provided a new definition ("starting with one and counting by ones"). Mr. Forrest and other students weighed in, unsure of whether this was an acceptable definition. At that point, Mr. Forrest directed attention toward the textbook definition, which ended the dialogic ATP. Although this interaction was dialogic, all shared ideas were not taken up in the discourse. In particular, one male student raised a question about "kinds of infinity" that was not addressed, and Amy's comment about zero possibly being a whole number was not discussed but was circumvented by appealing to the textbook definition. An exploration and determination with regard to zero would have been a further opportunity for ATP.

Discussion

We examined instances of ATP in the whole-class discourse in five secondary mathematics classrooms and focused on the univocal or dialogic nature of those instances. Some ATP instances were brief—for example, a teacher pointing out the need for labels in a graph or calling for the use of a specific term. Other instances were longer interactions focused on clarifying the transmission of someone's idea, many times the teacher's but sometimes a student's. Overall, the vast majority (93.6%) of ATP instances were univocal. Although past research (e.g., Stigler & Hiebert, 1999) led us to expect univocal discourse, its dominance was surprising for two reasons. First, due to grade level, topic, and teacher background variability, we expected at least some of the classrooms to exhibit more substantial dialogic discourse. As it turned out, even the class with the most dialogic instances (Mr. Forrest's) still had nearly 90% univocal. Second, many of the illustrations of the role that ATP can play in mathematics education (Koestler et al., 2013) involve dynamic, dialogic interactions wherein meaning is constructed through a collaborative process of critique, refinement, and extension. Such instances of ATP were rare in our data.

When dialogic ATP did occur, the meaning-making, by definition, was qualitatively different than in univocal interactions. A key question, then, is what spurred the dialogic discourse? The answer is certainly a multitude of factors and would require further analysis to parse out, but we can speculate that it was a confluence of the mathematical content or task available as a focus for the discourse, the teacher's discourse moves, and the students' contributions to the discourse. It is also important to consider ATP in particular, because it is possible that certain aspects of ATP—attention to labels, technical terms, etc.—lend themselves to univocal discourse because they are somewhat normative. Other aspects of ATP—the process of defining, clarifying reasoning—may naturally be better suited for dialogic discourse. This study provides preliminary evidence that such may be the case, but further research is needed.

Another important contribution from this study has to do with the role of students within ATP discourse. Many scholars have brought attention to the value of engaging students in dialogic discourse (e.g., Lobato et al., 2005; Otten & Soria, 2014). The present study highlights that involving students actively in the classroom discourse is not necessarily the same as engaging students in *dialogic* discourse. As we saw, even when students' ideas were the focus of ATP interactions, it often remained univocal because the purpose was to transmit or clarify those ideas to the rest of the class, which is distinct from engaging students in the collaborative meaning-making of dialogic discourse. Heeding Clarke's (2006) warning, we do not intend to set this as an unproductive dichotomy between univocal and dialogic discourse or of teacher-focused and student-focused discourse. Rather, this is meant to be a call to attend with precision to the different experiences provided to students in mathematics classrooms with regard to ATP.

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FOSTERING CLASSROOM COMMUNICATION ON REPRESENTATIONS OF FUNCTIONS

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Teaching students how to use and interpret various representations of functions remains an enduring challenge for educators. Providing students access to such representations is a key feature of providing them access to mathematical participation and communication. This study is based on a teaching experiment, which aims to provide students access to the discourse on representations of functions by eliciting students' discourses and making them explicit topics of reflection in a post-secondary classroom. The results indicate that the pedagogical approach used in this study has the potential to foster mathematical communication in the classroom as evidenced by students' awareness of the tacit aspects of their discourses that shape their thinking about representations of functions.

Keywords: Advanced Mathematical Thinking; Classroom Discourse; Instructional Activities and Practices; Post-Secondary Education

Representations mediate thinking about mathematical concepts and play critical roles in mathematical communication. Kaput and Rochelle (1999) argue that the emergence of new representational forms can help learners engage in powerful mathematical ideas that can otherwise remain inaccessible to them—an affordance they refer to as *democratization of access*. Arcavi (2003) also notes that visual representations can help students think about concepts and meanings which can be bypassed by symbolic representations. The representational system of a concept carries the underlying structure of that concept and the possible irrelevancies within a representational system "are dismissed or unnoticed by experts" (Arcavi, 2003, p. 232). For students who cannot see the underlying structure of a concept within a representational system, those irrelevancies can be significant since experts of mathematical discourse can "lose the ability to see as different what children cannot see as the same" (Sfard, 2008, p. 59). These arguments indicate that mathematical representations have the potential to provide learners access to mathematical communication. However, if their roles and use remain invisible to the students, they can also lead to miscommunication in the classrooms. Teachers play critical roles in making mathematical ideas transparent for students to enhance mathematical communication. This work is based on a teaching experiment that aims to provide students access to the discourse on representations of functions and explores whether this pedagogical approach has the potential to foster classroom communication.

Function is a central concept in K-12 and undergraduate mathematics and it is a challenging topic to learn due to the various notions associated with the concept (Eisenberg, 1991). In particular, many researchers argue that students have difficulties moving flexibly across graphical, algebraic, tabular, and verbal representations of functions (Monk, 1994; Schoenfeld, Smith & Arcavi, 1993; Sierpinska, 1992; Tall, 1996). Teaching students how to use and interpret various representations of functions remains an enduring challenge for educators and mathematics education researchers. Given the role functional relationships and representations play in mathematics and science as well as everyday interpretations of data, students' lack of access to the discourse on representations of functions can hinder their access to mathematical participation and communication.

This study is based on a teaching experiment that used a discursive approach to elicit students' discourses about representations of functions and made them explicit topics of discussion and reflection in the classroom. The goal was to teach students how various representations of the function concept are similar to and different from each other to address particular aspects of functional representations that can remain implicit for the students. The study addresses the following questions: What are the features of a discursive teaching approach that aims to provide learners access to the discourse on representations of functions and how can this approach foster communication in the classroom about representations of functions?

Theoretical Framework

This work aligns with the theoretical approaches that view learning as becoming a more fluent participant in mathematical communities of practice. Each community of practice leaves a historical trace of physical, linguistic, and symbolic artifacts as well as social structures that define the characteristics of participation (Lave & Wenger, 1991). From this perspective, learning to become a fluent participant in mathematical communities of practice involves learning to speak mathematically and learning to use the artifacts of the practice in the manner of full participants by engaging in mathematical activities. Lave and Wenger (1991) refer to the artifacts employed in any practice as *technology of practice* and argue that transparency of the technologies of practice with respect to their meaning and use is a critical condition for access. The visual representations that mediate mathematical communication (e.g., graphs, symbols, words) are among the technologies of mathematical practice. When the meaning and use of these representations are commonly agreed upon, they enhance mathematical communication. If their use and purposes remain invisible for the learners, then they may also hinder communication in the classrooms. For example, Güçler's (2014) earlier work demonstrates that merely presenting a mathematical representation to students is not sufficient for transparent communication in the classroom.

This study views mathematics as a discourse, where discourse refers to "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p.93). From this lens, providing access to the technologies of mathematical practice is tantamount to providing access to the *discourse* on those technologies. Sfard (2008) uses the term *meta-level rules* to refer to the elements of mathematical discourse that can remain implicit for learners and separates them from *object-level rules*. Object-level rules are about the behavior of the objects of mathematical discourse, whereas meta-level rules characterize the patterns in the activity of participants. Meta-level rules are "about the actions of the discursants, not about the behavior of mathematical objects" (Sfard, 2008, p. 201). For example, "the graph of the function $y = x^2$ is a parabola" is an object-level rule of mathematical discourse whereas the patterns in learners' actions when drawing that graph (e.g., using the assumption of continuity consistently when thinking about functions and their representations) constitute the meta-level rules in the learners' discourses. The meta-level rules of mathematical discourse are often tacit and, if not made explicit, learners can talk about the same mathematical object (e.g., a graph) in different ways, leading to possible miscommunication (Güçler, 2013).

Providing learners access to the technologies of mathematical practice requires the teacher to attend to the tacit aspects of their meaning and use in the context of the classroom. For this study, the tacit aspects of the technologies of practice refer to the meta-level rules in participants' discourses that shape their thinking about representations of functions.

Methodology

This work is part of a larger study that explored student thinking on functions, limits, derivatives, and integrals over the course of 13 weeks. The focus here is on the classroom discussions about representations of functions that took place during the first 3 weeks. The study followed a teaching experiment methodology as outlined by Steffe and Thompson (2000), which involves experimentation with the methods that can influence student thinking. This paper is on the features of the teaching experiment and the nature of discourse it elicited in the classroom regarding representations of functions. The participants were one pre-service and seven in-service high school teachers, hereon referred to as *the students*, taking a mathematics content course on calculus for their teaching licensure programs. The researcher was the instructor of the course and all of the students taking the course volunteered to participate in the study. The classroom sessions about functions were video-taped. The sections during which the instructor and students talked about representations of functions were transcribed. The transcripts included the utterances and actions of the participants.

Consistent with the theoretical assumptions of the study, a specific goal of the teaching experiment was to make the tacit meta-level rules in learners' discourses about representations of functions explicit topics of discussion and reflection in the classroom. In order to do that, it was important to bring forward the various ways in which learners used and talked about representations of functions. The activities on representations of functions were designed so that the students had the potential to act according to different meta-level rules, leading to different realizations of those representations and the function concept. Those instances were considered critical interms of eliciting students' existing discourses on representations of functions and examining which aspects of them were visible or invisible for the learners.There were also specific discussions about the similarities and differences among various representations of functions with respect to the different meta-level rules on which they are based. At the end of each activity, after eliciting the students' discourses, the instructor explicated the meta-level rules in their discourses. This was an intentional part of the teaching experiment with the goal of making transparent the different meanings and uses of representations of functions andproviding students access to these technologies of mathematical practice. Throughout the three lessons on functions, students worked on various activities on representations of functions, two of which will be presented in the next section.

While the visuals students used constituted the representations in their discourses on functions, *how* they used those representations (their discursive acts when visualizing functions) revealed the meta-level rules in their discourses. For the analysis of the meta-level rules in students' discourses, particular attention was given to the types of assumptions students used when thinking about representations of functions (e.g., assumption of continuity, regularity, discreteness). In the next section, the discussions on some of the classroom activities about representations of functions are presented with a particular focus on the meta-level rules in students' discourses. Students' awareness of the tacit meta-level rules shaping their discourses about the function concept and its representations were considered as indicators of enhanced classroom communication. All the student names used in the study are pseudonyms.

Results

The first classroom discussion about representations of functions took place during the first lesson on functions when students were asked to provide a definition of the concept in their own words. When multiple students mentioned *graph* as a definition of function, the instructor hypothesized that they may be using the assumption that a function is the same thing as its representation. She posed the following question to initiate a discussion: "If a function *is* a graph, then is a graph a function?" The students mentioned that "a graph is not always a function" but did not realize that they were using *function* and *graph* as equivalent words in their initial definitions. The instructor then asked "If a function *is* a graph, then can we also define a function as a table or algebraic expression?" The responses indicated that the students were comfortable with defining a function "as a graph or equation" but "not as a table". When asked to elaborate, they said that "a table is only a representation". The teacher asked whether, and how, a graph or algebraic expression was different than a table. The students quickly realized that all of those visuals were representations of functions. They were confused with their acceptance of some of those representations as a definition of function and rejection of others as signifying a definition. To help students resolve this conflict, the instructor asked if a function is the same thing as its representation. The students said they "always assumed continuity", which made it "natural [for them] to think about graphs and equations as functions". They also mentioned using tables to draw graphs of functions but they did not consider a table as a function because it was "just a set of values".

Although this discussion was about definitions of functions, it revealed some meta-level rules that shaped students' discourses about representations of functions as well. The students were using the assumption of continuity as a meta-level rule when thinking about a function as a graph or equation. The discussion revealed that they were also thinking about that equation as a single rule through the assumption of regularity. The students used the assumption of discreteness when talking about a table as a set of values and they considered a table as a tool to generate a graph. However, they did not realize that the meta-level rule in their discourses when talking about the tabular representations of functions (using the assumption of discreteness) was not compatible with the meta-level rule in their discourses when talking about graphs and algebraic equations (using the assumption of continuity). At that point, using the ideas students generated during the discussion, the instructor explicated the tacit meta-level rules shaping their thinking about functions and their representations (e.g., students' referral to a function as the same thing as one of its representations; their incompatible assumptions when transitioning from one representation to another).

The students worked on another activity on representations of functions during the second lesson on functions. They were given a tabular representation as shown in Table 1 and were asked "what can you say about *F(x)* based on this representation?"

Table 1: Tabular representation of *F(x)*

x	*F(x)*
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

Carrie initially mentioned that it should be the absolute value function, which she represented algebraically as $y = |x|$. Realizing that Carrie was thinking about *F(x)* through the assumption of regularity, the teacher asked students how they would translate this tabular representation to a graphical one. Fred said the prior class discussions made him think that he needed to know the domain on which the function was restricted. He argued that the algebraic representation of the function should also include $x \in \{-3,-2,-1,0,1,2,3\}$ and said the graph should consist of "a set of points" as shown in Figure 1(a). All of the students agreed that "the function" should be represented discretely as a set of points on the Cartesian plane. Since the students automatically assumed that *F(x)*is a function, the teacher then asked what would happen if the domain of *F(x)* in Table 1was $[-3,3]$. Figure 1(b) shows the graph students generated for the question.

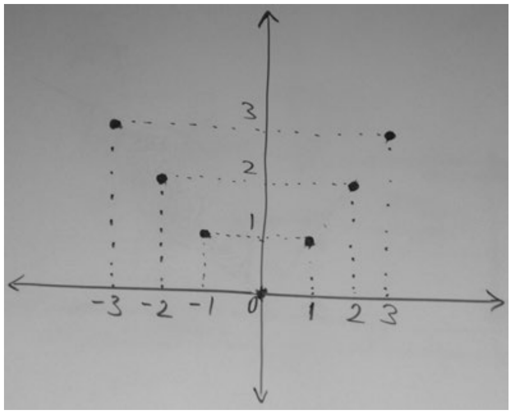

Figure 1(a): The first graph students generated during the activity

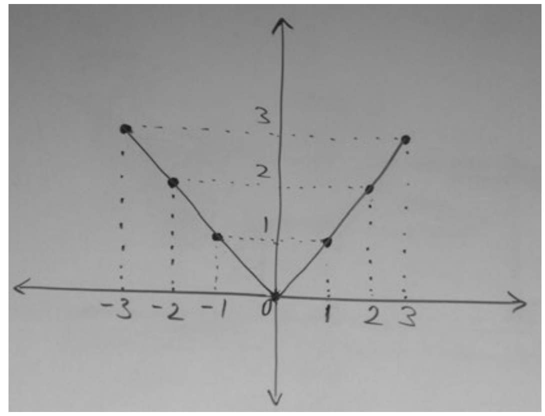

Figure 1(b): The second graph students generated during the activity

Note that, despite their realizations of the importance of the domain of a function based on previous classroom discussions, the students were still using the assumptionsof continuity and regularity as meta-level rules in their discourses when they generated the graph in Figure 1(b). They were thinking about *F(x)* as the continuous absolute value function over the interval $[-3,3]$, possibly due to the pattern they saw in the tabular representation in Table 1. Although the students recognized the tabular representation as consisting of static set of points using the assumption of discreteness when they generated Figure 1(a), they were using the assumption of continuity when they generated the graph in Figure 1(b). This clash in the utilization of different meta-level rules students used prompted the teacher to ask "how do you know that *F(x)* is a function?" In response to the students' puzzled looks, she drew two graphs as shown in Figure 2(a,b), which satisfied the conditions in Table 1 and asked students to elaborate on those graphs.

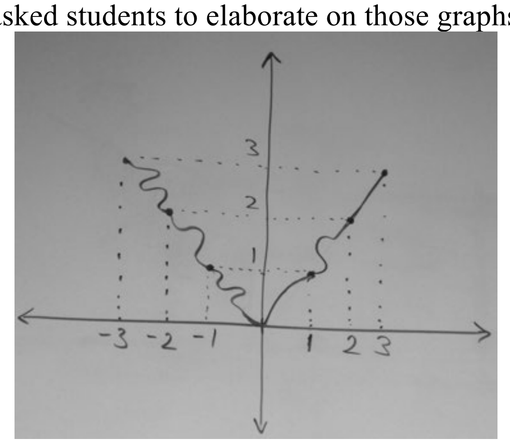

Figure 2(a): The first graph the teacher generated during the activity

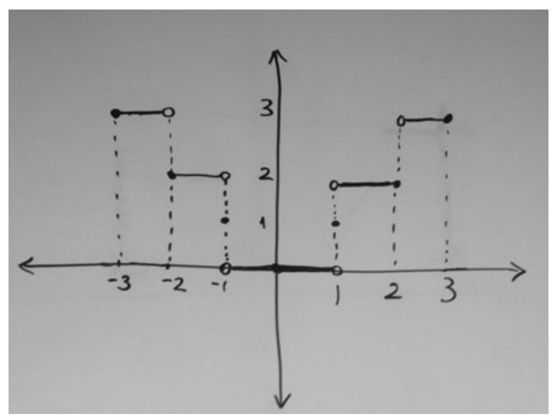

Figure 2(b): The second graph the teacher generated during the activity

The students quickly realized that the graph in Figure 2(a) did *not* represent a function on $[-3,3]$ and the one in Figure 2(b) was not continuous although both graphical representations were consistent with the set of values in Table 1. In addition, the students realized that Figure 2 (b) "does not represent a regular function that has a single rule". Sally mentioned that these challenges were occurring because they "only worked with continuous functions" in their education. Steve then mentioned that he used tabular representations every time he modeled continuous real-life phenomena. This led to discussions about how using different assumptions shape thinking about functions and their representations. During those discussions, the students explicitly mentioned that they were using two different assumptions—continuity and discreteness—which indicated that they became aware of those meta-level rules. At the end of the classroom discussion, the teacher explicated the connections and differences between a definition of function and the visual mediators that represent the concept. While doing so, she used students' considerations of function as a graph and rejection of any graph as a function to encourage them to think about how definitions of functions are formulated to avoid ambiguity.

The features of the teaching approach used during all the activities about representations of functions were consistent with those demonstrated in the aforementioned activities. Those features included (a) eliciting how students talk about functions and their representations, (b) listening to their responses carefully to capitalize on the instances in which students reveal the meta-level rules in their discourses by asking probing questions, (c) creating opportunities so that students act according to different meta-level rules, leading to communicational conflicts, (d) giving students opportunities to reflect on their discourses to resolve those conflicts, and (e) explicating the emerging meta-level rules in their discourses at the end of the discussions for further reflection.

The results of the study provide some evidence how a discursive approach to teaching has the potential to foster classroom communication, particularly with respect to students' awareness of the meta-level and tacit aspects of their discourses on functions and their representations. Although the focus of this paper is mainly on the classroom discussions due to space constraints, additional evidence regarding the affordances of the teaching experiment in fostering classroom communication is given in Table 2. Such evidence is based on students' weekly journal entries about functions in which they further reflected on any aspect of the classroom discussions that they found interesting.

Table 2: Examples from students' reflections on representations of functions

Sally: I didn't imagine that functions would be so complicated, there is much more to it than we are used to seeing…I'm on the mindset that the representation of the function doesn't define the function.
Martin: One theme that keeps coming up…is the notion of continuity. I feel I will need to develop strategies to address this issue. Students, just like us, have a tendency to assume continuity despite not being told or shown that a function is continuous.
Lea: The class activity highlighted the idea that both graphs and tables are merely representations of a function, and cannot always depict all the possible values in a function… However, even though the graph and table are limited, they are both still important in understanding functions as they offer students a unique visual representation of a function… This example, along with others, also demonstrated that functions are very often discontinuous, or piecewise and may also contain more than one rule to it.
Ron: In the some ways, it makes sense to gradually introduce different representation of a function over time as well as redefine a function as new concepts are presented. Through our class discussions, we talked about 8 different ways to represent a function... I think the big question is when we should introduce set notation and synthetic representations of a function.

Table 2 includes some representative examples of students' reflections on representations of functions. By the end of the lessons on functions, these students were aware of some of the assumptions they used (e.g., continuity and regularity) as meta-level rules in their discourses on function (Table 2, [2], [3]). They also talked about the difference between the abstract concept of function and its representations (Table 2, [1], [3]). Further, since these students were also teachers, the classroom discussions helped them think about how to teach these ideas to their students (Table 2, [2], [4]). These results suggest that the teaching approach used in the study helped students be aware of some of the meta-level rules shaping their discourses on representations of functions and the function concept. The study also confirms the tacit nature of meta-level rules and their role in mathematical communication since the students in the study mentioned that they never learned about those meta-level rules before this course.

Discussion

Representations of functions are among the technologies of mathematical practice and play important roles in classroom communication. However, if their meaning and roles are not shared by participants, they may also lead to miscommunication. This study demonstrated that some aspects of the discourse on such technologies of practice can remain invisible to learners—even to those who have been exposed to the technologies in their prior education. This finding is in accordance with Güçler's (2014) previous work and indicates that teachers should not take the communicative power of mathematical representations for granted; they need to make the discourse on those technologies of practice transparent for their students. The notion of transparency, which refers the visibility of the use and meaning of technologies of practice (Lave and Wenger, 1991), is a critical condition for providing learners access.

In this study, access to the technologies of practice was conceptualized as access to the discourse on those technologies through the use of Sfard's (2008) framework. This framework was useful in identifying and examining the features of classroom discourse that can remain tacit for the learners (e.g., meta-level rules) and served as a lens that helped shape the design of the teaching experiment used in this study. The results of the study indicate that a teaching approach that is responsive to the discourses of the students, which also elicits the meta-level aspects of participants' mathematical discourses, has the potential to foster classroom communication and transparency needed to make mathematical ideas clearer for the learners. Such a teaching approach elicits the various ways in which students think about technologies of mathematical practice to highlight which aspects of them remain invisible for the students in the context of the classroom.

This work may have some implications for teacher education. The participants of this study were pre-service and in-service high school teachers. Although they were considered as students in the content course they were taking, they were also the teachers of that content in high school classrooms. The results of the study indicate that these teachers' reflections on their discourses on functions and their representations also triggered their reflections on how to teach those ideas to their own students. Some of those teachers mentioned using the activities they worked on in the classroom with their own students. In this respect, this study offers educators some activities and ideas that may be useful in teacher education courses and professional development.

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TEACHER NOTICING STUDENTS' MATHEMATICAL STRENGTHS

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Research about teacher noticing of students' mathematical thinking has been an important and ongoing strand of research and practice in mathematics education. Our work extends this agenda by working collaboratively with teachers to learn together how to notice students' mathematical strengths. The lens on strengths runs counter to the prevalent culture in U.S. schools to overemphasize gaps in students' understandings. In this paper we describe a video club focused on identifying and naming students' mathematical strengths and the protocols that support this focus. We illustrate and discuss the important shifts in teachers' ways of noticing and talking about students' mathematical activity. We also discuss implications for further research and professional development focused on teacher noticing of students' math strengths.

Keywords: Secondary Mathematics; Equity and Diversity; Teacher Knowledge

Researchers and professional developers focus on teacher noticing because noticing informs practice, and teachers' practices are consequential for students' learning. Current video technology has made it possible to study and facilitate teacher learning to notice students' mathematical thinking and understanding (Goldsmith & Seago, 2011; Jacobs et al., 2011; van Es, 2011; van Es & Sherin, 2008) and instructional features of classrooms (Star & Strickland, 2008). Video also supports teachers to practice attending to important features and critical events happening within the complex setting of classrooms before they are faced with them in real time.

As the knowledge base for equitable mathematics teaching continues to grow there is arguably more in classrooms now than ever before to which math teachers must attend, including knowledge of mathematics for teaching (Ball, Thames & Phelps, 2008), knowledge about communities (Civil, 2007), knowledge about math identities (Aguirre, Mayfield-Ingram & Martin, 2013), knowledge about social and academic status (Featherstone et al., 2011), and an understanding of the critical relationships between math learning, math learners and broader sociopolitical structures (Gutiérrez, 2013).

In addition, teaching for equity requires a focus on resources (what students have) and students' potential rather than deficits (what students are lacking). When teachers focus on strengths, they position young people as competent learners (Cohen, 1997), support students to create positive math identities (de Abreu & Cline, 2007; Jilk, 2014; Martin, 2000), help them recognize and value peers as intellectual resources (Cohen, 1997) and expand school mathematics to include a rich set of skills, practices and understandings in which students can find themselves (Boaler & Greeno, 2000; Featherstone, et al., 2011). Maybe even more important, a strength-based classroom culture disrupts the dominant educational discourse focused on gaps and deficits (Gutiérrez, 2008) and provides a new and more realistic narrative about learning for young people who have traditionally been marginalized by school math.

Believing that students have strengths from which to connect and build is challenging work. Recognizing (and naming) these strengths in real time is even more difficult. Research has shown that even very well intentioned teachers who profess a stance towards teaching for equity are challenged to enact these beliefs in their day-to-day teaching practice (Ladson-Billings, 1994; Walker, 2012). Our experiences working with math teachers confirm these results. Even teachers who desperately want their students to positively identify as "math people" often struggle to know what "counts" as a mathematical strength or how to talk with young people about the strengths they have. We believe that there are valid reasons for such challenges.

Teaching is a cultural activity filled with taken-for-granted assumptions and shared convictions and values (Stigler & Hiebert, 1999). As U.S. educators, we are immersed in a culture that focuses on students' deficits and perpetuates unexamined habits of teaching as fixing students' problems and misconceptions. These daily practices tend to get in the way of re-imagining and inventing instructional moves that instead focus on accessing and building on students' strengths and multiple ways of understanding (Ladson-Billings, 1994).

In addition, professional noticing is also a cultural practice. Hand (2012) argues that "dispositions relate to what teachers do or do not notice" (p. 234) in their classrooms and therefore drive instructional decisions. We attend to the classroom events that we have been taught to value. The cultures in which we have been immersed essentially train us to see and hear what is important to us. It therefore makes sense that if teachers are now expected to reshape their ways of noticing and their repertoire of teaching practices, then they need repeated opportunities to practice seeing and hearing students' strengths in action and practice articulating these strengths to students in ways that are convincing.

The video club we report on here sought to challenge and disrupt our collective tendency to look for students' mathematical shortcomings. The video club was designed to provide a shared, local experience from which a heterogeneous group of teachers could learn to notice (Sherin, Jacobs & Phillip, 2011; van Es & Sherin, 2002) students' math strengths in ways that could be useful and usable in their classrooms. A focus on drawing attention to students' intellectual strengths is a noteworthy feature of equity teaching practices in general and of *Complex Instruction* in particular, which is the instructional approach that the teachers in this video club had committed to learn and use in their classrooms. Next we share key features of the context and design of this video club in order to then share our analysis of shifts in teachers' noticing and their ways of talking about students' mathematical activity.

Context and Background of Video Club

The teachers who we describe in this paper are part of a professional development network that has six mutually informed learning spaces. The learning spaces include a weeklong course about Complex Instruction, In-Classroom Support, Common Planning Time with course teams, a monthly video club and Teacher-leader meetings. The learning spaces are connected in several ways for the goals of developing coherence with common themes and making connections across spaces. An important goal for this network is to re-culture math departments with empowering professional development experiences in ways that build and strengthen teachers' capacity to take up and sustain Complex Instruction as their equity pedagogy across the entire department.

Complex Instruction, at its core, requires a belief that all students come with intellectual, social, and cultural resources and are able to learn rigorous content (Cohen & Lotan, 2014; Cohen & Lotan, 1997). Complex Instruction (CI) creates a classroom "social system" that directly attends to and addresses problems of social inequality inside the classroom. Based on status generalization theory (Berger, Rosenholtz & Zelditch, 1980), CI methods are deliberately designed to "disrupt typical hierarchies of who is 'smart' and who is not" (Cohen, 1994; Sapon-Shevin, 2004, p. 3).

Multiple ability treatments and assigning competence are two specific CI strategies used to address status issues in classrooms (Cohen & Lotan, 1997, 2014; Tammivaara, 1982). A multiple ability treatment makes visible the array of intellectual abilities, skills and competencies that are required to be successful with a given task. Assigning competence is a teaching move that also disrupts students' perceptions of competence of self and peers by creating a mixed set of expectations for participation and success by publicly names the strengths students use when they are learning together. Hence, both of these practices rely heavily on teachers' abilities to notice and name students' mathematical strengths in real time. In short, successful implementation of Complex Instruction pedagogy requires an ability to look at students' mathematical activity and interactions with peers, listen to their sense making in real time in order to monitor their participation and understanding and find ways to further their learning.

It is important to clarify that we are not advocating for "feel good" teaching practices that emphasize compliments and empty praise to help students feel better about themselves. Nor are we suggesting that teachers lower their standards for academic rigor or cognitive demand by searching for anything a student does or says that is remotely mathematical. Not everything we see or hear in classrooms is worth attending to. In a strength-base video club we consider a rich and expanded view of mathematical understanding that is conceptually demanding and includes both content knowledge and learning practices.

The video club component of the professional development network was developed because program facilitators and math teachers needed an opportunity to observe the same classroom event simultaneously. The video club provided this experience along with a collective capacity to create a shared language about math strengths that could be used not only by individual teachers, but by *all* teachers, across grades and schools, thereby providing continuity and coherence to students' school math experiences.

The video club met monthly for two hours. Participants included all of the mathematics teachers who were members of the larger network, any student teachers who worked alongside these teachers, and the administrators and instructional coaches for math from participating schools and districts. A typical monthly video club meeting drew approximately twenty-five teachers from grades 6-12 from three different schools in two urban school districts. In the 2013-14 school year, there were 17 female and 9 male teachers who attended video club. Of these, 23 were White, 2 were Black and 2 were Asian. Table 1 reports the demographics for the public schools in which the teachers worked.

Table 1: School Demographic Data (U.S Department of Education).

	Male	Female	Multi-racial	White	Black	Latino	Asian Pacific Islander	American Indian	Free & Reduce Lunch	ELL
HS A	50.5%	49.5%	3.2%	4.4%	41.2%	11.8%	37.2%	2.2%	75.0%	12.2%
HS B	52.8%	47.2%	.9%	15.7%	34.8%	16.8%	30.8%	1.0%	56.9%	15.0%
Middle School	50.7%	49.3%	3.9%	12.1%	35.0%	20.0%	27.7%	1.3%	75.0%	15.0%

We share this data for several reasons. First, this particular information contributes to readers' understanding of the context in which we work and how it informs the examples we share about teachers' accomplishments in shifting any aspect of their teaching practice. Second, the meaning of "urban school" varies across the country depending on the local context in which schools and readers are situated. We want to be clear about the meaning of "urban" in this particular locale as it relates to teacher and student demographics. Finally, we provide this data, because culture matters. Skin color, gender, language, and SES, among other factors, shape our lived experiences, and these experiences act as lenses through which we notice and interpret classroom events. Since most of the teachers in this program are members of the dominant culture, and the young people with whom they work are poor students and students of color, it is reasonable to imagine these teachers might be additionally challenged to perceive and interpret moments of classroom activity as strengths and potential resources for learning (Chazan, 2000; Hand, 2012; Ladson-Billings, 1994).

The video clips we use come directly from the classrooms of teachers who participate in the network. For the purpose of supporting teachers to notice students' mathematical strengths, video clips show one group of 3-4 students working cooperatively for 8-10 minutes without interruption.

This kind of video clip makes several important things available for noticing students' strengths, including looking at students when working unassisted by the teacher, their resourcefulness and inventiveness, how they wrestle with ideas that they have not yet studied, and opportunities to watch and hear students get stuck and unstuck as they make progress on a group-worthy task (Lotan, 2003).

Supporting Teachers' Noticing of Students' Strengths

In addition to carefully selecting videos for video club, careful coordination of video club structures and attention to potential status issues from skilled facilitators is important. The norms, protocol and focus questions developed to guide the video club meetings include: learning requires participation, we all have something intellectually valuable to contribute, we all have something to learn, and we are smarter together. Unsurprisingly these norms were shaped by the pedagogical principles of Complex Instruction providing facilitators with opportunities to model, practice and reinforce strategies for promoting equal-status participation and learning.

Articulating strengths rather than deficits was initially challenging for teachers. The strengths teachers most often noticed at first were articulated in relation to state standards or learning objectives. For example, a 7th grade math teacher might state that her students could "use proportional relationships to solve multistep ratio and percent problems" (NGA, 2010). This grade-specific objective does not easily translate into language that makes visible *the actions* of students as they make sense of ratio and percent problems or what they did and said to figure out such problems. In other words, the shift towards noticing strengths in students' meeting particular learning objectives was a good start but not enough to make visible the important math thinking and actions the students had engaged with when working on those tasks.

In response to this challenge, program facilitators developed a structured evidence-based protocol to disrupt patterns of deficit talk about students' mathematical activity and to support teachers to generate descriptions and talk about students' strengths. In addition to a focus on strengths, the video club protocol aims to promote safety and intellectual risk taking and support teachers to make connections between the video footage and events in their own classrooms.

Unlike other projects that use small group discussions to support teachers' learning with video cases, this video club used a general Go-Round structure (McDonald et al, 2003), sometimes referred to as a Round Robin Protocol, to organize how the teachers shared ideas for each focus question. There are a few things to note about this protocol that are important for facilitating this kind of video club. First, this protocol is inclusive. It affords each person an opportunity to share ideas and it supports the expectation that each of us participates. The Go-Round structure also gives us access to an expansive set of ideas about student strengths that no one person would notice alone. Finally, the sentence frame we ask teachers to use in conjunction with the Go-Round Protocol offers explicit guidance for how to talk about students with a strength-based lens. Our assumption is that teachers are learning to speak and gain fluency with a new language (language of strengths) and this takes time and practice.

What Teachers Noticed about Students' Strengths

In order to appreciate students' mathematical activity and support the noticing of students' strengths, each video club starts with teachers doing math together. In this first phase of the meeting, teachers work cooperatively in groups of 3-4, most often with colleagues from different sites and grade levels to complete parts of the task that will be shown in the video. Doing math together affords a later conversation about learning objectives and provides teachers with the entire group-worthy task (Lotan, 2003), from which they might consider students' prior knowledge, potential common mistakes and misconceptions, the flow of mathematical ideas throughout a task, task development and the inclusion of Complex Instruction structures that support participation, autonomy and accountability (Cohen & Lotan, 2014).

The group then moves to phase two to consider students' math strengths in relation to the learning objectives for the lesson. They watch the video clip for the first time and attend to students' understanding of math content by pointing to and naming what students said and did that had mathematical potential and then provide a strength-based interpretation (Prompt #4 in the Protocol is reproduced in Figure 1 here).

> ***What did students do or say that was mathematically smart?***
>
> I think it was smart when <u>(name of student)</u> did/said <u>(evidence from the video),</u> and I think this because <u>(how does this strength support students' learning?)</u> .

Figure 1: Sentence Frame for Noticing Students' Strengths

The following are some examples of strengths teachers noticed using the sentence frame to name a student's strength and justify its connection to learning. These examples come from different video club sessions throughout a school year with teachers who work in grades 6-12.

- Damarius <u>**translated**</u> .40 into "4 tenths," and then he was able to write it as fractions, 4/10 and $\frac{2}{5}$, because three representations of the same number helped him figure out how to move between the different forms.
- Rashida <u>**created a system**</u> for organizing and making zeros and then keeping track of them in an algebraic expression, because it allowed Rashida and her group to keep track of their terms and combine them correctly.
- Lydia <u>**noticed a pattern**</u> for how to use parenthesis to group terms in an expression, because this pattern allowed her to see the like terms before she combined them.
- Tian <u>**hypothesized**</u> that the similar figure would be bigger and not smaller, because then her group decided to try multiplying the lengths of sides with the scale factor instead of dividing.

Many would consider it enough for teachers to notice that Damarius understood how to convert decimals to fractions or that Rashida correctly combined like terms. However, referencing topics, objectives or standards is not sufficient if teachers are to create classroom systems in which students choose to actively engage in learning, support each other in the learning process and create positive math identities. As we have noted earlier, equity-oriented teaching pedagogies such as Complex Instruction require an understanding of what students should come to know along with the processes, skills, and actions (verbs) they use as they come to know these things, so connections can be made to prior knowledge and leveraged for new learning. Damiarius and his team will soon believe in his strengths as a math learner and likely come to rely on these strengths much more often if Damarius's teacher notices and names his strategy for translating multiple representations of number and explains how this strength contributes to Damarius' learning. An orientation towards strengths and the specific strength-based language will support students' expectations for competence and produce more equal-status interactions between students when they are engaged in learning math.

In the third phase of video club the focus turns to the norms for participation that students enact that move their math understanding and groupwork forward. Norms for participation are "learning practices" (Cohen & Ball, 2001), ways in which we expect students "to go about the work of learning" (p. 75). Specific to Complex Instruction, norms are intended to promote autonomy and interdependence in groups and to foster mathematical learning (Cohen, 1994). Teachers must be transparent about participation norms, provide students with opportunities to practice them, and notice and assign competence to these practices to make them a more normal way of going about the business of learning math. To this end, we provide teachers with an opportunity to notice these behaviors in action (replayed back in the video clip) and practice talking about them as strengths. Below are some examples of the norms for participation that teachers noticed. Again, each teacher statement below used the sentence structure: *I think it was smart when_________________, because___________________.*

- Tamika **pressed for clarification** from her group about "what to go up by" when scaling the x-axis, so all of the group data could fit on the graph. I think it was smart, because then Dariana had to explain how she scaled the y-axis, and Jason talked about the range of their data set. It helped the whole group learn more about how to create an accurate graph.
- TJ **expressed confusion** about the different meanings of minus in both the geometric and algebraic representations, because knowing these different meanings will help her in other contexts that use minus.
- Asad **made sure that everyone in his group understood the directions** before they started the task, because then the whole group could get started together and consider more than one way to do the problem.
- Sierra **took a huge risk** by sharing her ideas about combining like terms with the entire class, because learning requires intellectual risk taking and people usually learn more when they are willing to try something new.

Implications for Research and Professional Development of Teacher Noticing

> "I just start talking this way. These sentences [from the video club protocol] help me focus my thinking on students and their strengths when I'm back in my classroom" (math teacher).

The small shifts we document here about the ways teachers are learning to notice and talk about students' strengths in the context of video clubs are hugely important because they carry into classrooms. Teachers who are participating in these video clubs *are* thinking and speaking differently about their students when they go back into their classrooms. We have evidence from working with teachers in the other learning spaces of this professional development network that they are seeing and hearing strengths more often in real time, and they are more willing to speak a language of strengths with their students. Perhaps more importantly teachers themselves are noticing their own transformation as the quote above suggests.

Collectively, the math teachers are often reminded of the power they have to help students notice their own mathematical strengths, change their participation and learn mathematics. The *Feature Teachers,* the teachers whose classroom videos are used, often reflect positively on video club after they have the rare experience of listening to their colleague's talk about the many ways in which their students are smart. Feature teachers usually return to their classrooms feeling rejuvenated by this feedback and convinced that teaching practices grounded in strengths are well worth investing in.

Changing how we frame students and their participation in math classes is not easy. It takes a concerted effort to shift perceptions about students and learning in which we have been immersed for many years. Even after one or two years of participating in this program, teachers still find it challenging to articulate math strengths in real time. The heterogeneity of the video club community addresses this challenge. Participants come with a range of experiences teaching different courses and grades. This means that they notice different things and contribute to a more expansive set of ideas about what counts as a strength, how different student behaviors might be interpreted as strengths, and how to name strengths in real time. Additionally, the diversity of teaching experiences affords articulation that is rare and powerful and often contributes to improved course design and program development in teachers' home-sites. High school teachers hear middle school students make sense of math in ways they rarely consider. Middle school teachers get glimpses of former students on film demonstrating math practices they never thought possible. Teachers often report feeling more hopeful about their ability to impact students' learning when they have opportunities to talk across grade levels.

Finally, and perhaps most importantly, we have also noticed shifts in the ways the teachers talk about themselves and their colleagues. They often assign competence to each other, sometimes playfully, but always with intent to bring attention to particular strengths. They might highlight something new they have learned from a peer and describe how it impacted their teaching. Sometimes they mention a particular way a colleague might draw out different strengths from others. These new ways of being professionals are not surprising. In addition to video club, these teachers are immersed in a culture of professional development in which the norm is to work from strengths rather than focus on deficits. We notice what we *can* do before addressing what we have yet to improve. We practice naming our resources so we know what we have to offer. We ask our colleagues to show up and speak up and share their many ways of being smart so we can be more successful together. This is the kind of transformative learning that all teachers deserve so as to support and sustain their equity work with students.

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NOVICE ELEMENTARY TEACHERS' INSTRUCTIONAL PRACTICES: OPPORTUNITIES FOR PROBLEM-SOLVING AND DISCOURSE

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The purpose of this study was to examine the mathematics instructional practices of 75 second-year elementary teachers (K-5) in terms of the learning opportunities provided to their students. On average, each teacher completed instructional logs for 43 days across the school year. Select items were analyzed in order to better understand the elementary students' opportunities to engage in problem solving and discourse. Results indicated frequent opportunities for discussion but limited opportunities for engagement with more open-ended tasks and explanation. Implications for future research and mathematics teacher education are discussed.

Keywords: Classroom Discourse; Problem Solving; Instructional Activities and Practices; Elementary School Education

Purpose

Elementary teacher preparation programs strive to prepare high-quality teachers in the field of mathematics by increasing content and pedagogical knowledge through methods courses (Burton, Daane, & Giesen, 2008) and providing beneficial field experiences (Darling-Hammond, Chung & Frelow, 2002; O'Brian, 2007). However, all elementary teachers, including early-career teachers, continue to struggle to enact standards-based mathematics instruction due to variety of reasons including knowledge deficits (Mewborn, 2001), anxiety with mathematics (Bekdemir, 2010; Bush, 1989), and deep-rooted beliefs about the nature of the discipline (Raymond, 1997; Wilson & Cooney, 2002). Early-career teachers, whom we will call "novices" in this paper, also face challenges associated with induction into the profession such as learning to manage a classroom of students with varying needs and developing lessons on new topics (Feiman-Nemser, 2003).

All of these aforementioned factors affect instructional choices of novice elementary teachers; therefore, it is important to understand their enacted mathematics instructional practices in order to inform the work of elementary teacher preparation in mathematics. Little is known about the mathematics instructional practices of novices; existing research has looked at instruction generally rather than focusing on mathematics (e.g., Ingersoll & Strong, 2011). The current study aims to begin to fill this void in the literature by examining the mathematics instructional practices of elementary teachers during their second year of teaching. Specifically, the research questions guiding this study were:

1. What instructional practices, relative to problem solving and discourse, do novice elementary teachers utilize during mathematics lessons?
    a) What instructional practices are utilized in the primary grades (K-2)?
    b) What instructional practices are utilized in the upper elementary grades (3-5)?
2. How do instructional practices of novice teachers, specifically in problem solving and discourse, vary across their own mathematics lessons and in comparison to other novice teachers?

Theoretical Framework and Related Research

This study examines mathematical instructional practices using the *Opportunity to Learn* framework (OTL). That is, we are interested in the learning opportunities of elementary students during mathematics lessons. Initial research utilizing OTL emerged from evaluation work analyzing curriculum coverage as a measure of the opportunities students have to engage with certain topics (McDonnell, 1995; Wang, 1998). In more recent work, OTL has been applied to international comparison research (Floden, 2002), analysis of instructional strategies (Bell & Pape, 2012), and influences of diversity on opportunities for students (Tate, 2005). This study explores the opportunities given to students in novice teachers' classrooms to engage in particular mathematical practices or processes.

These mathematical processes were outlined in standards released by the National Council of Teachers of Mathematics (NCTM) in 2000. Then, the 2010 release of the Common Core State Standards for Mathematics (CCSSM) included a continued emphasis on students' mathematical practices as a critical component of K-12 mathematics instruction (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Two processes that are evident in these standards documents include students' opportunities to engage in *problem solving* and *discourse*, the foci of this study.

Problem solving refers to students' opportunities to grapple with mathematical problems to which they do not already have a solution method to use (NCTM, 2000). Furthermore, when students are given opportunities to demonstrate more than one method for solving a problem, they have the chance to demonstrate their ability to think flexibly. The nature of the tasks given to students translates to their opportunities to engage (or not engage) in problem solving (Stein, Smith, Henningsen, & Silver, 2009).

Past research in student discourse has shown that student interactions are a means of engaging as mathematicians (Nystrand, 2003; Herbel-Esienmann, 2009). Subsequent work by Bell & Pape (2012) analyzed the opportunities to learn that were created through social interactions, and this current study extends that line of research by exploring the opportunities students have to engage in discourse about their mathematical work.

Giving students opportunities for problem solving and discourse during mathematics has been proven to be challenging for elementary teachers (Walkowiak, 2010), but it can be particularly difficult for novice teachers. Research has shown that novice teachers are more likely to be concerned with management (Melnick & Meister, 2008), creating a different context for instructional decision making. However, research concerning enacted practice is limited when considering the novice teacher population. Much of the literature documenting novice teachers' practices are couched in induction and mentoring research and only looks at small samples of teachers. Also, this literature base examines teaching practices with broad strokes and not with a specific lens on mathematics education (Ingersoll & Strong, 2011).

Methods

Participants

The participants were 75 second-year elementary teachers, from a southeastern state in the United States. All teachers graduated from a public teacher preparation program within the state and were employed by a public elementary school. Propensity score matching was used to match teachers on aptitude scores (SAT, ACT) and other college-entry characteristics. Of the 75 participating teachers, 47 taught primary grades (K-2), and 28 taught upper elementary grades (3-5).

Measure

As part of a larger grant-funded project (Project ATOMS), the Instructional Practices Log in Mathematics (IPL-M) was created to measure the extent (as a proportion of time) to which certain instructional practices were present in a mathematics lesson. Within the log, teachers used a 4-point Likert scale (not today, little, moderate, and considerable) to respond to items beginning with the question stem, "During today's instruction, how much time did the students in the target class…"

Teachers used the scale response of "not today" to indicate when students did not engage in an instructional practice, and they used "little" to indicate when the practice was used for a brief amount of time. On the other hand, teachers chose "moderate" when the practice was used for a substantial amount of time but not the majority of the lesson, and teachers used the response "considerable"

Table 1: Percentage of Elementary Lessons Utilizing Instructional Practices

		Not Today	Little	Moderate	Considerable
Nature of the Task					
Work on exercises specifically for practice or review K-5		22.9	27.8	25.6	23.6
	K-2	27.1	26.7	22.8	23.5
	3-5	14.6	30.1	31.3	24.0
Listen to me explain the steps to a procedure K-5		42.1	42.0	14.0	1.9
	K-2	44.4	40.8	13.2	1.6
	3-5	37.6	44.4	15.5	2.5
Perform tasks requiring ideas or methods already introduced to the students K-5		11.0	23.7	28.4	36.9
	K-2	10.1	24.0	29.6	36.4
	3-5	13.1	23.1	25.9	37.9
Perform tasks focused on math procedures K-5		51.9	20.8	16.3	11.0
	K-2	59.2	20.0	13.8	7.0
	3-5	37.1	22.6	21.1	19.0
Perform tasks requiring ideas or methods NOT already introduced to the students K-5		55.1	23.2	14.3	7.4
	K-2	57.4	23.2	13.3	6.0
	3-5	50.4	23.2	16.4	10.2
Work on problem(s) that have multiple answers or multiple solution methods K-5		58.3	17.7	15.3	8.7
	K-2	59.9	17.1	14.8	8.2
	3-5	54.9	18.9	16.4	9.8
Demonstrate different ways to solve a problem K-5		53.9	23.8	15.2	7.1
	K-2	56.3	23.8	13.3	6.5
	3-5	49.1	23.6	19.0	8.3
Discourse					
Discuss ideas, problems, solutions, or methods with other students in small groups or pairs K-5		26.6	30.9	26.4	16.2
	K-2	32.1	30.0	22.9	15.0
	3-5	15.3	32.7	33.5	18.5
Discuss ideas, problems, solutions, or methods in large group K-5		20.6	35.3	28.4	15.6
	K-2	21.3	34.7	27.9	16.1
	3-5	19.2	36.6	29.4	14.7
Explain orally his/her thinking about mathematics problems K-5		31.9	36.7	26.1	5.4
	K-2	33.8	37.1	24.8	4.3
	3-5	28.1	35.7	28.7	7.5
Talk about similarities and differences among mathematical representations K-5		50.9	28.6	15.8	4.7
	K-2	54.2	28.1	14.2	3.6
	3-5	44.2	29.4	19.2	7.1
Talk about similarities and differences among various solution methods K-5		66.6	29.9	10.9	2.6
	K-2	70.3	18.9	8.8	2.0
	3-5	59.2	22.0	15.0	3.8

when a practice was used for more than half of the lesson. In addition to student activities, teachers also responded on items about the time and content focus of the lesson. As part of validation work on the log, a Kappa coefficient of .69 (Z=6.30, p<.001) was calculated between live observer and teachers' log responses to indicate there is evidence that teachers are reliably reporting the practices as stated in the items.

Instructional logs, like any measure of instruction, provide affordances for studying classroom instruction along with limitations (Rowan & Correnti, 2009). When compared to observational measures, which are often seen as the gold standard in the measurement of teaching practices, logs allow for sizable increases in number of lessons that can be examined. Additionally, in light of educational surveys that require teachers to reflect on instruction from previous weeks or months, daily logs reduce the error that comes with retrospection. Lastly, while logs are criticized due to self-report, past work has shown that with training and proximity of logging to time of instruction, teachers can reliably report on their instruction (Rowan, Jacob, & Correnti, 2009).

To support the reliability of log responses, each teacher attended a regional, two-hour training on the IPL-M. During this training teachers were provided with detailed explanations of items that were vulnerable to misinterpretation. In addition to item explanations, teachers practiced with scales and the online interface. Also, teachers were required to log as soon after instruction as possible and utilized a user's manual to support their understanding of the items.

Data Collection and Analysis

The teachers logged at three different time points throughout the 2013-2014 school year, with approximately 15 days per time point. Teachers logged for a range of 11 to 58 total days (M=42.83, SD=8.99) with each day corresponding to one mathematics lesson. A total of 2,741 mathematics lessons were logged with 1,837 being K-2 lessons and 904 being 3-5 lessons.

Descriptive statistics were analyzed using SPSS for items on the log. Lessons were analyzed to determine what percentage of lessons included opportunities for problem solving and discourse as demonstrated by the extent to which certain practices were or were not utilized in the lessons. Next, lessons were aggregated and analyzed based on primary (K-2) and upper elementary (3-5) grade bands.

Lastly, Intraclass correlations (ICCs) were calculated using a multi-level model approach within SAS to determine the amount of within- and between-teacher (Raudenbush & Bryk, 2002). That is, due to the nested nature of the data, a multi-level approach is necessary to account for how an individual teacher's instructional practices vary from lesson to lesson and how his/her practices vary from other teachers.

Results

Description of Instructional Practices for Full Sample of K-5 Teachers: RQ #1. Table 1 presents the percentage of all lessons in which students engaged in the instructional activity detailed in the item. The items are organized based on the *nature of task* and *discourse.* The items within these two constructs provide insight about students' opportunities (or the lack thereof) to engage in problem solving and talk about the mathematics. As we present the results, we focus on students' opportunities to engage in the instructional practices for a substantial amount of time by reporting the percentage of lessons in which the practice occurred for a moderate or considerable amount of time

Some of the log items within *the nature of the task* category describe instructional activities that more prescribed in nature such as "perform tasks focused on math procedures" and "work on exercises specifically for practice or review," while other items such as "Work on problem(s) that have multiple answers or multiple solution methods" are more indicative of tasks of higher cognitive demand with opportunities for problem solving. Two of the log items within the *discourse* category describe general opportunities for students to talk about the mathematics such as "discuss ideas, problems, solutions, or methods with other students in small groups or pairs." Other log items in this category such as "talk about similarities and differences among representations," provide more detail about the aspect of the mathematics in which the discussion is focused.

In 65.3% of K-5 lessons, students were engaging with tasks that they had already learned an idea or method to use to solve for a substantial amount of the lesson (moderate or considerable amount of time). In contrast, only 21.7% of lessons involved a longer span of time allocated for students to perform tasks without having a predetermined way to solve it (perform tasks requiring ideas or methods NOT already introduced to the students). Furthermore, 24% of lessons involved students working on problems with more than one answer or way to solve, and 23.2% of lessons included students demonstrating these different ways. These two items describe problems that were more open-ended in their solutions or solution methods. These types of problems are more amenable to elevating the cognitive demand based on the different ways students can approach the problem and possible comparison of solution methods (Stein, Lane & Silver, 1996); therefore, when utilized, the opportunity is present for higher levels of cognitive demand.

As shown under the discourse section of the Table 1, teachers reported 42.6% of lessons involved a substantial amount (moderate or considerable) of small group or pair discussion, and 44.0% of the lessons involved a moderate or considerable amount of whole group discussion. However, lower percentages of lessons are reported that utilized moderate to considerable amounts of time for student explanations (31.5%), talk about similarities and differences among representations (20.5%), and talk about similarities and differences in solution methods (13.5%).

Description of Instructional Practices of K-2 versus 3-5 Teachers: RQ #1a&b. Table 1 also presents the percentage of lessons by grade bands in regard to the use of instructional practices. For several of the practices, such as "perform tasks requiring ideas or methods already introduced to the students," primary and upper elementary lessons look similar in terms of the extent of the lesson utilizing this instructional practice. Other items show differences in the grade bands, such as "perform tasks focused on math procedures" and "discuss ideas, problems, solutions, or methods with other students in small groups or pairs." Upper elementary lessons have a higher percentage of lessons involving these practices, which can be expected due to content goals focused on mastery of algorithmic processes for addition, subtraction, and multiplication, and the increase in maturity and attention span to engage in small-group conversations with peers.

Also, as aforementioned, few lessons included the opportunity for students to discuss similarities and differences in representations and solution methods. When further aggregating the lesson percentages, K-2 lessons are limited (17.8% and 10.8% respectively) in the opportunities for these types of discussions.

Examining variance within and between teachers: RQ #2. Table 2 presents the Intraclass correlations (ICC) for the instructional practice items on the mathematics log. The ICCs represent the proportion of variability in how that item was reported between teachers. The higher the ICC, the more variability can be attributed to differences in teachers. The lower ICC indicates that more of the variability can be attributed to the differences an individual teacher's lessons across time. For example, the item, "Perform tasks requiring ideas or methods already introduced to the students," has an ICC of 0.12, which represents that 12% of the variance in that items response rate is attributed to differences between teachers (i.e., between-teacher variance) while 88% of the variance is attributed to the variation across lessons for individual teachers (i.e., within-teacher variance).

The items related to the opportunity for problem solving have a wider range of ICC values, .12-.48. Items "Perform tasks requiring ideas or methods already introduced to the students" and "Perform tasks requiring ideas or methods NOT already introduced to the students" have lower ICCs (.12 and .17 respectively), and therefore seem to vary between lessons rather than between teachers. The discourse items presented in Table 2 range in ICC values from .23-.33, with 33% of the variance

Table 2: Intraclass Correlations for Log Items

Log Item	Intraclass Correlation
Perform tasks requiring ideas or methods already introduced to the students	0.12
Perform tasks requiring ideas or methods NOT already introduced to the students	0.17
Work on exercises specifically for practice or review	0.20
Discuss ideas, problems, solutions, or methods in large groups	0.23
Work on problem(s) that have multiple answers or multiple solution methods	0.23
Discuss ideas, problems, solutions, or methods with other students in small groups or pairs	0.24
Demonstrate different ways to solve a problem	0.25
Listen to me explain the steps to a procedure	0.26
Talk about similarities and differences among mathematical representations	0.30
Talk about similarities and differences among various solution methods	0.31
Explain orally his/her thinking about mathematics problems	0.33
Perform tasks focused on math procedures	0.48

in the responses to the item "explain orally his/her thinking about mathematics problems" being between teachers.

Discussion

These analyses provide a glimpse into novice teachers' mathematics classrooms and the opportunities students have to engage in various practices. The instructional log used in the study shows promise in its ability to document the practices teachers are using, as seen by the distribution of responses and evidence of variability even with a relatively homogenous sample of teachers (novice, formally prepared teachers). Logs have been used in past research (Rowan, Harrison & Hayes, 2004; Stecher, 2006), but the IPL-M was carefully designed to align with mathematical practices and processes (CCSSM, 2010; NCTM, 2000). The ability to collect data on a large amount of lessons in a relatively efficient manner makes the log an advantageous tool for teacher educators and researchers to understand teachers' instructional practices during mathematics.

In looking at the descriptive log data, three themes emerged. First, it seems that majority of lessons are focused on tasks that more prescribed in nature. These lessons utilized methods that have already been taught or focused on review and practice. This aligns with findings from Rowan, Harrison, & Hayes (2004) that approximately 70% of elementary lessons from a more experienced teacher sample involved direct teaching with known ideas. Although international comparisons imply that U.S. teachers need to engage students in more opportunities to grapple with mathematics, it seems there is still a tendency to over structure our students' learning opportunities by presenting the mathematical procedures that they then need to practice. While it is important not to detract from the value of practice and the need to review, we need to be simultaneously critical of the proportion of lessons devoted to these goals. For a novice teacher, this might be especially difficult due to the newness of navigating pressures of curriculum pacing and student assessment.

The second theme that emerged from analysis of descriptive information is the presence of student talk in novice teachers' lessons. Teachers reported that majority of lessons included small group or whole group discussion for a substantial amount time in the lesson. This paints the picture of interactive classrooms, in contrast to the lecture-style environment with which traditional mathematics is often associated. Also, ICCs of the items indicating whole group and small group discussion were .23 and .24, respectively, meaning that over 20% of the variance in utilizing these modes of discourse is between teachers. Future steps will be employed to try to account for the contextual factors or teacher characteristics that explain this variance. Understanding why some novice teachers implement more discussion than others would provide important implications for elementary mathematics teacher educators.

The third theme emerges from a closer look at the opportunities for student discourse. Although a majority of the lessons involved some level of whole or small group discussion, other items provide supplemental information about the nature of the discourse in these discussions. Approximately 30% of lessons involved student explanation for a substantial amount of time, and an even smaller proportion of lessons had students discussing similarities and differences among representations and/or solution methods. It seems that students have the opportunity for classroom discussions, but they may be limited in their opportunities to engage in discourse specific to mathematical representations and ways of solving problems. This seems supported by the lower percentage of lessons involving open-ended tasks or multiple solution methods. Furthermore, the ICCs of items detailing the nature of the discussion (explain, discuss similarities in representations and/or solution methods) were each approximately .30 indicating that about 30% of the variation in these items can be attributed to the teacher. So while lessons vary in the opportunities for students to engage in specific discourse practices, overall some teachers are engaging in these practices more than others.

As preservice and inservice teacher educators strive to equip teachers to implement high-quality mathematic instruction, this work provides valuable insight into the instructional practices that novice teachers are utilizing. While it is encouraging that teachers are engaging students in opportunities for discussion, methods courses might further strengthen teachers' instruction by making more explicit how teachers can specifically prompt students to discuss aspects of mathematical ideas such as explaining or comparing solution strategies. Our next steps with this data and ongoing data collection are to begin to analyze the contextual factors that might account for teacher variance on the use of certain practices. This will provide teacher educators with important information to help support novice teachers and help break the detrimental cycle of reverting back to practices based on past experiences and anxiety.

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ON FRAMING TEACHER MOVES FOR SUPPORTING STUDENT REASONING

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Supporting students' mathematical reasoning is an important goal of mathematics instruction, but can be challenging for many teachers .We report the results of a study aimed at better understanding and identifying the ways in which teachers support student reasoning when provided with conceptually rich tasks. This study resulted in the Teacher Moves for Supporting Student Reasoning (TMSSR) framework, which organizes moves vis-à-vis their function and their potential for fostering student thinking. We describe the TMSSR framework, illustrate its affordances for studying teacher practices, and highlight its utility for teachers, teacher educators, and researchers.

Keywords: Instructional Activities and Practices; Algebra and Algebraic Thinking

Introduction

An important goal of mathematics instruction is to support meaningful and productive student reasoning; however, this is a goal that many teachers can find challenging (e.g., Rasmussen & Marrongelle, 2006). Two essential means for achieving this goal are the implementation of conceptually rich tasks and teachers' abilities to support and foster student engagement in such tasks. We report the results of a study aimed at better understanding and identifying the ways in which teachers can support student reasoning. This study is part of a larger project (http://tinyurl.com/badgerellis) that aims to a) help students develop deductive reasoning competencies in algebra through quantitative reasoning opportunities, and b) support teachers in achieving this goal. To scale up the findings from small-scale teaching experiments to whole-class settings, we partnered with practicing mathematics teachers to implement research-based curricular units in their classrooms. Analysis of a middle school classroom yielded the Teacher Moves for Supporting Student Reasoning (TMSSR) framework. Below we present the TMSSR framework, illustrate its affordances for studying teacher practices aimed at supporting student reasoning, and highlight its utility for teachers, teacher educators, and researchers.

Theoretical Background

Frameworks Investigating Teacher Moves

Various frameworks exist for investigating teacher moves during classroom instruction. While some frameworks focus on the questions teachers ask (e.g., Driscoll, 1999; Frey & Fisher, 2011), others focus on discursive moves (e.g., Herbel-Eisenmann, Steele, & Cirillo, 2013; Hufferd-Ackles, Fuson, & Sherin, 2004; Krussel, Edwards, & Springer, 2004). Yet others take a broader approach to include teacher questioning as well as other moves that teachers make in the course of instruction (e.g., Lampert et al., 2013; Staples, 2007). Taken together, these frameworks outline the general teacher moves and questions that occur in classrooms while teachers are eliciting, encouraging, and responding to students during instruction.

Additionally, there are frameworks that focus on student thinking. For example, Stockero et al.'s (2014) MOST framework for analyzing productive mathematical student thinking seeks to identify the most productive student thinking instances that warrant further teacher response. We build on these frameworks by investigating the moves teachers employ in quantitatively-rich contexts and the potential these moves have for supporting student reasoning. Like others have noted (e.g., Franke et al., 2009), teacher questioning is often used to help students make their thinking more explicit. However, in contrast to a focus exclusively on the questions teachers ask, a framework that includes additional practices that appear to support student thinking can provide a more complete picture of how teachers can foster student reasoning when engaged in conceptually rich tasks.

Quantitative Reasoning

Quantities are individuals' conceptions of measurable attributes of objects or events, such as length, area, volume, or speed. Relying on situations that involve quantities that students can make sense of, manipulate, and investigate can foster their abilities to reason flexibly about dynamically changing events (Carlson & Oehrtman, 2005). Reasoning with relationships between quantities has been found to support students' understanding of algebraic relationships and to encourage deductive argumentation (Ellis, 2007; Smith & Thompson, 2007). We therefore designed quantitatively rich tasks in a series of small teaching experiments, which we then provided to the teachers for implementation in their classrooms.

Methods

The study we report here occurred in an 8th grade mathematics classroom at a public middle school and consisted of ten days of instruction on linear relationships grounded in a context of gear ratios. We provided the teacher with a set of research-based tasks for exploring and identifying relationships between gears rotations; the teacher also had the liberty to make modifications to the tasks as she saw fit. All sessions were videotaped and transcribed. Additionally, field notes, student work, and an interview with the teacher provided supplementary data.

We began analyzing the transcripts (which also included the images of student work and written work on board) of the observed lessons via open coding, without any particular framework in mind. As we progressed into the data analysis using the constant comparison method (Glaser & Strauss, 1967), our attention focused on the teacher moves that supported student reasoning, and we eventually developed an emergent coding scheme through multiple passes of open coding. After the initial development of the coding scheme, we also analyzed the literature base to identify the ways in which our codes for teacher moves intersected with existing descriptions reported in the literature. After reaching a fairly stable coding scheme, we proceeded with focused coding (Saldaña, 2009) and two researchers independently re-coded the entire data set. Through constant comparison and discussion of each researcher's coding, the coding scheme was further refined by way of revising some definitions as well as delineating the functions the teacher moves served to support student reasoning.

Findings and Discussion

The TMSSR framework identifies and organizes teacher moves into four categories based on the function they serve in supporting student reasoning (i.e., eliciting, responding to, facilitating, and extending). In addition, teacher moves within the same category differ in their potential to support student reasoning. For instance, although *correcting a student error* and *prompting a student error correction* are both moves teachers make in response to student reasoning, prompting a student to correct her error has the potential to lead to a greater learning opportunity for the student than if the teacher had merely corrected the error (Speer, 2008). Drawing both from Speer's discussion about teacher moves offering different potential for supporting student learning and an analysis of how the teacher's moves affected student reasoning (as inferred from students' responses), we place teacher moves along a continuum for the potential each move has for supporting reasoning. More specifically, moves that offer greater potential are located towards the right hand side of each category. These moves occur during whole class discussions as well as when the teacher is working with students in small groups or individually. We begin by describing each category of the framework and then present an analysis of one teacher's classroom with the TMSSR framework, who we call Ms. L.

Focusing in on the TMSSR Framework

Tables 1-4 present the categories of the TMSSR framework. Although related teacher moves are organized along a continuum (signified by rows in the table), it is important to note that the continuum represents the *potential* each teacher move has for supporting student thinking. How the teacher enacts a move and the students' responses determine the *actual* affordance for supporting student reasoning. In some cases (e.g., re-voicing, encouraging student re-voicing, and re-representing) more than two teacher moves are organized within the same row to signify their related nature. Due to page constraints we are not able to demonstrate how all of the moves are placed along a continuum in the framework. We focus on a row from tables 1 and 4 since the moves in Tables 1 and 4 encourage students to take a more active role in the discussion (as opposed to Tables 2 and 3 where the teacher has a more prominent role).

Table 1: Teacher Moves for Eliciting Student Reasoning

<table>
<tr><td>Eliciting Answer: Teacher asks a question geared at eliciting the answer to a given task or problem.</td><td>Eliciting Ideas: Teacher asks a question geared at eliciting students' ideas for a solution strategy.</td></tr>
<tr><td>Eliciting Facts or Procedures: Teacher asks questions geared at eliciting students' recitation of facts or procedures.</td><td>Eliciting Understanding: Teacher asks questions geared toward assessing what students understand and how they are reasoning.</td></tr>
<tr><td>Asking for Clarification: Teacher asks a question to clarify the student's meaning because teacher genuinely does not know what the student meant.</td><td rowspan="2">Pressing for Explanation: Teacher asks student(s) to elaborate on their thinking, explain their reasoning, or reflect on and share their reasoning.</td></tr>
<tr><td>Figuring Out Student Reasoning: Teacher is trying to figure out a student's solution, or understand a student's explanation or reasoning.</td></tr>
<tr><td>Checking for Understanding: Teacher asks a question to assess students' understanding of the mathematical ideas that are currently under discussion.</td><td></td></tr>
</table>

The moves presented in Table 1 enabled Ms. L to elicit students' reasoning while implementing the quantities based tasks. These moves served to engage students in sharing their thinking and often occurred at the beginning of a discussion about a particular problem. These teacher moves commonly occurred when Ms. L worked with students in small groups as well as when she facilitated whole class discussions. The excerpts below demonstrate the potential difference that two related teacher moves, *Asking for Clarificatio*n and *Pressing for Explanation,* have for supporting student reasoning. As seen in the following excerpt, when Ms. L asks for clarification, the student's response (i.e., Leigh) is often minimal, with little or no elaboration:

Leigh: You could just plug in numbers so the middle gears teeth equals, the middle gears teeth is 12 and the big gears teeth are 16. So you need 3/4 times 12 and if it equals 16, if it equals 16, then…

Ms. L: So you are wondering this[*writes* $\frac{3}{4}(12)\overset{?}{=}16$ *on board*]?

Leigh: Yes.

Asking Leigh what she was wondering about may have clarified for Ms. L Leigh's current thinking about the task. However, if Ms. L had instead pressed Leigh for an explanation, as she did with Hope in the following episode, Leigh would have had more potential to think through and articulate her own strategy. For example, when Ms. L pressed Hope for an explanation (i.e., asking where the ratio 2/3 exists in Hope's work), this move encouraged Hope to think more about why her strategy made sense. Thus, Hope was able to start moving from a procedural explanation towards a more conceptual explanation:

Hope: I made the table again. So like, so this is the first one, so Lewis' formula yesterday was to divide the smaller number which is the bigger gear by two. So I wrote, so it's 2.5 *(points to 5 on table)(writes 5: 2= 2.5)* and then instead of writing a whole new one I worked by like continuing, so I added 5 even though it isn't proper and it said 7.5 and then it equaled this one (*points to 7.5 on table*) so you know it works because of the 2/3.
Ms. L: Okay, so can you maybe elaborate a little bit more? Which thing is the 2/3?
Hope: Okay, so like this is like you're reducing it down a different way.What I did is I would say, like this, you're just making 'cause, I don't know how to say this, okay so like 7 and a half like you're trying to find 2/3 of 7.5.*(points to 5)*

Table 2: Teacher Moves for Responding to Student Reasoning

***Validating a Correct Answer**:* Teacher actively confirms the student's idea by re-voicing, or re-wording in her own words, or adding a bit to the student's idea or response.	
***Re-voicing**:* Teacher repeats student ideas (verbally or written) in order to make those ideas public.	***Re-representing**:* A form of re-voicing in which a teacher provides her own representation as a way to publicly share a student's idea, work, or strategy. The teacher may organize, re-frame, or formalize the student's statement or work.
***Encouraging Student Re-voicing**:* Teacher asks students to re-voice other student ideas or solutions.	
***Correcting Student Error**:* Teacher corrects a student error or supplies the correct answer more generally.	***Prompting Error Correction**:* Rather than correcting the student, the teacher prompts the student to address an error herself.

Due to space constraints, we primarily focus on the organizational structure of Tables 2 and 3, which present the teacher moves for responding to and facilitating student reasoning, respectively. The moves in Table 2 often occur after a teacher has already elicited student reasoning and s/he is trying to make students' reasoning more public or amend a student's statement. Teacher moves in Table 3 also generally occur after the teacher has elicited student reasoning and is now trying to assist students in developing their reasoning through various forms of guidance and explanations. These moves may help students engage with a task or summarize students' contributions before moving on to a new task. Although the teacher moves described in Table 3 are common, the *Topaze Effect* is worth noting. Stein, Grover, and Henningsen (1996) describe this move as reducing, however, we build from Brousseau's (1997) description of how a teacher breaks a task into smaller parts and thus significantly alters how a student conceptually engages in the task. In the following excerpt Ms. L asks Laura a question about the relationship between the two gears. However, before Laura has a chance to respond, she immediately asks an easier question. The second question reduces the original question (in which Laura would have to determine a fractional amount of a rotation) down to a yes/no response (whereby Laura merely has to identify whether the gear made an entire rotation).

Ms. L: (*To Laura*) So if you turned the small gear once, how far around would the big gear go? (*Brief pause*) Would it make it all the way?
Laura: No.

Table 3: Teacher Moves for Facilitating Student Reasoning

<table>
<tr><td colspan="2">Cueing: Teacher cues students' attention by indicating that they should focus on a particular aspect of a problem, task, idea, solution, etc.</td><td colspan="2">Providing Guidance: Teacher provides hints, ideas, a potential strategy, or another type of conceptual scaffolding of the problem without outlining the solution structure or otherwise shutting down students' opportunities to reason on their own.</td></tr>
<tr><td colspan="2">Topaze Effect: Teacher breaks a task into smaller parts, reducing the complexity of the task by asking easier and easier questions, thereby reducing students' opportunity to engage in authentic problem solving.</td><td colspan="2" rowspan="2">Building: Teacher builds on students' earlier contributions to support new understanding, or encourages students to build on one another's contributions.</td></tr>
<tr><td colspan="2">Funneling: Teacher asks questions that move students down a specific path (e.g., through leading questions).</td></tr>
<tr><td rowspan="2">Providing Procedural Explanation: Teacher provides a procedural explanation for how to solve a problem. This move includes telling students a priori how to solve the problem by outlining the solution structure (or some other way).</td><td colspan="2">Providing Summary Explanation: Teacher summarizes for the class final thoughts about a task or problem, or a summary of information or discussion about the task.</td><td rowspan="2">Providing Conceptual Explanation: Teacher provides an explanation that has a conceptual basis, often focused on explaining why something works. This move can also be thought of as demonstrating logic.</td></tr>
<tr><td colspan="2">Providing Information: The teacher provides new information relevant to doing mathematics generally rather than information about a specific problem or task.</td></tr>
<tr><td></td><td colspan="2">Encouraging Multiple Solution Strategies: Teacher encourages a proliferation of solution strategies, including pressing students to come up with a different way to solve a problem.</td><td></td></tr>
<tr><td></td><td colspan="2">Providing Alternative Strategy: Teacher initiates a new or different way of solving a problem after students have shared their solution strategies or solutions.</td><td></td></tr>
</table>

Table 4 presents the teacher moves that were used to extend student reasoning. These moves usually occur after students have worked through a task for some time and have made some progress into the solution. To further extend students' initial reasoning, the teacher pushed students to provide complete answers rather than vague responses, to make connections to the context, to think about the underlying concepts involved in the task, and to justify their ideas. The excerpts below demonstrate the potential difference that two related teacher moves, *Topaze for Justification* and *Pressing for Justification,* have for supporting student reasoning. In both situations, Ms. L asked students to explain why an idea, solution, or strategy works; however, in the case of *Topaze for Justification*, Ms. L did not allow students enough time to grapple with the initial prompt to justify their ideas. Instead, she reduced the complexity of the question by following up with an easier question or by narrowing the question's focus. As an example, in the following excerpt Ms. L asked Gert to show why there was not a relationship between the gears, but she then immediately suggested testing with numbers. By doing so, Ms. L unwittingly prevented Gert from devising her own way to justify her claim:

Ms. L: What'd you come up with for a reason for part a?
Gert: I said because there's no relationship.
Ms. L: Good. Can you show why not? Can you show like with numbers?

Without the space to think of her own justification, Gert agreed to use Ms. L's suggestion, but the emphasis on numbers shifted the conversation to a calculational explanation. In contrast, Ms. L was able to better advance students' reasoning when she persistently pushed students to justify their ideas, allowing students enough time to think and solidify their reasoning. The following excerpt exemplifies the *Pressing for Justification* move:

Ms. L: Okay. So nine goes here is what we are saying (*writes 9 to "B" column*). So does that seem correct that if the medium gear spun twelve, the big gear would spin nine?
Students: Yes.
Ms. L: Okay. Yes, you are saying yes. (*spinning gears*) Anyone see a proof of why that works? Something you can use for evidence. Laura?
Laura: Okay. Well the ratio was three fourths
Ms. L: Uh-hum.
Laura: So then if you, um if you... Yeah, so the ratio is three fourths and now it's like, you could say it is nine twelfths. And then if you divide nine by three its three and if you divide twelve by three it is four.

Table 4: Teacher Moves for Extending Student Reasoning

Pressing for Precision*:* Teacher encourages student(s) to provide an exact rather than vague answer, to check his or her work for accuracy, or to quantify a qualitative statement.	***Encouraging Reasoning :***Teacher encourages students to think about the task conceptually, for instance by thinking about why a strategy makes sense, by thinking about where the numbers connect to the quantitative situation, etc.
	Encouraging Reflection: Teacher asks students to reflect on provided answers or explanations (either from the teacher or from another student).
Topaze for Justification: Teacher initially pushes for justification, but then immediately downgrades her question to a less-sophisticated why question by heavily leading students into justification via easier questions.	***Pressing for Justification****:* Teacher asks students to explain why something works or to justify (logically, conceptually) their idea, strategy, or solution.
	Pushing for Generalization: Teacher encourages students to generalize their reasoning, either through formulating a rule, describing a process in general terms, or making connections across problems, numbers, cases, or events.

Analysis of Ms. L's Classroom Using the TMSSR Framework

Table 5 illustrates Ms. L's moves for supporting student reasoning while implementing the quantities-based unit. Frequency counts for each move are listed in parentheses. It is important to note that more than one teacher move may occur at the same time (e.g., a teacher often elicits facts or procedures while she is funneling). Shading corresponds to the proportion of a specific move compared to all moves in the table, with the darker cells representing the moves that occurred more frequently. When comparing the four functional categories in the TMSSR framework, the table suggests that Ms. L spent more instructional time eliciting student reasoning compared to any one other category. Eliciting moves occurred most frequently because the other three categories represent moves that generally occur after student reasoning had been elicited. Given that the data come from Ms. L's first implementation of the research-based unit, it is not surprising that her moves for supporting student reasoning were on the left hand side,with less potential for supporting student reasoning. As a teacher becomes more familiar with the tasks and the moves that have greater potential for supporting student reasoning, we would expect to see more moves located on the right hand side of the continuum. Although the TMSSR framework focuses on teacher moves, it is important to note that such moves are also related to the classroom environment (for instance, it had already been established that students were routinely encouraged to share their reasoning and were viewed as responsible learners by themselves and the teacher) and students' ability to engage with the tasks. Because the TMSSR framework focuses on the potential support teacher moves have for fostering student reasoning, two teachers could have similar illustrations (e.g., both could look similar to Table 1) but their students' development of reasoning could be different.

Table 5: Illustrating Ms. L's Moves with the TMSSR Framework

Eliciting Student Reasoning		Responding to Student Reasoning	
Eliciting Answer (53)	Eliciting Ideas (13)	Validating a Correct Answer (85)	
Eliciting Facts or Procedures (144)	Eliciting Understanding (37)	Re-voicing (27)	Re-representing (26)
Asking for Clarification (25)	Pressing for Explanation (46)	Encouraging Student Re-voicing (4)	
Figuring Out Student Reasoning (46)		Correcting Student Error (26)	Prompting Error Correction (38)
Checking for Understanding (10)			

Facilitating Student Reasoning			Extending Student Reasoning	
Cueing (47)	Providing Guidance (40)		Pressing for Precision (17)	Encouraging Reasoning (20)
Topaze Effect (60)	Building (11)			Encouraging Reflection (25)
Funneling (73)			Topaze for Justification (12)	Pressing for Justification (13)
Providing Procedural Explanation (74)	Providing Summary Explanation (28)	Providing Conceptual Explanation (21)		Pushing for Generalization (5)
	Providing Information (14)			
	Encouraging Multiple Solution Strategies (26)			
	Providing Alternative Solution Strategy (9)			

Conclusion

We propose a framework for teacher moves that support student reasoning by organizing moves vis-à-vis their function and their potential for fostering student thinking. Although the teacher moves were made while implementing quantities-based algebra units, many of the moves(e.g., building, eliciting, re-voicing) are similar to those that have been presented in other frameworks(e.g., Herbel-Eisenmann, Steele, & Cirillo, 2013; Lampert et al., 2013; and Staples, 2007). Therefore, we posit that these moves are not unique to these classrooms and can serve others working with teachers and investigating the moves that they make to support student reasoning.

By examining teacher moves holistically, we can better understand the various ways teachers support student reasoning. The TMSSR framework classifies teacher moves into functional categories and locates these moves along a continuum based on the potential support that a move has for supporting students' reasoning. Ideally, we would like to see teachers more frequently employ moves for extending student reasoning. However, we also caution that only attending to the frequency of teacher moves across categories may lead to incorrect conclusions about a teacher's practices. Our analysis revealed that some teacher moves have more potential for supporting student reasoning than others within the same category (e.g. *Topaze for Justification* and *Pressing for Justification)*;thus, placing moves along a continuum is helpful for better assessing teacher practices for their potential for supporting student reasoning. Teachers who are interested in informally assessing their teaching may benefit from thinking about both the four categories of moves and the ways in which teaching actions are organized within each category. Similarly, researchers and teacher educators studying teachers' practices could use the TMSSR framework to identify specific areas in which teachers may benefit from additional support.

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THE TELLING DILEMMA: TYPES OF MATHEMATICAL TELLING IN INQUIRY

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Teacher telling continues to be poorly understood within inquiry. In this paper I extend prior efforts to reimagine telling within contemporary pedagogical thought. Using a case study, I investigated a well-regarded teacher's use of mathematical telling while supporting groups and individuals working on tasks. The teacher used seven unique types of mathematical telling: assess, interpret, qualify, clarify task, guide, disclose, and validate. This mathematical telling framework aids in the identification of subtle telling, the recognition of implicit telling, and the acknowledgment of explicit telling. Telling practices should be conceptualized and evaluated contextually.

Keywords: Classroom Discourse; Instructional Activities and Practices

Introduction

At the heart of mathematics teaching lies an enduring dilemma of how and when to tell. This dilemma surfaces within a broader struggle to privilege the spontaneous thought and activity of the child while simultaneously cultivating and enculturating the child into a scientific and socialized society. Contemporary pedagogical practice is driven by progressive reforms advocating student-driven inquiry, on the one hand, and increased standardization and accountability to prescribed conventional knowledge of mathematics, on the other. These demands require a give-and-take approach since students cannot just independently discover everything they are expected to know, nor do they simply absorb foreign ideas that are explained to them. To teach, then, is to skillfully manage the intersection of the child and the curriculum. The telling dilemma repeatedly surfaces in that work. Teachers grapple with how to manage learning through tasks (Henningsen & Stein, 1997; Simon, 1995), how and when to steer class discussions (Ball, 1993; Chazan & Ball, 1999), how to reorient student inquiry that is misguided or trivial (Ball, 1993), and how to help diverse students access and benefit from implicit forms of knowing and learning in classroom discourse practices (Lubienski, 2002).

The issue is not whether always to tell or never to tell, but rather when and how to tell (Baxter & Williams, 2010; Chazan & Ball, 1999; Lobato, Clarke, & Ellis, 2005). Nevertheless, explicit teacher telling of mathematical *content* remains taboo as teacher practice is framed through less mathematically obtrusive constructs. Teaching has been framed in terms of orchestrating discourse (Lampert, 1990; Rittenhouse, 1998; Staples, 2007; Stein, Engle, Smith, & Hughes, 2008; Wood, 1998), facilitating collaboration and interaction (Clarke, 1997; Cohen, 1994; Dekker & Elshout-Mohr, 2004; Webb, 2009; Webb et al., 2009), and posing high-quality tasks (Henningsen & Stein, 1997; Simon & Tzur, 2004). When research does acknowledge and discuss teacher moves related to telling, either these telling moves are believed to compromise the quality of learning (Chiu, 2004; Dekker & Elshout-Mohr, 2004; Tyminski, 2010), or the mathematical substance originating from the teacher is absent or downplayed (e.g., Chazan & Ball, 1999; Smith, 1996; Staples, 2007).

Arguably, such treatment of teacher telling is overly conservative given that the teacher's mathematical discourse can be a tool for promoting student sense making (Baxter & Williams, 2010; Ding, Li, Piccolo, & Kulm, 2007; Lobato et al., 2005). Further understanding is needed of how teachers tell or speak assertively with students about mathematics, especially while they grapple with the major challenges of teaching within inquiry.

Although teacher telling might occur in various instructional formats, this study focused on telling during teacher interventions with collaborative groups. In the whole class setting, teachers can rely on student contributions of content to advance class discussions (e.g., Stein et al., 2008) whereas beforehand in the collaborative group setting, teachers must prepare students so they are able to make those contributions and access the contributions of others. During students' struggles with tasks, teachers may encounter opportunities to engage mathematically with particular students in response to specific needs. The research question of this study was, "What types of mathematical telling does a well-regarded inquiry-based teacher use to support students while intervening with small groups and individuals working on mathematical tasks?"

Framework

To address my research question I briefly discuss inquiry before framing telling more rigorously. The term "inquiry" in this study is meant as a broad descriptor of instruction that aspires to privilege student thinking and activity during the cultivation of curricular standards. Without claiming that inquiry is a well-defined and homogenous mode of instruction, I argue that the teacher in this case study upheld the basic aspiration of inquiry as she enacted various practices. The teacher varied her instructional format between whole-class discussion, small-group work, and individual work. She introduced non-routine mathematical tasks and monitored student progress. She facilitated students in sharing and critiquing one another's reasoning. The teacher helped students form complete and correct justifications of their ideas. She gave detailed feedback on their work in and out of class. These practices are believed to support student inquiry into mathematics (see Clarke, 1997; Staples, 2007; Stein et al., 2008; Webb et al., 2009).

To study mathematical telling, I followed Lobato et al. (2005) in framing telling as a phenomenon of verbal discourse with three main attributes: form, content, and function. These three attributes are discussed below.

The form of discourse refers to its organization and grammatical structure. Discourse is produced into questions, statements, requests, commands, and so on, by selecting from a complex variety of verb tenses and grammatical syntax. Although an utterance derives meaning in part from its grammatical form, the form does not uniquely determine whether that utterance is telling. Questions can tell and statements can question (Lobato et al., 2005, p. 9).

The content of discourse refers to the objects and ideas denoted by particular words and phrases. Although language does not literally transmit fixed meanings from speaker to listener (von Glasersfeld, 1995), the references of an utterance are important indexical markers for objects and their negotiated meanings. A mathematical reference is an indicator of telling even if the internal mental referent of the speaker's utterance is inaccessible to the listener.

The function of discourse refers to its purpose in a situated activity. Function is simultaneously determined by a speaker's intentions, a listener's interpretations, and the nature of the activity itself that is enacted through the discourse (Lobato et al., 2005). Although these three elements may or may not align with one another and are often difficult to infer, I managed this complexity by limiting my research focus to the verbalized discourse of the lesson. My inferences were therefore centered on the observable function of the discourse during an unraveling event rather than on the layers of meanings for the participants.

In coordinating these three attributes of discourse, I did not use form directly, but I was sensitive to form while making sense of content and function to overcome unwarranted prejudices about telling based on form (e.g., "questions don't tell"). I defined mathematical telling, then, as discourse that contained mathematical *content* (indexical references) and served the *function* of inserting something new mathematically into the conversation. During later analysis the "something new" came to mean the insertion of mathematical ideas, structure, constraints, or acts that were not in play before the utterance.

Method

I conducted a case study of a well-regarded teacher educator teaching one class at a large four-year private university. The course was a mathematics content course (the second of a two-course series) designed for pre-service elementary teachers and included topics such as fractional reasoning, probability, and statistics. The class met twice weekly for 110 minutes per class. I observed and analyzed two full curricular units of instruction, each with seven lessons. I typed field notes to index the events of each class and recorded conversations in abbreviated form. All classroom activities were filmed, and subsequently all interactions between the teacher and students during task-work were transcribed. These transcriptions formed the core data for analysis, supplemented by my field notes and the task-sheets and handouts from the course.

I first coded for mathematical telling using the definition articulated earlier. The unit size for a mathematical telling act was generally a teacher turn. I then started differentiating mathematical telling acts from one another by attending to their mathematical content and the relationship of that content to the students' inquiry. I analyzed small subsets of data and generated provisional categories of telling. I then analyzed my set of categories, working to find general overlaps, inconsistencies, ambiguities, and so on. Changes to the categories were then taken back into the data. I continued this process until a provisional set of codes had surfaced. I briefly explained the codes to an informed peer who coded a small portion of data. The questions and discrepancies from this exercise helped me to modify, invent, dissolve, and reorganize codes and improve their written theoretical definitions. I also picked two fresh lesson transcripts to code twice, with a one-week interval between coding. After comparing my first and second reading of each transcript, I identified potential technical overlaps at the boundaries of codes and updated code definitions to set precedents for how to handle similar cases. I had used about half of my data to solidify this coding scheme, and I coded the remaining half comfortably.

Results

There were seven telling types in the eventual framework. As I interpreted the results, two telling types (*guide* and *disclose*) contained significant internal variation that I sorted into four purposes each in order to understand them better. For presentation, I ordered the seven types roughly from the least obtrusive to the most obtrusive.

Assess

The first type of mathematical telling is to assess. Not every assessment is telling (such as "How did you solve the problem?"), so a telling assessment is a mathematically structured or constrained request beyond just an open elicitation of student thinking. Even though a telling assessment does not reveal the answer or solution, it imposes a new idea or constraint and is therefore a form of telling. A responding student shares not only his or her own thinking but shapes it in order to either conform or object to the constraint embedded in the assessment.

A typical example occurred frequently during a particular lesson. The teacher would assess students' understanding of two fraction images (i.e. conceptual representations) by asking of their reasoning, "Is that partitioning or iterating?" Despite seeming innocuous, there is considerable mathematical structure embedded in such discourse. The assessment restricted students to those two images despite there being reasonable alternatives that at least one student was observed to use. Furthermore, it created a forced choice between the images even though many students at times believed they were appealing to both images within a single justification. Other assessments were even more complex when they implied potential connections (e.g., between fractions and the operation of division) that students had not yet considered.

Interpret

The second type of mathematical telling is to interpret students' mathematical formulations. The teacher clarified or characterized a student utterance by rephrasing, summarizing, generalizing, condensing, or inferring unspoken pieces of a student thought. Even though the teacher attributed such interpretations to students, they were nevertheless filtered through the teacher's own conceptual grid of meanings and brought something new into play.

Qualify

The third type of mathematical telling is to qualify the mathematics. The teacher qualified a mathematical part of the conversation according to human experience such as student feelings, motivation, or common sense. Because some students saw mathematics as threatening, irrelevant, contrived, difficult, or tedious, the teacher attended to these issues. For example, the teacher downplayed student errors to minimize embarrassment, attached value to the activity or task, acknowledged interesting contributions, and characterized (valid) justifications as awkward or natural. The discourse in these telling acts mediated the students' relationships with the mathematics and consequentially communicated something new about mathematics. This telling type was a surprising result and suggests that how individuals think and feel *about* mathematics is inextricably related to what mathematics *is* to them.

As a brief example, a student worked on a probability question about drawing colored balls repeatedly from a bag. The students had not yet developed a formula or procedure and were calculating the probabilities by making lists of possible outcomes and determining the fractional part of the outcomes that answered the question. One student began to find the activity tedious:

Student: So like, what's the point of listing all the combinations? What's it teaching us?
Teacher: What's this teaching you? Well for a lot of— *(not coded mathematical telling)*
Student: Like to be patient.
Teacher: No, no, no [laughs]. It's not teaching you to be patient. It's first of all helping us think about the situation, like what's really involved here in the situation. We have these shortcuts. We can't make meaning of the shortcuts if we don't know what's actually going on. And so I know you have these nice shortcuts like multiplication, but if the shortcuts don't make sense in terms of the situation then they are meaningless. But writing out the combinations can help us see where the shortcuts are going to come in. *(qualify)*

Here the teacher qualified the value of the task by affirming that doing mathematics is about making meaning of situations rather than executing shortcuts.

Clarify Task

The fourth type of mathematical telling is to clarify the task. The teacher clarified the mathematics of the task, question, or activity without addressing the actual solution. Two main purposes for clarifying the task were to provide basic instructions for student engagement and to clarify the mathematical meanings prerequisite to engaging in the task.

Guide

The fifth type of mathematical telling is to guide. The teacher guided students while they developed solutions, constructed justifications, discussed concepts and addressed errors. The key characteristic of teacher guidance was that the mathematical substance it contributed remained partially incomplete or unresolved, requiring the student to act on, complete, or incorporate it into his or her work. The teacher guided students for four main purposes (see Table 1): focus toward or away from an idea, lead students into productive ways of thinking, address reasoning errors, and give helpful hints and suggestions.

Table 1: Purposes of Guidance

Focus	Direct student attention, encourage or discourage student approaches, point to parallel examples or previous experiences
Lead	Pose questions or next steps, structure student justifications or explanations, ask rhetorical questions, help students draw a conclusion or inference
Address reasoning errors	Identify a student contradiction, invalid assumption, or deficiency; give counter-arguments; workshop a student justification; locate or correct an error
Give hints, suggestions	Suggest a method, bound or estimate the solution, interpret the problem, state principles or guidelines, explicate the criteria for an acceptable solution

In the example that follows, a student was solving the following question with Cuisenaire rods: “If purple is 2, what is the value of black?” She had placed a purple rod adjacent to a black rod, and then lined up seven small white rods alongside them as shown in Figure 1. The student first expressed confusion with the problem, using language such as, “The purple is four out of seven, the black is seven out of seven.” The teacher guided the student by first *focusing* her attention toward the given information in the problem (that the purple rod has length 2), but the student could not reconcile this given information with her propensity to think of the little white blocks as units or ones (she continually referred to purple as “four”). The teacher *addressed the reasoning error* by explicitly inviting the student to reconcile her naming of purple as four and the problem’s given assumption that purple is two. When the student was unable to do so, the teacher followed up with a series of crucial *leading* moves (presented below) that marked the climax of the interaction and enabled the student to subsequently reason her way to the solution:

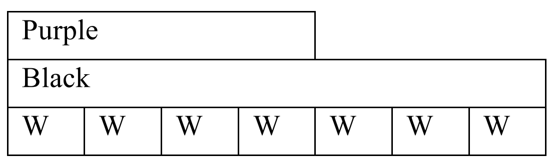

Figure 1. Student’s arrangement of purple, black and white Cuisenaire rods.

Teacher: So if this [purple] is, if this is two, what’s this? [picks up one white block] *(Guide-Lead)*
Student: A half.
Teacher: Okay what, how did you get that? Why do you know it’s a half? *(Not mathematical telling)*
Student: Because one half multiplied by four is two.
Teacher: Okay, so what would one be? *(Guide-Lead)*
Student: One would be, [shows the amount of two whites with fingers]
Teacher: Yeah. Okay, so now use this to figure out what black is. So you know that these [whites] are not one. *(Guide-Lead)*

Soon after this, the student correctly identified black as three and a half. The teacher’s leading questions were not just non-mathematical process help (see Dekker & Elshout-Mohr, 2004), nor were they “funneling” questions that trivialized the mathematical concept for the student (Wood, 1998). The first leading question firmly established the given assumption that purple is two as the premise from which to make a new deduction about the quantity that one white represents. This information helped the student to stop thinking of purple as four and white as one, and was the pivotal turning point after which the student began to reason appropriately about the situation. Guidance such as this was a frequent tool for this teacher and is a powerful form of telling whose mathematical substance should not be downplayed.

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Disclose

The sixth type of mathematical telling is to disclose. A disclosure, unlike guidance, was more mathematically complete and resolved and usually revealed a solution component, an alternative solution, an explanation, a justification, a norm, or a convention. The teacher disclosed this information while discussing complex concepts, answering student questions, helping students construct solutions and justifications, and refining student work. Four main purposes of disclosure emerged (see Table 2): amplify student input, explain mathematical concepts, model appropriate reasoning, and provide norms and expectations.

Table 2: Purposes of Disclosure

Amplify student input	Elaborate on a student answer, finish a student's thought, rephrase and modify mathematical language, validate a response, synthesize input from multiple students
Explain	Explain idea or concept, distinguish concepts, clarify method, illustrate with an example, contrast problem structures
Model reasoning	Justify a solution or algorithm, correct flawed student work, provide solution component, provide alternate solution
Norms & expectations	Explicate norms such as mathematical conventions, notation, terminology, expectations for justifications, or the intended form of an answer

An example illustrates two of these purposes. During one group discussion students were trying to understand the range of a set of shoe sizes (that varied from size 6 to 12) in a shoe store by counting the available sizes. They had some trouble deciding whether the range was six or seven (by whether they counted the smallest size), as well as whether one would need to count half sizes. To avoid this quandary one student joked that her store just wouldn't offer half sizes! The teacher decided to disclose the following explanation to help clarify range:

Teacher: You don't offer half sizes!? So, okay, so it's like—So, it sounds like, okay, it sounds like these are two different numbers. When you're talking about the range, um, that's saying I go—I, like, I encompass this amount. Whereas, if I say I offer fourteen sizes, that's a count of the number of sizes you offer. And I think those are two different numbers. *(Disclose-Explain)*

In something of a tentative follow-up, one student ventured her own definition of range, included below. The teacher amplified this student's input:

Student: Is the range the one number that's how many numbers are in between the smallest and the greatest?
Teacher: Yeah, it's the distance between the smallest and the greatest. So, that's a six. *(Disclose-Amplify)*

The teacher amplified the student's input by strategically modifying the mathematical imagery and accompanying language. Instead of the problematic phrase "how many numbers" (there are actually infinitely many numbers between any two real numbers), she used the more precise term "distance." In addition, she modeled appropriate usage of her definition to provide the answer to this particular example—the range of shoe sizes varying between 6 and 12 is six.

Validate

The seventh and final type of mathematical telling is to validate a student expression of mathematics as correct or incorrect. Validation was unique in that it could occur as the only function of a telling utterance or in combination with one of the other six telling functions (for example, many disclose–amplify utterances also validated student input). Validation was the least reliable code to discern in the data. At times it was ambiguous due to words such as "yeah, okay, uh-huh" and "alright" that were also social conversational markers. Other times validation occurred in the discourse simply by repeating student language with positive emphasis or negative skepticism. Other implicit interactional norms or gestures may have served as validation and gone unnoticed. I conservatively marked discourse as validate only when the speech before and after the incident in question showed evidence that the teacher and students treated it as validation. Even so, validation was a frequent type of telling, occurring in well over half of the mathematical conversations the teacher held with groups and individuals.

Discussion

Three important findings across the seven telling types distinguish this study from prior work in inquiry. First was to recognize the frequently overlooked mathematical contributions in subtle acts such as assessing and interpreting student thought. Second was to make mathematical contributions of indirect mathematical aid, such as guidance, more transparent. Third was to acknowledge that overt forms of telling, such as disclosing items and validating student work, were pervasive and integrated components of the teacher's inquiry-based instruction.

Conclusion

The primary contribution of this study is to empirically expand the construct of telling as it occurs in contemporary pedagogical spaces such as inquiry. Telling is more than "simply telling students whether their answers are right or wrong or giving students correct answers" (Chazan & Ball, 1999, p. 2). Mathematical telling comprises a rich vocabulary for talking about some of the mathematical content and pedagogical functions of teacher discourse. Telling as a conceptual space was enriched to include less conspicuous forms of telling whose embedded mathematical structures have frequently been eclipsed by the glaring omission of more conspicuous mathematical information.

Inquiry has informally been set apart from more direct modes of teaching by its pervasive lack of "telling." The descriptive case study presented here suggests that this need not be the case. The examples of mathematical telling presented in the framework were a consistent and integrated component of the teacher's inquiry-based practice. Naturally, not every instance of mathematical telling was ideal, and the study should not be taken to justify the indiscriminate use of telling. However, mathematical telling as a practice is not automatically harmful, unwarranted, or inferior to less mathematically saturated discourse simply by virtue of initiating mathematical ideas (see Lobato et al., 2005).

The main implication of these considerations is the need to take a more nuanced and contextual orientation to telling. Mathematical telling practices should be viewed as one set of many available instructional tools for creating structure and managing student activity, along with mathematical tasks, tools, physical environments, and collaborative interactional norms.

Further research should investigate teacher awareness of and intentional use of telling practices, the influence of mathematical telling on student thinking, the use of telling practices in whole-class discussions and other instructional settings, and the situated relationship of mathematical telling to the other parts of the teaching environment as a whole.

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A MIXED METHODS STUDY OF ELEMENTARY TEACHERS' EXPERIENCES WITH AND PERSPECTIVES ON THE CCSS-MATHEMATICS

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This mixed methods study explored elementary teachers' (n=73) experiences with and perspectives on the newly implemented CCSS-Mathematics at a high-needs, urban Professional Development School in a state where roll-out of the standards has been fraught with opposition. Analysis of the survey, questionnaire, and interview data reveals the findings cluster around: familiarity with and preparation to use the standards; implementation of the standards, including incorporation and teacher change; and tensions associated with enactment of the standards. Notably, the teachers believed in the merit of the standards but were constrained by their inadequate content knowledge, limited aligned curricular resources, lack of student readiness, and a perceived mismatch with ELLs. The results illuminate the professional needs of teachers during this critical time of transition and also add to the scant research in this area.

Keywords: Elementary School Education; Instructional Activities and Practices; Policy Matters; Teacher Knowledge

Purpose

As a means of improving the mathematics education of students in the U.S., teachers in 43 states are now expected to utilize the academic standards of the Common Core State Standards for Mathematics (CCSS-Mathematics) in their daily classroom instruction (CCSS, 2015). The CCSS-Mathematics is intended to provide more rigor and depth of the mathematics for students, while potentially requiring increased specialized content knowledge and fundamental changes in instructional practices of teachers (Schmidt, 2012). Ultimately, the difficulty of transitioning to the CCSS-Mathematics lies in putting the standards into classroom practice, with teachers having control over how this will play out (Dacey & Polly, 2012).

For the state in which this study was conducted, the roll out of the CCSS-Mathematics was and continues to be fraught with uncertainty and opposition, contributing to a general climate of anxiety and unrest for teachers. Given this challenging context, coupled with the widespread acceptance of the CCSS-Mathematics across the U.S., careful scrutiny of these new standards is warranted. As teachers are the ultimate force on how these standards are implemented in classrooms, a close study of their perspectives is needed, particularly in light of the scant research in this area. Accordingly, this mixed methods study explored teachers' experiences with and views on the newly implemented CCSS-Mathematics, including awareness of and preparation to use the standards, integration of the standards into classroom teaching practice, and tensions associated with the standards, including affordances and constraints linked with enactment. This study adds to the much needed research base and also provides insights into the professional needs of teachers during this critical time of transition to the CCSS-Mathematics.

Related Perspectives

The CCSS-Mathematics represents a major overhaul of the standards previously used in most states adopting these new standards. The standards include 11 critical areas of mathematics for grades K-8 in order to provide a coherent and focused curriculum built around big ideas (CCSS-Mathematics, 2010). The standards go beyond specifying mathematical content and also include eight Standards for Mathematical Practice, with an emphasis on applying mathematical concepts and skills in the context of authentic problems and understanding concepts rather than merely follow a sequence of procedures. The standards were created with strong consideration for the research base about the development of students' understandings of mathematics over time (Cobb & Jackson, 2011). As such, the topics at particular grade levels are different, with this re-sequencing reflecting current research on learning trajectories (Sztajn, Confrey, Wilson, & Edgington, 2012). In addition, summative assessments have been created that align with the more rigorous and in-depth expectations. Given these significant shifts proposed by the CCSS-Mathematics, its introduction will require a significant revamping of mathematics education in many schools (Schmidt & Houang, 2012).

Whether or not K-12 students learn the CCSS-Mathematics depends upon teachers' instructional expertise (Schmidt & Houang, 2012). The introduction of the CCSS-Mathematics requires many teachers to change what and how they teach and therefore calls into question their readiness for implementing these standards. Phillips and Wong (2012) suggest ". . . that moving from the standards on paper to the deep changes required in practice will be a significant challenge" (p. 31). For example, many standards designated for a particular grade may be reintroduced unnecessarily over the course of several years and spanning different grade levels (e.g., 4th grade standards may be taught in classrooms from 2nd through 6th grades) if teachers continue to rely on past standards' implementation schedules (Gewertz, 2012). In addition, Schmidt and Houang (2012) suggest that many teachers view the CCSS as predominantly the same content as their state's previous standards and this lack of awareness poses significant difficulties. Further, the CCSS-Mathematics proposes that teachers focus on fewer "big" mathematical ideas so students will: build conceptual understanding, achieve procedural skill and fluency, and learn how to transfer what they know to solve problems in and out of the mathematics classroom (Phillips & Wong, 2012). In order to develop these student understandings, Ewing (2010) contends, "Teachers must have deep and appropriate content knowledge to reach that understanding; they must be adaptable, with enough mastery to teach students with a range of abilities; and they must have the ability to inspire at least some of their students to the highest levels of mathematical achievement" (para. 6), highlighting some of the necessary teacher competencies for teaching the CCSS-Mathematics.

Method

This descriptive study used a mixed methods research design, with data collection occurring via a survey, open-ended questionnaire, and individual interviews. Specifically, a "concurrent triangulation" (Creswell, Clark, Gutmann, & Hanson, 2003, p. 223) approach to mixed methods was used, which implies: (a) concurrent collection of quantitative and qualitative data, (b) equal prioritization of quantitative and qualitative data, and (c) integration during the analysis and interpretation phases. Both types of data were collected in an attempt to cross-validate findings within a single study.

Participants and Context

This study involved 73 teachers at a large, urban elementary school in the southeastern U.S. The school had partnered as a Professional Development School (PDS) with the researchers' University since 2005. During data collection, teachers reported demographic information, which reveals years of teaching experience ranged from 40% with 5 years or less, 25% with 6 to 10 years, and 35% with more than 10 years. The educational background of the teachers includes 68% with at least a Master's degree. The PDS perpetually faces challenges associated with a high rate of teacher turnover and also student mobility. With regard to student demographic information, the PDS is a Title I school with 95% of the students eligible for free or reduced lunch at the time of the study. The student population was highly diverse, including 69% Hispanic, 22% African American, 5% Asian, and 4% Caucasian. Seventy-two percent of the students were non-native English speakers, and the English as a Second Language (ESL) program served 55% of the student population. In

2002, this school was removed from the state's failing schools list and has achieved adequate yearly progress (AYP) goals for the subsequent years.

The PDS relationship included a close partnership between the University and elementary school focused on: improving the preparation of new teachers, enhancing faculty development, inquiry directed at improved practice, and increasing student achievement; one of the researchers had done significant work at the PDS across the past 8 years supporting these goals. Given this sustained, collaborative partnership, there was ready entry for this research project, as well as established rapport and trust between the researchers and PDS teachers.

At the time of data collection, the teachers were mid-way into their second year of implementation of the CCSS-Mathematics. The department of education for the state in which the PDS is located adopted the standards in 2010, and the roll-out included communication and administrator training during 2010-2011 and teacher training during 2011-2012. Initial classroom implementation of the standards was expected in fall 2012, with full implementation in fall 2014 (doe.k12.ga.us, 2014).

Data Collection

Quantitative data were collected via a survey, and qualitative data were collected via an open-ended questionnaire and individual interviews. Participants completed the survey and open-ended questionnaire on the same day during their grade level Common Planning Time. All interviews were conducted within two weeks of this initial data collection at the convenience of the interviewees and at the PDS site.

All teachers completed a survey focused on their experiences with and perspectives on the CCSS-Mathematics, as well as an open-ended questionnaire designed to provide insights into the survey items. At the time of this study, there were no published surveys or questionnaires addressing teachers' perspectives on the CCSS that emphasized mathematics. The survey includes 22 items, some of which are multi-part. Some of the items were written by the researchers, and some were modified items from EPE Research Center's (2013) national survey of teacher perspectives on the Common Core. The domains of the survey items cluster around: teachers' reported experiences with the standards, including familiarity with, preparation for, and implementation of them. Additional items focus on teachers' perspectives on the standards, including the potential of the standards to influence or change their instructional practices and students' learning, as well as mathematics education in general.

After completing the survey, all participants completed the open-ended questionnaire intended to illuminate the survey items; the questionnaire contains eight multi-part questions. Six randomly selected teachers participated in individual, semi-structured interviews, and the interview protocol was identical to the open-ended questionnaire. The interviews ranged from 30 minutes to 1 hour in duration.

Data Analysis

Data from the surveys were analyzed using individual response analysis by examining the scores for each item; data from the interviews and open-ended questionnaires were analyzed using constant comparative methodology (Lincoln & Guba, 1985). Specifically, the researchers individually analyzed the qualitative data through line-by-line open coding which generated numerous categories. Categories represented observed phenomenon found in the data (Strauss & Corbin, 1998). Researchers met and discussed the categories to reach consensus on meanings related to the categories. This discussion and analysis of all interview and questionnaire data resulted in a coding manual representing the relationships within codes. The researchers then engaged in data reduction by recoding interviews and questionnaires using the coding manual for guidance in comparing and refining codes. Coded categories were collapsed and renamed until themes were identified. The analysis of the survey data was triangulated with the themes from the qualitative data sources, resulting in final themes on teachers' experiences with and perspectives on the CCSS-Mathematics.

Results

Familiarity with and Preparation to Use the Standards

Teachers reported relative familiarity with the CCSS-Mathematics. On the item, "Please rate your overall level of familiarity with the CCSS-Mathematics," only 22% reported they were very familiar, while 67% indicated they were somewhat familiar. And, though mid-way thru the second year of implementation of the CCSS-Mathematics, 25% reported no professional development on the standards. Of the 75% indicating professional development experiences, 56% reported three days or less, while 29% indicated over five days. When asked to describe the format, teachers indicated the most frequent as "collaborative planning time with colleagues", with the next two frequent as "structured, formal setting (seminars, workshops, conferences)" and "Professional Learning Communities". Given these accounts, on the item, "To what extent do you agree with the following statement? Overall, my professional development and training on the CCSS-Mathematics have prepared me to teach the CCSS-Mathematics," only 7% strongly agreed, with 65% agreeing and 20% disagreeing. The qualitative data illuminated these data, and notably there was an emphatic call for more professional development:

> It's particularly challenging because there's been no support or staff development. . . I know myself and many teachers who are doing everything we can to reach our students. But, we don't necessarily feel that we really know what we are preparing them for and how to be prepare them. . . The most important thing I would say that I need or would like to have is some preparation and some support, which I don't and am not getting from my school and school system. And, I don't know if that's because it's not available or they're in the same boat that we are because they don't know either. But, this is an area that's going to have to be addressed if the implementation of this program is going to be successful.

Teachers proffered professional development should involve: "modeling of lessons", "unpacking the standards", and understanding differences between "same lesson taught using CCSS-Mathematics and not CCSS-Mathematics".

In addition, when considering preparation, teachers had varied responses based on different groups of students. Mean scores reveal teachers felt least prepared to teach students with disabilities and most prepared to teach low-income students. The qualitative data show preparation for teaching varying groups of students as a concern: "I don't necessarily feel that I am prepared to successfully reach all of my students and prepare them for math understandings."

Implementation of the Standards: Incorporation and Teacher Change

When considering implementation of the standards, the data reveal two subthemes: incorporation into teaching practices and changes in teaching practices. On the item, "To what extent have you incorporated the CCSS-Mathematics into your classroom teaching practice," 39% indicated incorporation into some areas of teaching but not other areas, while 57% reported full incorporation into teaching. Teachers were also asked about their incorporation of the eight Standards for Mathematical Practice into their classroom instruction. The data show the two most included as: "Make sense of problems and persevere in solving them" and "Use appropriate tools strategically". The least incorporated were: "Construct viable arguments and critique the reasoning of others" and "Look for and make sure of structure".

In regard to implementation, teachers reported the new standards necessitate a change in their instruction. For example, 73% strongly agreed or agreed the standards require them "to do things differently as a teacher of mathematics". On a similar item, 68% strongly agreed or agreed the standards necessitate them "to change their classroom teaching practices". Threaded across statements about changing teaching practice is the placement of students at the center—valuing and emphasizing students' thinking, reasoning, representation, and explanation, with less teacher direction. Reported changes include: "moving away from teaching a standard algorithm to having the students explain their work, and they're working more with manipulatives and coming up with models"; "It's so much more in-depth. And, it's definitely trying to get them to do, not just like know, for instance, the formula for area, but all the different ways to get area". A teacher described her changes in instruction as: "We didn't have discussions. It was more so, it's wrong. . . Where now, I'm like, how they did get that, and they can explain it." Similarly, another teacher described her shift in teacher practice as:

> More student-centered than teacher-related. . . I've had to back up a little bit because the students are kind of exploring and finding different strategies to use. So, that's a different practice for me. I kinda want to give them something but have to back off and say well, if that strategy worked for them or they're finding strategies that maybe I didn't know, then I give them the freedom to explain or teach it to the class.

This shift away from teacher as central during instruction is also described as, "Now they're the owner of what is being said. . . I enjoy it more because I have given more responsibility to the child." Interestingly, teachers reported more inclusion of student explanation, but of the eight Standards for Mathematical Practice, the one focusing on students constructing arguments was reported on the survey as the least incorporated.

Tensions Associated with the New Standards

The teachers identified tensions with their perceptions of the new standards and enactment. The teachers overwhelmingly believed the standards would improve their instruction and benefit student learning. But, they identified several challenges for implementation, and these competing affordances and constraints generated tensions for teachers.

Tension #1: Affordances for Teachers and Constraints. The teachers believed the new standards would make them better teachers of mathematics. On the item, "To what extent do you agree or disagree with the following statement? The CCSS-Mathematics will help me improve my classroom teaching practice", 83% strongly agreed or agreed. A teacher stated, "I feels like to me that the new standards are just good teaching." However, the teachers identified several constraints with incorporating them into classroom teaching practices. These tensions can be linked to: lack of *mathematics knowledge for teaching* (MKT) and inadequate curriculum materials.

Mathematical knowledge for teaching (MKT) is multi-faceted and includes in part common content knowledge and SCK for teaching mathematics (Hill & Ball, 2009). The teachers identified a struggle with the mathematical content in the new standards. Sample interview comments include: "One area I struggled with was math. . . now with the CCSS-Mathematics standards I have to go deeper, and I do not feel comfortable;" and "I am having to relearn math to be able help my students." The teachers also reported a struggled with what has been defined as SCK for teaching mathematics, which includes in part, teachers' abilities to analyze and interpret students' mathematical thinking and ideas. Teachers' struggling to understand children's invented solutions strategies was commonplace, as a teacher stated, "And half the time, I've seen something, I'm like how did you do that? And, then I have to look at it, and I'm like, oh, okay," and with another reporting, "I had a hard time conceptualizing how different thinkers think different ways. In the hardest part of my lesson I was trying to connect all those different ways of learning and the way that different thinkers think. As I think about math in a very different way than Johnny or Billy Bob might."

Another constraint identified by the teachers was a lack of curricular resources aligned with the new standards. The teachers recognized the need for changes in their curriculum, as 88% strongly agreed or agreed that the CCSS-Mathematics requires "new or substantially revised curriculum materials and lesson plans". In addition, of the options that would help teachers to be better prepared to teach the new standards, one of the top choices (44%) was "access to curricular materials aligned to the standards". The qualitative data supported this: "Teachers have not been given any curriculum materials, anything that aligns with the standards, and there really isn't much out there that is aligned… So, the challenge here is that teachers like myself are doing the best we can to learn these new standards. Not only are we having to learn new standards, but we are having to create everything we are doing and hoping that we are understanding."

Tension #2: Affordances for Students and Constraints. The teachers largely held a positive view of the new standards, with notable beliefs about the benefits for students and their learning. They believed the CCSS-Mathematics provides a positive direction for mathematics education, as on the item "The CCSS-Mathematics will improve mathematics education in the U.S.", 73% strongly agreed or agreed, with 19% reporting "I Don't Know". On a similar item, "The CCSS-Mathematics is more of a positive step than a negative step in mathematics education in the U.S.", 80% strongly agreed or agreed, with 19% indicating "I Don't Know". Further, the teachers perceived the new standards to be of benefit to their own students, as on the item, "The CCSS-Mathematics will improve my students' learning", 34% strongly agreed and 44% of agreed, though 16% reported "I Don't Know".

The interview and survey data offer insights into benefits for students. The teachers appreciated the emphasis in the new standards on mathematics as a sense-making activity, including a focus on conceptual understanding, explanation and justification, and connections. For example, a teacher asserted, "I think that it's preparing them to be better thinkers when it comes to math," with another stating, "I call them [students] microwaves because they want the answer now, but Common Core forces them to work it out and really just dig into it. . . It will have a great impact on deepening their knowledge and really getting them to understand why math is math." Another teacher stated:

> It will help students' learning because instead of just telling them to do it, they know why they're doing it. Why it's important. When things become more meaningful, it seems more real to them and their brains can connect the concepts better than when they are just memorizing. . . This is why the area formula is what it is. And, wondering things like perimeter or area and how they can connect and see how it all works together. It's not all isolated incidents that have no meaning in relation to each other. . . Students have to be able to explain why math is the way that it is. Students have to explain why formulas are the way they are. Students can explain why we do math the way that we do and not just use rote memorization to solve problems. I really like how it's supposed to make students think more critically.

Though teachers believed in the value and emphasis of the new standards for students, there were associated constraints related to enactment that generated tensions for teachers, including lack of student readiness and a perceived mismatch of the standards with ELLs. The teachers believed students were not ready for the new standards, with gaps in content and skills linked with past ways of learning mathematics. Interestingly, when considering student preparation to learn the standards, on a scale of 5 as "very prepared" and 1 as "not prepared", the findings reveal a mean score of 3.2, thus the perception that students had mediocre preparation. One teacher aptly stated, "It's almost like we're going back, undoing and unteaching what they have been taught." Another declared, "A lot of my students have been working with algorithms for the most part since they have been in school. It's difficult to try to go back and teach them really why the algorithm works, just understanding why they're doing it." Another teacher stated, "Students have a hard time explaining how they got the answer. They just say things like I know that 3+3 =6. . . It's hard for them to grasp the words to communicate what they understand. They have been so used to just memorizing facts it's very confusing for them. . . Students have to able to do it and explain their thinking as opposed to just answering or recalling facts." One teacher identified particular challenges with students in the upper grades, as she noted that with younger children, ". . . it's easier because you're teaching them what we would consider the proper way and they can go from there. . . The higher the grade level the harder it will be because they've learned a certain way and now they have to learn a new way. . . Because they have to go backwards, it tends to frustrate them." This teacher went on to propose a phasing in of the CCSS-Mathematics, starting with the primary grades: "A line should have been drawn to like, okay, phase one, implement K-2 and then phase two, 3-5, instead of everybody getting it at the same time. Instead of boom, you're in 5th grade but you need to learn how to do this."

In addition, the teachers had salient concerns about the misfit of the CCSS-Mathematics with ELLs. When asked about their preparation to teach the new standards to different groups of students, ELLs had the second to lowest mean score. In particular, this school has a large ELL population, and teachers voiced concerns with the emphasis on communication and explanation as posing difficulties: "Out of my 29 students, I have 26 ELLs and it's very challenging for them. . . Definitely with the Common Core, across the board it is always explaining why." Teachers also lamented multi-step tasks or word problems involving several parts that require higher levels of reading comprehension from ELLs, with one teacher asserting she has "learned to break the tasks or the activities down for them, and I find that works."

Conclusion and Discussion

Though the state adoption and implementation of the CCSS-Mathematics has been highly contentious, becoming a political issue and encountering parental opposition, notably, the teachers held decidedly positive views on the standards. They believed the standards improve their teaching of mathematics and benefit their students' learning, with this perspective linked in part with the emphasis on mathematics as a sense-making activity. This optimism is remarkable, considering this is the third set of academic standards for K-12 education in this state across the past 10 years, as one teacher lamented, "We have had three or four sets of standards and each time we are told these will be around for a long time only to see them changed every few years. This can be very frustrating for teachers… teachers are just so tired of change". Despite this revolving door of standards, such a hopeful view can go a long way with adequate teacher preparation and aligned curricular resources—of which, in general, teachers in this study seemed to need more. Allowing time for teacher change is crucial, particularly with the uncertainty about forthcoming assessments, and the rapidly approaching third year for when assessment scores are consequential seems premature.

Several constraints were identified. As indicated by others (Schmidt, 2012), content knowledge, particularly SCK, was a barrier for enactment. Teachers struggled to understand, interpret, and respond to children's thinking and invented solution strategies. The importance of well-developed mathematical knowledge for teaching is undisputable. Professional development for teachers should provide ways of concurrently building SCK while studying children's thinking, with one such option being the professional development materials from the Cognitively Guided Instruction (CGI) Project (Carpenter, Fennema, Franke, Levi, & Empson, 1999; 2014). CGI is an approach to teaching and learning mathematics that focuses on teachers using knowledge of children's mathematical thinking to make instructional decisions, which can simultaneously develop SCK. In addition, concerns about using the new standards with specific groups of students should be noted and in this particular study, teachers did not feel prepared to do so with students with disabilities and also ELLs, which are prevalent at this PDS site. The implications for the new standards differentiated for the learning needs of different groups of students must be considered and addressed. In addition, lack of student readiness was of concern, linked with their past experiences as learners of mathematics. Student explanation was one such challenge, with teachers needing ways to develop classroom norms for engaging in discourse and social mathematical norms that help students understand what constitutes a good mathematical solution and explanation. Guidelines such as these could help students to persevere when solving problems and forming mathematical arguments. In sum, the findings of this study illuminate both the tremendous potential for positive change provided by the CCSS-Mathematics and accompanying barriers. As teachers are ultimately the deciding factors on how the standards play-out in classrooms, this close study of their perspectives can hopefully provide insights into ways of better quipping them for teaching the standards and in turn benefiting students' learning.

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ATTRIBUTES OF STUDENT MATHEMATICAL THINKING THAT IS WORTH BUILDING ON IN WHOLE-CLASS DISCUSSION

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This study investigated the attributes of 297 instances of student mathematical thinking during whole-class interactions that were identified as having the potential to foster learners' understanding of important mathematical ideas (MOSTs). Attributes included the form of the thinking (e.g., question vs. declarative statement), whether the thinking was based on earlier work or generated in-the-moment, the accuracy of the thinking, and the type of the thinking (e.g., sense making). Findings both illuminate the complexity of identifying student thinking worth building on during whole-class discussion and provide insight into important attributes of MOSTs that teachers can use to better recognize them. For example, 96% of MOSTs were of three types, making these three particularly salient types of student mathematical thinking for teachers to develop skills in recognizing.

Keywords: Classroom Discourse; Cognition; Instructional Activities and Practices

An enduring challenge in mathematics education is figuring out how to best support teachers' effective use of student mathematical thinking in their classrooms. For several decades reform documents (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000, 2014) have consistently called for teaching that focuses on developing students' abilities to reason mathematically. For mathematical reasoning to happen, the NCTM recommends that students engage in exploration of complex tasks, state and test conjectures, and build arguments to justify their conjectures. In response to this recommendation, many researchers have investigated issues around student thinking, such as students' abilities to think mathematically using tasks with high cognitive demand (Stein, Grover, & Henningsen, 1996), obstacles to students' learning (Bishop, Lamb, Phillip, Whitacre, Schappelle, & Lewis, 2014), challenges beginning teachers face when trying to use student thinking (Peterson & Leatham, 2009), important teachable moments created by student thinking made public during classroom instruction (Stockero & Van Zoest, 2013), and classroom instances that have potential for building students' mathematical understanding (Leatham, Peterson, Stockero, & Van Zoest, 2015). However, little is known about the nature of student thinking that becomes publicly available for teachers to use during instruction.

Our ongoing work investigates student mathematical thinking made public during whole-class interactions that, if made the object of discussion, has the potential to foster learners' understanding of important mathematical ideas—instances of student thinking that we call **M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking [MOSTs] (Leatham et al., 2015). The work reported here analyzes instances of student thinking that have been identified as MOSTs in order to investigate attributes of this high-leverage subset of student thinking. A better understanding of the attributes of MOSTs has the potential to support research on mathematics teaching in at least four ways.

First, using student mathematical thinking productively requires that the thinking be noticed (van Es & Sherin, 2002). Stein, Engle, Smith, and Hughes (2008) suggested that teachers might be able to orchestrate classroom discussion effectively when student work with potential to enhance learning is identified, attended to, and sequenced in a developmentally appropriate way. Understanding attributes of MOSTs may help teachers develop their skills for noticing student thinking worth building on and thus improve their ability to orchestrate classroom discussion that fosters student learning. Second, Ball, Lewis, and Thames (2008) described students' mathematical thinking as "both underdeveloped and under development" (p. 15) and identified students' mathematical thinking as "the raw materials for building justified mathematical knowledge" (p. 25), but did not characterize the nature of the raw materials in student responses. Using students' mathematical thinking as a cornerstone for subsequent construction of student mathematical understanding requires an understanding of the nature of that thinking. Understanding attributes of MOSTs, a critical subset of student thinking, has the potential to provide insight into Ball et al.'s (2008) "raw materials" and contribute to the development of "justified mathematical knowledge" (p. 25). Third, Carpenter, Fennema, Peterson, Chiang, and Loef (1989) found that giving teachers access to different strategies students employ to solve problems positively affected teachers' beliefs about learning and instruction, their practices, their knowledge about students, and students' achievement. Giving teachers access to attributes of MOSTs may have similar positive effects on teachers because it would give them information about the nature of student mathematical thinking available to them in their classrooms and better equip them to use that thinking productively. Finally, Hiebert, Morris, Berk and Jansen (2007) argued that teaching should be assessed based on how teachers make use of student responses in classrooms to foster understanding of mathematical ideas rather than on the presence of recommended instructional features. Identifying attributes of student responses that are MOSTs might enhance the development of ways to assess teaching in this manner.

Theoretical Framework

Leatham et al. (2015) defined MOSTs as occurring in the intersection of three critical characteristics of classroom instances: student mathematical thinking, significant mathematics, and pedagogical opportunities. For each characteristic, these authors provided two criteria that can be used to determine whether an instance of student thinking embodies that characteristic. For student mathematical thinking the criteria are: "(a) one can observe student action that provides sufficient evidence to make reasonable inferences about student mathematics and (b) one can articulate a mathematical idea that is closely related to the student mathematics of the instance—what we call a mathematical point" (pp. 93-94). The criteria for significant mathematics are: "(a) the mathematical point is appropriate for the mathematical development level of the students and (b) the mathematical point is central to mathematical goals for their learning" (p. 97). Finally, "an instance embodies the pedagogical opportunity characteristic when (a) the expression of a students' mathematics creates an opening to build on student thinking to help develop an understanding of the mathematically significant point of the instance and (b) the timing is right to take advantage of the opening" (p. 103). When an instance satisfies all six criteria, it embodies the three requisite characteristics and is a MOST. We see analysis of MOSTs as a means toward identifying important attributes of high leverage student mathematical thinking that might be used to help support teachers in developing their skill at productively using such thinking.

Stockero and Van Zoest (2013) investigated and categorized "instances in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 127)—what they called pivotal teaching moments (PTMs). We see PTMs as a subset of MOSTs, and thus we used the PTM categories to inform our thinking about attributes of MOSTs. In particular, these researchers identified five categories of PTMs: (1) *extending*—students make connections to create a much deeper lesson from what was planned; (2) *incorrect mathematics*—student incorrect mathematical thinking becomes public; (3) *sense making*—students are trying to make sense of the mathematics under consideration; (4) *contradiction*—student responses have competing interpretations; and (5) *mathematical confusion*—students clearly state mathematically what they are confused about. These categories and the work related to the development of them provided a starting point for our exploration into attributes of MOSTs.

Methodology

This study is part of a larger project focused on understanding what it means for teachers to build on students' mathematical thinking (see LeveragingMOSTs.org). We selected 11 videotaped mathematics lessons from the MOST project that reflected teacher diversity (race/ethnicity, gender, experience, teaching style), mathematics diversity (6-12th grade, topic, textbook), and classroom diversity (region of the US, community type, race/ethnicity). The unit of analysis for identifying MOSTs was an instance of student thinking—an "observable student action or small collection of connected actions" (Leatham et al., 2015, p. 92) that had the potential to be mathematical. StudioCode (Sportstec, 1997-2015) was used for three passes of coding. In the first pass, classroom context and other relevant information were noted on the Studiocode timeline and instances of student thinking were identified and transcribed. During the second pass, the MOST Analytic Framework (see Leatham et al., 2015) was used to determine which instances were MOSTs. We identified 297 MOSTs in the 11 lessons; these MOSTs served as the data for the current study. The third pass of coding, completed for the current study, focused on identifying attributes of these MOSTs.

We coded the 297 MOSTs for seven attributes that fall into two groups: Locus and Cognition (Figure 1). The Locus group encompasses attributes that locate a MOST within the mathematical and lesson terrain and includes what immediately preceded the MOST (Prompt), whether the MOST was based on earlier work (Basis), and the distance of the mathematical idea of the MOST from the day's lesson (Math Goal). The Cognition group focuses on the expression of the student's thinking and includes whether the MOST was a question or statement (Form), whether the student thinking was correct (Accuracy), the extent to which the intellectual need is obvious (Intellectual Need), and the nature of the MOST (Type). To illustrate our coding and the nature of our results, we discuss four of the attributes (Basis, Form, Accuracy, and Type—bolded in Figure 1), as well as interactions between them.

Basis refers to whether the student mathematics (SM) in the MOST is based on earlier work (*Pre-thought*) or on in-the-moment thinking (*In-the-moment*). A MOST is coded *Pre-thought* when the student appears to be sharing thinking from previous work. Although this previous work could be from homework or another class, typically it is from small group or individual work completed during the lesson. A MOST is coded *In-the-moment* when the SM stems from students' in-the-moment thinking. This thinking might be in response to a follow-up request from the teacher or to another students' thinking or question, or it might be seemingly spontaneous.

Locus	Cognition
Prompt (Spontaneous, Open Invitation Spontaneous, Open Invitation Selected, Targeted Invitation) **Basis (In-the-Moment, Pre-Thought)** Math Goal (Lesson, Unit, Course, Math)	**Form (Question, Declarative, Tentative)** **Accuracy (Correct, Incorrect, Incomplete, Combination, N/A)** Intellectual Need (Obvious, Translucent, Hidden) **Type (Incorrect or Incomplete, Sense Making, Multiple Ideas or Solutions, Other)**

Figure 1: MOST Attribute Codes and their Categories by Groups

Form refers to the way in which the student thinking is expressed (*Question, Tentative Statement* or *Declarative Statement)*, regardless of its correctness or completeness. A MOST is coded *Question* if the student thinking is shared as a question or with the intent to question. *Declarative Statement* is used when students appear to be confident in what they are saying and *Tentative Statement* is used when the student appears to be making a conjecture or is wondering about something. Tentativeness is typically indicated by the student's voice inflection when making the statement, but it can also be indicated by their use of hedge words such as "maybe" or "I'm not sure."

Accuracy is used to categorize a MOST based on the validity of its SM. A MOST is *Correct* if its SM is a correct mathematical statement; *Incorrect* if its SM is an inaccurate statement; *Incomplete* if the SM is not incorrect, but it has gaps or ambiguities that keep it from being completely correct; *Combination* if it involves a complete statement(s) that falls in multiple *Accuracy* categories; or *N/A* if it is not possible to determine its correctness (e.g., if it is a question).

Type is used to categorize what about the SM made the instance a MOST. There are four *Type* categories: *Incorrect or Incomplete*, *Sense Making*, *Multiple Ideas or Solutions*, and *Other*. A MOST is coded *Incorrect or Incomplete* if it was compelling because its SM is inaccurate or missing critical components of the mathematical idea being expressed. A MOST is coded *Sense Making* if it was compelling because the SM implies that the student was trying to make sense of the mathematics, or they had comprehended an idea with which the class had been struggling. A MOST is coded *Multiple Ideas or Solutions* if it was compelling because the SM created an opportunity for comparison of multiple ideas or solutions.

To illustrate the attribute codes, consider the SMs from four MOSTs (see Figure 2). All four MOSTs came from class discussions based on tasks students had solved beforehand in small groups. SM1, SM2, and SM4 were in response to what was currently being shared rather than something they had done earlier, thus were coded *In-the-Moment (Basis)*. In contrast, SM3 was a reporting out of a student's earlier work, thus received the code *Pre-Thought (Basis)*. The first two MOSTs involved statements, rather than questions, thus received the code *Declarative Statement* (*Form)*. The third was also a statement, but the student preceded it by expressing a lack of confidence in her answer, thus it was coded *Tentative Statement.* The fourth was a question and was coded *Question*. In the *Accuracy* category, the first MOST received the code *Combination* because although it includes the correct idea that the rate of change is a constant $2.50, the language suggests that the slope is increasing at that rate rather than that the slope is that constant rate. The second MOST was coded *Incorrect* because it is possible to divide by a fraction. In SM3, also coded as *Incorrect*, the class had already agreed to the convention of putting the independent variable on the x-axis and the dependent variable on the y-axis and the problem the students were exploring implied that "money" was the dependent variable and "weeks" the independent variable. The fourth MOST was coded *N/A* for *Accuracy* because questions, by their very nature, do not have a truth-value. The compelling aspect of the first and fourth MOSTs was a student grappling with a mathematical idea—the difference

Student Mathematics of four MOSTs	**Coding (*Basis*, *Form*, *Accuracy*, *Type*)**
SM1: The slope is increasing at a constant rate. The slope is not going any faster. The slope is always going up $2.50.	*In-the-Moment, Declarative Statement, Combination, Sense Making*
SM2: You can't divide by a fraction.	*In-the-Moment, Declarative Statement, Incorrect, Incorrect or Incomplete*
SM3: I put the money on the x-axis and weeks on the y-axis.	*Pre-Thought, Tentative Statement, Incorrect, Incorrect or Incomplete*
SM4: Doesn't solving sometimes include simplifying?	*In-the-Moment, Question, N/A, Sense Making*

Figure 2: Coding Examples

between an increasing graph and a graph with an increasing slope in SM1 and the difference between solving and simplifying in SM4—thus they were both coded *Sense Making (Type).* The compelling aspect of the second and third MOSTs was that the students had introduced incorrect ideas, thus these MOSTs were coded *Incorrect or Incomplete.*

Three research assistants individually coded the 297 MOSTs and then reconciled them as a group. If they were not able to reach agreement, the issue was brought to the attention of the principal investigators and either the codes or the code definitions were modified to resolve the issue. We then determined the frequencies of the codes and interactions between them for each of the 11 lessons and compiled all the results into a spreadsheet that allowed for within and across lesson comparisons. We used that information to search for patterns among the results that would lead to a better understanding of the attributes of MOSTs.

Results and Discussion

Figure 3 provides the percentages of MOSTs in each category of the four attributes *Basis*, *Form*, *Accuracy,* and *Type*. Roughly 20% of the 297 MOSTs in our data were based on work that students had completed earlier in the class, thus were available for the teachers to identify by monitoring students as they worked. This percentage speaks to the benefit of teachers developing skills such as the five practices for orchestrating classroom discussion identified by Smith and colleagues (e.g., Smith & Stein, 2011). The finding that 80% of the MOSTs were based on student thinking that occurred during whole-class interaction speaks to the importance of teachers also developing skills for carefully listening and responding to evolving thinking.

BASIS		**FORM**		**ACCURACY**		**TYPE**	
In-the-Moment	80	Question	16	Correct	40	Incorrect or Incomplete	31
Pre-Thought	20	Tentative	7	Incomplete	8	Sense making	50
		Declarative	77	Incorrect	24	Multiple Ideas or Solutions	15
				Combination	9	Others	4
				Not Applicable	19		

Figure 3: Percentages of MOSTs in Attribute Categories

The vast majority of the MOSTs were declarative statements (77%) as opposed to questions (16%) or tentative statements (7%). This means that it is insufficient to focus only on expressions of student mathematical thinking that are intuitively suggestive of thinking worth building on (such as questioning or wondering).Instead, teachers must develop more sophisticated ways of recognizing which student thinking has this potential.

Knowing the accuracy of an instance of student mathematical thinking is also insufficient to determine whether it is a MOST, as there was no predominate *Accuracy* category. There were, however, more correct (40%) than incorrect (24%)MOSTs. This is particularly interesting given that the project team had initially hypothesized that it would be difficult for correct student mathematical thinking to meet the MOST criteria.

Half of the MOSTs (50%) occurred when students were grappling to make sense of a mathematical idea. The next highest *Type* category involved instances of student thinking that were incorrect or incomplete (31%). Student thinking that led to multiple ideas or solutions being available for students to consider occurred in 15% of the MOSTs and 4% did not fit in the three main categories of *Type*. Although *Incorrect or Incomplete* and *Multiple Ideas or Solutions* had lower frequencies than *Sense Making*, they may be easier for teachers to recognize. Thus, it seems important for teachers to attend to all three main categories of *Type*. The fact that 96% of the MOSTs were captured by these three categories is encouraging as it suggests some parameters for developing teachers' abilities to recognize MOSTs.

Figure 4 considers interactions between the attributes. All but one of the MOSTs in the form of questions (see the Q column in Figure 4) were compelling because the student was grappling to make sense of a mathematical idea (see, for example, SM4 in Figure2). The fact that all questions in this data that qualified as MOSTs involved sense making suggests the need for teachers to consider the potential of a question to determine the nature of their response. If the question is a MOST, the most effective teacher response might be to provide an opportunity for the class to join the student who asked the question in making sense of the idea.

	FORM			BASIS		ACCURACY*				
	Q	T	D	ITM	PT	CR	ICR	INC	COM	N/A
TYPE										
Incorrect or Incomplete	1	12	87	80	20	0	66	22	11	1
Sense making	32	5	63	83	17	58	1	3	3	35
Multiple Ideas or Solutions	0	2	98	71	29	58	16	0	24	2
Others	0	18	82	91	9	55	9	0	9	27
FORM										
Question (Q)				98	2	0	0	0	0	100
Tentative (T)				91	9	27	45	5	23	0
Declarative (D)				76	24	49	27	10	10	4
BASIS										
In-the-Moment (ITM)	20	8	72			36	25	9	7	23
Pre-Thought (PRE)	2	4	94			55	19	5	17	4

*Accuracy Abbreviations: Correct (CR); Incorrect (ICR); Incomplete (INC); Combination (COM); Not Applicable (N/A)

Figure 4: Percentages of MOSTs in Interactions between Attribute Categories

The MOSTs that were compelling because they provided an opportunity for students to consider multiple ideas or solutions were predominantly declarative statements (98%). As might be expected, MOSTs in which students shared thinking from previous work were also typically declarative statements (94%). Still, 76% of the declarative statements resulted from in-the-moment thinking, as did 98% of the questions and 91% of the tentative statements. There were also no extreme differences among which *Types* result from in-the-moment thinking, with *Sense Making* having the highest frequency (83%) of in-the-moment thinking of the three main *Type* categories and *Multiple Ideas or Solutions* having the lowest (71%). Thus the earlier stated need for teachers to skillfully respond to evolving thinking stands, regardless of the form in which that thinking is expressed or what made it compelling.

Figure 4 also shows a much higher percentage of the tentatively stated MOSTs were incorrect as were correct (45% vs. 27%); nearly the reverse was true of the MOSTs that were declarative statements (27% vs. 49%). This finding suggests that although there was some correlation between students' confidence in their thinking and the accuracy of it, the relationship was not strong enough to be counted on. That is, in the context of MOSTs, relying on tentative thinking to be incorrect and confident thinking to be correct would cause one to be wrong much of the time. Likewise, although pre-thought SM was more likely to be correct than SM that was generated in the moment (55% to 36%), both types of SM were also often incorrect (19% and 25%, respectively).

Finally, with the exception of MOSTs that were compelling because they involved incorrect or incomplete thinking (e.g., SM2 and SM3 of Figure 2), there did not seem to be a strong relationship between *Accuracy* and *Type*. For example, although 58% of MOSTs that provided the opportunity for students to consider multiple ideas or solutions were based on correct SM, 24% were based on SM that had both correct and incorrect elements, and 16% were based on SM that was incorrect. Again, it seems that accuracy of student mathematical thinking is not a useful predictor of MOSTs.

Conclusion

This study set out to contribute to our developing understanding of how to best support teachers' effective use of student mathematical thinking in their classrooms by investigating attributes of MOSTs—a high leverage subset of student thinking. The results provide insight into claims about the complexity of responding to students' mathematical thinking on the spot (e.g., Choppin, 2007; Jacobs, Lamb, & Philipp, 2010). We now know that surface features of thinking, such as how it occurs, the form in which it is expressed, and how accurate it is, are not sufficient to determine whether the thinking should be pursued. Rather, responding effectively to student mathematical thinking requires careful attention to the content of the thinking to discern the underlying mathematical idea and what it might offer as the object of a class discussion. For example, some student questions may be best answered directly, but those that reflect a student's grappling with important mathematical ideas provide rich opportunities to engage the class in the type of mathematical activity advocated by current reforms (e.g. NCTM, 2014). Calculation and other surface mistakes may be dispensed with quickly, but errors in students' thinking are often worth building on. Similarly, correct answers may be an indication to continue, or they may provide an opportunity to stop and engage the class in consolidating important mathematical understandings.

Despite the lack of easy answers about which thinking is worth building on in whole-class discussion, this work does provide some parameters that may make the process more manageable. For example, correct student thinking that does not involve sense making or multiple ideas or solutions is not likely to be worth pursing in a whole-class discussion. Being aware of this pattern can help teachers avoid initiating unproductive discussions.

In general, this work supports the need for teachers to have criteria they can use for evaluating which student thinking is worth building on. The MOST Analytical Framework (Leatham, et al., 2015) is one such set of criteria. Such criteria, in conjunction with the parameters contributed by this study, provide a starting place for designing teacher education and professional development to support teachers in developing the teaching practice of productive use of student thinking.

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DO YOU NOTICE WHAT I NOTICE? PRODUCTIVE MEDIUMS FOR TEACHER NOTICING

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This article is a report on the findings of three case studies that focused on elementary teachers' in-the-moment noticing across a month of instruction. Extending Jacobs and her colleagues' framework (Jacobs, Lamb, and Philipp, 2010), this articles categorizes the mediums (i.e., written accounts, verbal interactions, physical strategies) by which the teachers attended to their students' mathematical reasoning. Even though the teachers taught similar lessons, they attended to students' actions through different mediums. These mediums reflected and often aligned with individual teaching practices and routines. More interestingly, the teachers' preferred medium was also the most productive for them in terms of discussing students' understanding and informing instructional decisions.

Keywords: Teacher Knowledge; Teacher Beliefs; Mathematical Knowledge for Teaching

Background

In our everyday lives, we each attend to and notice different things. When walking down a city street, some of us might be drawn to the items displayed in a storefront window, others to the aroma of a local restaurant, and some may be drawn to something that they have seen before or perhaps something that has changed. We each make sense of our world in different ways and may not notice the same details. As Schoenfeld (2011) states, "noticing is consequential-what you see and don't see shapes what you do and don't do" (p.228). What we notice and make sense of shapes our daily decision-making processes.

In a similar sense, what teachers notice and make sense of in the classroom is often different depending on the teacher. Erickson (2011) found that teacher noticing is selective (attending to some events, but not to others), multi-dimensional, and influenced by prior experience. He also noted that teachers most often noticed events in the classroom that required immediate attention or action on the part of the teacher. Even though teachers notice many things during instruction, how to respond to what they notice is sometimes difficult. Multiple researchers (Berliner, 2001; Jacobs, Lamb, & Philipp, 2010; Mason, 1998) have documented that teachers often have difficulty devising instructional responses based on what they notice.

Even though noticing and responding to students' thinking is a difficult task, researchers have shown, with professional development, it is a learnable skill (Jacobs et al., 2011; Santagata, Zannoni, & Stigler, 2007). Using mediums such as classroom artifacts (Goldsmith and Seago, 2011), student video cases (Jacobs et al., 2010;), and videos of teachers' in-the-moment noticing, researchers noted growth in teachers' abilities to attend to students' mathematical thinking and form instructional responses.

With regard to in-the-moment noticing, researchers have devised ways for teachers to capture moments as they occur using a Déjà Vu camera and then later reflect and respond to these moments. This camera attaches to the teacher's body and she can press a button to capture 30 seconds of video at any time. Sherin, Russ, and Colestock (2011) discussed that teachers often use these cameras in multiple and unexpected ways (i.e., using it as a still life camera). Star, Lynch, and Perova (2011) took this idea a step further and categorized what pre-service teachers noticed during practice (i.e., classroom environment, classroom management, tasks, mathematical content, and communication).

When using the Déjà Vu camera and analyzing their own teaching, teachers have the freedom to choose what is significant to them. By using this camera, researchers can gain insight into how teachers attend to and make sense of their students' mathematical reasoning. There is little to no research examining the types of events teachers choose in their daily practices and why these events are significant. It important to investigate the ways in which teachers naturally attend to students' actions and reasoning.

Within this paper, I categorize the mediums used by three teachers to attend to their students' mathematical thinking across a month-long unit of instruction. In art, a medium is the substance the artist uses to create his or her artwork. Within this paper, medium refers to the way in which what was noticed was conveyed to the teacher (i.e., drawing a picture, counting using fingers). The paper describes a framework for categorizing mediums as well as the significance of specific mediums for individual teachers. Within the paper, I address the following research questions:

- By what mediums do teachers attend to students' mathematical thinking in the classroom?
- In what ways, if any, do these mediums aide in teachers' instructional decision-making?

Theoretical Framework

This study is grounded in the theory of teacher research. Cochran-Smith and Lytle (1993) defined teacher research as "systematic, intentional inquiry by teachers about their school and classroom work" (p.24). Teacher research usually stems from issues and questions that arise within the practice of teaching. It is grounded in the epistemological belief that teachers should have a voice and a presence in the research community and acknowledges that the teachers' perspectives are critical when implementing and evaluating research.

This paper embraces this theory by examining teacher noticing from the teachers' perspectives. When we, as researchers, investigate teacher noticing, we often have an end goal in mind or something we hope teachers notice and respond to. By allowing teachers to choose important events, we gain insight into their natural practices and how they use these practices to guide decision-making.

Methods

Participants

Three teachers participated in the study: Mrs. Grey, Mrs. Purl, and Mrs. Brownstein. Mrs. Grey and Mrs. Purl were both fourth-grade teachers while Mrs. Brownstein taught fifth grade. The teachers had 10-18 years of teaching experience and were second-year participants in a two-year professional development (PD) focused on measurement topics. Throughout the PD, each of the teachers had experiences watching and writing about videos of their teaching in terms of students' thinking and instructional implications. These teachers were chosen as case studies from the PD because they taught similar grade levels at the same school, Terrace Elementary.

Terrace Elementary

The study took place at Terrace Elementary, located in an urban city in the Midwestern United States. At the time of the study, 554 students attended Terrace and of these students 54% were boys, 46% were girls. They identified according to the following ethnicities: 54.7% Black, 23.3% White, 10.3% Multiracial, 9.6% Hispanic, 1.6% Asian, 0.4% Native American, and 0.2% Native Hawaiian. The average class size of the school was 18 students and the student to teacher ratio was 12 to 1. At the time of this study, 80% of the students came from low-income households and were eligible for free or reduced-price school meals.

Data Collection

During a month-long unit on measurement topics, each teacher was asked to wear a Déjà Vu camera to record events that she noted as important. This camera was a small device pinned to the teacher's shirt that connected to a collection box clipped to the teacher's pants. When the teacher pressed a button on the collection box, the camera collected and saved a 30-second video clip of the events prior to her pressing the button. Each teacher was asked to press the button on the camera when she felt that she noticed something important about students' thinking or when she made an important decision based on student thinking. All of the lessons were videotaped with a secondary camera, as well.

At the end of each day, I watched the videos and marked time stamps on the main video for each of the events that the teachers found important. These clips were then used as the main focus of a one-on-one interview. On a daily basis, each teacher and I would sit down for an interview in which we discussed the clips that she noted. For each clip, the teacher was asked to describe what the student(s) was doing, why she chose the clip, what she thought the student was thinking or understood, and in what ways this clip might inform her instruction going forward.

Following each teacher-researcher interview, I transcribed each conversation verbatim. This resulted in over 300 pages of written transcripts (100 pages per teacher).

Data Analysis

To examine what and how the teachers noticed, I began with the teacher noticing framework (Jacobs, Lamb, & Philipp, 2010) as my primarily analytical tool. However, the construct of teacher noticing has been primarily used in clinical interview settings and I experienced limitations in analyzing the data I had collected for this study. Using qualitative methods (Miles, Huberman, & Saldana, 2014), I instead envisioned the three tenets of teacher noticing as broader analytical categories to be explored. I incorporated provisional coding (Miles, Huberman, & Saldana, 2014) to expand and elaborate codes to better fit the aims of this study.

I decided to call each episode that the teacher noticed an event. The event was composed of four components: what the teacher noticed, interpreted understanding, implications for teaching, and significance to the teacher. It is important to note that an event could be composed of multiple student actions because a child could be doing two things simultaneously (e.g., drawing and counting area units).

I began the analysis by first reading the transcripts and taking descriptive, qualitative notes about each of the events. I recorded where an event began and ended and notes related to each of the constructs. I repeated this process for each of the teachers. Following the note-taking phase, I used an event-listing matrix (Miles, Huberman, & Saldana, 2014) to record the events the teacher noted in a chronological manner. I devised a matrix for each teacher in which the columns were the analytical categories of the event (what the teacher noticed, interpreted understanding, implications for teaching, and significance to the teacher) and the rows represented a description of each event in terms of those categories.

Because of the brevity of this piece and the focus on mediums of teacher noticing, I will only describe how what the teacher noticed was coded. For each event, I began by coding what the teacher noticed within the event and what task the action centered around. After each event was coded, I scanned the events again to see if codes were similar in nature or could be collapsed. I repeated this process for each lesson and each teacher. For this piece of analysis, the codes usually centered on specific tasks or routines within the classroom. As new codes emerged in the analysis, I reanalyzed each event for each teacher looking to see if the code had been missed.

For each teacher, I created codes that described what the event pertained to. In the results section, each of these codes are described in detail in the mediums framework (See Fig 1).

Results

What the Teacher Noticed		Description
Written Accounts	**Drawing Strategy**	The event involved a student's drawing a picture to represent or help find their answer.
	Written Answer	A solution or student's work recorded during instruction.
	Journal Account	Written reflections, definitions, and homework recorded by students in their journal.
Verbal Interactions	**Statement**	The student makes a claim or states a response in class.
	Question	The student poses a question to the teacher.
	Student to Student Interactions	Interactions, discussions, conversations made by students to each other within groups or pairs.
	Language Comprehension/Usage	The way in which students use and interpret language. Example: The teacher notices the student is confusing the words area and perimeter.
Physical Strategies	**Length/Area/Volume Counting Strategy**	The way in which the child counts length, area, or volume. Example: For area, students may count individual units, skip count by rows, or use the algorithm.
	Length/Area/Volume Measurement Strategy	The student uses tools to find length, area, or volume. Example: laying tiles down and counting them to find the area of a rectangle.
	Building Strategy	Students using tools to construct something. Example: Building a rectangle with a certain area or a prism with a certain volume.
Visualization Strategies	**Visualization Strategy**	A statement, the student makes, involving how they see volume or area. Example: Students states that the volume of a prism is like an elevator visiting multiple floors.
Non-Mathematical Behaviors	**Behavior**	Mainly non-mathematical events such as students rushing to complete a problem or not attempting a task.

Figure 1: Mediums by which the teachers attended to students' actions

When using the Déjà vu camera to indicate important events, the teachers noticed different types of student actions related to length, area, and volume measurement. These actions centered around five themes: written accounts, verbal interactions, physical strategies, visualization strategies, and non-mathematical behaviors (as shown in Fig 1). The types of actions the teachers attended to varied by teacher (as shown in Figure 2). Mrs. Purl primarily attended to verbal interactions, Mrs. Grey to written accounts, and Mrs. Brownstein to physical strategies. The teachers' classroom norms and practices often aligned with the ways in which they attended to students.

Written Accounts. Written accounts included when teachers noticed students' written strategies, drawings, or journal accounts. Of the three teachers, Mrs. Grey most often noted students' written accounts. Mrs. Grey reflected that she found it difficult, at times, to manage her classroom and implemented the concept of a math journal. On a daily basis, Mrs. Grey asked students to write down thoughts, explanations, and definitions in a personal journal. The math journal allowed her to maintain control while also allowing students to write and reflect on their mathematical understanding. Mrs. Grey viewed the journal as a window into her students' mathematical reasoning.

When asked to use the Déjà Vu camera, Mrs. Grey often used it as a still life camera to take photos of students' work or drawings. In the transcript below, Mrs. Grey had asked students to write their own definition of area, prior to instruction. She noticed one student's definition changed over time and speculated what the student might be thinking.

> **Researcher:** What stood out to you at this moment?
>
> **Mrs. Grey:** I think it was in her definition that she wrote down. When I looked at it, her definition was "area is the inside of a triangle, rectangle, or square". So that was when I asked her about the inside of a circle. Then, when we talked about changing our definition today, as a class, and I noticed she wrote "area is the inside of every shape and every shape has area".
>
> **Researcher:** So what do you think she was thinking, initially?
>
> **Mrs. Grey:** Um, it looks like maybe, from her drawing, that the units we use to fill in a shape don't fit the right way into a circle or oval because she is drawing square-ish shapes.

For Mrs. Grey, students' drawings and written explanations were powerful in helping her to see the ways in which students were reasoning about mathematics.

Verbal Interactions. Verbal interactions included statements, questions, language comprehension, and student-to-student interactions. Of the three teachers, Mrs. Purl primarily attended to students' verbal interactions. Mrs. Purl's classroom was designed as ELL (English Language Learners). During interviews, she described the need to help students explain their thinking aloud as well as explore the meaning of different words. She listened carefully to students' discussions, purposefully arranged students in groups or pairs, and encouraged them to share ideas aloud. During instruction, Mrs. Purl circulated the classroom and listened to different groups as they worked. For example, when measuring the area of a rectangle using tiles, Mrs. Purl noticed that two girls were using rulers as place markers to help structure rows and columns. The place markers took up space and Mrs. Purl noted the conversation that unfolded between the girls. She stated,

> I loved the interaction between Stephanie and Marlaysia. Stephanie thought using the rulers to mark rows and columns was a great idea, but she was missing the space under the ruler where she used them. Marlaysia realized and explained to Stephanie that the ruler took up space so their measurement was off.

Mrs. Purl often noted the questions students posed as well as the wording they used. She thought that this could tell her more about the ways in which students were thinking mathematically and what remained unclear to the class.

Physical Strategies. All of the teachers used square tiles, cubes, and measurement tools to help students study length, area, and volume. They attended to students' physical strategies as they measured, counted, and built areas and volumes. Mrs. Brownstein, a math and science teacher, frequently used physical materials as a starting point in the lesson. She noted how students counted volume units and the strategies they used. For example, when asked to find the volume of a rectangular prism that was 10 units high, 10 units wide, and 5 units long, Mrs. Brownstein noticed Promise's counting strategy.

> She was able to count 50 in a layer by then she ended up focusing on rods (vertical sections) of 10 and skip counting by 10. I think she is moving from seeing rods to seeing layers, but she isn't using the layers to find the total volume. Last class period she was able to count 50 (cubic units) in a layer but when I asked her to find the total volume she focused in on the rods of 10.

Beyond counting strategies, physical strategies also involved the ways in which students used a ruler, how they used tools like tiles to measure area, and how they built different figures with specified areas or volumes.

Visualization Strategies. Teachers also noted visualization strategies during instruction. These strategies were statements that included how students imagined area or volume or metaphors for area and volume. During a hands-on volume task, Mrs. Brownstein noticed students discussing the idea of volume as an elevator, a hamburger, and a waterfall. She stated:

> When I heard him say elevator, he was describing this kind of motion (moving her hand up and down). From what he said, I think that he is imagining a cart or an elevator going up and going down the prism.

It is important to note that visualization strategies were always coded with another strategy. Students had to speak or write down how they imagined the concept for the teacher to notice it.

Non-Mathematical Strategy. Lastly, teachers sometimes noticed students' non-mathematical behaviors in relation to the task. This included teachers describing that students had worked too quickly through the task, or they did not put enough thought into it. Since teachers were asked to focus on what they noticed mathematically, these events did not arise often.

Mediums by which the teacher attended to students' strategies.		Teachers: Mrs. Purl 37 Events 48 Student Actions	Teachers: Mrs. Grey 24 Events 31 Student Actions	Teachers: Mrs. Brownstein 27 Events 29 Student Actions
Written Accounts	**Drawing**	4	6	0
	Written Strategy (in class)	0	1	2
	Journal Account	0	6	0
	Total	**4 (8%)**	**13 (42%)**	**2 (7%)**
Verbal Interactions	**Statement**	8	6	1
	Question	1	0	0
	Student to Student Interaction	3	1	2
	Language Usage/Comprehension	2	1	5
	Total	**14 (29%)**	**8 (26%)**	**8 (28%)**
Physical Strategies	**Length Measurement Strategy**	2	0	N/A
	Length Counting Strategy	0	0	N/A
	Area Measurement Strategy	10	4	0
	Area Counting Strategy	9	5	0
	Building Strategy	1	0	0
	Volume Measurement Strategy	N/A	N/A	4
	Volume Counting Strategy	N/A	N/A	8
	Total	**22 (46%)**	**9 (29%)**	**12 (41%)**
Visualization Strategies	**Area Visualization**	6	0	0
	Volume Visualization	N/A	0	6
	Total	**6 (13%)**	**0 (0%)**	**6 (21%)**
Behaviors	**Non-Mathematical Behavior**	2	1	1
	Total	**2(4%)**	**1 (3%)**	**1 (3%)**

1 Implication Arose	2 Implications Arose	3 Implications Arose

Figure 2: Mediums that led to instructional implications

Significance of the Mediums. Each of the teachers attended to students' actions through different mediums, as shown in Figure 2, even though they enacted similar lessons. Mrs. Purl primarily attended to students' physical strategies (46%) and verbal interactions (29%), while Mrs. Grey attended to students' written accounts (42%) and physical strategies (26%). Mrs. Brownstein noticed students primarily through physical strategies (41%) and verbal interactions (28%). It is important to consider these mediums because they can act as pathways into how the teachers see and understand their students.

As prior researchers have noted (Jacobs et al., 2010), it is difficult for teachers to enact the cycle of noticing students' actions, reasoning about the students' thinking, and describing implications for instruction. Teachers often see students doing something but are unsure of what the student is thinking and how the students' thinking can be addressed in instruction. For example, the teachers within this study were only able to complete this cycle for 22% to 25% of events noticed (Mrs. Grey, 25% of events, Mrs. Purl, 25% of events, Mrs. Brownstein, 22% of events).

In figure 2, the cells that are shaded show student events in which the teacher was able to describe students' thinking as well as implications for instruction. Cells that are darker indicate a greater number of events led to instructional implications. The darkest cell indicates that three of the events led to instructional implications while the lighter cells indicate two events and one event. For example, in the table, we can see that Mrs. Grey noticed six events that involved students drawing. Of these six events, three of them resulted in an implication for instruction.

As shown in Fig 2, the student actions that teachers attended to most frequently were the most productive for describing students' thinking and instructional implications. For example, Mrs. Purl, who primarily attended to students verbal interactions, had five incidences where she was able to describe students' understanding and instructional implications related to verbal interactions. This was the highest of any of the categories. The same is true for Mrs. Grey in terms of written accounts and Mrs. Brownstein in terms of physical strategies.

Discussion and Concluding Remarks

In considering the cases of Mrs. Brownstein, Mrs. Purl, and Mrs. Grey several themes emerge. The data highlights that teachers notice in different ways when teaching. The mediums by which teachers notice are often dependent on the tasks they pose, classroom norms and practices, and beliefs. Strategies that teachers use on a daily basis (i.e., math journals, encouraging discussions) can be productive mediums and pathways for teachers to glimpse students' mathematical thinking.

Secondly, the types of actions more frequently noticed were also most productive in reasoning about students' thinking and instructional decisions. Each of the teachers had 10 or more years of teaching experience and noticed in ways that aligned with their teaching. This result seems logical because the teachers had many years to think and reflect on their teaching and its ramifications. For example, Mrs. Grey used mathematics journals for several years and had practice reading and reviewing students' journals. It made sense that she could see things about her students through these journals that perhaps she had witnessed before.

This article highlights that teachers learn about their students through multiple and varied mediums. As researchers, it is important to consider this phenomenon when working with teachers and choosing professional development tasks. Examining teachers' current practices can aide in determining how best to help teachers understand students' thinking and reasoning. It is also important to expose teachers to multiple and varied mediums by encouraging multiple forms of expression in the classroom.

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YOU DON'T ASK PAUL SIMON TO DO A DUET WITH NICKELBACK": EXAMINING MATHEMATICS TEACHER COLLABORATION

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Utilizing narrative inquiry, this study documents the experiences of one middle level mathematics teacher (Andrew) as he works to design mathematics lessons focused on student empowerment and power-sharing. We share snapshots of Andrew's three-year story—a story focused on engagement, push-back, collaboration, and Andrew's decision leave his school after three years. Implications for teacher education and support programs are shared.

Keywords: Curriculum; Teacher Education-Preservice; Teacher Education-Inservice (Professional Development; Instructional Activities and Practices

Background Information and Relevance to PMENA Audience

Andrew and I are sitting together over drinks and cheese fritters on a Tuesday in February; he came directly from school so it is relatively quiet in the bar. Andrew is in his second year of full-time teaching at a local middle school and in his third year of participating in interviews with me—we have met regularly from the beginning of his university student teaching experience until now. We engage in casual conversation first and then he updates me a bit on his new position at Parkview as a 6th and 7th grade teacher teaching all four content areas; he's still overwhelmed at times but managing. I probe a bit more and ask about his recent mathematics lessons—he prides himself on creating "outside the box" projects—and he hesitates; "I'm leaving," he says. "I got my contract for next year…I didn't sign it."

Andrew was a part of a larger study focused on mathematics pre-service teachers' experiences throughout university coursework, but we were struck by his case in particular. In his words, Andrew taught lessons with one goal in mind: "to produce informed, driven, fulfilled individuals capable of making an impact on the world." From the beginning of student teaching he talked frequently about his desire to engage in "authentic power sharing" with students. Intrigued by Andrew's curriculum design process and also the struggles he faced as he engaged in this sort of teaching, we decided to follow him throughout his first few years of teaching. Using data from interviews, observations, and coursework artifacts, our study centered on one main research question: *What are the experiences of a new middle level mathematics teacher engaged in "against the grain" (Simon, 1992) teaching practices?*

Brief Literature Review

Research on new teacher induction concludes that nearly half of all new teachers in the U.S. exit the classroom within their first five years (AACTE, 2010). In urban schools, it only takes about 3 years for half of all new teachers to leave. This high rate of attrition often results from challenging working conditions and the absence of a supportive professional culture. For mathematics teachers in particular, this attrition may also be related to curricular issues as teachers are increasingly placed in schools where a predetermined curriculum dictates what mathematics is covered and how it is taught. Through the adoption of specific mathematics textbooks, pacing charts, or state and national frameworks, districts are mandating curriculum materials and curricular frameworks as a strategy for improving student achievement (Corcoran, 2003). As mathematics teacher educators, we need to understand how these mandates position new teachers as they engage in teaching while still learning how to teach.

Inquiry and Analysis

We utilized narrative inquiry throughout this study; we “began with the experiences as expressed in lived and told stories” by Andrew (Creswell, 2007). Closely following Creswell’s (2007) process for implementing narrative inquiry, we gathered data through the collection of Andrew’s stories and reporting of his individual classroom experiences and “chronologically ordered the meaning of those experiences” (p. 54). Drawing on Clandinin and Connelly’s (2000) procedural guidelines for narrative inquiry, we spent considerable time gathering Andrew’s story through multiple types of information: interviews (eight formal recorded interviews and multiple informal conversations each year); written artifacts from Andrew including lesson plans, reflections on teaching, and statements of teaching philosophy; and stories about Andrew from others close to him. All formal interviews with Andrew and veteran educators who worked with him were audio-recorded and transcribed for inclusion in the data set, along with all written reflections and philosophies. We also created research memos after informal conversations or meetings with Andrew and others when audio-recordings were not used. As described by Creswell (2007), our narrative inquiry describes the story of Andrew “unfolding in a chronology of [his] experiences, set within [his] personal, social, and historical context, and including the important themes in those lived experiences” (p. 57).

Andrew’s Story

When he entered his teacher education program, Andrew was several years older than most students in his cohort. He talked about his path to education and his road to finding passion:

> It took me ten years to get through a degree. I was out doing other things. Looking back on it was trying to discover what a passion for me would be. Then I decided to give education a shot. I walked in my first class and my professors came in and gave this impassioned rant about oppression and training and—I mean, really, he gave a step-by-step account of my life in education thus far. I was convicted and inspired and thought, “Okay, something feels different.” I’d finally found that one thing that I have to do.

During student teaching, Andrew continued to talk about his passion:

> What drives my passion is that I feel like I was failed by my education. It wasn’t that I wasn’t good at it; I graduated top of my class. I was able to do what they wanted me to do. I figured out pretty early that I could give back what was asked of me and do it well. It doesn’t sit right with me to know that millions of others are coming up the same way.

Andrew’s curriculum appeared to come out of a space of frustration with what he encountered as a student, and also his desire for his students to develop a passion for and reason to engage in mathematics. During student teaching, when Andrew was required to create and teach a 4-week unit around several standards related to ratios and proportions within the 7th grade mathematics curriculum, he talked about wanting his students to encounter life and math “more naturally” and decided to design a math unit around fear:

> Okay, we’re taking the next four weeks of math to study proportions, but more so talk about our fears, and how we can decide whether or not our fears are rational. Are we okay with our fears being a little unreasonable if it means it keeps us out of danger? We can explore how fear might be mongered purposefully by media in order to get something. Through all that, we encounter proportions; we encounter ratios; we encounter a mathematical thing that yes, is going to be tested on the [end of grade test], but encountering it that way, it’s more meaningful, and I think there’s room for that.

After teaching this unit, Andrew and others (his mentor teacher and many students) reported success. Andrew discussed his students’ initial apprehension when he positioned himself as a facilitator, someone off to the side allowing student voice and choice to dictate the classroom environment and activities. Andrew wrote the following reflection at the end of student teaching:

> I am convinced that if the purpose of education is to produce informed, driven, fulfilled individuals capable of making an impact on the world, then authentic power sharing is absolutely necessary in our schools, and this sharing must be prominent in relationships, in what content is taught, and in how content is taught.

Andrew received an offer to come on as a lead teacher the following year, and he frequently cited his fear unit as the reason he got the job. Like most new teachers, Andrew found himself struggling at the beginning of his first year of full-time teaching, explaining that "I don't feel like I belong in my classroom right now." However, different from most new teachers, Andrew cited collaboration as one of his biggest frustrations and felt like there are "all these demands and all these emphases on collaboration that force me to be on a particular pace." Andrew related this to Paul Simon and Nickelback:

> I went to see Paul Simon speak earlier this year and the topic was "The solo artist in an increasingly collaborative culture." What I took away from it was that you've got people who are gifted. He is a gifted guy. And you wouldn't ask Paul Simon to do a duet with the guy from Nickelback, right? …. You wouldn't ask Mozart to collaborate with Beethoven. They are both fantastic—but they have their own way of doing things and it would likely be disastrous. That's kind of the way I feel about collaboration right now.

When talking about collaboration with other teachers, Andrew also talked about "having had more freedom as a student teacher." He explained further:

> I'm not playing into my strengths, you know. That's where the not having a sense of belonging comes from. My strengths of sitting and going through this organic process of taking math content and figuring out, okay, what can we learn from this? What is the big idea that may not even be content related, but we'll use the content to get there. That's where I'm at my best and I'm not—I just can't do that right now.

As Andrew moved into his third year of teaching at Parkview, he interviewed for and accepted a position as a 7th grade teacher tasked with teaching all content areas. When we met in October, he continued to talk about a lack of opportunity to create math lessons "my way" and about how hard it was to teach lessons that he did not design: "When I design a lesson or unit, I know the purpose of each piece of the lesson, and I can more easily modify in the moment. It's much harder to do that when using someone else's plan." Despite this frustration, he thought it might get better as thought about how to integrate the subjects. He will not, however, have the chance to find out. When we met in February he told us he had resigned. When asked what his plans were next he responded: "I do not know what I'll be doing next year, but I will not be at Parkview. I haven't been able to talk about it yet and it's hard to say out loud, but I'm done."

Findings and Discussion

Andrew talked a lot about the overall purpose of schooling during the three years we worked with him. He talked about the importance of developing close relationships with students. He spoke frequently about his desire to design math lessons that would empower students to take control of their own learning. Considering carefully the role of mathematics in this quest, Andrew spoke of the importance of designing lessons around "bigger" questions about culture, society, and topics of particular interest to middle schoolers. He felt quite certain that it was then, as students engaged in projects around these topics, that they would learn the math.

Despite this belief, Andrew found himself quite often using daily lessons designed by other teachers—lessons that (1) did not align with his belief that there should be some larger goal in mind when teaching math and (2) lessons that Andrew felt would not position him to be "at his best" in the classroom. In short, Andrew's recognition of this inconsistency between actions (his use of other teachers' lesson plans) and beliefs (that the lesson plans he designed himself designed to empower students were better) caused cognitive dissonance (Festinger, 1957).

Cognitive dissonance theory (CDT) posits that dissonance is resolved in three basic ways: change beliefs, change actions, or change perceptions of actions (Festinger, 1957). When analyzing Andrew's narrative through the lens of CDT, it is interesting to consider the choices he had as he tried to resolve dissonance and move towards internal consistency. First, Andrew could have *changed his belief*—he could have concluded that designing empowering lessons around larger questions was no longer important. This happens to many new teachers in the field; they believe in the power of new curricular innovations learned during university coursework but then their beliefs about what could happen in a mathematics classroom change when they are in the field. This move to alleviate cognitive dissonance was unlikely for Andrew. He felt strongly that math should be taught differently and did not waiver in that during his three years teaching.

Alternatively, Andrew could have *changed his actions.* This was also unlikely for Andrew and for most teachers faced with competing responsibilities and assignments in their first few years in the field. Changing his actions to align with his original beliefs and cognition would mean that Andrew would have had to design all lessons on his own. In order to teach in the way he wanted to teach, Andrew would have likely had to work nights and weekends designing these new curricular units; there was simply no time in the day to do that work.

Finally, Andrew's third choice was to *change his perception of action*. Andrew's narrative made it clear that he tried to justify his actions in his first and second years as a lead classroom teacher, at least for a short period of time. He found himself reconceptualizing his decision to utilize other teachers' lessons because he wanted a life outside of work. He rationalized his decision to collaborate and use others' lessons because his administration asked him to work with other teachers and share lessons. For Andrew and other new teachers, policies (and politics) in place in his school—like a severe lack of planning time, additional responsibilities often placed upon teachers at charter schools, etc.—could be seen as levers that push teachers towards changing their perceptions of actions designed to relieve dissonance—actions made in their classrooms that do not align with their beliefs. Even worse, like in Andrew's case, teachers leave teaching all together. For Andrew, who no longer felt that his passion—the thing he "could not *not do"*—was teaching, leaving was the only option.

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EXPLORING TEACHERS' EXPERIMENTATION AROUND RESPONSIVE TEACHING IN SECONDARY MATHEMATICS

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Experimentation has been proposed as a key mechanism by which teachers learn through experience. However, few studies have illustrated what experimentation looks like or how teachers think while experimenting during instruction. This paper investigates the experimentation used by two secondary mathematics teachers that exhibit responsive teaching practices using point-of-view observations to gain access into teachers' in-the-moment thinking. Both teachers engaged in deliberate experimentation around how to influence student thinking. Similar to qualitative methods of inquiry, this experimentation is focused on developing contextually-relevant models of the way that student thinking is influenced by instruction.

Keywords: Teacher Knowledge; Teacher Education-Inservice; Metacognition; High School Education

Supporting teacher learning and growth is essential for improving student outcomes. However, the processes by which teachers are able to improve their teaching, particularly from a cognitive perspective, have been overlooked in empirical studies of teacher change. Researchers have suggested that experimentation with teaching practice may be an important process by which teachers improve their teaching (Hiebert, Morris, & Glass, 2003; Little, 1993). This paper takes a first step at investigating the role that experimentation plays in changing teaching practice and the teacher thinking involved during this experimentation.

Experimentation and Teacher Growth

While there is an intuitive idea that teachers improve through experience, research about the impact of teacher experience on student learning has been inconsistent. Additionally, this work largely ignores *how* teaching experience influences teaching and the different ways that teachers may learn through experience (Sullivan, 2002). The framework of teacher noticing suggests that the experiences teachers attend to and the way they make sense of them should have a profound impact on the way that teachers grow through experience (Sherin, Jacobs, & Philipp, 2011).

One proposed way that teachers are thought to learn through experience is using experimentation (Hiebert et al., 2003; Little, 1993). While research has documented that teachers report experimenting with their practice (Grosemans, Boon, Verclairen, Dochy, & Kyndt, 2015; van den Bergh, Ros, & Beijaard, 2015), and examined hypotheses developed by pre-service teachers from classroom video (Yeh & Santagata, 2015), there has not been a focus on understanding how the process of experimentation unfolds in practice. This study investigates the experimentation that teachers engage and examines the teacher thinking that is at play for teachers that exhibit consistent responsive teaching practices. In other words, these teachers typically attend and base instruction off of the substance of student thinking (Hammer, Goldberg, & Fargason, 2012; Pierson, 2008). This selection of teachers allows this work to examine whether experimentation could support the development or refinement of responsive teaching practices, which are complex, difficult for teachers to enact, and take time to fully develop (Hammer et al., 2012).

Methods

Participants

This paper focuses on two secondary mathematics teachers, Rachel and Mary, who have consistently used responsive teaching practices. Both teachers were videotaped several times since the 2003-2004 school year, and most recently during the 2013-2014 school year. Examining two videotaped lessons from each teacher, one from the beginning and one from the most recent data collection, both teachers were found to have stable responsive teaching practices (Dyer & Sherin, in press). Specifically, the majority of their turns of talk during whole class discussions included substantive probes of student ideas (Pierson, 2008), invitations for students to comment on each other's ideas or "student uptake" (Lineback, in press), and substantive comments on students' ideas or "teacher uptake" (Lineback, in press; Pierson, 2008).

Data

The data used include interviews with the two teachers from the 2013-2014 school year. The interviews come from a series of point-of-view (POV) observations (Sherin, Russ, & Colestock, 2011) with the teachers. During these observations, the teacher captures video from their perspective using a small wearable camera. The camera records the previous minute of action when the teacher presses a button on a remote. Teachers were asked to save moments they "wanted to reflect on or think about later", After the lesson, the teachers were interviewed about each of the saved clips (referred to as moments) in order to access what the teacher was thinking. The discussions focused on (a) the reason the moment was saved, (b) what the teacher noticed in the moment, (c) the teachers' interpretation of the moment, and (d) what implications the teacher drew from moment. Fifteen POV observations were completed with the teachers, 10 with Rachel and seven with Mary. These observations provided a total of 91 discussed moments.

Analysis

The discussion with a teacher around a saved clip or moment was the unit of analysis. The first phase of analysis identified discussions where the teacher implemented or proposed a change to their teaching. In the second phase, the process of using an experience as feedback in each discussion was characterized through open coding. Deliberate experimentation was one of several processes identified. In deliberate experimentation, the teacher purposefully tried or proposed to try multiple ways of achieving an instructional goal. This process was defined by meeting two criteria: (1) the teacher describes or references experiences that use different methods of achieving an instructional goal, and (2) the effectiveness of the methods are compared. These methods or tests could happen during the same lesson or across different lessons, as well as with similar students or across different groups of students.

Findings

The analysis found that both teachers engaged in deliberate experimentation, both during the same lesson and across different lessons (see Table 1). The results also indicate a difference in the frequency by which the two teachers experimented. Across Mary's discussed moments, most of the changes she proposed or implemented were a part of experimentation. Rachel, by contrast, seemed to use her experiences in a variety of ways to inform changes to her teaching.

The following example from Mary is used to illustrate an instance of deliberate experimentation. In this example, Mary considers what questions she could ask different groups to help them accurately model the snow shoveling situation students were working on in her AB Calculus class. The students were developing an equation that modeled the total amount of snow that had been shoveled from 6AM to 7AM when they were given the rate of snow shoveling was 0 ft^3/hour from 12AM to 6AM and 125 ft^3/hour from 6AM to 7AM.

Table 4: Distribution of Deliberate Experimentation Across Lessons

	Mary							Rachel								
Observation Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Total**
Moments Discussed	15	17	13	5	8	4	2	7	3	2	3	4	1	4	3	91
Deliberate Experimentation	2	2	3	2	1	0	0	0	0	0	0	0	1	0	1	12
During one lesson	0	1	3	0	0	0	0	0	0	0	0	0	1	0	0	5
Across multiple lessons	2	1	0	2	1	0	0	0	0	0	0	0	0	0	1	7

Mary explains the experimentation that was at play when discussing the moment, "[I knew] that this is where they were going to screw this up. This group had the 125*t* and the 108*t*…and initially, some of them aren't doing subtraction, [they] divide by 6 first. So they know that there's something that you do with 6…So, and, I think I just experimented with different ways, like, how I interact with the groups, like, sort of, the best way to get them to understand that it was t minus 6 as opposed to just t [which was asking students how much snow had been shoveled as specific times between 6 and 7]." In this example that Mary explicitly mentions experimentation in the ways that she interacted differently with multiple groups in the same lesson. Additionally, she mentions her goal in this experimentation of trying to find the best way to help students understand the equation. Mary also considered how asking students to respond to the idea that the time term in the equation is *(t-6)*, but decided that method was less "cognitively demanding".

One striking feature of all of the experimentation identified was that both teachers were experimenting around how to influence student thinking. This experimentation involved complex teacher reasoning. First, the teachers interpreted how students are thinking during the multiple tests as well as before and after each test. In addition to these interpretations, the teachers make hypotheses about how that student thinking was changed or influenced. In the discussions, teachers mentioned a variety of aspects of the classroom environment that influenced student thinking, including the design of mathematical tasks, the wording of questions, or the representations used in diagrams or graphs. Overwhelmingly, the teachers created hypotheses that were contextualized in the specific thinking, specific students, or environment of the classroom. This thinking is well aligned with responsive teaching in that these hypotheses identify the different ways that student thinking can be influenced, which is the basis for responsive teaching.

This was apparent in Mary's example described above when she went on to describe the way the questioning helped students understand the equation:

> What I think right now is the best way to approach it is [a round of questioning about] "how much snow has she shoveled at 6?" And you know, they say "0." "How much snow has she shoveled at 6:30?" And they say, "125 divided by 2, 125 times one half". "How much snow has she shoveled at 6:45?" Asking them those questions for them to get to the *t-6*, um, but then still kind of affirming that "yeah, 125*t* is good, like, you're thinking antiderivative, and like, it's a constant rate, therefore like, *G(t)* would be, you know, 125*t*," but then a nuance is the *t-6*, and then the *t-7*, and so on for the next part.

In this quote, Mary explains how this line of questioning relates to what students are initially thinking about the equation, namely that students start with 125*t* for the equation. Throughout this quote and the previous excerpts, she makes three different interpretations of student thinking, (1) the initial idea that the situation is modeled by 125t, (2) the idea that the time variable in the equation should be divided by 6, and (3) the idea that time variable should be shifted by 6. Mary also develops a model of how the questioning influenced students to think in a different way. In particular, she recalls how students are able to determine how much snow has been shoveled at discrete points in time, and how that progression can be leveraged to off of the initial thinking that $125t$ is an accurate way to model the snow shoveled at 6AM.

Discussion

In many ways, the findings reflect much of the theory that has been proposed about the existence of experimentation. However, these findings suggest that making conclusions about an experiment and developing hypotheses for future experiments, seen in the findings as models that teachers created for the ways that student thinking is influenced by teaching, are in many ways intertwined. These models may represent a key part of the teacher thinking that is involved in experimentation. This type of contextually-valid mental model is similar to how Hiebert et al. (2003) describes experimentation as a test of “a teacher’s local theory of how students learn and how instruction facilitates learning” (p. 208). The contextual nature of these models suggest that video-based studies of experimentation or learning through experience may not accurately reflect the process that happens in real classrooms.

Additionally, the findings show that experimentation happens on a moment-to-moment basis, suggesting that this micro-level time scale may be important level for looking at experimentation, and more generally mechanisms for teacher growth. Thinking about experimentation in small moments during a lesson also stands in contrast to many of the ways that experimentation is supported in current teacher education contexts, which often focus on more long-term experimentation and study of teaching practice. The findings suggest a more balanced approach where teachers are also supported in engaging in experimentation at a more micro-level that could be easily incorporated into everyday instruction.

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WHAT INFLUENCES A TEACHER'S WILLINGNESS TO CREATE OPPORTUNITIES FOR DISCUSSION IN A GEOMETRY CLASSROOM?

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This study presents an analysis of geometry teachers' willingness to create opportunities for student discussion in the classroom as revealed through the use of a multimedia survey instrument. We presented 42 geometry teachers with 8 multimedia narratives and asked them to choose between a normative instructional action which closed off discussion and a less typical action that encouraged student discussion. Our analysis provides insight into the professional obligations that teachers use to justify their departure from the norm in order to encourage student talk as well as the background variables that are associated with such decisions. We found that while experienced geometry teachers are more likely to make a normative decision than those who are less experienced, there are some cases in which experienced geometry teachers will justify their departure from the norm with their obligation to the discipline.

Keywords: Geometry; Teacher Knowledge; Classroom Discourse; Teacher Beliefs

The importance of discussion in the mathematics classroom has been argued by a number of researchers (O'Connor, 2001; Stein & Smith, 2011) whether it takes place in the context of whole class interactions or within small groups. However, the positive benefits of discussion can only be realized if a mathematics teacher creates opportunities for such interactions in the classroom. The persistence of lecture and recitation format in mathematics classes, however, suggests the importance of understanding the conditions under which a mathematics teacher is willing to choose discussion as an appropriate thing to do. This paper presents initial results from the use of a scenario-based instrument that probes the decision-making of secondary mathematics teachers and, in particular, explores if and when teachers are willing to depart from customary instruction in order to give their students opportunities to discuss problems in small groups or as a whole class.

Theoretical Framework

Researchers have catalogued many benefits of small group and whole class discussions. These interactions provide unique opportunities for thinking and learning (Yackel, Cobb, & Wood, 1991; O'Connor, 2001; Ge & Land, 2003; Stein & Smith, 2011). These studies do not, however, look at what influences mathematics teachers' willingness to depart from customary instruction in order to encourage discussion in their classroom. Researchers who examine intact secondary instruction have documented the presence of customary ways in which teachers carry out their work, both at the large grain size of cultural teaching scripts (Stigler & Hiebert, 1999) and at smaller grain size in instructional situations customarily used to teach particular mathematical ideas (Herbst et al., 2009; Chazan & Lueke, 2009). This literature suggests that opening the classroom discourse to include more classroom discussion might require departing from some of the norms that characterize instructional practice. How can we gauge teachers' disposition to do so?

Most of the research on teachers' decision making is qualitative and is difficult to replicate at a larger scale. Outside the field of educational research, researchers have examined decision making in professional settings (Chan & Schmitt, 1997; Lievens, Peeters, & Schollaert, 2007) using written vignettes and video-based representations as a stimulus for a particular context-based decision. In mathematics education Herbst, Aaron, and Erickson (2013) have shown that animations of cartoon characters work just as well as live-action video clips for provoking teachers' thinking about content and pedagogy; and Herbst and Chazan (2015) have shown how storyboards realized with cartoon characters can be used to study decision making. Thus, we argue that a scenario-based instrument where the scenarios are realized with cartoon characters may provide a means of conducting larger-scale studies of teacher decision-making.

These instruments are grounded in the theory of practical rationality which posits that instructional norms, obligations to the profession, and individual teacher resources (such as beliefs, knowledge, and skills; Herbst & Chazan, 2012) come into play in teachers' in-the-moment instructional moves. We argue that teachers' decision-making can be understood as part of an instructional system wherein the teachers' role is to manage the terms under which students interact with the mathematical content to be learned. Further, we theorize that a practical rationality of instruction enables teachers to call on their professional obligations as sources for justification to support deviations from customary practice (Herbst & Chazan, 2012). In earlier work we gathered evidence that we can use scenario-based assessments to help determine whether teachers recognize departures from a norm in the situation of doing proofs, where those departures might increase students' opportunities to participate in mathematical discussions (Herbst et al., 2013); the data gathered in that pilot study supported the contention that these departures might be justified on account of professional obligations to individual students, the class as a whole, the discipline of mathematics, and the institution. From that work the question remained—if an individual teacher was given the opportunity to depart from a norm of doing proofs in ways that might increase chances for students' discussions, would they choose to do so? What could explain the variance of these responses?

Methodology

The Decision instrument was designed as a means of assessing teacher decisions in the context of an instructional situation. The participant is presented with a storyboard depicting a teaching scenario set in a geometry classroom where the teacher is about to assign a proof problem to their students. For each item, there is a choice between four actions where one action is consistent with a norm (e.g., the teacher provides the givens and prove statement) and the other three are breaches of that norm in favor of a participation structure that may or may not incorporate greater student talk: discuss the problem as a class, work on the problem in small groups, or work individually. After choosing one of those options participants are asked to choose justifications for their choice from a collection of ten statements which contained pairs tied to each of the four obligations respectively and another pair which served as distracters. The distracters are statements that make the decision a personal matter for the teacher (e.g., "This action best aligns with my beliefs about mathematics teaching"). The participant may choose as many of these statements as they wished which meant that we could give them a composite score between 0 and 8 across the four items corresponding to a norm for each obligation. In the following section, we describe some of the significant differences between items with respect to opportunities for student talk and with respect to the obligations chosen to justify teaching decisions. We also looked for relationships between these choices and whether a participant had more than 4 years experience teaching geometry (EGT; N = 22) or not (non-EGT; N = 20).

Results

While the modal choice for each of the eight decision items was either small group or whole class discussion, thus showing that teachers tended to prefer increasing opportunities for discussion, there were some items that evoked more responses encouraging discussion than others. For example, in item 61001, the teacher writes a proof problem without an accompanying diagram and a student asks for clarification. The participant can choose between rewriting the problem with a diagram or encouraging the students to work through the problem themselves, as a whole class, or in groups. Similarly, in item 61003, the teacher draws a figure and tells the students what they are going to prove. The participant can choose to clarify the givens and prove statement on the board or make moves to encourage student discussion. There were significantly more EGTs who chose the small group discussion option for the first item while there were significantly more non-EGTs who chose the small group option for the second item. Why were EGTs more supportive of discussion in one case than the other? Looking across the open responses to 61001 reveals that many of the EGTs who chose the small group option wanted students to draw the diagram while the non-EGTs either did not address the diagram or suggested that the teacher provide one. A review of the teachers who did *not* choose the small group option for 61003 suggests that the EGTs may have been more concerned with maintaining control of the conversation in order to clarify the problem for their students whereas the non-EGTs focused more on encouraging student contributions.

In order to learn more about how teachers justify departures from the norm, we looked for connections between the professional obligations that participants used to justify their decision and their personal characteristics as well as their responses to the decision items. We ran a two-tailed t-test and found that there was a significant difference in favor of EGTs for justifying decisions on account of the disciplinary obligation ($p < .0001$) and that there was a significant difference in favor of non-EGTs for doing so on account of the institutional obligation ($p < .05$). By way of contrast, there was no significant difference with respect to experience for the individual or interpersonal obligations. We conjecture that more experienced geometry teachers have a better conception of the demands of the discipline while teachers with less experience may feel more bound by the school curriculum. We also found that use of the interpersonal obligation to justify a decision was negatively correlated with the normative option and positively correlated with the whole class discussion option. These results have face validity since the choice of a whole class discussion suggests concern for the manner in which students communicate with one another.

Conclusion

We have demonstrated how scenario-based assessment of teachers' decision-making can be used to find out when mathematics teachers are willing to depart from an instructional norm in order to encourage student talk. Taking as a case the situation of doing proofs in high school geometry and looking in particular at the setting of a task that might encourage student discussion, our results suggest that a majority of high school teachers approve of actions that encourage discussion (either in small groups or whole class form) even if these actions may depart from common practice. We have further evidence about the nature of the professional obligations that these teachers see either justifying the encouragement of talk or else justifying a normative action. In particular, we found that appeals to the disciplinary obligation featured more often as justifications for those teachers who have more experience teaching geometry while the institutional obligation did so for teachers with less experience teaching geometry.

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WHAT IS HAPPENING IN CALCULUS 1 CLASSES? THE STORY OF TWO MATHEMATICIANS

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This study examined class practices of two Calculus 1 classes to understand what really happens in the class. The mathematicians' and students' questions as well as tasks used in class were analyzed to examine students' opportunities to engage in cognitively demanding tasks and discussions. The findings of this study indicate that students in these particular calculus classes did not seem to have enough learning opportunities to explore high-level cognitively demanding questions, to experience multiple representations and alternative solutions, and to engage in meaningful discussions.

Keywords: Classroom Discourse; Instructional Activities and Practices; Post-Secondary Education

Introduction

Recent studies show that work–force demands for students with science, mathematics, and engineering backgrounds have increased in the last few decades, but the number of students pursuing these majors in the U.S. and other countries either remains constant or is declining (Hurtado, Eagan, & Chang, 2010). One influencing factor of pursing a degree in these areas is students' experiences in undergraduate mathematics, especially calculus classes (Ellis, Kelton & Rasmussen, 2014). Since the nature of classroom teaching significantly influences how students learn mathematics (Hiebert & Grouws, 2007), it is important to know what transpires in calculus classrooms. Here are the research questions that we attempt to answer.

- What types of questions, in terms of cognitive demand and mathematical content, are asked by both the calculus instructor and students?
- What types of representations are presented in Calculus 1 class?
- What discourse patterns are found when various calculus concepts are introduced to students?

Related Literature

Studies on calculus teaching and learning

The overall consensus among calculus studies is that students do not have full conceptual understanding of various calculus concepts (Bezuidenhout, 2001). There has been criticism that calculus students' understanding is heavily procedural and algebraic without conceptual depth (Swinyard, 2011). In a recent calculus study by Ellis and her colleagues (2014) showed that "Good and progressive teaching" includes listening to students, presenting more than one method of solving problems, and more actively engaging students.

Theoretical Framework

It is very complex to understand classroom practices because it is both an individual and social process (Sfard, 2001). Mathematical tasks and questions should also be examined to understand class practices (Franke et al., 2007). Providing worthwhile tasks is a critical part of class practices (Boston, 2012). Teachers can provide productive discourse when tasks enable students to connect various mathematical ideas with multiple representations and alternate solution strategies (Franke et al., 2007).

Methodology

Setting, Data Collection and Analysis

The setting for this study was a Midwestern research university in the United States. Two mathematicians, Jenny and Tom[1], have a PhD in mathematics. All calculus classes were videotaped in 2014. In total, we collected 59 video clips. In all, we analyzed 602 questions by the mathematicians, 370 tasks, 32 student questions, and 91 calculus concepts.

Categories of Analysis

The analytic framework used for this study was based on previous studies that examined class practices and class observation theory (Boston, 2012; Franke et al., 2007).

Analyzing Questions

A two-dimensional framework was used to analyze each question – cognitive demand and mathematical content. We carefully analyzed the cognitive demand of each question asked by both mathematicians and students. In addition to cognitive demand, we examined the mathematical content (algebra or calculus) of each question. Lacking understanding of algebra concepts is one difficulty for calculus students (White & Mitchelmore, 1996). Tables 1 illustrate each example.

Table 1Questions by Code

	Mathematical Content	
	Algebra	Calculus
Higher level Cognitive Demand		Task: Understanding the derivative of a function. Question: What does it mean to have slope of tangent is zero?
Lower Level Cognitive Demand	Task: Find the slope of the tangent to $f(x) = x^2$ at $x = 2$. Question: (after graph was sketched) What is the y – coordinate?	Task: understanding anti-derivative. Question: What function would be an anti-derivative of x?

Representations and Alternate Solution Methods of Implemented Mathematical Tasks

Providing multiple representations to students has been emphasized in calculus teaching (Haciomeroglu et al., 2010). Thus, we paid attention to types of representations used to solve implemented tasks. Whether alternate solution methods were presented to students or not was also examined.

Presenting Concepts, Theorems, and Properties in Calculus 1

One interesting finding from a very well–known classroom-based study, the **Trends in International Mathematics and Science Study (**TIMSS) video study, is the way mathematical concepts are presented to students. In U.S., compared to other countries, large number of concepts (78 %) were presented to students without being developed (Stigler & Hiebert, 1999). We also paid attention to how these concepts and rules were presented to students.

Coding Reliability

After the codes were set for analysis, we went over each code carefully to fully understand their meanings. The second author and the third rater, a doctoral student in mathematics education, were trained to code the questions, tasks, and segments. Both authors and the doctoral student in mathematics education viewedthe video clips several times to code each question and task. Next, the three raters coded each segment independently and discussed the discrepancies. When the two authors disagreed, those items were coded based on majority rule using the third rater's codes. There were no questions or tasks in which all three raters disagreed. The percent agreement of the three raters was between 93% and 96% for questions and tasks.

Results

In this section, we present results of our analysis.

Analysis of Questions

There is a large difference in the number of questions asked by the two mathematicians. Jenny asked 17.5 questions per lesson while Tom only asked 1 question per lesson. The way Tom taught his classes, which were mostly lecture based, could account for this. Over 80% of their questions required low cognitive demand, meaning that many questions asked by both mathematicians were about simple recall and/or procedures. A question such as "What is the derivative of x^1?" only requires procedural knowledge of finding the derivatives of simple function. On the other hand, to answer "Can anyone explain why the derivative of a constant function is zero?", students need to think about rate of change, slopes, and the derivative.

In Jenny's lessons, algebra and calculus related questions were asked a comparable number of times, where 269 (46.6%) were algebra related and 276 (47.8%) were calculus related. About half of Jenny's questions were algebraic nature, which suggests that algebraic skills are necessary to solve some of the tasks in her calculus class. On the other hand, 76% of Tom's questions were about calculus; however, there were only 25 questions, about one per class, so it is difficult to come to a definite conclusion about Tom's questioning. A total of 27 questions were asked by Jenny's students in contrast to only 3 questions from Tom's students. These numberssuggest that regardless of cognitive demand and content, students had few opportunities to pose any questions. Out of the 27 questions asked by Jenny's students, none required high level cognitive demand and 25 were related to algebra concepts. Here is one question asked by Jenny's students. When the task, "Find the tangent line of $f(x) = \frac{(2x)}{x^2+1}$ $at\ x = 1$," was given, one student asked (after the derivative $\frac{(x^2+1)2-2(x)(2x)}{(x^2-1)^2}$ was found) "Why don't we cancel $x^2 + 1$?" This question is about finding the common denominator for rational expressions, an algebra topic. This indicates that the implemented tasks in their calculus classeswere about algebraic procedures or that they still lacked understanding of these algebra concepts. While we can say a few things about Jenny's students, it is very difficult to say anything about Tom's students because they asked only three questions. In Tom's class, opportunities to create norms by asking questions were too limited.

Analysis of Representations and Solution Strategies

The first notable finding from this table is the difference in the total number of tasks solved in each instructor's classes. The main reason that contributes to this is that Tom spent large portions of instructional time lecturing. Also, we can see that many implemented tasks were solved algebraically in both classes. It appears that opportunities to synthesize different representations were not often provided to these calculus students (21.6 %) . Furthermore, in both mathematicians' lessons, there were only few times, one time in Jenny's classes and three times on Tom's classes, in the semester that alternate solution strategies were presented to students. In Jenny's lesson, the problem was to differentiate $\sqrt{x^2+1}$. She first stated that there were two functions, $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, so that it can be solved by the chain rule, but in an alternate approach, she also noted that it can be written as $(x^2+1)^{\frac{1}{2}}$. Thus, it can be differentiated by the power rule. From these two ways, students not only saw how these two rules are related but they also reviewed the algebra topics of the law of exponents. However, such opportunities were very rare in this calculus class.

Concepts Presented without Development

In total, there were 46 calculus concepts, theorems, and rules presented to students during the semester in Jenny's class while 45 were presented to Tom's students. Among these, 29 (63%) were not developed and simply stated by Jenny while 30 (66.7 %) were developed in Tom's class

In all, there were differences in class discourse when various calculus concepts and rules were introduced. The fact that more than 60% of calculus concepts were not developed in Jenny's class and 98% of concepts were either developed *uni–directionally* or not developed (30 were developed *uni–directionally* and 14 were not developed) indicates that it is rare for students to express, reason, and evaluate their own or someone else's mathematical thinking when these concepts were presented in class.

Discussion and Conclusion

Results of this study reveal that calculus students may not have opportunities to answer high level cognitive demanding questions, experience multiple representations and alternate solutions, and engage in meaningful discussions. Moreover, many calculus concepts, theorems, and rules were introduced either not being developed in any way or developed *uni–directionally* by the mathematicians. Since opportunities to engage in meaningful discussions and tasks are limited, it may be difficult to establish norms − acceptable and adequate mathematical answers, interaction, and activities in classes. Finally, an important question we need to ask is, Can we find similar results from other calculus instructors' classes? More studies that examine other calculus instructors' classes will be needed to fully understand what needs to be done to improve calculus teaching.

Endnote

[1]Pseudonyms

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SUPPORTING FRACTION OPERATION ALGORITHM DEVELOPMENT: NUMBER SENSE AND THE ENACTMENT OF ADDITION

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Number sense is a key component in being able to make sense of problem situations that involve fraction addition. Positioning students to invent algorithms for adding fractions involves their ability to use number sense-based reasoning. This paper offers five forms of mathematical interaction that teachers engaged in during classroom discussions to support student engagement in number sense-based reasoning when estimating fraction sums.

Keywords: Rational Numbers; Classroom Discourse; Instructional Activities and Practices

This study of teacher practice is situated in the context of fraction operations with a specific focus on number sense-based estimation in preparation for fraction addition and subtraction algorithm development. It is founded upon arguments that practice should have more attention in an effort to unpack the professional work teachers do in classrooms (Grossman et al., 2009). In an effort to understand teacher practice and the orchestration of mathematical discourse in the domain of fraction operations the following question was studied: What were the key mathematical ideas teachers elicited or accepted from student reasoning that framed mathematical conversations about students' number sense-based work related to fraction operation algorithm development for addition?

Background and Theoretical Framework

The literature on fraction number sense and operation development has revealed the struggles students have had related to fraction addition and subtraction (Petit, Laird & Marsden, 2010). For example, when asked to estimate in computational settings, students often want to compute an exact solution and then round that to the nearest benchmark. Students may also view the numerator and denominator of a fraction as individual numbers that are unrelated. In turn, students struggle to engage in number sense-based reasoning with fractions. Gravemeijer and van Galen (2003) differentiate between engaging students in the learning of algorithms as a process of building number sense and the direct teaching of algorithms. Instructional activities that support the construction of a network of number relations are a key part of the process where algorithms emerge from the development of number sense. Engaging students in modeling and representation use are also important components in the development of fraction number and operation sense (Petit, Laird & Marsden, 2010).

The findings reported here draw from the initial days in a fraction operation unit that began with a three to five-day task sequence for approximating and estimating sums of fractions that preceded a task sequence to draw out algorithmic strategies for adding and subtracting fractions. During the estimation sequence, it was not uncommon for students to fail to fold back and use fraction number sense developed in prior instructional units. It is argued that these moments represent common scenarios that emerge in classrooms when asking students to engage in number sense-based reasoning related to fraction addition and subtraction. This research will share five forms of mathematical interaction that teachers routinely used to position and support students when they struggled to engage in number sense-based reasoning.

Methodology

The settings for this qualitative study were the classrooms of four experienced skillful sixth-grade teachers and their students. Each teacher used the Connected Mathematics Project (CMP) II instructional unit *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006). The instructional unit used a guided-reinvention approach (Gravemeijer & van Galen, 2003) allowing algorithms to arise through student engagement with both contextual and number-based situations. During the teaching of the *Bits and Pieces II* unit, classroom lessons were videotaped each day during the five to six weeks it took to cover the unit. In addition, the teachers wore an audio recorder during each lesson. When a teacher completed a lesson, they audio recorded a short five-minute reflection about their teaching and student development during the lesson. The first researcher visited each classroom approximately once every ten days. This timeline was repeated across two school years. Data analysis was guided by Erickson's (1986) interpretive methods and participant observational fieldwork, which addresses the need to understand the social actions that take place in a setting.

Findings

There were five forms of mathematical interaction (see Figure 1) that occurred routinely in teachers' mathematical conversations with students when they were working on the three to five-day estimation sequence. These five forms of interaction are in response to common ways that students struggle in this mathematical domain. They suggest fraction-specific ways that teachers can support and further students' ability to reason with fractions and to make sense of what fraction addition is an enactment of. Two classroom examples will be used to illustrate the forms of interaction.

- Prompting students to rename benchmarks (0, 1/2, 1) in terms of specific fractional denominators;
- Developing and eliciting levels of precision: closest to a benchmark, greater than or less than a benchmark, or how far from a benchmark;
- Drawing out explicit distinction between exact and estimated solutions;
- Pressing for consideration of how estimated sums and exact sums are related; and
- Highlighting when estimation is used as a reasoning tool.

Figure 1: Five Forms of Mathematical Interaction Framing Number Sense-Based Estimation Instructional Conversations

Example 1: Making Fraction Strips To Think About "How Large" and "How Far"

At the start of the fraction operation unit, most students were able to reason about the size of an individual fraction in relation to benchmarks and other nearby quantities. However, the main focus of the instructional sequence was to engage students with reasoning about the approximate size of the sum of two or more quantities. One teacher provided grid paper to students who needed support to visualize fractions in relation to benchmarks (see Figure 2). For example, when trying to estimate whether the sum of 7/10 + 2/5 would be closer to 0, 1, 2, or 3, a student was directed to make a tenths strip and then locate and mark 0, 1/2, and 1 on the strip. Next, they labeled 0/10, 5/10, 10/10 and 7/10 on the strip. The student then marked, and labeled a strip for 2/5. This allowed the student to visualize and count how far 7/10 and 2/5 were from neighboring fractions.

This teacher was working to support a student who had not yet developed a strong visual sense of where individual fractions were located in relation to nearby benchmarks. Without this reasoning a student would struggle to understand what estimating a sum would entail and to use location and magnitude when estimating. This discussion highlights the first form of mathematical interaction: prompting students to reinterpret benchmarks in terms of specific fractional units being used (i.e.; tenths or fifths). This episode also highlights the second form of mathematical interaction: determining how far a fraction is from a benchmark. In addition, this work supported conversations about what an estimate was and how reasoning with quantities as locations is important.

Teacher: How far is 7/10 away from one whole?
Student: Three tenths.
Teacher: How far is it away from the 1/2?
Student: Two tenths.
Teacher: So which is it closer to?
Student: 1/2.
Teacher: How far is 2/5 from 0?
Student: Two fifths.
Teacher: How far away from 1/2 is it?
Student: Half of a fifth.
Teacher: Is 2/5 closer to 0, 1/2, or 1 whole?
Student: To 1/2.
Teacher: So would 7/10 + 2/5 be closer to 0, 1, 2 or 3?
Student: 1/2 + 1/2 would be 1. Closer to 1.

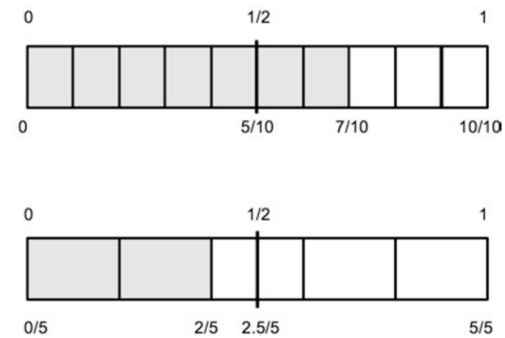

Figure 2. Discussion with Fraction Strips When Estimating Sum of 7/10 + 2/5.

Example 2: Plausibility of a Calculational Approach

This episode supports students to understand number sense-based estimation entails. In this episode students had been asked to think "about how big" the sum of 1/9 + 9/10 would be.

Teacher: Tell me where you would start. How big would the sum of 1/9 and 9/10 be?
Student: Well, first I would try to find a common denominator.
Teacher: I really don't want the exact answer. Can you tell me about how big it is?
Student: 10/20.
Teacher: Around 10/20 so about 1/2? Tell me how you did that.
Student: I thought 9 + 1 is 10.
Teacher: So you are doing 1+9 is 10.
Student: Yeah. Then I just thought on the bottom 9 + 10 was 19.
Teacher: Okay, about how big is 9/10?
Student: Almost a whole.
Teacher: And you think if I add something more to that [teacher points to 9/10] I am going to get around 1/2?
Student: Wait. No, that wouldn't [work].
Teacher: Yeah. It seems like it's going to be more than 1/2, doesn't it. So I think you were trying out a strategy like add these [numerators] and add these [denominators]. It didn't seem to work very well did it? Could you show this with some kind of model?

Another student then offered a strip model of 9/10. Pointing to the unmarked 1/10 of her fraction strip she said, "I knew that 1/9 is bigger than 1/10. So it would be over one. Maybe right here." She gestures to show how it would be a short distance beyond one whole.

At the beginning of the episode the first student showed awareness of how to calculate an exact sum using common denominators. He also correctly reasoned that 10/19 was about 1/2 and that 9/10 was about 1. In his struggle to understand what estimation involved, he tried adding numerators and adding denominators to calculate a sum. While he was able to think about magnitude and location for an individual fraction, he struggled to reason about the resulting location and magnitude of the fractional quantities when combined.

In this episode the teacher did multiple things to move the conversation forward and focus on using number sense-based reasoning. With the first student she utilized the third and fifth form of mathematical interaction. She made an explicit distinction between finding an exact sum and an estimate by stating, "I really don't want an exact answer". Second, she prompted the student to use number sense-based estimation as a reasoning tool to determine if his approach of adding numerators and adding denominators led to a reasonable result. When the second student offered reasoning that would allow one to consider how the estimated sum and the exact sum were related, the fourth form of mathematical interaction was utilized.

Discussion

The students highlighted in the cases represent learners a teacher might typically encounter. This includes students who are able to reason with individual fractions but not sums. The forms of interaction in Figure 1 offer ways teachers can support students to use number sense-based reasoning when adding fractions together. These interactions support the development of number relations. According to Gravemeijer and van Galen (2003), students' invented algorithms emerge from the development of a mathematical framework of number relations that they use as tools when solving problems. This focus on reasoning about the combining of fractions as quantities and as locations can help support reasoning needed if the goal is for students to develop valid computational strategies for adding fractions.

Acknowledgement

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PRESERVICE TEACHERS' CRITIQUE OF TEACHER TALK

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This study aimed to explore pre-service elementary teachers' (PSTs') conceptions of effective teacher talk in mathematics instruction through a two-phase task that involved analyzing and evaluating a sample teacher-student talk and developing alternative ways of communicating with students. Findings were interpreted in terms of PSTs' views of mathematics, their perceived roles in classroom discourse, and their conceptions of effective forms and functions of teacher talk. The outcome has implications for teacher education and future research in addressing this important aspect of mathematics teaching practice and opening up pedagogical possibilities for teacher educators.

Keywords: Teacher Education-Preservice; Classroom Discourse; Teacher Beliefs

Effective classroom communication calls for teachers to facilitate meaningful mathematical discourse and pose good questions. With this said, it is an important first step to identify teachers' conceptions of effective communication in a mathematics classroom. The purpose of this study is to explore PSTs' conceptions of effective communication in mathematics instruction through a series of tasks that involve analyzing and evaluating a sample teacher talk and developing alternative ways of communicating with students. Specifically, this study examines how the criteria PSTs used in analyzing, evaluating, and modifying a sample teacher-student talk reflect the following aspects: (1) PSTs' views of mathematics and teacher roles in classroom discourse and (2) PSTs' conceptions of effective forms and functions of teacher talk.

Theoretical Framework

Teachers' Views of Mathematics and Roles in Effective Classroom Discourse

Teachers' communications enacted in mathematics instruction are highly associated with their conceptions of teacher roles. Ernest (1989) categorized three different models of teaching mathematics by focusing on the teacher's role and intended outcome of instruction: (a) Teacher as an instructor: aims at skills mastery with correct performance, (b) Teacher as an explainer: aims at conceptual understanding with unified knowledge, and (c) Teacher as a facilitator: aims at confident problem posing and solving. Which of these roles the teacher adopts will have a great impact on the overall classroom communication and actions such as the teacher's questioning strategies or feedback patterns.

Forms and Functions of Teacher Talk

The analytical frameworks used in the previous research differ across various studies, and there is no prescribed equation for knowing what and when to say because teacher talk can be used for various purposes. In terms of cognitive level, higher-order versus lower-order thinking questions/prompts framework is frequently used. Whether the type of teacher question/prompt is open or closed is also frequently addressed in the prior studies. Generally, the more open questions/prompts, which are designed to ensure that multiple answers are valid and valued, the more they will identify the students' mathematical understanding and abilities (e.g., Blosser, 2000). However, studies also comment that posing open-ended questions alone is not a sufficient condition to initiate an explorative interaction for investigating the child's own understanding of mathematical knowledge. In this regard, researchers suggest that it is necessary to distinguish between form and function when analyzing and evaluating teacher-student interactions (Wells, 1999 cited in Mercer & Dawes, 2014).

Methods

Participants

Data from this study came from 46 PSTs who enrolled in a K-8 mathematics methods course at a Midwestern university in the United State. For most of PSTs, it was one of their last two semesters before student teaching.

Tasks and Data Collection

The following short transcription of the teacher-student talk was provided for PSTs to analyze and evaluate.

Given Question: The price of a video game is $60. You want to buy it using a 20 % off coupon. How much should you pay for the video game?

S_1: Do I have to worry about taxes?
T_1: No, we're just calculating the price.
S_2: Price is $60 and I have a 20 % off coupon…
T_2: So what are we supposed to do with that coupon? Multiply? Divide? Add? Subtract?
S_3: I have to find out what 20% of 60 is, right?
T_3: That's right! So how do we do that?
S_4: Multiply, I think. I need to do 60 times 20, right?
T_4: Well, sort of. If wanted to know what 20 times 60 was, yes...but what am I actually looking for? You just said it a few seconds ago.
S_5: I want to find what 20% OF 60 is.
T_5: Yes! So we need to do something other than just multiply them. Remember the rule, what way do we move the decimal?
S_6: To the left.
T_6: How far?
S_7: 2 places.
T_7: You got it!
S_8: But what if it is already a decimal, like .5%?
T_8: The rule never changes, still 2 spots to the left.
S_9: Ok, so I have to make 20% .20 and multiply it by 60. (When the student does the math, he sets it up correctly but gets 1200 and looks at it confused for a minute.)
T_9: So what's wrong with that?
S_{10}: It's way too big. How can it be bigger than the price of the game?
T_{10}: See now this is why I ask you guys all to show your work!! Let's look back at your math.
S_{11}: Ok. I multiplied zero by each number, then 2 by each number and I get one, two, zero, zero….wait, the decimal point! I forgot about it.
T_{11}: Yes! You got it, so where does it go?
S_{12}: Well there are two places to the right so I need to move it two places over, $12.00 is the discount then?
T_{12}: Yep! So how much would you be paying?
S_{13}: I need to subtract that now from the first price so it would be ….$48.00 for the game?

Using this sample teacher-student talk, PSTs were asked to engage in two tasks.

Task 1: Critique of sample teacher-student talk. PSTs were asked to comment on/evaluate each of the teacher's statements (T_1 ~ T_{12}). PSTs first indicated whether they evaluated it positively or negatively and then provided their justifications to explain their evaluation. They were also asked to suggest alternative statements for the negatively evaluated statements.

Task 2: Small group discussions for revisions of the sample teacher-student talk. Small groups were formed for this task. Each group needed to reach a consensus if they wanted to replace the original teacher statement. If a group was unable to reach a consensus, they were asked to note why it could not be resolved. Each group's final revision was shared with the researcher in a small group discussion setting.

Data Analysis

PSTs' written responses from Task 1 were analyzed using an inductive content analysis approach (Grbich, 2007). Thus, throughout the data set, each sentence was coded for whether it referenced in some way the categories identified. At the completion of coding, frequencies of coded themes were identified. The data from Task 2 were analyzed descriptively, highlighting the features of teacher talk on which PSTs agreed or disagreed.

Results

Table 1 shows the major themes found in the PSTs' evaluations of the sample teacher talk in Task 1.

Table 1:Aspects considered in evaluating the sample teacher-student talk

A. Interpersonal skill	A. Offering encouragement, compliments, reassurance (to provide emotional support)
B. Follow-up questions/prompts	B1. Asking for content-specific explanation (to press students to restate/ explain the math content)
	B2. Asking for procedure-specific explanation (to press students to recall mathematical procedures)
C. Format of questions/prompts	C1. Asking open-ended questions (to probe student thinking and make it explicit)
	C2. Asking close-ended questions (to get or guide to predetermined answers)
D. Language/word choices	D1. Using appropriate words/ expressions; non-mathematical in nature (to improve clarity/politeness)
	D2. Using correct math terms and concepts (to be mathematically precise)
E. Response to student work/question	E1. Offering clear directions (to avoid confusion)
	E2. Offering straightforward/ neutral responses (to avoid confusion, but not to be directive)
F. Information provided by teacher	E3. Confirming correctness of student work (to immediately confirm)
	F1. Refraining from talking or providing direct solutions (to encourage student's own thinking)
	F2. Offering detailed explanations (to offer more instruction and guidance)

Figure 1(a) shows the distribution of number of positive/negative comments on each teacher statement. T_4 and T_5 received the most comments from the PSTs, and the negative comments were the majority in the evaluation of these two teacher statements. While most statements received more negative comments than positive ones, there are several statements that received more positive comments than negative ones (e.g., T_3, T_7, and T_{11} for emotional support). Figure 1(b) shows the distribution of positive/negative comments per each theme. Theme A was most frequently used in the evaluative comments of the teacher statements. PSTs used this theme more to address positive aspects than negative aspects. Other themes used in positive evaluations also included C2 and E3. Most of the themes were more frequently used to negatively evaluate teacher statements, especially themes D1 and D2.

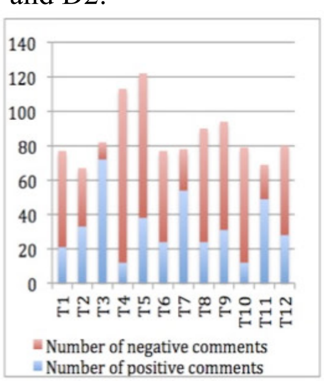

Figure 1: Frequencies per teacher statement

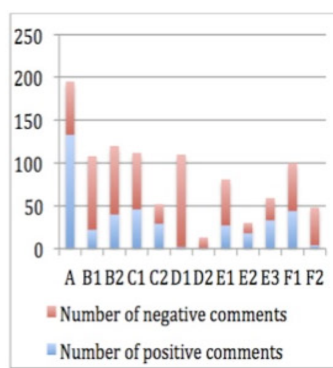

Figure 2: Frequencies per theme

There were several consenting improvements suggested in the group revision process (Results from Tasks 2). Those include the need to strengthen the following aspects: (a) checking for student prior knowledge, (b) increasing clarity/specifications, (c) asking for more explanations/justifications from the student, and (d) employing more positive tones. Several conflicts/disagreements noted in Task 2 were mainly focused on the following questions: (a) Who is responsible for learning? (b) What is the role of compliment? (c) When to stop probing questions in response to student struggles and confusions? (d) How to interpret students' mistakes?

Discussion and Implications

Teachers' Views of Mathematics and Roles in Effective Classroom Discourse

While the view of the teacher as a facilitator was a more supported view in Task 1, it was difficult to see the same pattern in Task 2. The small group discussion in Task 2 noted only minimal changes in the forms of teacher statement and also showed that even though PSTs shared similar views of mathematics and teacher role in mathematics communication, they revealed a wide range of differences on the quality and quantity of teacher talk and showed varied levels of struggles with deciding what should be the right amount of information or teacher assistance in specific contexts. One notable pattern in PSTs evaluations and modifications was that using vague languages were considered defects (e.g., $T_{4:}$ "Well sort of"). PSTs strongly revealed the 'teacher as a knower' view in this critique. The perception of "the traditional hierarchy of teacher as the autocratic knower and learner as the unknowing" (Fosnot, 2005, p. ix) seems to be still prevalent among the PSTs. Many PSTs noted that this kind of vague teacher statement would cause student confusion by implying confusion as a negative experience to be avoided.

Forms and Functions of Teacher Talk

Despite the acknowledgement of various forms/functions in Task 1, many PSTs did not fully demonstrate a deep understanding of the pedagogical moves, nor an ability to plan for effective alternative statements. Disagreements/conflicts within group discussions in Task 2 were usually raised by a small number of PSTs. This implies that although the PSTs have already been familiar with many features/forms of effective classroom talk and many have entertained the idea of facilitating effective classroom talk, they seemed to be unsure of what it actually looked like or how it functioned in context.

This study provides a window into the potential roadblocks for PSTs to shape the nature of their classroom talk. It is anticipated that the patterns identified in this study would provide teacher educators with insights into how they can better offer the opportunities to learn to teach. Reflection and critical discussion in the teacher education program that scaffold new understandings about the pedagogical knowledge teachers should have will eventually help PSTs develop the foundation for their professional career.

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BUILDING LEARNING OPPORTUNITIES: APPEALING TO CONVENTION AND COLLECTIVE MEMORY

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Using interactional analysis we examined the content of interactions during a whole group discussion in a 9th grade geometry classroom with the intent to identify the teachers' modes of intervention as he tried to engage students in criticizing two mathematical arguments raised in proving uniqueness of midpoint of a line segment. The teacher's multiple invitations for engaging students in identifying problems with the two arguments were not well received.

Keywords: Classroom Discourse; High School Education; Cognition; Reasoning and Proof

Introduction

It is widely agreed that peer interaction and group discussions can positively influence the growth of mathematical understanding among learners. A major interest in mathematics education community is describing modes of teacher intervention that allow for development of productive interactions leading to increased learning for all students. Analysis of teachers' practices according to the type of learning opportunities they attempt to create is far more useful than judging learning outcomes instantaneously, since time and space may be needed for learning to manifest.

In this work we examined the practices of a teacher and his attempts at structuring students' reflections on, and analysis of the validity of two arguments that were offered as proofs to a mathematical proposition. Two inter-related questions guided our analysis: What type of mathematical actions does the teacher try to enable in the course of his interventions during the large group interactions? What results from the teacher's interventions?

Literature Review and Theoretical Grounding

Kahveci and Imamoglu (2007) proposed that all instructional interactions have the purposes of increasing participation, developing communication, receiving feedback, enhancing elaboration and retention, supporting learner control and self-regulation, increasing motivation, team building, discovering, exploring, clarifying understanding, and/or closure (p.139). In synthesizing factors considered in studies concerning classroom interactions and mathematics learning the authors included: Structure of lessons, norms/context established and maintained for judging mathematical arguments, student motivation to learn, teacher goals and teacher knowledge, how the teacher interacts with students or leads discussion, and physical and social environment of the classroom. As it pertains to the activities of the teacher in the course of interactions, some studies have focused on unpacking teacher talk within the whole or small group settings (Cobb & Bausersfeld, 1995), while others have captured the strategies that teachers used (or did not use) or the knowledge bases from which they drew (or could draw) while doing so (Stein et al., 2008). Not all these reports though focus on the activity of teaching focused on supporting inquiry or conceptual development among learners. For instance, Sinclair (2005) reported on study conducted by Towers (1999) in which the investigator catalogued twelve intervention strategies used by teachers when interacting with students:(1) managing—carrying out administrative and disciplinary work, (2) checking—asking if the student understands, (3) enculturating—introducing students to terminology and processes used in the mathematics community, (4) reinforcing—stressing an idea, (5) clue-giving—deliberately pointing the student to the correct answer or path, (6) anticipating—trying to prevent the student from making a mistake, (7) blocking—stopping a student from following a particular path, (8) inviting—suggesting "a new and potentially fruitful avenue of exploration" (p. 201), (9) modeling—

providing an example of the teacher's own approach, (10) rug-pulling—deliberately introducing a puzzling idea, (11) retreating—walking away and allowing the student to think further, and (12) praising.(Sinclair 2005, page 90). According to Towers the strategies of "rug-pulling" and "inviting" were most effective in facilitating growth of understanding in mathematics (Sinclair, 2005, pg. 91). Nonetheless, Sinclair reported that the teachers in Towers' study were not particularly concerned about nurturing student centered learning.

Wagner (1994) defined identified two specific purposes associated with instructional interactions: to change learners and to move them toward an action state of goal attainment. (p.8). This spontaneous acting and reacting towards increasing understanding in mathematics classrooms is what Cobb and Bauersfeld (1995) characterized as co-learning, "a process of mutual adaptation wherein individuals negotiate meanings by continually modifying their interpretations" (p. 8). As such the teacher is recognized as a co-learner in the classroom enterprise. It is this perspective that guided our data analysis.

Methodology

The case on which we based our analysis for this report is grounded in data from one session in one 9th grade geometry classroom taught by Mr. Chris, a highly qualified teacher who stressed the importance of student-centered instruction and utilized it in his lessons daily. The session that became the focus of our analysis was a part of an instructional unit on logical reasoning and lasted approximately two weeks. During this session the students were asked to consider three propositions and to prove or disprove them (Diagonals of regular pentagons are of equal length; Angle bisectors of rectangles are concurrent, A line segment has only one midpoint). Approximately 20 minutes of the class time was devoted to small group work on the tasks. The whole group discussion of the first two propositions progressed quickly as all students had identified counter-examples to each of the two statements. The discussion of the third proposition however, became complicated due to the group's lack of willingness to conform to the teacher's goal of registering flaws in two arguments presented in class. The teacher's struggle to avoid direct instruction and the students' reluctance to comply to the teachers' requests provided a fruitful ground for careful analysis of interactions.

Data Analysis

Data Analysis consisted of three phases. First, an interactional diagram was constructed to capture the group's exchanges, depicting turns in discourse by the contributing members, highlighting episodes of "action-reaction" during the interactional event. Since the goal of our research was to catalogue the teacher's types of interventions during the whole group interactional episode as a starting point in our coding process we used Towers' (1995, cited in Sinclair) typology of interventional strategies. Particular Teacher's interventions that did not fit Towers' typology were identified accordingly. Lastly we considered the students' reactions to the comments that the teacher made so to trace how his comments influenced their thinking.

Results

The session included a total of 70 turns. Eleven of the 33 teacher utterances either *invited* students to *reflect on* arguments or aimed to *provok*e them to examine ideas more carefully. 8 of his comments challenged students' claims. 3 comments *reinforced* socio-mathematical norms of reasoning and thinking in his class. Two comments attempted to provide a structure for students' thinking as he asked them to recall specific events from previous class sessions. The remaining comments were either reflective of his own thinking or provided evidence of his reflecting on arguments raised in group. Here we also counted the teacher's prolonged silences as another interventional strategy since they were perceived as invitations for extended thinking.

Although the majority of the teacher's comments intended to gauge students towards identifying flaws in reasoning of peers his pleas were not well received. Throughout the whole group dialogues students reflected on each other's and their own ideas, provided additional explanations and refined their previously stated ideas. They referenced and verified knowledge and conventions by examining textbook definitions; they considered consequences of definitions and re-stated their ideas in presence of anomalous results from their search and yet throughout the episode students resisted the teacher's persistence that their peers' arguments be confronted.

Uniqueness of the midpoint

The whole group discussion began as one student (S1) presented her group's argument for why a line segment can have only one midpoint. Their method consisted of drawing 5 segments of varying lengths and then using a ruler they had marked the midpoint of each segment. The group then argued that since the midpoint moved with the segment then there was only one midpoint for each segment (S1). Although the teacher recognized that this mode of reasoning was not a proof he was unsure of how to respond to it during the group deliberations. During this segment the teacher's comments were primarily reflective (turns 2,3,4), disclosing his desire to learn more about how the students had resorted to this approach (turns 6), or his appreciation for the connections that the students had made to the work previously done in class (turns 8, 10). He also expressed his confusion about how to react to the argument and that he needed time to contemplate the idea (turn 13, 15). Indeed the first six minutes of the whole group discussion were devoted to the teacher seeking space to think about the proposed argument and in doing so he even appealed to the group for help (turn 10). Interestingly, despite the fact that the students did not object to the first group's argument they appeared to respect the teacher's need for thinking since a member of another group asked if they could present their method (turn 14). This delicate intervention resonated Towers' *retreating* strategy, though the strategy was enacted by the students, in reaction to their acknowledgement of the teacher's need for space to think.

During the second segment of the whole group discussion a second argument was offered: since in a triangle from each vertex only one median could be constructed and that since a median demanded a midpoint then it "proved" uniqueness of the midpoint. The teacher invited students to respond to the new argument while acknowledging that he was still thinking about the first group's reasoning (t 19, 21, 24) and yet the group defended the validity of their peers' arguments and expressed that they found both reasonable and convincing. The teacher was solitary in his quest to reveal flaws in reasoning and in doing so he attempted to provide learning opportunities for students to realize and share his mathematical goal. Indeed, among the 15 student utterances expressed during the second segment 9 were statement of peer approvals (t 18,20,21, 23, 26, 29). Rather than confronting the arguments of the two groups the teacher then resorted to a different strategy and asked if other students had used different approaches to proving the proposition (t 28). It was during the third segment of the episode that he became more explicit in his judgment of the arguments. He asked the students to consider specific assumptions that each group had made and in examining the assumptions he asked that they revisit definitions and conventions.

Refinement and Resistance

The third segment of the session marked the teacher's direct attempts at pointing out areas in at least one of the arguments that needed careful attention. He asked students to identify the specific assumptions that the second presenting group had made (turn 32). In response to this call, members of the first group offered further explanation for why their argument was true (turn 37). The additional explanations they provided indeed resembled proof by contradiction. Nonetheless, the discussion returned to whether uniqueness was a part of the definition of the median, as one student in defense of the second's group's argument stated (turn 38, 39). Pleading to the norms of practice in his classroom the teacher reminded students that in presence of even one disagreement then claims to definitions needed to be carefully verified and in doing so he reminded students that they must distinguish statements of definitions from consequences of those definitions (turn 51). Despite these efforts the group remains unsure of the teacher's expectation (turn 52, 53, 54, 56). Only one student responds to the teacher (turn 55) and yet his comment goes unnoticed by peers. During the last segment of the session, and as a final resort the teacher asked the students to recall what they had done in class the previous week and in doing so he appealed to their memory as a strategy for structuring their activities (turn 61, 64), which he ultimately used to engage the students in revisiting both arguments (turn 70). Students' responses to the teacher's comments indicated their awareness of what the teacher intended for them to consider as they stated specific memories they had attached to the mathematics discussed in class (turn 67, 67, 69).

Discussion

In the case we presented here, the teacher's desire to refrain from dictating what technique students may use to prove the proposition he had assigned certainly created an environment for students to think, reflect, and in some cases make connections and explore ideas they had not considered themselves. The learners were in tune with the arguments that their peers had raised and yet they were not responsive to the teacher's pleas for conforming to reason and convention; they were not alert to the mathematical performance they were expected to display. Their cognition was only shifted in presence of the teacher's plea to collective memory and remembering. Here, of course, we do not make a claim that the students came to identify theirs and their peers' arguments as less convincing or even flawed, only that the collective remembering provided them with a platform for understanding the teacher's expectations for their mathematical work. It remains unclear, considering the multitude of research on learners' difficulties with proofs and proving what modes of teacher interventions might be fruitful in advancing learning while supporting student autonomy. Studying classroom interactions using content as a lens for analysis of forms and types of interventions is highly needed.

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WHEN MATHEMATICS TEACHERS CONSIDER ACTING ON BEHALF OF THE DISCIPLINE, WHAT ASSUMPTIONS DO THEY MAKE?

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We discuss findings from a study of 226 K-12 mathematics teachers who rated the appropriateness of instructional actions in a scenario-based assessment. The teaching actions shown in the scenarios were designed to represent mathematics teachers fulfilling their obligation to the discipline of mathematics. Our analysis of the participants' follow-up responses provides insight into how other professional obligations interact with teachers' obligation to the discipline. We describe our method for detecting what we call conditional construals—assumptions participants regarded as critical for rating the appropriateness of instructional actions. In addition, we report preliminary findings about how those assumptions can be accounted for in terms of the professional obligations of mathematics teaching.

Keywords: Instructional Activities and Practices; Teacher Knowledge; Teacher Beliefs; Research Methods

Introduction

Scenario-based assessments are widely used to assess judgment in professional fields, such as medicine, police work, army tactics, and human resources (Weekley & Polyhardt, 2006). In this study, we analyzed a set of open-ended responses to scenario-based assessment items of K-12 mathematics teaching. Open responses from participating teachers followed forced-choice ratings of instructional actions shown within scenarios of K-12 mathematics classrooms. There were two parts to the analysis. The first part of the analysis involved validating a procedure for identifying when a response contained linguistic markers that indicated that the appropriateness of an instructional action was dependent on an element of the scenario (e.g., traits of the students, the amount of available time, the content that had been previously covered) that was unspecified in the representation that they viewed. We call such markers *conditional construals.* For the first part of the analysis, we analyzed responses from 226 teachers to 15 items (*n*=3405) designed to probe teachers' professional obligation to the discipline of mathematics (Herbst & Chazan, 2012). The second part of the analysis was a qualitative examination of the kinds of additional information participants stated they would need to know about a scenario in order to rate the appropriateness of an instructional action. The qualitative analysis examined the responses to 2 of the 15 items that had the highest incidence of responses that contained a conditional construal. In this paper we report on the results of the preliminary qualitative analysis of the responses to these 2 high construal items.

We looked for additional aspects of the represented scenario that teachers considered critical in deciding whether an action was appropriate. The following research questions guided our inquiry: 1) What types of assumptions do teachers make in order to justify instructional decisions that respond to a professional obligation to the discipline? 2) How do the different professional obligations support or constrain teachers' departures from "business as usual" in favor of their obligation to the discipline of mathematics? Our purpose with this study is to investigate the types of assumptions teachers make when considering instructional actions that respond to the professional obligation to the discipline of mathematics.

Theoretical Framework for Research

Herbst and Chazan's (2012) account of the *practical rationality of mathematics teaching*attempts to synthesize rational actor theories of teacher decision making (e.g., Calderhead, 1996; Schoenfeld, 2010) with accounts of teaching as a sociocultural activity (Stigler & Hiebert, 1999). Herbst and Chazan (2012) propose that while teachers have personal resources such as knowledge and beliefs, teachers also play roles in activity systems (e.g., algebra instruction) that have customary norms (Much & Shweder, 1978). Within such activity systems, teachers are accountable to four professional obligations—to the discipline of mathematics, to individual students, to the classroom community, and to the institutions of schooling (Herbst & Chazan, 2012). While actions following customary norms may be enacted without reflection, a teacher's deviations from those norms, whether motivated by individual resources or in response to social demands, need justification (Buchmann, 1986). Herbst and Chazan (2012) argue that the four obligations provide a basis for the sources of professional justification. For example, while the obligation to the discipline of mathematics would require a teacher to limit the range of symbolic strings that she allows herself to call *equations* (e.g., $2x = 5$ is an equation but $3x -7$ is not an equation), this same obligation could also justify her calling *equation* a symbolic string like $2^x = x^2$ in spite of the fact that the beginner's algebra curriculum might not include it under the equations that algebra students learn to solve (Herbst & Chazan, 2012).

While much work has been carried out to establish links between teachers' beliefs and conceptions about mathematics and instructional practices (Skott, 2001; Stipek et al., 2001), the notion of *practical rationalit*y suggests that teachers' decision-making is not driven by individual resources alone. Rather, teachers' decisions are context-dependent and a fuller understanding of the resources that they can employ for departing from normative practice can be achieved through the use of instruments that represent the situated nature of practice. As such, we approach the work of understanding teachers' decision making by presenting them with scenarios of instruction in which the represented teacher departs from instructional norms in order to act on behalf of the professional obligation to the discipline.

Methodology

We conducted our study on data collected from K-12 inservice mathematics teachers with a 15 item scenario-based instrument designed to measure recognition of teachers' professional obligation to the discipline of mathematics. Within each item, participants examined a scenario in which a teacher departs from an instructional norm in order to attend to one of the professional obligations. Participants were asked to rate the extent of their agreement (using a 6-point Likert-like response format from strongly disagree to strongly agree) with the teacher's action. Participants were then prompted to comment on their ratings. In order to identify conditional construals in these responses, we located *circumstances of contingency* (Halliday & Matthiessen, 2004, p. 271) which are accompanied by linguistic markers such as "depending on", "as long as", and "assuming". In particular, we examined the nominal group introduced by the circumstances of contingency. These nominal groups are usually "a noun denoting an entity whose existence is conditional, a noun denoting an event that might eventuate, or a nominalization denoting a reified process or quality" (Halliday & Matthiessen, 2004, p. 272). We present an example with the following response, "This depends on timing. I think it can be valuable information for students to see common errors played out in a solution, but time will dictate whether or not that can happen in any given scenario." We argue that "timing" is marked here as a circumstance of contingency by the prepositional phrase "this depends on". We then categorized those circumstances by professional obligation. In this example, we placed the conditional construal in the *institutional* category because the participant indicates that the appropriateness of the action is contingent on the amount of time available, and time constraints are imposed by the institution.

Findings

We found that teachers consider the other obligations—to the students as individuals, to the classroom community, and to the institution—as they consider whether to respond to their professional obligation to the discipline of mathematics. We illustrate these categories of conditional construals by sharing teachers' responses from the two items with the highest number of responses that contained conditional construals. The analysis reported below qualitatively assesses the content of these conditional construals, with respect to the professional obligations.

In the first item, participants view a scenario in which the teacher announces that students seem to be on the right track and, rather than spend time on practice problems (which we hypothesize would be the normative action in such a situation), the teacher decides to discuss mathematical theory related to the topic at hand. Participants were asked to rate the statement: "The teacher should give students additional practice problems, rather than elaborate on mathematical theory." Of the 226 participant responses, 16% indicated that their decision depended on assumptions they had made about the scenario. In the second item, participants viewed a scenario in which a student makes a mistake while working a problem at the board; rather than point out the mistake directly (which we hypothesize would be the normative action in such a situation), the teacher allows the student to build on the mistake. Such an action could be justifiable on the disciplinary grounds that the teacher is creating an opportunity for a contradiction or anomaly to arise from mathematical reasoning that builds on the mistake. Participants were asked to rate the statement: "The teacher should correct the mistake observed, rather than ask students to build on mistaken work," and we found that 7% of the participants indicated that their decision depended on assumptions they made about the scenario.

These items highlight the variety of factors that teachers consider in order to make instructional decisions that respond to the disciplinary obligation and demonstrate how the role of the other professional obligations can vary depending on the scenario. For example, while there were 30 participants who referenced the individual obligation in their responses to the first item, the individual obligation was used in different ways. One perspective suggested that knowing the "level" of the class is essential for determining whether or not a particular instructional action is appropriate; another perspective suggested that it was of primary importance to determine whether students had been taught relevant information. While these are both manifestations of the obligation to the individual student, they have different implications for how the teacher perceives the agency of the students and the responsibility of the teacher. Such differences manifested themselves across items as well: In the case of the first item, there were 30 references to the individual obligation, but there were only 9 references to the individual obligation for the second item. Ongoing analysis is being to conducted to perceive patterns in the way each of the obligations are manifested within participants' construals.

Study's Significance

Mathematics instruction is beholden to the discipline of mathematics as the source of legitimacy of the knowledge at stake in the classroom as well as the practices that govern the development of that knowledge. Nonetheless, there is often a tension between the type of instruction advocated by mathematicians and mathematics educators and the everyday reality of K-12 education (Schoenfeld, 2004). We have demonstrated how we have been able to collect a corpus of data from K-12 teachers of mathematics describing the circumstances that dictate whether they approve of a fellow teacher departing from customary classroom practice in order to respond to their perceived professional obligation to the discipline of mathematics. Our analysis of that corpus suggests that, for these teachers, their professional obligation to the individual students (e.g. students' understanding), to the discipline of mathematics (e.g. mathematical value), to the classroom community (e.g. class norms), and to the institution of school (e.g. time) all have a bearing on their approval of such decisions. It remains an open question for us whether individual differences account for part of the variance in participants' conditional construals. To explore this, we could examine how proxies for such differences, such as mathematical knowledge for teaching and experience, are related to the ways that they see the professional obligations impinging on their practice.

While surveys of teachers' beliefs about mathematics provide one way of conceptualizing how the discipline of mathematics may influence instructional decisions, our work presents a different approach by suggesting that there are aspects of the professional position (for example, the need to teach a given curriculum, a limited allowance of time, the characteristics of the class being taught, the various characteristics and needs of individuals one is assigned to teach) that may mediate a teachers' willingness to respond to the discipline regardless of their beliefs about the subject. As a better model is developed of the way that the professional obligations interact both with one another and with personal resources of the mathematics teacher, and as the different aspects of the professional obligations themselves are further analyzed as we have begun to do here, we will be closer to being able to understand the hidden factors that obstruct or facilitate the take-up of instructional practices that depart from the norm in order to align with the discipline of mathematics.

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TRACING STUDENTS' ACCOUNTABILITY AND EMPOWERMENT IN AN ONLINE SYNCHRONOUS ENVIRONMENT

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This study traced students' chat and whiteboard interactions in an online, synchronous environment as they collaborated to solve cognitively demanding, open-ended mathematical problems. The objective was to document the emergence of discourse that showed accountability to a community of practice when doing thoughtful mathematics. Findings also revealed that the students transitioned into empowering stances about their ability to think mathematically. The pedagogic conditions under which the students worked and how those influenced students' development as a learning community are also discussed.

Keywords: Classroom Discourse; Problem Solving; Technology

Purpose

As Ernest (2002) indicates, there is a need to research teaching practices that encourage student accountability and empowerment in mathematics education. With the increased demand for online learning, it is critical to understand how these attributes can be fostered in online synchronous environments (Stahl, 2009). The online learning environment can be designed for students to engage in the mathematics on their own terms. Yet issues and challenges may arise as students figure out how to collaborate and, ideally, form a learning community. The purpose of this study was to engage students from diverse backgrounds in doing thoughtful mathematics by working to solve open-ended problems and justify their approaches and solutions. Through capturing students' chat and whiteboard representations, researchers traced their discursive interactions and mathematical reasoning to seek evidence of empowerment and accountability.

Theoretical Perspective

This study is rooted in sociocultural theory that highlights the meditative role of social interaction in the development of individual knowledge (Vygotsky, 1978).This perspective acknowledges that learners actively construct knowledge and that their knowledge is constantly modified by their social interactions. It is further posited that (a) students need to be engaged to support the transfer of knowledge; (b) engagement is best accomplished through discourse and a sense of belonging to a community of learners; and (c) this can be an empowering experience for students. Through practicing mathematics collaboratively, students increase their own accountability, and that leads to a strengthened empowerment over the subject of mathematics.

Michaels, O'Connor and Resnick (2008) propose three forms of accountability in the classroom: accountability to the learning community, accountability to accepted standards of reasoning, and accountability to knowledge. Their research also draws on constructivist and sociocultural principles that emphasize the importance of learning through discussion. In accountability to the learning community, participants listen to each another, build on each another's ideas, and question each other to expand upon their own individual ideas. Accountability to accepted standards of reasoning emphasizes logical connections and the drawing of logical conclusions that occur during interactive reasoning and discussion. The third and most complex form, accountability to the knowledge, depends upon the accurate use of facts and knowledge in the discussion. One focus of this study was on the students' accountability to a learning community as they shifted away from individual, competitive work and more towards a collaborative group sharing of mathematical ideas.

The second focus, empowerment, can be divided into two domains: mathematical and epistemological. Mathematical empowerment is a social process of acquiring facility in talking about mathematics, using its specialized language, and gaining proficiency in using and applying mathematics (Ernest, 2002). Mathematical empowerment occurs over time as a result of social interactions and achievement in a mathematics classroom. Epistemological empowerment concerns the individual's growth in confidence: the development of a personal identity so as to become a more personally empowered person. As Ernest (2002) contends, students cannot be autonomous learners or think critically without first becoming empowered. To foster student empowerment, an environment must first allow them to overcome internal inhibitions and perceptions of inadequacy. Classroom practices can facilitate the process of encouraging students to assume responsibility for their learning (Stinson, 2004).

Methods

The data for this report is a subset of a larger data corpus. Total participants consisted of 17 seniors from two different high schools in New Jersey. Eight of the students were from an urban public high school, and the remaining nine students were from a suburban private high school. Twice a month, during their regular class time, they participated in the research project by solving mathematical tasks in the online environment, called Virtual Math Teams (VMT). The VMT project was a NSF-funded research program designed to investigate sustained collaborative problem solving in a computer-supported environment. Its two main interactive components are a shared drawing board, referred to as a workspace, and a chat window. All VMT sessions were observed while the students worked on tasks, as well as afterward, and chat discussions were downloaded for coding. Students worked on open-ended combinatorial tasks not seen in their traditional mathematics classroom.

Online chat discussions were coded for accountability to their learning community and mathematical and epistemological empowerment. Coding identified the nature of the chat in order to observe whether there were shifts in their behavior during the six sessions. Chat statements that built on the efforts of the group as they solved the mathematical task were coded as accountable to the learning community. Examples include (a) encouraging one another to join or continue efforts in the problem task; (b) agreeing with the current approach; and (c) asking clarifying questions that show positive participation towards the group effort of solving the mathematical task. Coding for epistemological and mathematical empowerment highlighted moments that displayed the individual's growth in confidence within the group as they solved the mathematical tasks. Instances of showing epistemological empowerment include: (d) taking the lead, (e) redirecting, and (f) posting an answer to the chat and/or workspace. Mathematical empowerment (g) was evidenced during the creation or validation of the knowledge.

Results

This study focused on analyzing data from Team Two, which consisted of four male students: Pedro, JohnC, John, and Jordan (not their actual names). The following narrative reports online interactions among Team Two and how they evolved over six sessions. During the opening 12 minutes of the first session, the students socialized and questioned their virtual teammates about gender, sports, and music. Pedro initiated the lead by encouraging his teammates to shift from their initial socializing and begin the task. There were statements of apprehension from each member of the team as they commented that they were unsure how to proceed with the mathematics. Jordan persuaded Pedro to enter his work onto the whiteboard, claiming they he and John had an idea but would like to see Pedro's solution strategy first. Pedro submitted the first solution onto the whiteboard and to that, John responded, "I had the same idea but was doing it on paper b/c I didn't know how to put it on the computer." John then immediately submitted his own solution, which contained an error, directly to the final solution tab and requested that Pedro write up the summary. Pedro complied with the directive. The team accepted John's final solution and did not question why it was different from Pedro's solution. JohnC's only contribution was several apprehensive comments in the beginning about how to proceed but he did not engage in any of the problem solving. Jordan was the last to log off, thanking Pedro for writing up the summary. At first it appeared that Pedro was the leader and everyone was extending him a turn to show his work. It quickly changed when John posted his work and positioned himself as the one who provided the solution. Jordan encouraged the team to produce solutions but did not submit any work of his own. For three teammates, the session ended on a positive note, however, for JohnC, the session might not have been as positive. Not only did he not engage in any of the problem solving, he drew pictures on the whiteboard that were erased by his teammates without discussion, and he logged off several minutes before the session ended.

As the sessions progressed, routines and rituals emerged that were similar to the first session. Pedro often began his chat statements with the words, "we," and "let's," as he encouraged the team to work together. He was tenacious in his desire to always contribute a solution; however, he rarely stated that he was sure of his strategy. Most of his statements began with, "I think this is it," or, "I am not sure." John initially did not consider his teammates' solutions or strategies. He would show his frustration by entering one word at a time into the chat log, commanding his teammates to agree with his solution to the task. However, as the sessions progressed, he began to display evidence of accountability by asking whether his teammates agreed with his solutions and, in the sixth session, he considered a solution other than his own. Jordan's chat entries were not frequent, yet the content of his contributions were pivotal to the community coming together. At first his accountability was exclusively slanted towards his co-located teammate John. Yet, as the sessions progressed, Jordan began to value solutions other than John's and expanded his notion of community to the four teammates. In contrast, JohnC generally entered negative comments and used the shared whiteboard to occasionally doodle. The other team members allowed him to stay adrift, and erased his doodle without mention. Yet in the fourth session, JohnC tried to work on the task and argue his point contrary to John's solution. However, JohnC's solution was never accepted and he appeared to harbor anger from this point forward.

A very interesting exchange occurred during the last session as John assigned tasks to Jordan and Pedro. Jordan's task was to tally John's listed items as he entered them into the chat and Pedro's task was to re-enter them into the workspace. The process was seamless as the three boys worked, however, two times John entered a duplicate into the chat, which was counted by Jordan, but not transferred to the workspace by Pedro. He intentionally left out the duplicates thus correcting John's mistakes. He did not at any time mention his corrections or his teammates error. Interpretation of these results is discussed below.

Discussion

Accountability to community means that group members listen to each other, build on each another's ideas, and question each other to expand on their own individual ideas (Michaels et al, 2008). Pedro displayed a strong sense of accountability to the community from the first session to the last. He continuously encouraged the team to work together and solve the tasks assigned to them. John and Jordan's accountability to community evolved during the study, in the sense that they expanded their definition of community. John displayed a modest shift in his accountability as he eventually considered solutions other than his own and Jordan expanded his view of the team from himself and John to a team of four. JohnC struggled with his accountability to community, notably supported by evidence of his involvement being oppressed by the other members. It appeared that lack of contribution was construed as inattentive by his teammates.

Sharing mathematical ideas is an empowering experience and from this empowerment, a student's mathematical identity is strengthened (Ernest, 2002). Pedro's and John's mathematical empowerment was evidenced in their submitted solutions. Initially each student created his own solution in the workspace and neither discussed nor justified their reasoning. However, in the last session, both used the heuristic of controlling for variable to create two similar lists that were better organized, and each were able to follow the others' work because of the organization.

Epistemological empowerment is the individual's growth in confidence in using, creating and validating mathematical knowledge (Ernest, 2002). Pedro's epistemological growth was evidenced when he was confident enough to fix John's solution without mention. John epistemological shift was most poignant as he developed a sense of himself in relation to the group as he began to value other members of the team. His epistemological growth is evidenced in how he managed and improved in his leadership skills. Team Two went from a group of four boys individually solving combinatorial tasks to a cohesive team under John's leadership.

In summary, the analysis revealed that the students displayed accountability to their learning community, as well as epistemological and mathematical empowerment, as they engaged in thoughtful mathematics in an online synchronous environment. As they worked in a small group to solve open-ended mathematics problems, they shifted away from individual, competitive work to more group cooperation and collaboration. They developed their own unique practice, expectations of one another, and most importantly, how to work together as a unique learning community. The data also indicated that this accountability was an empowering experience for the participants, one that may affect their relationship with the subject of mathematics. This study indicated that the environment provideda virtual and empowering forum for diverse students to construct mathematical ideas and reason collaboratively, even when remotely located.

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SUPPORTING PEER CONFERENCES IN INTRODUCTORY CALCULUS

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This paper focuses on the nature of student talk during peer conferences about calculus problems. Conversations were studied in the context of Peer-Assisted Reflection (PAR), an activity structure that supports communication and conceptual understanding through peer assessment. Despite a wealth of research on peer assessment, relatively little has been published on the specifics of students' conversations as they discuss each other's work. This paper introduces a coding scheme for analyzing such conversations, and applies it to illustrate the impact of a systematic training procedure on improving student conversations.

Keywords: Classroom Discourse; Design Experiments; Metacognition; PostSecondary Education

Calculus is an area of persistent challenge for students pursuing STEM careers (Bressoud, Carlson, Mesa, & Rasmussen, 2013). This paper focuses on student conferences in Peer-Assisted Reflection (PAR), a promising activity for improving student outcomes; PAR increased pass rates (completing the course with a C or higher) by 13% and 23% during two phases of study (Reinholz, in press-a, in press-b). PAR is built around peer assessment, a core part of formative assessment (Black & Wiliam, 2009). Broadly, formative assessment is focused on eliciting information about student thinking and using it to modify learning activities, which improves student outcomes (e.g., Black, Harrison, & Lee, 2003). In peer assessment, students collaborate and explain their reasoning to each other, developing self-assessment skills as otherwise invisible assessment processes become more explicit and transparent (Reinholz, in press-b).

Despite considerable work on peer assessment (e.g., Falchikov & Goldfinch, 2000), little is published on student talk during peer conferences. As a result, it is difficult to design to improve such discussions, because they are not well understood. This paper focuses on student talk during two phases of a design experiment (Cobb, Confrey, Disessa, Lehrer, & Schauble, 2003), exploring the impact of systematic training included in Phase II. It addresses two questions:

- What was the focus of student conferences on calculus problems?
- What was the impact of systematic training on student conferences?

Method

Design

Each week, students completed a challenging PAR problem as a part of their homework (14 problems total). After completing an initial solution, students came to class and exchanged their work with a peer. Students read each other's work silently for five minutes before discussing it together for five more minutes. After conferencing, students revised their work and turned in a final solution. This cycle of activities was implemented during both Phase I and Phase II.

During Phase II, a systematic training procedure was added. Each week, the instructor led a whole-class discussion in which students analyzed three sample solutions to a part of the PAR problem and discussed how to improve them; these discussions focused on how the solutions explained and communicated the mathematical concepts of the PAR problem.

This paper focuses on conferences from three PAR tasks: PAR06, PAR10, and PAR14. These tasks were chosen to span the duration of the semester. These tasks had: a low floor and high ceiling, multiple solution paths, and required explanation (cf. Schoenfeld, 1991); PAR6 explored the difference between radians and degrees with sine and cosine functions, PAR10 focused on the approximation of complex areas using simple shapes (as a precursor to Riemann sums), and PAR14 involved creating a bead as a solid of revolution (the napkin ring problem).

Participants and Data Collection

Students in a semester-long introductory calculus course at a research university attended four 50-minute class periods each week. Phase I (409 students) took place in the fall, while Phase II (336 students) took place in the subsequent spring semester. There were a total of ten parallel sections each semester (taught by 8-9 different instructors) with a common curriculum. PAR was implemented in a single experimental section each semester; this paper focuses on only these two sections (not the comparison sections). Student PAR assignments were collected and scanned, and 4-7 student dyads were randomly chosen each week to be audio recorded during their peer conferences. A total of 155 conferences were recorded (66 during Phase I, 89 during Phase II).

Analysis

A total of 44 conferences were transcribed and de-identified. The transcripts were coded randomly to avoid systematic bias across tasks or phases of study. Each conversation was coded by assigning each sentence of talk to one of seven dimensions. Using a single sentence as a unit of analysis, it was possible to assign each sentence uniquely to a single category.

The categories were: (1) communication (focused on how mathematical ideas were expressed), (2) comparison (of different solutions or multiple parts of the same solution), (3) concepts (the underlying mathematics of the problem), (4) procedures (computational fluency), (5) task (clarifying the parameters of the task), (6) other (mathematical talk in none of the above categories), and (7) unrelated (talk not related to the problem). These categories were drawn from the framing of PAR (asking students to talk about communication and correctness), the nature of the tasks (focused on multiple solutions, allowing for comparison), and an iterative process of working with transcripts to develop a minimal set of codes. Because students often introduced themselves to start a conversation and said "thank you" to end the conversation, "unrelated" talk from the beginning and end of conversations was not analyzed. This also meant that off-topic talk that took place after students finished their discussion was not analyzed.

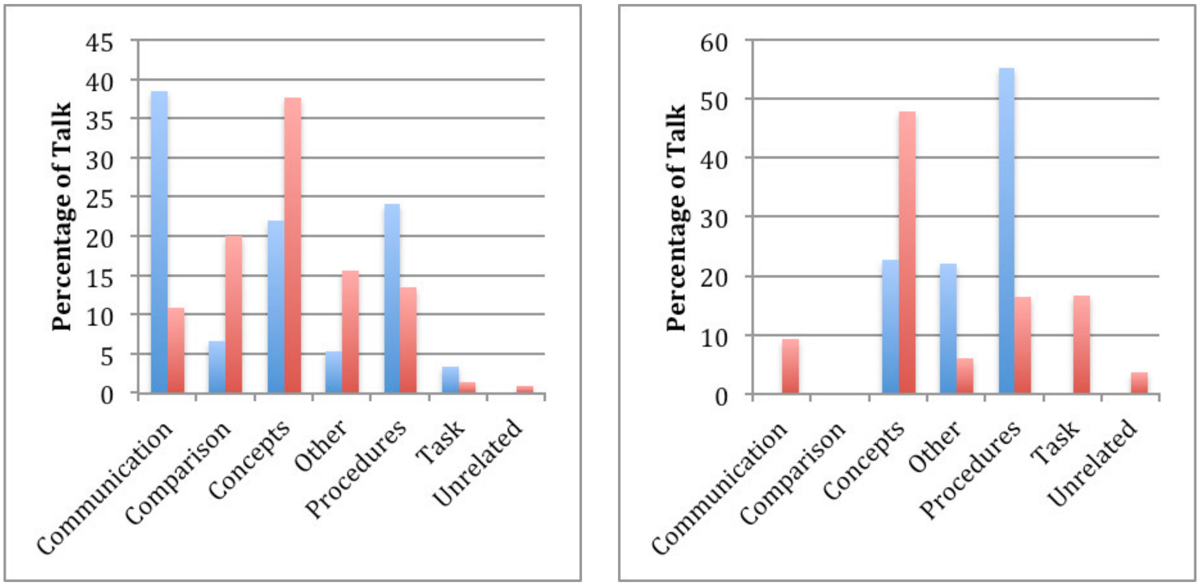
Results

Student conferences during Phase II were more than twice as long as those during Phase I (see Table 1). Conversations were also analyzed for percentage of on-topic talk (categories 1-6); overall, 98% of talk during Phase I and 97.8% of talk during Phase II was categorized as on-topic. This indicates that students during Phase II spent more than twice as much time as the Phase I students discussing the mathematics of the PAR problems.

Table 1. Length of Conversations (Number of Sentences)

	Phase I	Phase II
PAR06	18.8	31.9
PAR10	12.8	43.7
PAR14	17.0	44.8

Figure 1 gives the breakdown of talk for PAR06 and PAR14, averaged over all student conversations (PAR10 omitted due to space). During Phase II, students focused more on concepts, while Phase I students focused more on procedural computations. Nevertheless, students during both phases used peer conferences to deepen their understanding in a variety of productive ways,



PAR 06 PAR 14

Figure 1. Breakdown of Talk (Phase I in Blue, Phase II in Red)

connecting their solutions, improving their explanations, and discussing mathematical procedures and concepts.

Typical Phase I and Phase II discussions for PAR14 are given in Figure 2. Most of the Phase I discussions focused on computations involving solids of revolution (4 of 5 conversations):

> Ben: This is where I screwed up. You're going to want to substitute in 1/2 h, that quantity squared…like 1/2 h square root equals that. So when I kept subbing it kept screwing up. I kept on putting in 1/2 h to the 3/2, but it's gotta be this quantity squared in that.
>
> Sam: Gotcha.

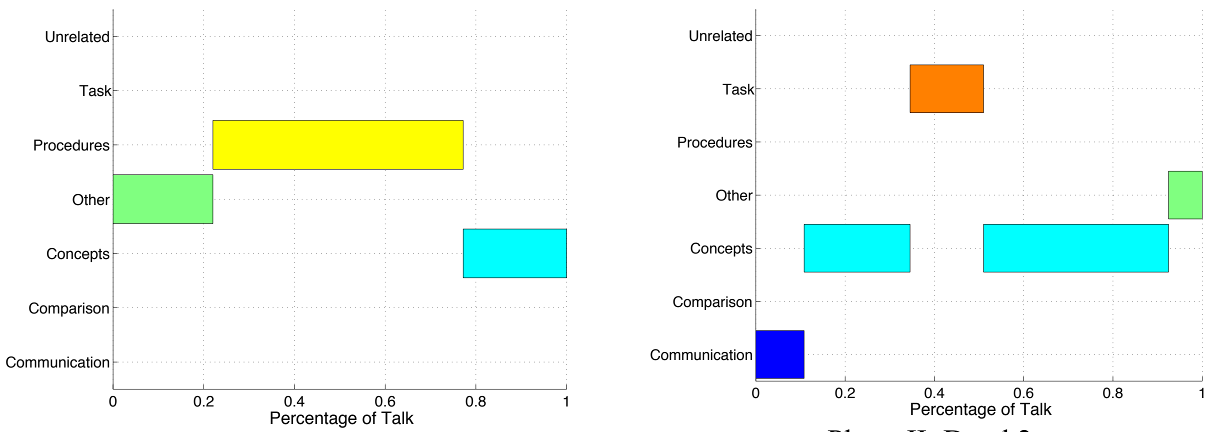
In contrast, Phase II students focused more on concepts. Moreover, while discussing concepts in the problem, Dyad 2 realized that they were confused and needed to clarify the nature of the task. While Tim was discussing the bounds for their integral, Bethany realized that she was not sure how to read the diagram in the problem:

> Tim: So your *x* bounds are *r* and big *R*. Uh, integral of that and then just the 2 times pi…
>
> Bethany: Wait, I'm confused; can I ask you about this thing? Is this saying that *h* is this whole thing?

The discussion continued on and the students resolved the confusion, and returned to discussing concepts. Returning to the task in this way mirrors how skilled problem solvers may return to a problem statement and reevaluate their approaches in the middle of problem solving (Schoenfeld, 1985). During Phase II, only 2 of 6 conversations were mostly procedural.

Discussion

This paper illustrates an approach to analyzing peer conferences and reports preliminary findings about the nature of student conversations. Students discussed a variety of aspects of the problems, such as communication, the nature of the task, procedures, and concepts. Moreover, the results suggest that the systematic training activities had a considerable impact on student discussions. Phase II students discussed more than twice as much mathematics as the Phase I students, and the nature of the conversations was qualitatively different; the students focused more on concepts than procedures.



Phase I: Dyad 1 Phase II: Dyad 2

Figure 2. Comparison of Pairs (PAR14)

This shift is consistent with the nature of the training exercises that focused on discussing concepts. This shift may also be due in part to the deeper understanding that the Phase II students developed (Reinholz, in press-a), which allowed them to move beyond the surface aspects of the problems. Future work will focus on establishing reliability with multiple coders and analyzing all of the PAR problems in more depth.

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QUALITY INSTRUCTION, TEACHERS' SELF-EFFICAY, AND STUDENT MATH ACHIEVEMENT IN KOREA AND THE UNITED STATES

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This study compares and contrasts student, teacher, and school factors associated with student mathematics achievement in Korea and the United States, using data from the Trends in International Mathematics and Science Study (TIMSS) 2011. We examine factors linked to teachers who deliver quality instruction with high self-efficacy the association between teachers with high-quality instruction and high self-efficacy and 8th graders' math achievement. Years of teaching experience and several student background characteristics such as parent's level of education and the number of books at home were associated with achievement in both countries. In addition, teacher's perceived academic emphasis was associated with teachers who claimed to provide high-quality math instruction with high self-efficacy. However, the two countries' results differed in the effect of teacher's opportunities to learn in professional development programs on teachers.

Keywords: Teacher Knowledge; Affect and Beliefs

Introduction

Students' mathematics achievement is often associated with the future economic power and competitiveness of a country. Accordingly understanding and identifying factors that may have meaningful and consistent relationship with mathematics achievement has been of great interest to policy makers and educators in the United States but also all around the world including Korea. Researchers have identified multiple variables that have an impact on student learning, including family background and socio-economic status (e.g., Bress & Mirazchiyski, 2010), peer influence (e.g., Guiso et al., 2008), gender differences (e.g., Else-Quest et al., 2010), and personal characteristics (e.g., Chiu & Klassen, 2010). However, growing evidence shows that teacher quality is a crucial contributor to student learning which in turn can lead to the development of highly knowledgeable and skilled workers. Effective teachers have the intersection of three important components including teacher knowledge, pedagogical skills, and dispositions resulting in a strong sense of self-efficacy.

In this study, we hypothesize that teacher's self-efficacy is a basis for creating quality instruction where teachers promote students' higher order thinking in mathematics. Based on the data collected in TIMSS 2011 (eighth-graders), we examine the factors that are associated with mathematics achievement of secondary students in Korea and the United States. In doing so, we first examine the factors that distinguish teachers who provide quality instruction with high self-efficacy. According to Cochran-Smith and Zeichner (2005), effective teachers need to provide not only cognitively demanding instruction but also possess high self-efficacy. Thus we first examine whether such teacher characteristics is a significant factor associated with student achievement. The factors that distinguish teachers who provide quality instruction with higher self-efficacy from other teachers are explored. Then, we examine the association between other teacher, school, and student factors that are related to students' mathematics achievement. In particular, by comparing and contrasting student, teacher, and school factors that are associated with student mathematics achievement in Korea and the United States, we intend to provide suggestions to mathematics education in both Korea and the United States. Research questions that guide this study are: (1) What factors are associated with teachers who provide quality instruction with high self-efficacy (i.e., teacher math instruction and self-efficacy typology) in the United States and Korea? and (2) To what extent teacher mathematics instruction and self-efficacy typology is associated with students' math scores, after controlling for student backgrounds, other teacher factors, and school contexts in the United States and Korea?

Method

This study used data from TIMSS 2011. Conducted on a regular 4-year cycle, TIMSS has assessed mathematics and science at the fourth and eighth grades since 1995. This study focused eighth graders, their mathematics teachers, classrooms, and schools in the United States and Korea. Note that total sample sizes are, 10,445 students, and 537 teachers in the US, 5,170 students, and 375 teachers in Korea, respectively. Figure 2 shows our conceptual framework for the study.

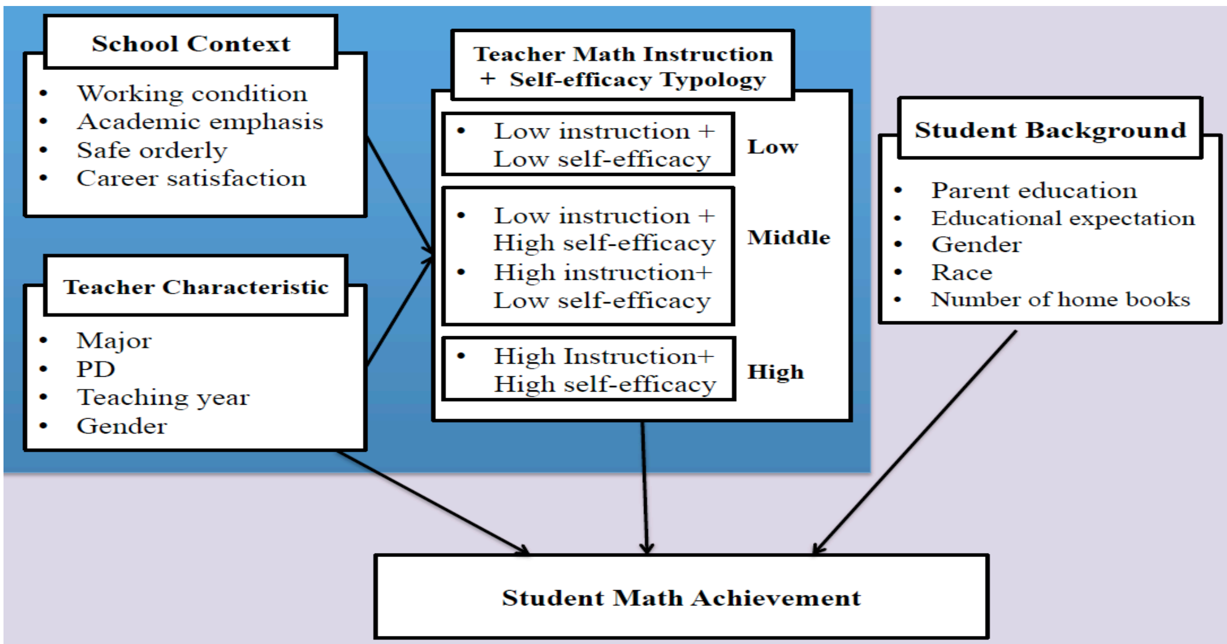

Fig 2. Conceptual basis for the study

The outcome measure for this study is students' math achievement scores. Teacher mathematics instruction and self-efficacy typology is selected as an independent variable for the study. Teacher background variables such as a teacher's major, gender, years of experience; student background variable including gender, race, number of books at home, and educational expectations; and school factors based on teachers' perceived collaboration and their perceived academic emphasis are selected as control variables for the study. We tried to input same background variables into both of two county models, however, two variables, race and teacher major, were considered differently. First, in the analysis of Korea, race variable was excluded because Korea is well-known as ethnically homogenous society (Kang, 2010). Second, when it comes to teacher major, four categories are used to represent US teachers whereas two categories are used for Korean mathematics teachers. For example, in the United States model, teacher major was classified into four categories including (1) no major in education and math; reference group, (2) major in education but math, (3) major in math but education, and (4) major in both education and math. In contrast, Korean teachers were classified into two categories—(1) Mathematics major or non-mathematics major. Because there are only few sample in "no major in education", and "major in both education and math" in Korea, we combined the cases. This study employed multiple imputations (MI) due to missing data on teacher and student background variables.

Analytic strategy

To answer the first research question, we first created three different types of math teacher typology; (1) a teacher with high-quality instruction and high self-efficacy (High group), (2) a teacher with high-quality instruction but low self-efficacy or low-quality instruction but high self-efficacy (Middle group), and (3) a teacher with low-quality instruction and low self-efficacy (Low group). The measure of math instruction is derived from items asking "*in teaching mathematics.* Next, we classified teachers into two groups: (1) above the median as teachers with high-quality instruction and (2) below the median as teachers with low-quality instruction. 49.6% and 50.4% of teachers are categorized as "low instruction teacher" and "high instruction teacher", respectively. The measure of teacher self-efficacy is derived from items asking *in teaching mathematics to this class, how confident do you feel to do the following;* (1) *show students a variety of problem solving strategies* (2) *provide challenging tasks for capable students* (3) *adapt my teaching to engage students` interest* (4) *help students appreciate the value of learning mathematics*." We summed teachers' responses on these 4 items and then classified teachers into two groups: (1) a teacher who reports the highest value for all 4 questions as a teacher with high self-efficacy and (2) a teacher reports all other values as a teacher with low self-efficacy. Next, multinomial logistic regressions were used to examine the factors that are associated with teachers who provide the quality instruction with high self-efficacy (i.e., teacher math instruction and self-efficacy typology). To the second research question concerning the degree to which teacher math instruction and self-efficacy is associated with students' math scores, after controlling for student backgrounds and other teacher factors and school contexts, we conducted OLS regression analyses. According to the findings from research question 2, the third research question was interpreted to provide implications to mathematics education in United States and Korea.

Summary of Results

Factors associated with teacher math instruction and self-efficacy typology

We found that several factors are associated with teachers' mathematics instruction and self-efficacy typology. In Korea, teachers' educational backgrounds are not significantly associated with teachers who deliver high-quality math instruction with high self-efficacy (high group), whereas in the United States data, teachers who majored in "education and mathematics" are about 2.4 times likely to be classified into the "high group" compared to teachers who did not major in education or mathematics. We also found that several school contexts including perceived academic emphasis and perceived safe orderly are positively associated with teacher math instruction and self-efficacy typology in the US data. One unit increase in the perceived academic emphasis index (e.g., from medium to high emphasis or from high to very high emphasis) is associated with a 2 times likely to be teachers who deliver high-quality math instruction with high self-efficacy in the United States. In addition, PD opportunities are found to significantly distinguish teachers in the middle group (i.e., a teacher with high-quality instruction but low self-efficacy or low-quality instruction but high self-efficacy) from teachers in the low group. This finding suggest that teachers in both high and middle groups tend to report that they participated in PD opportunities focusing on curriculum and inquiry methods more often that teachers in the low group who provide cognitively low demanding instruction with low self-efficacy.

Factors associated with 8th graders' mathematics achievement

There is the significant positive association between teacher math instruction and self-efficacy typology and student math achievement in the United States, even after controlling for other student, teacher, and school backgrounds. Nonetheless, we found no significant relationship between teacher math instruction and self-efficacy typology and student math achievement in Korea. Specifically, only US students who taught by high group teachers (i.e., teachers with high-quality math instruction and high self-efficacy) have 10.26 higher math achievements scores than students who are taught by teachers with low quality of math instruction and low self-efficacy. Interestingly, no significant difference in mathematics achievement was observed between students who taught by "high group teachers" and those with "low group teachers" in Korea. In addition to teacher's cognitively demanding instruction with high self-efficacy, teachers' year of teaching experience, their perceived academic emphasis, and their major were found to be significantly associated with students' mathematics achievement in the US context. Furthermore, we also found that several individual-level characteristics were associated with student math achievement.

Discussion and Implications

This study contributes to the current literature on student learning and teacher education. The current study expands the prior research by creating teacher math instruction and self-efficacy typology and finding the factors that are associated with teachers who deliver quality instruction with high self-efficacy. In the US context, teachers' major (i.e., education and mathematics), teachers' years of teaching experience, their perceived academic emphasis, and professional development opportunities are collectively associated with teachers who emphasized cognitively demanding instruction with high self-efficacy. The findings of this study highlight the importance of cognitively demanding instruction but also improvement of teachers' self-efficacy in the US context to improve students' mathematics achievement. Future research is needed to examine the degree to which professional development and teacher evaluation are associated with student achievement because current national and state policy in the US emphasize teacher evaluation to improve teacher quality.

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UNCOVERING TEACHERS' GOALS, ORIENTATIONS, AND RESOURCES RELATED TO THE PRACTICE OF USING STUDENT THINKING

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Improving teachers' practice of using student mathematical thinking requires an understanding of why teachers respond to student thinking as they do; that is, an understanding of the goals, orientations and resources (Schoenfeld, 2011) that underlie their enactment of this practice. We describe a scenario-based interview tool developed to prompt teachers to discuss their decisions and rationales related to using student thinking. We examine cases of two individual teachers to illustrate how the tool contributes to (1) inferring individual teachers' goals, orientations and resources and (2) differentiating among teachers' uses of student thinking.

Keywords: Classroom Discourse; Research Methods; Teacher Beliefs; Teacher Education-Inservice

Researchers and teacher educators need to better understand teachers' reasoning about the practice of using student mathematical thinking in order to support teachers in enhancing that practice. In previous work (e.g., Peterson & Leatham, 2009; Stockero & Van Zoest, 2013), we have used classroom observations and recordings of instruction to understand teachers' responses to student ideas; however, these methodologies have been insufficient in two important ways. First, we have found that using video of instruction to analyze teachers' responses to student thinking does not provide sufficient data to make certain inferences, particularly when teachers do not respond to students' ideas—we have no way of knowing whether the lack of response was deliberate, or whether the teacher did not notice the importance of an idea. Second, we have found it difficult to make comparisons among the practices of teachers who are teaching different content, in different contexts, with different student responses, and thus, have different opportunities to use student ideas. To provide a mechanism for better understanding how different teachers respond to student mathematical thinking in similar contexts, we developed a scenario-based interview (Scenario Interview) as a tool to further understand teachers' use of student thinking. Here, we describe the interview tool and illustrate the type of information it avails us in regard to teachers' goals, orientations, and resources (GOR) in the context of using student thinking.

The Instrument

Schoenfeld's (2011) theory of goal-oriented decision making describes teachers' decisions as being shaped by their GOR. A goal is "something that an individual wants to achieve, even if simply in the service of other goals" (p. 20). Goals can be short- or long-term, and may relate to the learning of specific content, to broader outcomes for students or to teacher actions. Goals may or may not be conscious to the teacher, meaning that one cannot simply ask teachers to state all of their goals for their students and their classroom. Orientations are defined to include teachers' "dispositions, beliefs, values, tastes and preferences" (p. 29) of which, like goals and resources, teachers may not be explicitly aware (Leatham, 2006). Resources include everything a teacher could access to support instruction. They include not only physical materials, but also teachers' knowledge (of, for example, things such as mathematics content, teaching strategies, and typical student conceptions). Schoenfeld

(2011) stipulated that, together, teachers' GOR drive their behavior in the classroom. Understanding GOR in the context of using student thinking is critical to understanding why teachers engage with student mathematical thinking as they do and what might help them further develop this teaching practice. The Scenario Interview was created to provide insight into how a teacher thinks about attending to student thinking during instruction and to infer teachers' GOR in the context of using student thinking.

Throughout the Scenario Interview the teacher is presented with statements from eight individual students—four each from algebra and geometry contexts—that represent a range of thinking, including statements of answers, explanations of solution processes, a suggestion to modify a problem context, and unclear thinking. The interviewee is situated as a classroom teacher and asked to describe what they might do next were the student statement to occur in their mathematics classroom. Although normally a classroom teacher would know details of the context of the situation in which the student thinking occurred (e.g., the task that students are working on, what prompted the student statement, etc.), the Scenario Interview initially does not reveal to the interviewee any contextual information. Rather, after being presented with a student statement, the teacher is given an opportunity to ask questions about the context, which offers insight into possible GOR that the teacher relies on to make decisions. The interviewee is then given five more opportunities to reveal their GOR as they are asked to: (1) describe what they would do immediately after the student's statement was made, (2) explain why they would respond in that way, (3) articulate assumptions they were making that informed their decision, (4) explain their reason for wanting to know the contextual information they asked about, and (5) describe how their response may have been different had they known additional contextual information.

Usefulness of the Instrument

Here we describe two teachers who were chosen to highlight different GOR that might affect teachers' decision making with respect to using student thinking. Although the interviews revealed several GOR for each teacher, we limit our discussion to just one main goal for each (along with one related orientation and resource) to illustrate how the Scenario Interview allows us to infer the reasoning that underlies each teacher's use of student thinking. After discussing each teacher individually, we look across the teachers to highlight differences in their GOR that the interview revealed and discuss how this knowledge might inform the work of researchers and teacher educators.

Ms. Shaw

Ms. Shaw's main *goal* for having students share their thinking is to engage them in making sense of the mathematics behind the thinking. She wants to engage the sharing student in sense making through questions that highlight important mathematical ideas in the thinking, such as, "What are you assuming when you're giving me 4 pi? How did you come up with that band as 4 pi?" and "Why are you using [2] as your radius?" She also wants to engage the whole class in making sense of student ideas. For example, in a situation where a student modified a problem, she proposed turning that modification to the class and asking questions such as, "What would my table look like [in this new situation]? What would my graph look like? How does [this modification] change those two representations?"

One of Ms. Shaw's *orientations*—that an important part of student learning is providing students ample opportunity to think about mathematical ideas—helps her achieve her main goal by ensuring the presence of plenty of student thinking to ground the class discussion. She values providing students individual "think time" before hearing others' ideas about a problem, as well as time to individually "process" ideas that surface during whole-class discussion. Furthermore, Ms. Shaw believes in providing opportunities for students to collectively think about and, when appropriate, compare ideas during whole class discussion: "I want the class to look at the two methods… tell me what they're doing differently. What is similar?"

Consistent with the above goal and orientation, Ms. Shaw considers student mathematical thinking a *resource* for making instructional decisions and helping students make sense of the mathematics in a lesson. She thinks through what students might be able to contribute, and wonders "Can they talk through it?... Is someone going to bring that up for me?" For Ms. Shaw, having a student clarify their ideas provides a means to help all students make sense of the mathematics. For example, she sees student responses to her probing questions as opportunities for all students to "start to see" the compelling mathematics behind student thinking.

Mr. Mead

Mr. Mead's main *goal* for having students share their thinking is to position students as thinkers. He wants students to elaborate on their thinking, both "to see what they think about [a] situation" and to encourage that thinking by "grab[bing] onto that thinking". Although he sometimes has his own ideas about what students might be thinking, he still wants to "dig into" student ideas to "try to figure out what… the student [is] actually thinking," thus positioning students as mathematical thinkers.

Not surprisingly, Mr. Mead's *orientations* include the belief that students can learn mathematics through exploration—"developing their understanding through [a] problem." He is comfortable leaving ideas that surface early in a lesson unresolved, knowing that students will have opportunities to think about them further during the lesson: "If this was a launch of the lesson then… I would say, 'Ah, okay, that's interesting Chris. So how did you get that?'… and maybe get a couple others. If I didn't get any other ideas, then I would just go ahead with the lesson. I wouldn't even address the fact that Chris is wrong at this point."

Mr. Mead uses student thinking as a *resource* to develop the mathematical ideas in a lesson and to tie ideas together. For example, after two students shared different methods of solving a problem, he suggested that he would position the lesson as "more of a verification of what [they] did for us today, so… let's verify whether… the methods… are correct or not, and that would give me something to go back to at the end as well." In another instance, he positions a student idea as a resource for "helping me [and] the students with this context—understand this," because "it's… just a little bit ahead of where we wanted to go at that point, but it seems like it's great to take an idea like that and to run with it." Viewing the student's idea as an opportunity to help the class increase their mathematical understanding, even though it is "a little bit ahead" helps Mr. Mead meet his goal of positioning students as mathematical thinkers.

Discussion and Conclusion

The Scenario Interview helped us to infer subtle differences in teachers' GOR. For example, Ms. Shaw and Mr. Mead both viewed student thinking as a resource for helping students develop an understanding of the mathematics in the lesson. Their main goals for having students share their ideas, however, are slightly different. Ms. Shaw's primary goal is to engage students in making sense of the mathematics behind the thinking that is shared—a content related goal. By contrast, Mr. Mead's primary goal is to position students as mathematical thinkers—a goal more closely related to identity formation. Ms. Shaw's belief in the importance of providing students ample opportunity to think about mathematical ideas individually or as a group supports her goal of using student thinking as a vehicle for students to make sense of the content. Mr. Mead's orientation that students can learn through mathematical exploration positions them as the "doers" of mathematics, and thus supports the development of their mathematical identity. These two cases illuminate the value of the Scenario Interview for inferring GOR at a level that one can make distinctions among teachers—even those who use student thinking in similar ways. If we observed Ms. Shaw and Mr. Mead questioning students about their ideas, we might conclude that they had the same goals and orientations related to using student thinking as a resource.

However, the Scenario Interview revealed subtly different purposes for similar teacher actions. Because GOR are often implicitly held, they cannot be accessed merely through observation or direct questioning (Leatham, 2006). The Scenario Interview appears to be an effective mechanism for prompting teachers to discuss teaching practice in ways that contributes to revealing their GOR. For example, requiring teachers to ask for the contextual information they feel is relevant for deciding how to respond to an instance of student thinking provides information about what resources they draw on and what they value in making decisions.

Prompting teachers to talk about the same varied collection of instances of student thinking provides multiple opportunities to infer their GOR, allowing us to develop themes that characterize each teacher's GOR while also providing the similar context needed to identify important differences among teachers in terms of the GOR that underlie the practice of using student thinking. The Scenario Interview appears to be a valuable addition to our methods of collecting data that allow us to confidently infer teachers' GOR and thus better understand teachers' practice of using student mathematical thinking. Understanding the current state of this practice for particular teachers has the potential to support researchers and teacher educators in designing learning opportunities to enhance this teaching practice that are responsive to teachers' current thinking.

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ONE TEACHER'S UNDERSTANDINGS AND PRACTICES FOR MAKING REAL-WORLD CONNECTIONS IN MATHEMATICS

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Recent scholarship in mathematics education has increasingly supported the power of connecting mathematics lessons to students' lived experiences. This case study, drawn from a larger multi-year study, traces the reflections and pedagogical practice of a middle school mathematics teacher who regularly connected her lessons to real-world contexts. We highlight how the teacher connected a fractions lesson to the context of making soup for her family to accomplish several goals including: (1) sharing stories to learn more about students, (2) moving beyond numbers to build understanding, (3) building students' mathematical confidence, and (4) making space for students to connect mathematical ideas. These findings provide insight into how making real-world mathematical connections may impact students' understanding.

Keywords: Instructional Activities and Practices; Teacher Knowledge

Mathematics educators increasingly agree that connecting school mathematics to experiences, situations and contexts outside of school, including students' own experiences and understandings of the world, is a critical element important to student learning (Ladson Billings, 2009; Turner, Aguirre, Bartell, Drake, Foote, Roth McDuffie, 2014; Turner & Strawhun, 2007). This argument is supported by studies that suggest that the knowledge and experiences that students bring from their everyday lives can serve as resources for learning mathematics (Civil 2002, 2007), as well as evidence that student learning is enhanced when concepts and skills are connected to realistic contexts and situations (Boaler, 2008). In this paper, we refer to teaching that includes connections to real-world contexts and situations, including students' own experiences outside of school, as *real world mathematics teaching*. While some prior research has documented examples of how mathematics teachers make real world connections, still needed are in-depth analyses of teachers' reasoning about these connections, and investigations of how these connections play out in teachers' instruction.

In one study of secondary teachers' understandings of real world mathematics teaching, Gainsburg (2008) found that although the teachers had a wide range of practices they classified as real world, the fleeting connections that teachers made did not often require students to actively engage or think deeply about the mathematics. Moreover, some teachers believed that real world mathematics learning should occur after students have mastered mathematical skills and concepts. Gainsburg also found that although some teachers saw great worth in rigorous tasks, others were apprehensive that challenging and poorly executed real world mathematics tasks might overwhelm students. Gainsburg's study illustrates the complexity and challenge that real world mathematics teaching poses for classroom teachers, and raises questions about what it might mean for teachers to connect their mathematics teaching, in meaningful and sustained ways, to students' experiences outside of school.

In this study we analyzed one early career 7th grade teacher's understandings and practices related to real world mathematics teaching, guided by the following research questions:

- What are the teacher's (Evelyn's) understandings about real-world mathematics teaching?
- How does Evelyn plan for, enact and reflect on connections to real-world contexts and situations, including students' own experiences outside of school?

Methods

Evelyn, an early career middle school mathematics teacher, is a participant in a larger ongoing research project, [TEACH MATH]that follows preservice teachers from their preparation programs and into early career classrooms. Across three years, we observed Evelyn teach 22 mathematics lessons; two observations occurred during student teaching and the remainder during her first two years as a classroom teacher. Evelyn was purposefully selected for this case study because she frequently made connections in her mathematics lessons to real world contexts, and her own and students' experiences in and out of school.

Data included transcripts of recorded pre and post observation interviews, field notes detailing the observed mathematics lessons, classroom handouts, and sample of student work. During the first round of analysis, the three authors reviewed all data for instances of real world connections during mathematics lessons, including Evelyn's planning or reflections on such connections. Then, through iterative analysis, secondary codes were developed that focused on specific instructional moves that Evelyn used to a) elicit and incorporate her students' knowledge and experiences, b) introduce a real world context for a lessons, or c) to share about her own experiences. Codes also attended to key ideas in Evelyn's reflections on connecting mathematics to real world contexts and/or students' home and community funds of knowledge. From this broader analysis, one specific lesson, "Making Tortilla Soup for my Family" was identified as a case of Evelyn building a mathematics lesson around her own experience. This lesson was then analyzed to build the following case study.

Findings

We first overview the "Tortilla Soup" lesson. We then use this lesson to highlight key features of Evelyn's understandings and practices related to real-world mathematics teaching.

Evelyn began her second year of teaching seventh grade mathematics at a different school and with a different curriculum. Although Evelyn saw merit in the curriculum she was expected to teach, she worried about the lack of real world examples and how/if her students would personally connect to, and see the relevance of the content they would be learning.

Evelyn recognized that her students had a limited understanding of multiplication of fractions. After teaching a "very numbers-based lesson," she realized students lacked conceptual understanding, leaving them apprehensive and unsure about following the procedure they had been taught. Although Evelyn felt pressure to "move" through the curriculum she was reluctant to do so, and opted instead to reteach the lesson. She considered presenting a "hands-on" activity from the textbook to support students' learning, but decided to veer from the text. She planned a lesson around cooking and scaling recipes that she believed might support understanding of multiplication of fractions in a "real way." Evelyn's choice of context was informed by a beginning of the year survey in which students reported using mathematics in family cooking and/or baking activities.

Evelyn began the reteach by presenting a handwritten version of her family's favorite tortilla soup recipe. She shared her experience of cooking soup and adjusting the recipe for groups of different sizes, hoping to link her experiences cooking to those of her students. Evelyn drew students' attention to the different ingredients, as well as how many people the recipe would serve (16). She then posed the task to students: Adjust the recipe for smaller and larger groups of family and friends. For example, how much of each ingredient is needed to serve 8, 32, 40, or 82 people? Evelyn's goal was for students to understand how to scale up or down the quantities in the soup recipe by multiplying each quantity by a fraction or mixed number.

As students worked on the task, Evelyn probed their thinking, asking questions such as, "What did you do to get this number?" or "What happened when you were feeding more people?" Students used multiple strategies to adjust the recipe, envisioning themselves cooking the soup, and questioning if their adjusted ingredients made sense. Some students made connections between multiplying ingredient quantities by a fraction and their prior understanding of operations with decimals and percents (i.e., multiplying by ½ is like taking 50% of a quantity). In summary, the lesson helped students connect the real world experience of adjusting a recipe to the mathematical concept of multiplying by a fraction.

Shared experiences leads to learning about students

A key feature of Evelyn's real world mathematics teaching was that she created spaces for students to share their experiences during her lessons. One way that Evelyn invited students to share about themselves was by telling stories from her own experience. She reported:

> I get [students] an hour each day, and it's been a lot more harder to make those connections. I say, "Hi" in the hall all the time and stuff like that to try and get them to feel comfortable with me. When I have something like [the recipe] that's sharing a part of me, they're willing to share a part of themselves. I just feel like everybody's a little bit more open to sharing and taking risks.

When Evelyn introduced her family's tortilla soup recipe, several other students shared food they made with their families and/or enjoyed eating (i.e. the making of enchiladas, the enjoyment of corn tortillas). Evelyn used this practice of sharing her out of school experiences as a means to discover more about her students' own experiences.

Moving from numbers to meaning

A key motivation underlying Evelyn's use of real world contexts in her mathematics lessons was that the connections would help students make sense of the mathematical tasks. Evelyn used gesturing to illustrate the concept of doubling the recipe of all the ingredients into two separate pots. When one student expressed confusion about how to adjust the recipe, and wondered whether he should add the quantities of all the ingredients together, Evelyn drew the student's attention to the activity of cooking soup to support his sense making. She noted:

> Yeah, but if we're cooking – I was like, "Sure we put it in the pot [the ingredient] but if you add them, what does that number mean?" He goes, "Oh! Okay!" Then it was like he thought about it in a real way.

This aspect of Evelyn's practice suggests she recognized that providing students with a familiar real world context (the making of soup) might help students move beyond seeing only numbers in mathematical tasks towards sense making and understanding.

Stepping up to do the mathematics

Additionally, Evelyn thought that real world connections in mathematics lessons would enhance students' engagement, particularly for students who tended to be less confident and more reluctant to share their thinking. In reflecting on the soup lesson, she explained:

> There was just this level of confidence from the start because they could connect to what we were doing. It wasn't just the digits and operations and think about it as math. It was, "This is an experience, and you guys know how to do it, so how do you do it?" Then it was more of them using the math to help them, but they had this huge experience that they could draw from to help them work with the numbers.

Evelyn reported that because students could relate to the context of cooking "A few people that are normally really quiet all of a sudden felt like they were experts, and they 'stepped up." Students' confidence appeared to flourish, as they used the experience of cooking (something they were familiar with) to make sense of the task of adjusting the recipe. Evelyn reflected that making these kinds of connections in her lessons seemed to support more equitable participation.

Making space to connect mathematical ideas

Finally, Evelyn saw the use of real world contexts in mathematics lessons as a means for students to create connections among mathematical ideas. For example, Evelyn found that when she invited students to share experiences and to make connections to lesson tasks, this opened a space for students to connect a range of mathematical ideas and understandings. In the soup lesson, as students were reasoning about multiplying fractions, they made connections to other mathematical operations. She explained:

> Then, one thing that I was impressed with [is that multiplying fractions] wasn't so much of a struggle. So many of them used a reference to money, or a reference to just decimals in general. One student changed, I think, all of his numbers to decimals, and then it was so easy for him to figure out what half of that was, and he just did it.

Evelyn felt that part of the value of real world mathematics teaching was that it encouraged students to make connections and to utilize multiple mathematics strategies to solve the tasks.

Discussion

This study provided a glimpse into one middle school teacher's understandings and practices related to real world mathematics teaching. More specifically, Evelyn's case highlights the importance of teachers sharing their own experiences with students as one entry point to connecting with students and eliciting students' experiences. Additionally, our findings suggest that teaching through familiar real-world experiences is important not only for student engagement and understanding, but also a way to address issues of confidence and status that may surround mathematics learning. Finally, our findings contribute understandings about the ways that teachers might plan, enact, and reflect on real-world connections in their teaching, which may inform mathematics teacher educators in their efforts to support other teachers in this complex and challenging practice.

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IMPLEMENTING THE CORE TEACHING PRACTICES TO MAKE MATHEMATICAL THINKING VISIBLE USING STUDENTS' GENERATED MODELS

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The following case study details the analysis of a Lesson Study where middle grades teachers focused on the core teaching practices to teach and understand the different student generated models for a task on proportional reasoning. We analyzed the significant mathematics teaching practices that teachers engaged in during the planning, teaching and debriefing phases to depict ways in which teachers used students' representational models as a discursive tool to make mathematical thinking visible and instrumental in moving the mathematics agenda forward.

Keywords: Classroom Discourse; Teacher Knowledge; Rational Numbers; Instructional Activities and Practices

Theoretical Framework

In the *Principles to Actions*, NCTM (2014) calls for the mathematics education community to focus our attention on the essential core teaching practices that will yield the most effective mathematics teaching and learning for all students. Over the years, mathematics education scholars have challenged the profession of teaching to identify a common set of high-leverage practices that yield effective teaching. Ball and Forzani (2010) describe high-leverage practices as "those practices at the heart of the work of teaching that are most likely to affect student learning" (p. 45). Authors of the *Principles to Action* (NCTM, 2014) responded to this challenge by defining eight research-based core teaching practices that include: 1) Establishing mathematics goals to focus learning; 2) Implementing tasks that promote reasoning and problem solving 3) Using and connecting mathematical representations; 4) Facilitating meaningful mathematical discourse; 5) Posing purposeful questions; 6) Building procedural fluency from conceptual understanding; 7) Supporting productive struggle in learning mathematics; and 8) Eliciting and using evidence of student thinking. In the design of our professional development project, we used the eight listed Mathematics Teaching Practices outlined in the *Principles to Action*, through our content institute and our Lesson Study cycles. The intentional design of using a school-based Lesson Study as a follow up to the professional development institute was so that we could examine how teachers employed these eight teaching practices *in the enactment of the lesson*. Lesson Study (Lewis, 2002) allows for collective professional *"noticings"*(Mason, 2011)during Lesson Study that attends to children's strategies, interpretations and response to student learning. The focus of this research report is to share how professional development designed around these important teaching practices impacted teachers' knowledge, beliefs and practices for in-service teachers. By focusing on the research lesson we examine how these core teaching practices played a role in enhancing the teaching and learning of mathematics.

Method for Our Study

Research Questions

To explore how teachers engaged in the core teaching practices as they collaboratively planned and delivered a research lesson on proportional reasoning, we asked: 1) In what ways do teachers use the core teaching practices to make mathematical thinking visible through student-generated models? 2) What is the nature of teachers' reflection on a) their practice, and b) their beliefs as they engage in Lesson Study focused on the uses of students' representational models as a discursive tool?

Data and Procedures

Teachers' reflections, video clips and researchers' memos were collected of the preplanning, teaching and debrief sessions. This case study was part of a larger project involving forty elementary and middle grades teachers from grades 5 – 8 who met for eight weeks for a fall content institute and continued to meet as school teams over four face-to-face meetings before conducting Lesson Study by the end of the first semester. In order to sustain teachers' professional learning through the fall semester, the instructors met with teachers after the eight-week content institute in small vertical teams of five to six multi-grade teachers. In the following sections, we present the analysis of a Lesson Study focused on proportional reasoning called the "Candy Dilemma" problem as shown below.

> I bought a box of candy for myself last week. However, by the time I got home I had eaten ¼ of the candies. As I was putting the groceries away, I ate ½ of what was left. There are now 6 chocolates left in the box. How many chocolates were in the box to begin with? Be sure to show and explain all of your reasoning.

Data Analysis

To begin analyzing the themes, we used the document analysis technique using teachers' reflections from the Lesson Study, the group lesson plan and the researchers' memos. To display and organize the collected data, we systematically analyzed data by developing initial codes then used the method of axial coding to find categories in such a way that draws emerging themes (Miles & Huberman, 1994). To verify and compare recurring themes and categories, the research team worked individually on coding the documents before comparing preliminary codes in order to agree upon recurring themes from the reflections.

Results

I. Taking Stock of Student Generated Work to Make Mathematical Thinking Visible for Worthwhile Discussions

For the first research question, In what ways do teachers use the Core Teaching Practices to make mathematical thinking visible through student-generated strategies, we examined the videotaped lesson, debrief and reflections from teachers. While teachers focused on the core practices of using and connecting math representations to engage in math discourse, we found explicit attempts to use student thinking to adapt their instruction *in situ* to enhance and extend student learning. Teachers used student-generated models to take stock of what their student generated work revealed about their understanding while highlighting efficient strategies or novel approaches to build collective knowledge in the classroom; and question or incorporate manipulatives to repair misunderstandings.

We found that teachers planned very carefully so that they could take stock of student- generated work by creating the space for students to show their thinking. The host teacher in the Candy Dilemma Problem offered students a "work mat" to display student thinking. "I chose to give each student a "work mat" (large piece of construction paper). The teachers observing my lesson thought this was a great way to show all possible strategies with plenty of room to work and large enough to observe problem-solving abilities" (Kate's reflection). Initially, students hesitated to start the problem but then began by rereading the problem to themselves quietly before attempting to solve. All students attempted to solve the problem by using numbers and were not eager to try a different method and needed prompting. Additionally, students needed prompting to justify their answer, which in turn had them question their answer or pushed them to explain what the numbers meant in terms of this problem. The host teacher who adopted the *Five Practices for Orchestrating Math Discourse* (Smith & Stein, 2011) during the course referred to it in her reflection as she selected, sequenced and connected student strategies.

> "After looking at the student work around the room, I chose students to come up to the board to show their solution/s. I had ordered them for specific reasons according to what they had shown on their work mats. The second student asked up, documented their strategy and then talked about how theirs connected to the strategy of the person's before them. I have used this procedure a lot and find that it is a good way to further solidify student learning and understanding."

Teachers noticed that students were relying mostly on guess and check and although they had learned the multiplication of fractions, some did not use that operation. Students also did not know how to make sense of the whole number that was left over. This made teachers think about the next steps in their teaching, where they would introduce more related problems and see how the conceptual understanding in these work backward problems can connect to students working through some of the operations they are learning.

II. The Nature of Teachers' Reflection on the Core Teaching Practices

For the second research question, what is the nature of teachers reflection on the core teaching practices in terms of their a) their teaching and classroom norms, and b) their beliefs as they engage in Lesson study focused on students' generated models as a discursive tool, we found two important recurring themes of the importance of setting up mathematical norms and teachers experiencing their own unique productive struggles.

Teaching practices and learning expectations need to be set up as classroom norms.

Teachers focused on the new practice of connecting students' strategies but also recognized that engaging students in the "math talk" was not something that happens over night. The host teacher shared how she spent the beginning of the school year setting expectations for students to show their thinking in more ways than one. It was a mathematical norm/expectation set up in her classroom. She also was thoughtful in creating collaborative groups that would be able to share their thinking despite their diverse abilities. The mathematical norms set up in Kate's classroom was observable by the other teacher observers. Ashley, in her reflection noted how skillfully Kate sequenced the strategies.

> At the end, Kate asked several students to come up to the board and show/explain their strategy. Each subsequent student who came up was to do the same, and then explain how it connected to the previous ones. They saw that they did the same thing, but with a different representation, or working backwards from what someone else had (Ashley's reflection).

Nature of teachers' reflection on their beliefs-Teachers experienced their own productive struggles.

Implementing these core teaching practices were new to some of our teacher participants. Engaging in professional development that focused on these eight core practices provided our teacher participants with the opportunity to not only reflect on these essential practices but enact these practices during their research lesson. One teacher reflected on how watching the host teacher's use of these practices convinced her to try to incorporate problem solving and create productive struggle in her own classroom.

> I like seeing the process of problem solving and watching most of the kids struggle, get a little frustrated, and then have the moment where they figure it out. Rewarding! I am working to make my classroom a place with more of those opportunities and participating in this lesson study allowed me to see how I can make that work in my room (Jessica's reflection).

Conclusion

Implementing these eight Mathematics Teaching Practices outlined in the *Principles to Action* provided opportunities to engage teachers in high leverage practices during our professional development project. Our study addressed ways in which teachers enacted these core teaching practices during their research lesson. In many ways, teachers who engaged in enacting these teaching practices also experienced the "productive struggle" that we try to create for our students through the problem-solving task. In other words, teachers also "grappled" with their own unique "problem of practice". Although teachers voiced the challenges of the constraint of time and the pressures of standardized test as a hurdle in providing the time and space for students to engage in problem solving, they also shared how the productive struggle and reasoning created good mathematicians in their classroom. By analyzing student thinking through a problem-solving task such as the *Candy Dilemma* allowed teachers to learn more deeply about how students approach problem solving and some misconceptions that arise in understanding fraction concepts. Using student-generated models to engage in the core teaching practices made students' mathematical thinking visible and an important discursive tool in their mathematics classroom.

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NATURE AND UTILITY OF TEACHER QUESTIONING: A CASE OF CONSTRUCTIVIST-ORIENTED INTERVENTION

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This study examined question types utilized by a researcher-teacher to facilitate children's knowledge of unit fractions through a constructivist-oriented mathematics intervention. Data were derived from six tutoring sessions; analysis examined the nature and utility of employed questions. Preliminary analysis shows the teacher employed four main types and nine sub-types of questions. Future analysis will delineate the how questioning shifted to correspond with the evolution and solidification in children's conceptions.

Keywords: Rational Numbers; Learning Theory; Instructional Activities and Practices

This case study examined question types utilized by a researcher-teacher in a constructivist-oriented intervention as a means to foster conceptual understanding of students with learning disabilities (SLD). Development of conceptual understanding for SLD is critically important and presents an enduring challenge for researchers and practitioners. Current questioning methods used throughout much of the special education literature are rooted in Reductionist models (Poplin, 1988). Not surprisingly, interventions created for SLD use questioning based on student responses to teacher demonstration (e.g., Fuchs et al., 2014), as opposed to teacher's responses to student thinking. The issue we take with such interventions types is that although they are considered to yield procedural efficiency, they disallow SLD to cognitively reorganize and extend their informal thinking into abstracted, transferable conceptions (Hiebert & Grouws, 2007). Nonetheless, there is little research with respect to the nature of alternative questioning teachers might employ in interventions that have as their aim SLD's construction of conceptions. Initial depictions of questioning might support later studies that reveal how varying questions may be used to support thinking and, ultimately, how teachers may employ responsive interventions in the classroom. The research question was, What types of conceptually-driven questions were utilized by the researcher-teacher during a constructivist- oriented mathematics intervention used to promote unit fraction knowledge in one fifth grade SLD?

Conceptual Framework

One constructivist-oriented notion of knowing and learning that can guide teacher questioning is the Reflection on Activity-Effect Relationships (Ref*AER) framework (Simon, 2004). This framework defines the beginning of a mathematical conception as the assimilation of a situation into a child's existing conceptions. The perceived experience sets a goal for the child's learning that regulates his or her goal-directed, mental activity in a problem situation. In such activity, the child's mind makes two types of reflections. The first, within problem situation reflection (i.e., Type-I reflection), occurs when the child notices and reflects upon what they anticipated would occur as a result of their activity versus what actually occurred. The second, across problem situation reflection (i.e., Type-II reflection), occurs when the child reflects on and compares their effects of their activity across similar problems and begins to notice commonalities in the relationship between the mental activity and its effects. The child begins to anticipate that similar activity will result in similar effects as it did in past situations, and can thus use this anticipation to figure out effects in novel situations. Activity-based conceptions are participatory or anticipatory (Author, 2004). A learner who has participatory conceptions may know what activity leads toward an intended result, but he or she is relying on prompts in order to call upon the anticipation. Conversely, a learner who has anticipatory understanding of a concept has abstracted the anticipation resulting from their mental activity – learners can then apply abstracted conceptions to new situations.

The Ref*AER framework entails an adaptive, constructivist-oriented instructional approach comprised of facilitative activities (Author; Simon, 1995) that (a) identifies the child's goal- directed activities, mathematical objects they may operate on, contexts familiar to children, and effects they may notice and (b) focuses children's activity and reflection based on conjectures about how the child's activities and reflections may bring forth the intended learning. An essential part of focusing children's activity and indirectly influencing the child's advance or development from participatory to anticipatory conceptions involves "orienting students' noticing of differences between their anticipated and actual effects and interjecting prompts [questions] that orient reflection across the learners' mental record of activity-effect dyads" (p. 2). More explication is needed in the literature with respect to the types of questions teachers employ within constructivist-oriented interventions that utilize an Adaptive model.

Methods

Participants and Data Sources

Lia (11-year old, grade-5) attended elementary school in the Northwestern United States. She was purposively chosen to participate in the teaching experiment and subsequent tutoring sessions because she was identified by her school system as having a learning disability specific to mathematics performance. According to her Individualized Education Plan (IEP), Lia was 1.5 grade years behind her peers in terms of her mathematics performance, and had failed the district's state mandated testing in mathematics. Data collection was facilitated through a constructivist teaching experiment (Steffe & Thompson, 2000). We worked closely with Lia once a week for 60 minutes (fixed-time period) over six, non-consecutive weeks between February and April. Data collected for each teaching episode consisted of transcribed video-recordings and observation field notes taken independently by two observers during the episode.

Analysis

Constant comparison analysis (Glaser & Strauss, 1967) was used to delineate codes and themes from the data as to the main types of questions the researcher-teacher utilized in the tutoring sessions. The unit of analysis in this study was a speaking turn, that is, a tri-part interchange consisting of a) the child's utterance or action, b) the researcher-teacher's question in response to the child's utterance or action, and c) the child's responding utterance or action. Emergent coding was utilized within the constant comparison method across two dimensions: a) the overall type of each question asked by the researcher-teacher (e.g., to assess the child's understanding; to foster Type I reflection) and b) any varying utility or subtype (e.g., asking for more explanation; requesting clarification). To categorize the utility of varying questions found within each overall question type and the level of cognitive complexity the question was meant to elicit, a coding theme adapted from Webb's (1997) Depth of Knowledge (DOK) was used as a deductive framework (Leech & Onwuegbuzie, 2007). Additionally, researchers performed classical content analysis to obtain percentages of overall question types and subtypes across the six tutoring sessions. Additional analysis is currently underway to delineate how the utility of questioning changed across sessions to align with how the child's conceptions evolved.

Results

Preliminary analyses show the teacher employed four main types of questioning and nine utilities (or subtypes) of questions within and across overall question types (Table 1 illuminates themes and codes). Preliminary analysis shows that Assess the Child's Understanding questions were most used by the researcher-teacher across the tutoring sessions, followed in frequency by Focus Type-II Reflection, Focus Type-I Reflection, and Invite Application of Concept questions. In terms of question subtypes, Make a Prediction/Formulate a Plan (DOK Level 3), Explain and Defend (DOK Level 3), Explain (DOK Level 2), and Evaluate Cause and Effect (DOK Level 3) were the most often noted utilities across the tutoring sessions.

Table 1. Researcher-teacher's questions in constructivist mathematics intervention

Types of questions and their codes	Examples
1. Assess understanding	
Define/Recall	OK. I'm going to take one of your pieces and pull it out of there. Let's label that. What would we call that piece?
Clarify	Do you mean that it takes three of these, if I wanted to make the whole back up, that I would have to have three of these?
Explain	One of the whole? Say more.
Explain and Defend	One-third. Why do we call it one-third?
Defend reasoning	So we can call this one 11 elevenths, and we can call this one nine ninths, even though the numbers are different, it's still the same thing? How can that be?
Critique	Couldn't you just put that entire amount of the [shortage/overage] on your next estimate? Why would that not work?
2. Invite Type-I Reflection	
Cl: Clarify	Was your estimate too long or too short? Too long? The piece should have been shorter?
Ex: Explain	What happened with your second estimate?
CE: Cause/Effect	When you repeated your estimate, is that about where you thought it would stop?
DrC: Defend reasoning/conclusions	You said it wasn't big enough. Why wasn't that one [estimate] big enough?
3. Invite Type-II Reflection	
Cl: Clarify	What do mean when you say make the piece a little bit bigger?
Ex/DrC: Explain and Defend	How did you know how much longer to make it?
DrC: Defend reasoning/conclusions	Smaller than the one-fourth you made. Can you say how you knew it was going to be that much smaller?
FP: Make a Prediction/Formulate a plan	How much less will you have to put onto each to cover it all? Convince me of the amount before you do it.
4. Invite Application of Concept	
Ex: Explain	The more you're giving away, the smaller the size gets? Talk about that more. What's that mean?
Crt: Critique	There's an eleventh. And there's a ninth. You're right. But when I count, 11 is a bigger number than nine.
Ana: Analyze	Which one's bigger, a ninth or an eleventh? Say why.

Although analyses are preliminary, two discussion points can be offered. First, Assess the Child's Understanding questions seemed predominant, and consistent with a two-fold constructivist teacher's purpose—inferring the child's current conceptions and inviting the child to voice their reasoning (elaborate on and make public her ideas). This is an important departure from Reductionist models of teaching and learning often provided for SLD. Assess questions facilitate the teacher's inferences about an SLD's mental activity and subsequent instruction builds on these inferences to foster the child's conceptualization of the intended mathematics. Such an assessment is based on the teacher's response to the child and her present conceptions to guide subsequent questioning to support learning. In contrast, Reductionist-framed mathematics interventions focus on measuring the child's responsiveness to the teacher's thinking as a finite gauge of learning or knowledge (Author, 2004; Poplin, 1988).

Second, an associated utility of Assess questions involve an effort to reorient the child's disposition in her own mathematical learning toward taking ownership of and justifying her own mathematical reasoning. Utilizing questioning in this manner originates from teaching approaches consistent with constructivism (von Glasersfeld, 1995), and help situating the child's notion of "doing" mathematics. This valuing of children's reasoning stands in contrast with Reductionist questioning utilities that seem to render the child a passive, compliant participant in mathematics learning (Poplin, 1988), and submit SLDs to scripted instruction of rather minute skills through a series of targeted, teacher-directed questions and rapid child responses (Fuchs et al., 2014). We contend that, if the goal of mathematics intervention is the SLD's conceptual reorganization and growth, then questioning needs to shift so a teacher responds to the child's available and/or forming notions of mathematics, situating questioning as a formative, dynamic mechanism within and a basis for instruction.

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MATHEMATICAL CLASSROOM DISCOURSE IN THREE MIDDLE LEVEL SCIENCE CLASSROOMS

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This study concerns a discourse analysis of questioning and mathematical development in three middle level science classrooms. Teachers who received the same professional development on a mathematics and science integrated curricular unit adapted and assimilated the materials and content differently when implementation occurred. Each teacher's lesson enactment of measuring distances between objects in the sky was examined at length to determine the ways in which questioning and mathematical discourse were developed and how this impacted student responses. Two of the three teachers had high levels of questioning and highlighted mathematical content such as ratios, angles, measuring tools, and graphical representations. Higher level mathematical questioning led to higher occurrences of classroom interactions that promoted students' mathematical reasoning.

Keywords: Middle School Education; Measurement; Classroom Discourse

Objective and Theory

In this study, discourse analysis was conducted on three middle level teachers' science classrooms as they implemented a mathematics-science integrated curricular unit. All teachers received approximately 20 hours of professional development utilizing the curriculum and its materials. Teachers adapted and assimilated the unit within their individual classrooms differently when implementation occurred. We argue classroom discourse influences how well students learn mathematical content. Specifically, we examined the teacher's use of questioning and mathematical reasoning within their classrooms. Our research questions were: In what ways did classroom discourse drive the implementation of curricular materials by teachers? How did questioning promote productive mathematical discourse in STEM classrooms?

Duschl and Osborne (2002) emphasized that adequate instruction cannot occur within a STEM classroom without having students engaged and actively learning through means of questioning. They claimed that teaching is "a process of enquiry [*and*] without the opportunity to engage in argumentation, the construction of explanations and the evaluation of evidence is to fail to represent a core component of the nature of science or to establish a site for developing student understanding" (p. 41). Walshaw and Anthony (2008) similarly insisted that 'discourse' is not simply a one-way conversation, but entails the teacher and students actively engaged in back and forth questioning and evaluative discussion. In terms of mathematical discourse, White (2003) conducted a year-long study of an IMPACT (Increasing the Mathematical Power of All Children and Teachers) camp and found several key features to productive classroom interactions. These features include: (a) Valuing and highlighting students' ideas, (b) Exploring student answers, (c) Incorporating students' background knowledge, and (d) Encouraging student-to-student communications.

Enyedy and Goldberg (2004) argued that "social frameworks and microcultures established in the classroom have a direct impact on *what* students learn" (p. 928). So even when teachers receive the same professional development and materials, classroom culture, teacher beliefs, and teacher content knowledge will drive the instruction and classroom discourse that results in how and what students explore, connect with prior knowledge, and communicate to their peers.

Research Methods

Participants

Research subjects were three sixth-grade science teachers and their students from one south-central US middle school (ButternutMiddle School). Three student groups (*N* = 229) were taught by Mr. Land and Ms. Apple (both 2nd year teachers), and Ms. Roling with 16years teaching experience. Butternut middle school demographics showed 74% of the students as White, 10% Black, 8% Hispanic, 5% Asian, and 3% other races, with 26% of the students eligible for reduced lunch. *Pseudonyms were used for the school and teachers.*

Data Collection and Analysis

In order to determine how classroom discourse drove implementation and how teacher questioning promoted mathematical discussion, we implemented a qualitative research design. Three science teachers' implementation of one heavily integrated math/science lesson was videotaped and transcribed. We coded video clips using mathematical concepts and questioning as parent codes, then refined our questioning code into high and low level types of questions. The transcripts were analyzed separately by each researcher and then traded and reviewed. Finally, they were jointly coded in agreement. The inter-rater reliability was relatively high as most of the coded material agreed amongst the three researchers. The few instances of non-agreement initially were on the level of a question.

The lesson that was taught by all three teachers concerned measurement with a focus question of *How do I measure the distance between objects in the sky?* The intended lesson design goes as follows. The lesson begins with students using body parts such as their thumbs and fists to measure lengths of objects in the classroom (from their seats). Discussion ensues in terms of the requirement of consistency in their measurements (extended arms) and why each person would have similar numbers even when people's fists are different sizes. Constant body ratios are discovered as students plot data of arm lengths versus fist widths. Students then determine that one fist width per person is equivalent to 10 degrees as they utilize a protractor in their measuring. Finally students determine that a fist (with arm extended) can be used to measure the distance between objects in the sky as well as the altitude distance of the Moon from the horizon. Azimuth and altitude angles together can determine the location of an object in the sky. Students determine the number of fists required to go from horizon-to-horizon as well as full circle around their body. Students further confirm that each fist width equals approximately 10° after experiencing these activities utilizing body movements (arms extended while measuring). Students document altitude and azimuth angles of various objects in the classroom or outdoors.

Results

Although teachers received the same materials and professional development on the lesson enactment, each utilized their individual strengths, comfort levels with mathematics, and

Table 1. Number of Questioning (High and Low) and Mathematical Episodes

Teacher	Years of Teaching Experience	Total Questioning Instances	Low Level Questioning Instances	High Level Questioning Instances	Mathematics Instances
Ms. Apple	2	52	29	23	7
Mr. Land	2	34	18	16	12
Ms. Roling	16	6	4	2	4

instructional beliefs when implementation occurred. Instances of questioning (high and low) and mathematical episodes are illustrated in Table 1.

Two teachers (Apple and Land) tended to use more questioning and mathematics in their classrooms. Ms. Roling did not use as much of either category. If a question occurred that required only for the students to respond with a simple yes or no, we coded it as a low-level question. When a more substantial thoughtful response was required, we coded this as a high-level question. An example of a high-level question occurred with Ms. Apple's class when she asked students to consider how a scientist might determine the location of an object in the sky.

Apple: So raise your hand if you have an idea of how scientists measure and communicate the location of objects in the sky.
Student: Um, azimuth and the altitude?
Student: Satellites
Student: Knuckles? They used their knuckles to measure.
Student: Perhaps, like, stack their hands … one hand's equal to or equivalent to so many miles

This is a high-level question, since students drew from their real world background knowledge to speculate how scientists might determine the location of an object in the sky. The students who claimed that "knuckles" or "stack their hands" could be used as units of measurement were not far off since they were soon going to be learning how to use their fists as a measuring device.

A mathematical episode as well as high-level questioning was observed in Mr. Land's teaching. Students just finished plotting the class' arm lengths versus fist widths.

Land: All right let's look at the data now, so let me get…trend line. We'll add a linear trend line, so what's happening here? Somebody explain this slope. Is this positive or negative?
Class: Positive
Land: Positive, so what's happening, what does this slope mean? What can I extrapolate from this data? What can I take away from this data? Okay, what do you think?
Student: Okay, um, like the longer their arm to eyes are, um, wait never mind
Land: So the longer their arm is, what?
Student: Yeah, um, like the longer the arm is the or the, bigger their fist

Mr. Land requested that students explore the data set and determine a mathematical relationship between arm length and fist width. Students observed the linear relationship and were able to link the slope of the trend line to the body ratio of arm length to fist width.

Along with high-level questioning, teachers also asked a good deal of low-level questions. Ms. Roling tended to teach by telling for much of her class instruction. She explained to students the difference between an azimuth angle (cardinal direction where North means 0°, East means 90°, South means 180°, and West means 270°) and an altitude angle (angular measurement from the horizon). She followed each lecture with "Does this make sense to you?" Students responded by shaking their heads or saying "yes".

In contrast to Ms. Roling's azimuth explanation, Mr. Land had his students experience azimuth angles as they determined the number of fists that would go around one's body (360°) with arms extended.

Land: …360° in a circle, make sure you write down a little circle mark that represents degrees. Okay, because if you just write down 360, how the heck do I know what you're talking about? You could be talking about 360 pigmy elephants!
Student: But Mr. Land, it says how many degrees.

Land: It does, but generally speaking you always put the unit with the number. Okay, if there's 360 in a circle, and my 10, and my fists is how many degrees?
Class: 10
Land: 10 degrees, how many fists should I be able to do?...I want you to stand up and try it. Calculate how many fists it takes to go around your body. In one full circle.
Class: 36!!

Ms. Apple and Mr. Land emphasized mathematical terminology and concepts, such as degrees, ratios, data collection and graphing, slope, and angular measurements. They used these concepts embedded in science as a way of contextualizing the mathematics in order to make it more meaningful. Ms. Roling did not utilize much questioning or mathematics, and when it was used, it was at a low-level.

Discussion and Conclusion

Students' responses in the classroom corresponded with the level of discourse for which the teacher probed. The ways teachers used discourse as they enacted the measurement lesson influenced student learning as determined by their responses. In a related study, students of Land, Apple, and Roling were given a pre/post test that assessed both mathematical and scientific understandings. Figure 1 displays student gain scores by teacher with students of Land and Apple achieving higher gains than Roling's students from pre-instruction to post.

Within our qualitative study, we found that the teachers' approach to questioning and how they implemented the measurement lesson related to the amount of mathematical discussion in the classroom. Mathematical concepts tended to emerge when classrooms also had higher questioning episodes. Mr. Land and Ms. Apple incorporated White's (2003) discourse categories in terms of highlighting and exploring students' ideas and incorporating students' background knowledge. It is clear there was a discrepancy in how individual teachers enacted an identical curricular lesson. As described by Enyedy et al. (2004), these differences in implementation and ultimately in student learning can be explained by the microcultures each teacher established within their own instructional spaces. Even though all three teachers in our study were from the same school, Land and Apple appeared to have developed classroom cultures where students were encouraged to communicate their ideas and use language of the disciplines, rather than sitting at their seats being passive recipients of knowledge.

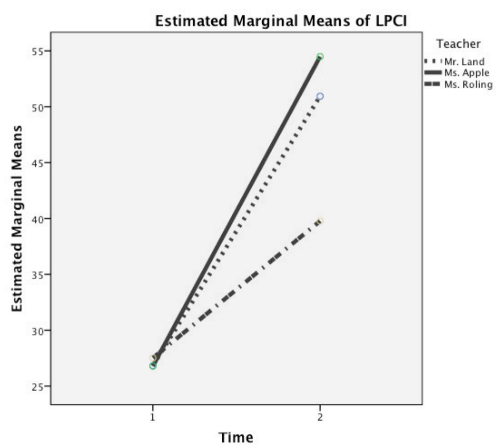

Figure 4. Student gain scores by teacher.

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LISTENING TO STUDENTS' THINKING: TEACHERS SETTING ASIDE THEIR OWN PREFERENCES FOR APPROACHING A MATHEMATICS PROBLEM

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The NCTM (2000) suggests that students should be "actively building new knowledge from experience and prior knowledge" (p.11) and that teachers should be "listening carefully to students' ideas" (p. 19). The idea of teachers listening to students' thinking to help students build on prior knowledge is key within the enduring theme of teaching as responsive to various conceptions of mathematics. Our studies investigated how teachers support students to build on prior knowledge, paying particular attention to what teachers do as they listen to students' thinking. Here we explore these results, and make inferences regarding how the NCTM recommendations on students' prior knowledge and teachers' listening can be implemented widely and in tandem.

Through several recent projects, we have been considering the notion of 'setting aside' – that is, that teachers may guide or encourage students to 'set aside' some ideas and build on other ideas that have more potential in a current situation. Here, we explore the notion of setting aside as it relates to teachers' preferred approaches to a mathematics problem and expectations of students' thinking. Our data analysis draws upon Davis' (1996) three types of listening: hermeneutic, interpretative, and evaluative. Evaluative listening focuses on listening to student thinking to identify the correctness of a solution. Interpretative and hermeneutic listening focus on listening to make sense of students' thinking. Hermeneutic listening involves the teacher being "a participant in the exploration" (Davis, 1997, p. 369) and is often associated with a listener's learning.

We present classroom instances from a grade four lesson involving probability, and a post-secondary lesson for pre-service teachers that considered multiplication strategies. Analysis of our data suggests that teachers, while listening to students' thinking, 'set aside' some of their own preferences and build upon some of their students' preferences for ways of approaching/solving a mathematics problem. For example, to estimate 250x97, the teacher preferred (and also expected students) to compute 250x100. As she listened to a student explain that he divided 100 by 4 and then multiplied it by 1000, she set aside her own preference, learnt a new strategy for that question, and began to think about 250x36. The two classroom instances we share will include a description of the teachers' preferences that were set aside, the thinking that students and pre-service teachers shared, and the teachers' learning that took place. This poster adds to the literature on mathematics teacher listening and it is hoped that the notion of setting aside preferences for approaching/solving a mathematics problem will be helpful to teachers in enhancing their classroom practices to more fully engage with student thinking.

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INVESTIGATING RELATIONSHIPS AMONG TEACHERS' MATHEMATICS BELIEFS, EFFICACY BELIEFS, AND QUALITY OF INSTRUCTIONAL PRACTICES

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The relationship between teachers' beliefs and practices has been the focus of previous investigations because teachers' deeply held beliefs influence their decisions about what and how they teach, and how they design their learning environments (Fives & Buehl, 2008). To augment prior research on how teachers' beliefs shape their perceptions and practices, the main aim of this study is to gain a deeper understanding of teachers' beliefs about mathematics and the links between their beliefs and the mathematical quality of their classroom instruction. Accordingly, this investigation was guided by three main research questions: whether teachers' mathematics-related beliefs are reflected in their instructional practices, whether teachers' self-efficacy beliefs play a role in their practices, and whether the alignment between teachers' beliefs and practices brings about high-quality mathematics instruction.

My research, as part of a long-term project, was a collective case study involving data from six participants; six individual cases were compared and contrasted to explore the relationship between teachers' beliefs and practices, and then two selected cases were analyzed in terms of their quality of mathematics instruction. Interview transcripts, videos of classroom instruction, and responses to self-efficacy surveys were examined for evidence of the teachers' beliefs and practices in teaching mathematics. A priori codes (i.e.. productive beliefs, unproductive beliefs) were determined before examining the interview data in hand (NCTM, 2014). Also to examine teachers' classroom practices, I used an observational instrument, the Mathematical Quality of Instruction (MQI; Learning Mathematics for Teaching Project, 2011). Data from the teacher self-efficacy survey were analyzed by conducting descriptive analysis.

My findings indicated that, with the exception of two cases, the teachers expressed beliefs that could be considered as both productive and unproductive as the NCTM suggests. Their professed beliefs were in fairly close alignment with their instructional practices although they described varying productive and unproductive mathematics beliefs, which affected their instruction accordingly. I see that the alignment between teachers' beliefs and practices did not guarantee high quality mathematics instruction. Although there was a quite close alignment between teachers' beliefs and practices, no clear linear relationship between them could be established, and other factors, such as the nature of their beliefs, that influence how teachers perceive and enact their roles in the classroom might have affected the mathematical quality of their instruction.

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RETHINKING MATHEMATICS TEACHING WITH PAPERT, BROWN AND OTHERS

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This project aims at rethinking and exploring mathematics teaching theoretically and empirically. Papert's (1980) suggestions about what teaching mathematics should be diverge from usual views that center on attaining predetermined goals/objectives and keeping the attention of the class precisely on them while teaching. In Papert's view, mathematics teaching is about allowing students to *be* and *act* in mathematics (as mathematicians do and act), rather than focusing on teaching specific mathematical content. He proposes that students do mathematics by exploring them, asking new questions, developing new ideas, and so forth, which leads teachers to work and explore mathematical objects without having predetermined end points to reach. These ideas have also been explored by Brown & Walter (e.g. 2005), on issues of problem posing; they propose that mathematics teaching occurs through, and is grounded in, students' questioning on a given problem. Starting from a specific theme provided by the teacher, the intention is to make *mathematics* emerge in the classroom by working with students on/from their ideas and questions, which leads to mathematical explorations where students do mathematic; this acting thus as the central goal. Other like Kieren, Davis, Lampert, and Lockhart, to name a few, have also discussed similar ideas about mathematics teaching. However, these ideas have not been integrated into an encompassing framework that grounds theoretically and offers a coherent organization of these teaching ideas and their implications. This study aims to develop this grounding framework, which is to be developed by exploring Holton's (1988) three nested dimensions: the empirical (i.e. observations to develop meaning); the analytical (i.e. theories to explain the phenomena); and the thematic (i.e. reflections about the underlying assumptions, meanings, vocabularies, and methodologies). This poster presents preliminary results gathered from combining the analytical and empirical dimensions, taken from a case study of a teacher who adopts these views about teaching mathematics; the results in return lead to reflections on the third dimension, that is, the thematic. These preliminary analyses underline significant characteristics, three of which are: (a) *Author/ity* (e.g. Povey & Burton, 1999), where the teacher and students work together to develop a mathematics community in which all have a personal voice and are makers of knowledge; (b) *Openings* (e.g. Remillard & Geist, 2002), which refers to important unpredicted instances that emerge when doing mathematics, which expand and deepen the mathematics under study and demonstrate by their presence that learning happens and that mathematics evolves; (c) *Complicity* (e.g. Davis & Sumara, 1997) refers to the fact that the teacher becomes complicit in students' knowledge in shaping what is learned, from mathematics itself to their views/philosophies of mathematics. These characteristics and others will be presented/discussed in light of the thematic issues they underline about mathematics teaching.

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MATHEMATICS TEACHERS' PERCEPTIONS OF FACTORS AFFECTING THE USE OF HIGH COGNITIVE DEMAND TASKS

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Mathematics teachers' difficulties implementing high cognitive demand tasks is well documented in the mathematics education literature, although teachers' perspectives on this issue are largely absent from such literature (e.g., Stein, Grover & Henningsen, 1996). In this study, I examined factors *teachers* discussed as inhibiting or promoting the use of high cognitive demand task. My intention was to give voice to middle school mathematics teachers who were trying to implement these tasks. This study can inform those providing professional development in relation to high cognitive demand tasks.

I conducted a multiple case study (Yin, 2003) with the seventh grade team of three teachers at Yellow Brick Middle School in the Southeastern United States, all of who had less than five years experience. The theoretical framework used in this study was the task implementation framework developed by Stein and colleagues (1996).I planned with the teachers for the specific implementation of two high cognitive demand tasks, observed the lesson, and interviewed each teacher after. The goal for data analysis was to gain teachers' perspectives of implementing high cognitive demand tasks. I realized the factors the teachers mentioned, not included in the task implementation framework, specifically related to events that happened before planning tasks. I used the new factors I identified, along with those in the task implementation framework to code the interviews accordingly making note of where the teachers had common barriers and supports, and then identified factors each teacher individually pointed to as being a barrier or a support.

During the course of this study, I looked at a more broad view of the task implementation process and gained teachers' perspectives on the entire process from planning, to implementing or not implementing a high cognitive demand task, and then into the classroom for the implementation of a high cognitive demand task. I found factors did not fit in with the task implementation framework (Stein et al.,1996) that included teachers' perceptions of students, time, district requirements and curriculum, and parental dispositions. I labeled these factors collectively as *factors influencing teachers' use of high cognitive demand tasks* and suggest these factors be attended to before the teachers implement high cognitive demand tasks. By knowing what factors teachers' perceive as affecting the decision to implement high cognitive demand tasks, researchers can help find ways to overcome perceived obstacles and support mathematics teachers. Teachers should realize how important and beneficial implementing high cognitive demand tasks is for their students and thus make the decision to find and implement high cognitive demand tasks. It is my intent this study will be a catalyst for further conversations about how to support mathematics teachers implementation of high cognitive demand tasks.

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TEACHERS' PERCEPTIONS AND USES OF THE COMMON CORE STANDARDS FOR MATHEMATICAL PRACTICE

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This study examined middle school math teachers' perceptions of the Standards for Mathematical Practices (SMP) in the Common Core State Standards for Mathematics (CCSSM), which are the de-facto intended U.S. national curriculum (Porter, McMaken, Hwang, & Yang, 2010). The SMP have their foundation in the NCTM process standards and strands of mathematical proficiency identified in the National Research Council's report, *Adding It Up* (Common Core State Standards Initiative, 2010). The authors of the CCSSM indicated that the SMP are intended for students to exhibit while engaging in mathematics. Therefore this analysis operated from the perspective that teachers' perceptions of the SMP will influence their enactment of the SMP in their planning and instruction. The background interview data were collected as part of a larger study with NSF funding (DRL-746573 & DRL-1222359) from 48 teachers from eight states. This analysis considered teacher responses that indicated the SMP represented a significant change from previous standards. These questions guided the analysis:

- What are the teacher's perceptions of the SMP their role in CCSSM and how should the SMP be enacted in the classroom?
- How are teacher's perceptions of the SMP evident their planning and daily instruction?

The primary finding is that teachers characterized the SMP as a measuring stick with regard to the level of rigor required by the CCSSM. One teacher expressed that the SMP showed the performance expectations associated with the CCSSM; "[SMP] gives us an opportunity to look at what we're expecting students to do and raise those expectations" (Granville, NY, all names are pseudonyms). Teachers stated that the SMP increased the rigor emphasizing a deeper level of engagement with mathematical content. For example, Ross (NY), stated that the SMP required "a lot more in-depth, a lot more deep thinking about math and explaining your reasoning and how you got to these conclusions, the number is no longer just enough." Teachers also indicated that the language of the SMP provided justification for their current instructional practices, which were not validated in their prior state standards. In addition, the SMP provided a mathematical authenticity to the content in CCSSM. For example, Tomar (NH) stated "that's probably a major innovation like we're defining what a math classroom looks like, what the practices of a mathematician look like." The implications of these findings provide a structure for further analysis of teachers' usage of SMP for instruction and formative assessment decisions, and for professional development designers providing learning opportunities.

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SECONDARY TEACHERS' BELIEFS OF TEACHING MATHEMATICS IN DIVERSE SCHOOL SETTINGS

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The body of research in teacher-held beliefs in mathematics education is wide and varied, with a legacy spanning more than four decades. Research topics in this literature base include teachers' beliefs regarding the use of manipulatives, incorporation of constructivist learning principles in instruction, and the degree of affiliation of beliefs and practice. It is not just important, but *critical*, to note that this body of research explores mathematics teachers' beliefs largely without consideration of race of the teacher or of the students. I problematize this in two ways. First, the absence of race of student and of teacher implies a generalizability of the research to all races, all ethnicities, all marginalizations. Second, the absence of race establishes a monolithic teacher and a monolithic student, each without color or culture

My study creates spaces for understanding conceptualizations of race in purportedly race-neutral mathematics, exposing uninterrogated aspects of secondary teachers' beliefs. In this session, I report on the preliminary results of my research focusing on secondary mathematics teachers' beliefs about race in conjunction with their beliefs about mathematics and mathematics education. In this qualitative multicase study of two white and two Hispanic secondary mathematics teachers in New Mexico, I explored two questions: (1) what are the intersections of teachers' beliefs of 3 areas: mathematics, teaching/learning mathematics, and race; (2) what are the origins of their beliefs? Participants engaged in four one hour-long interviews over a period of one month, and also provided three journals in which they responded to researcher-selected prompts on mathematics and mathematics education. Data from the interviews, journals, and researcher's field notes were coded and analyzed using the constant comparative method.

Preliminary results from the study indicate the complexity of teachers' beliefs in working with diverse student populations. Beliefs about the nature of mathematics and of methods of instruction varied substantially. Of the nature of mathematics, participants presented a wide range of thoughts from mathematics as independent of humans, and was discovered rather than invented, to mathematics as a purposeful tool to approach problems and find appropriate solutions. Beliefs of teaching methods also varied, from a traditional lecture model to constructivist only methods. Discussions of teaching methods were often in tandem with discussions of classroom management. Issues of performance in mathematics were often associated with student drive, work ethic, perseverance, determination, and grit. Success in mathematics was often presented as a binary construct of trying vs. quitting. Race and ethnicity were factors that white participants did not readily bring into the conversation, but were prevalent in discussions with one Hispanic participant, and evident to a lesser extent in the data from the second Hispanic participant. For white participants, student diversity was not a significant factor in planning for instruction. Survival mode as a teacher was a concern among participants that arose several times. Origins of beliefs were occasionally detailed, although probes into the sources of beliefs often returned vague replies, or the lack of specific episodes from which beliefs emerged.

CONSIDERING STUDENTS' RESPONSES IN DETERMINING THE QUALITY OF TEACHER'S QUESTIONS DURING MATHEMATICAL DISCUSSIONS

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Previous researchers have analyzed the types of questions teachers ask during mathematics lessons and have generated frameworks for categorizing teachers' questions (Boaler& Humphreys, 2005). This body of research has identified specific types of questions as effective in eliciting students' mathematical thinking and reasoning and other types of questions as effective in eliciting rote, procedural, or factual mathematical knowledge. In this study, we aim to extend the literature to provide a better understanding of the close connection between teacher questioning and student engagement. Specifically, we investigate the extent to which question types identified as higher-order questions actually serve to generate high-level student responses during whole-group discussions in mathematics lessons. We hope to make a contribution to the field by providing empirical evidence to support previous theoretical frameworks for teachers' questioning.

Data have been collected from 6 mathematics teachers in a large urban school district. Eight mathematics lessons (two teachers recorded twice) were analyzed from video recordings of 90-minute block class periods for: 1) types and levels of questions the teacher asks, 2) types of responses students provide, and 3) the length of students' responses (number of words used). Teacher-student dialogues were transcribed from the whole-class discussion part of each lesson. Frameworks used in the analysis include Wimer's (2001) categories of higher order and lower order questioning, Boston's (2012) rubrics for high vs. low-level student responses, and Boaler and Humphreys' (2005) questioning types: Gathering information, checking for a method (G), Inserting terminology (IT), Exploring mathematical meanings and relationships (EMM), Probing (P), Generating discussion (GD), Linking and applying (LA), Extending thinking (ET), Orienting and focusing (OF), and Establishing context (EC).

Results from the study indicate that the level of teacher questioning significantly impacted the level of student response ($\chi^2 = 39.61$; $p < .0001$).Consistent with Boaler and Humphreys' (2005), we found: 1) the majority of questions were Gathering Information (G), and these questions almost always (94%) generated low-level student responses; 2) low frequencies of questions (38 of 120; 32%) considered to support students' mathematical thinking and reasoning (e.g., EMM, P, LA, ET); and 3) Probing (P) and Linking and Applying (LA) were likely to generate a high-level student response. Last, the mean number of words in students' responses is significantly higher for higher-order questions than for lower-order questions ($t\ (118) = 3.17$; $p < .0001$ [one-tailed]). Hence, the type and the level of teacher questions affect how much students engaged in providing responses.

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TWO SECONDARY MATHEMATICS TEACHER CANDIDATES' ENACTMENT OF DISCOURSE MOVES AND QUESTIONING PRACTICES

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Ambitious teaching practices provide opportunities for all students to engage with mathematics. Within the broader challenge of preparing future teachers to enact more ambitious teaching practices, teacher educators must specifically prepare future teachers to facilitate productive classroom discourse. Facilitating discourse-intensive situations is complex and requires practice and reflection. It is therefore important that teacher educators provide opportunities for teacher candidates (TCs) to enact more ambitious teaching practices through coursework and to provide them the opportunity to reflect and refine their practice. This case study research follows two teacher candidates enrolled in year-long, post-baccalaureate certification programs. Specifically, it addresses the following research question: what discourse moves and question types do two teacher candidates utilize during instruction?

To address this research question three video-recorded lesson enactments were analyzed. The first video recording took place in June 2014 at the completion of the TCs' first mathematics methods course. This recording was of an *approximation of practice* (Grossman, Compton, et al., 2009), during which TCs taught a 30-minute lesson to their peers. The second video was recorded in November 2014 at the completion of the TCs' second mathematics methods course. The third video was recorded in March 2015 as part of a performance-based assessment. The last two video enactments were recorded in the TCs' field placements with actual middle school students.

Transcriptions of video recordings were coded for specific discourse moves such as revoicing, restating someone else's reasoning, applying someone else's reasoning, or using wait time (Smith & Stein, 2011). The questions each TC posed during instruction were classified as one of nine-question types identified in Boaler and Brodie's (2004) study of secondary mathematics classroom instruction. The nine question types are as follows: (1) gathering information, leading students through a procedure, (2) inserting terminology, (3) exploring mathematical meanings and/or relationships, (4) probing, getting students to explain their thinking, (5) generating discussion, (6) linking and applying, (7) extending thinking, (8) orienting and focusing, and (9) establishing context. Initial findings show that both TCs' were successful in applying a variety of question types and using some discourse moves, even in the earliest video recording.

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OBJECTIFICATION-SUBJECTIFICATION DIALECTIC IN MATH DISCOURSE

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Keywords: Algebra and Algebraic Thinking; Classroom Discourse; Equity and Diversity; Learning Theories

Intro & Background. This study explores the dialectic between the semiotic resources (e.g., gesture and language, as well as conventional tools such as tables/graphs) to which students have recourse to make mathematical assertions, and the hierarchical subject positions that participants co-construct through these multimodal interactions. I focus on processes of objectification (Radford, 2003) and subjectification (Foucault, 1977) as a method of exploring this dialectic. I present a detailed analysis of a 35 min. span of data involving three high school students engaged in a collaborative generalization task. I observe that objectification and subjectification are co- constructed through mathematical discourse. Below is a sample transcript segment and analysis.

Ailani: [*counts on fingers*] Eleven, twelve, thirteen… Figure ten will be… nineteen. [*10 sec pause*] (Figure ten).. nineteen.
Thalia: [*gazes down at her sheet; indicates her table of values*] Figure ten will be fifteen.
Ailani: [*responds to Thalia*] Nineteen.
Thalia: [*gazes down; shakes head*] Figure ten will be fifteen.
Ailani: [*loudly*] Figure ten will be nineteen!
Thalia: [*gaze down; indicates entries with her finger as she talks*] It goes two-three-six, then it goes two-three- seven, then it goes two-three-eight, then it goes two- three-nine, then it goes two-three-ten, then it goes two- three-eleven, twelve, thirteen, fourteen, fifteen. [*keeps gaze down, continues working*]
Ailani: I say nineteen. [*4 sec pause*] I'm not even sure I counted that right, but who cares?

Results. Ailani's semiotic means of objectification consisted of a counting strategy and verbal speech (with a certain illocutionary force) to assertively express a partial solution, in the form of $f_3(10) = 19$. In contrast, Thalia too resorted to verbal speech, yet she also used rhythm, gesture, and repetition, and a mathematical table as semiotic resources. Entering a rhythmic cadence (Line 6) enabled Thalia to objectify a recursive *generalization*, in the form of $\{f_1(n), f_2(n), f_3(n)\} = \{2, 3, f_3(n-1)+1\}$ which, in this context, is more informative than Ailani's solution.

Whereas both students made statements that tacitly positioned themselves as having the correct answer, Thalia resorted to a broader arsenal of semiotic resources to make her point. Additionally, Thalia did not make eye contact with Ailani, instead keeping her gaze on her work. This social-mathematical power encounter resulted in differentiated status positions, and an opportunity was missed to engage in dialog and collaborate on the shared goal, to determine Figure 10. In this way, Thalia's and Ailani's respective semiotic means of objectification also functioned, simultaneously, as semiotic means of subjectification. Thus Thalia gained *mathematical ascendency* over Ailani, which is a construct that I claim is a version of Foucault's (1977) theory of power-knowledge, as a co-constructed, semiotic-based status hierarchy.

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DIFFERENTIATING MATHEMATICS INSTRUCTION WITH MIDDLE SCHOOL STUDENTS

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Today's middle school mathematics classrooms are marked by increasing cognitive diversity—an enduring challenge in the field. Traditional responses to cognitive diversity are tracked classes that contribute to opportunity gaps (Flores, 2007) and result in achievement gaps. Differentiating instruction (DI) is a novel but untested response to cognitive diversity in which teachers proactively plan to adapt curricula, teaching methods, and student activities to address individual students' needs in an effort to maximize learning for all students (Tomlinson, 2005).

The purpose of this poster is to present preliminary findings about what characterizes differentiating mathematics instruction for cognitive diversity with middle school students. In the first two years of a 5-year project to study DI we conducted three 18-session design experiments, each with nine cognitively diverse 7th and 8th grade students. In keeping with design experiment methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), we made changes between experiments in order to foster the most possible progress while studying our interventions. For example, we knew that it was important to develop norms (Tomlinson, 2005) with students to help them learn to work autonomously and to honor differences among classmates. However, we found that we needed to help not just individuals but small groups work autonomously, and so we developed a focus on small group functioning.

Our preliminary findings include the following. When aiming to differentiate instruction:

(1) The teacher needs to use formative assessment to develop working models of students' ways of thinking and organize those ways of thinking into a network (Ulrich, Tillema, Hackenberg, & Norton, 2014). Planning for this network involves a "web-like" approach in which the teacher anticipates multiple (2-4) broad pathways for students. Implementing these plans requires a continual cycle of posing tasks, asking questions and listening, and making interpretations and adaptations for students moving along and/or across each of these pathways.

(2) The teacher needs to help structure students' exposure to different ways of thinking in order to help students navigate interaction with, and potential learning from, others. Doing so may help students make sense of and try out different ways of thinking without being overwhelmed by them. One aspect of this structure involves explicit conversations about how to respond to another's way of thinking. We view these findings to be components of a theory of differentiating mathematics instruction for middle school students.

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A FRAMEWORK FOR TEACHER RESPONSIVENESS

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Mathematics instruction that builds on children's mathematical thinking is advocated for within the larger mathematics education community (Carpenter et al., 1989; NCTM, 2014). This type of classroom instruction relies on what Pimm (1987) describes as genuine conversations in which students share the responsibility for determining the direction of lessons; however, established classroom routines in the U.S. favor IRE exchanges (Mehan, 1979). We describe instruction that incorporates children's thinking as *responsive* to children's mathematical thinking. Hiebert and Grouws(2007) claim that, "Documenting particular features of teaching that are consistently effective for students' learning has proven to be one of the great research challenges in education" (p. 371). In this session we propose a framework that documents a key feature of instruction that will allow researchers to specify relationships between responsiveness and other established constructs, to refine current theoretical models of teaching and learning, and to provide teachers with a tool to describe one aspect of their classrooms.

The framework is based on earlier research and has been revised based on recent data from over 160 lessons across 11 classrooms. *Responsiveness* reflects the extent to which student thinking is taken up and built on during mathematics instruction. Students' mathematical ideas must be expressed in order for teachers to be responsive. As such, a characterization of students' mathematical ideas is a necessary component of characterizing responsiveness. Thus, we holistically categorize segments of discourse in terms of two interrelated features of whole-class discussion in mathematics classrooms: (a) students' mathematical contributions, and (b) teacher responsiveness to these contributions. Categories of students' mathematical contributions express increasingly sophisticated contributions ranging from recalling facts and performing calculations (minimal); to sharing mathematical information without justification (limited); to participating in mathematical argumentation (substantial). Categories of teacher responsiveness reflect the extent to which students' mathematical ideas are made public, taken up, and the basis for instruction: low (ignoring, brushing-off, or evaluating student contributions), medium (revoicing student ideas, asking probing questions), and high (directing students to engage with others' mathematical ideas). Applications of this framework are presented on the poster.

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THE INFLUENCE OF TEACHERS' BELIEFS, GOALS, AND RESOURCES ON INSTRUCTIONAL DECISIONS

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The work of teaching is complex. There are numerous, not readily observable components involved in the decisions teachers make while carrying out a lesson that builds on students' mathematical thinking (Ball, 2008). One important decision involves deciding how to use evidence of student thinking during instruction. There are a number of factors that impact this decision. Developing goals for lessons is part of the work teachers do and help to drive the decisions they make during instruction (NCTM, 2014). Teachers' available resources also impact decisions. (Schoenfeld, 2011). Teachers come to a classroom with an already developed set of beliefs about how children learn mathematics (Vacc & Bright, 1999). These beliefs have an impact on the decisions teachers make in planning and implementing a lesson.

This research was part of a larger study in the domain of fraction operations when using a guided-reinvention approach to algorithm development. The purpose of this study was to examine the influence teachers' goals, resources, and beliefs about teaching mathematics have on the decisions they make during the planning and teaching in this domain. Data was gathered from three fifth-grade teachers and one sixth-grade teacher who were observed teaching an initial lesson in a unit on fraction multiplication. The lesson engaged students in solving contextualized problems where students bought fractional parts of partially full pans of brownies. All teachers, at some level, were observed eliciting student thinking in this lesson. However, how each teacher made use of students' reasoning in relation to supporting students' developing understanding of fraction multiplication differed. Through observation and post teaching interviews we also considered how beliefs, goals and resources impacted the mathematics developed as a result of the lesson. The portraits developed for each teacher revealed the importance of particular fraction multiplication questioning frameworks, how teachers interpreted and used students' reasoning in relation to their goals as revealed in these questioning frameworks, and how past teaching experiences, curricular and school-based resources were related to the mathematics developed in the lesson.

Acknowledgement

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PROMOTING EFFECTIVE SMALL GROUP LEARNING IN THE MIDDLE GRADES

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Background and conceptual framing. Peers Engaged as Resources for Learning (PEARL) integrates understandings of small group learning environments using three frameworks: Mathematics Task Framework (Stein et al., 1996), Math Talk Learning Communities (Hufferd-Ackles et al., 2004), and Peer Cultures of Effort and Achievement (Hamm et al., 2012). Small groups offer opportunities to address goals for all students in *Principles to Actions* (NCTM, 2014) and *Common Core State Standards for Mathematics* (NGA, 2010).

Research questions. The PEARL project's initial research questions are: "In current practice, what strategies do teachers use as they enact tasks, promote discourse and manage peer cultures of effort and achievement, and for what purposes? What challenges do teachers experience as they focus on tasks, discourse, and peer cultures?"

Research design and findings. The PEARL study combines examination of mathematics instruction for conceptual understanding and adolescent classroom peer processes through investigation, design experiment, teacher learning experiences, and experimental study. This presentation focuses on investigation using naturalistic field study. A preliminary study of existing data reinforced the importance of all three frameworks, leading to hypotheses about productive equilibrium, balancing areas such as exploration with sense-making; individual with group responsibility; and natural with guided peer interactions. These hypotheses informed the naturalistic field study of 11,6-9th grade teachers' classrooms, involving lesson observations, interviews with teachers, and focus groups with students. Findings illustrate productive equilibrium in small groups, pinpointing ways that teachers support productive equilibrium, and depict ways that small group learning environments can lack or lose productive equilibrium. The study indicated sparse explicit attention with students to establishing, supporting, and reflecting on small group functioning as an environment for learning mathematics.

Conclusion. PEARL aims to build stronger theoretical and practical framing of small group learning environments to inform teachers' professional practice in three domains: mathematics tasks, student discourse, and peer cultures. Situated authentically in classrooms, initial findings reveal opportunities and challenges for teachers to produce and maintain balance within and across these three domains to deliver on the promise of small group work.

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EXPLORING BASE 10 COMPLEMENTS THROUGH MATHEMATICAL GAMES

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This study about mathematical games concerns non-traditional way of teaching for improving and deepening students' experiences of mathematics. In addition to arguments in favor of using games to increase students' motivation as well as classroom interactions, researchers like Bragg (2003) or Peltier (2000), using pre- and post- tests and interviews, show students' improvement on mathematical content like arithmetic, geometry and probability through using mathematical games. However, few studies have analyzed in details what takes place *during* mathematical games in a classroom, that is, what are the actual learning opportunities and mathematical activity that students are immersed in while actually playing these mathematical games. This poster presentation presents such an analysis, taken in the context of an experimentation conducted in a Grade 1-2 elementary classroom.

There are many definitions in the literature about what a mathematical game is. For example, it can be described as a stage in the child development, as a social activity, as a fun teaching tool to develop knowledge, as an educational activity aimed at developing specific knowledge, and so forth. For this study, a mathematical game is defined as a teaching material explicitly designed to work on and explore a specific mathematical content. The game studied here is a memory game about base 10 complements in which two students, playing one against each other, have to match two cards picked from a stock of 30 face-down cards to make 10. Various representations are used (numeral, literal, box of ten, etc.) and numbers vary between 0 and 12 (11 and 12 are used for subtraction with 1 and 2). Figure 1 presents examples of possible combinations to make 10.

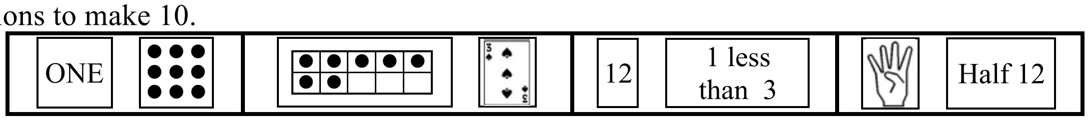

Figure 1: Examples of base 10 complement pairs

This poster presents a description and an analysis of how this game took place in a Grade 1-2 classroom, how the students interacted with it and what was the nature of the mathematical activity they were immersed in. For example, students engaged in a variety of strategies *to operate*, as some counted the numbers illustrated on the cards, either with dots or drawings, to get to 10, whereas others used their hands or even counted mentally without any physical support. Students also developed strong *associations* between a number and its complement. E.g., when kids picked 8, it was as if they also picked up a 2 "at the same time", where 8 was not considered alone but came with 2 as well. In that sense, when picking up a card, many knew what number to look for or anticipated the card to find (often by "cheating" through looking into the not-so-opaque cards!). Others students opted, at times, to decide for the value of the cards chosen (e.g. blank or joker cards, previously assigned the value of 0 in the game, were re-assigned the needed value to make 10 with the other chosen card). Students also demonstrated *fluency* and *flexibility* with the diverse representations: the various representations of numbers were not problematic at all to students, as they navigated through all of them and mixed them to make 10. These strategies (and others) will be shared and discussed in the poster presentation.

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RESPONDING TO STUDENT THINKING: MAKING IN-THE-MOMENT TEACHING DECISIONS

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The value that the mathematics education community places on student thinking is clear. What is less transparent in mathematics education research are the nuances of how a teacher can talk about, and enhance, the classroom practices by emphasizing and using student mathematical thinking in-the-moment. The NCTM principle, "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM 2000, p.11) has inspired us to think about 'taking up' student thinking as a teaching practice. For us, taking up student thinking involves an emergent, cyclical, continuous process where students share what they think, teachers listen to student thinking, and then teachers make in-the-moment decisions about how to respond to student thinking. This research speaks to the enduring challenge of teaching as responsive to various conceptions of mathematics, and is grounded in the question: What kinds of in-the-moment decisions do teachers make in response to student thinking?

This study builds on research that examines teaching interventions from an enactive perspective (Towers & Proulx, 2013) and emerging research related to in-the-moment pedagogy (e.g., Mason & Davis, 2013). We specifically use notions of in-the-moment pedagogy and ideas from Towers and Proulx's (2013) work to identify, describe and characterize teachers' decisions that were made while taking up students' thinking. Similar to some of the research that this study builds upon, our ideas are framed by enactivism (a theory of cognition). Teaching and understanding, in this framework, have an in-the-moment quality. In-the-moment teaching decisions are neither easy nor predictable, as they stand in a reciprocal relationship with student thinking, perpetually influencing, and influenced by, one another.

We present findings from a project that occurred in the context of student-generated multiplication strategies and involved the authors co-teaching and co-planning with an in-service teacher. Students in a grade four classroom were asked to complete twenty multiplication questions, identify 'easy', 'easier' and 'harder' questions, and to explain their thinking/strategies. In our findings we will describe students' thinking and two types of teaching decisions that were made in response to what students shared. Drawing on Towers and Proulx's (2013) constructs, we will be investigating how these in-the-moment decisions *oriented students' attention* for a doubling strategy to emerge, and *coordinated the possible* for a strategy that was grounded in the associative property, but did not only involve doubling, to unfold. We hope to offer practical insight into, and generate interest around, the broad discourse of in-the-moment pedagogy.

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USE OF EXAMPLES IN TEACHING CALCULUS: FOCUS ON CONTINUITY

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Giving examples in mathematics instructions is a widespread instructional practice, but not a straightforward process. How mathematics teachers use examples can have significant effects on students' opportunities to learn because students can interact with abstract mathematics concepts through given examples in classrooms. Thus, it is necessary to observe instructors' use of examples to reveal its possible influences on students' learning in calculus classroom. The purpose of this study is to investigate mathematicians' use of examples in their calculus courses. In this case study, use of examples to teach the concept of continuity was observed.

In this research, the definition of an example is aligned with those suggested by a couple of previous studies; an example is referred as mathematical object serving a cultural mediating tool between a person and mathematical concepts or theorems (Goldenberg & Mason, 2008). An example in instructions should allow learners to mentally interact to mathematical objects with abstract mathematical ideas in educational settings (Zodik & Zaslavsky, 2008).

We observed two mathematicians teaching calculus 1 at a Midwest university. Data were collected by classroom observation, initial and reflective semi-structured interviews, and field notes. Video-recorded lessons were mainly analyzed with constant comparison to interviews and field notes. Based on the definition of example in this research, analysis units were identified in videotaped lessons. All data were analyzed using existing codes for examples in previous research (Goldenberg & Mason, 2008;Zodik& “ “ <Zaslavsky, 2008) and modified in a data analysis process. Member checking was conducted with participants in reflective interviews in order to ensure the accuracy of analysis.

Remarkably, both mathematicians used *incorrect* examples of target concepts and theorems in order to check understanding of concepts or show impossibility (Goldenberg & Mason, 2008). In introduction of the definition of continuity, both mathematicians similarly employed examples of discontinuous functions after presenting the definition of continuous functions. These incorrect examples for the continuity concept allowed the instructors to check whether students not only understood the definition but also could apply it to a certain function.

However, the participants showed different patterns to teach the intermediate value theorem (IVT).one mathematician asked students to build non-existing examples as an informal proof of IVT and assessment of students' previous knowledge; a continuous function satisfying following conditions of (a) $f(1) = -3$, (b) $f(2) = 4$, and (c) the function never crosses the x-axis. In the recall interview, the mathematician mentioned that asking students to create impossible examples can be an intuitive way to begin proving theorems like IVT. This use of incorrect examples could be students' preliminary opportunity for mathematical formal reasoning. Therefore, use of incorrect examples on instructional purpose can be an appropriate way for formative assessment as well as a bridge between informal and formal proofs in college mathematics.

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ANALYZING DEVELOPMENT OF NORMS CONDUCIVE TO PRODUCTIVE DISCOURSE: PHASE ONE

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For several decades, teachers and researchers alike have sought to create mathematically *productive discourse* in classrooms, that is, discourse focused on sense-making, creating coherency by relating mathematical concepts, and appealing to mathematical reasoning as the authority. Despite the widely-recognized importance of productive discourse, efforts to create it have not always been successful. Many examples exist of teachers who have successfully increased the *quantity* of discourse without achieving productivity. Furthermore, discourse may superficially appear productive while creating low press for conceptual thinking (Kazemi & Stipek, 2001). Social and sociomathematical norms are useful constructs for analyzing and explaining regularities in discourse. Research has indicated that certain social norms seem to be correlated with productive discourse, such as the expectations that students explain and justify their thinking, collaborative in problem-solving, listen actively, and find multiple solution paths. Certain sociomathematical norms also seem to be correlated with productive discourse, such as explaining the rationale behind computations, and using explicitly defined criteria to differentiate between *different* solutions (Kazemi & Stipek, 2001). Many studies investigating norm development have focused analysis on an episode-level. While offering insight, such analysis does not consider episodes in light of previous or subsequent episodes. A longitudinal study offers this broader perspective on norm development. However, comparatively few longitudinal studies of norm development exist.

My study intends to longitudinally investigate the development of norms conducive to productive discourse. Thus far, the first phase, a pilot study, has been performed in a case study 5^{th} grade classroom. Video recordings and observations notes have been made twice per week for most of the 2014-2015 school year, confirming that productive discourse frequently occurs in this classroom. I intend to observe the case study classroom daily for the first two months of the upcoming school year, since research indicates that the majority of norms are established during this period (Yackel, Cobb, & Wood, 1991). Using grounded theory principles such as the constant comparative method, video footage will then be analyzed to identify specific social and sociomathematical norms that emerged. A chronology of each norm's development will be created. I will then identify teacher strategies, methods, and key interactions that contributed to norm development.

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TEACHING GEOMETRIC SIMILARITY FROM DILATING PERSPECTIVE: EMBEDDED FIGURES APPROACH

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Proportional Relationships and Geometric Similarity

One of the goals in mathematics teaching is to help students build connections among mathematical concepts/ideas and use them for future learning (NCTM, 2000). Geometric similarity and proportional relationships are two of these areas that students need to benefit from such connections. Studies reveal that students often struggle with recognizing the proportional relationships in similarity problems, and as a result, making inappropriate additive comparisons instead of multiplicative ones (Kaput & West, 1994; Lobato & Ellis, 2010). One possible reason for students' difficulties with recognizing proportional relationships is that they usually have trouble iterating or partitioning a composed unit (Lobato & Ellis, 2010). As a result, students often fail to consider coordinated changes of lengths in both dimensions.

Dilation is defined as "a transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor" (CCSS; Common Core State Standards Initiative 2010, p.85). According to CCSS, eighth graders should use ideas about similarity to define and analyze two-dimensional figures and describe how these figures change under some geometric transformations (i.e., translations, rotations, reflections, and dilations). Beckmann and Izsák (2015) suggest using the Dilating Perspective Approach (or Fixed Numbers of Variable Parts Perspective of Ratios) for developing students' understanding of geometric similarity from a transformational approach. The Dilating Perspective Approach suggests that "two quantities are said to be in the ratio A to B if for some-sized part there are A parts of the first quantity and B parts of the second quantity" (Beckmann & Izsák 2015, p. 21).

In this paper, we have examined the effectiveness of the Traditional Approach (including the Cross-multiplication) and the Dilating Perspective Approach with embedded representations of similar rectangles. The Traditional Approach does not necessarily lead to the understanding of proportional relationships, because it does not enable students to be aware of uniform stretching and shrinking between the sides that need to be compared. On the contrary, the Dilating Perspective Approach with embedded figures enables students to realize uniform stretching or shrinking in both dimensions by providing an easier visual comparison of similar figures.

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I-THINK: FRAMEWORK THAT IMPROVES PROBLEM SOLVING FOR ALL STUDENTS

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Keywords: Classroom Discourse; Problem Solving; Equity and Diversity; Elementary School Education

There is a recent shift in the United States, driven by the National Council of Teachers of Mathematics (NCTM) and the Common Core State Standards (CCSS), from mathematics instruction in an inclusive classroom that traditionally used teacher initiation, student response, followed by teacher evaluation to an approach that is more student centered. According to NCTM (2014) and the CCSS for Mathematical Practice (CCSSI, 2011) kindergarten through high school mathematics instructional practices should promote mathematical discourse for all students, including students with disabilities. The I-THINK framework promotes reflective and conceptual instructional practices, such as student justification of solutions and mathematical discourse, which helps students to become aware of and learn to monitor and evaluate their mathematical thinking (Lynch, Lynch & Bolyard, 2013).

A quasi-experimental control group design was used to determine whether the I-THINK problem-solving framework leads to greater problem solving performance versus a structured Think-Pair-Share (TPS) framework (Kagan, 1994). Participants included 118 elementary students from six fully included classrooms. Two classes from each grade level, 2^{nd}, 3^{rd}, and 4^{th}, were selected based on voluntary participation by the classroom teacher in a professional development session on discourse and problem solving frameworks. For this study, we utilized a nonequivalent control group design consisting of a pre-assessment, a six-week instructional cycle, and a post-assessment. One class from each grade was randomly assigned to the intervention group that received problem-solving instruction using the I- THINK framework. The other class was assigned to the alternative intervention group that received problem-solving instruction using the TPS framework.

Within the combined groups, participants in the I-THINK groups scored statistically significantly higher on the post-test, as opposed to the pre-test. Participants in the TPS groups scored statistically significantly higher on the post-test, as opposed to the pre-test. Post-test scores for second grade were statistically significantly greater in the I-THINK vs. the TPS group (p = .037). There was also a statistically significant difference in post-test scores between the frameworks for third grade, with statistically significantly greater scores in the I-THINK vs. the TPS group (p = .016). There was no statistically significant difference in the post-test scores between the frameworks for fourth grade. When analyzing the scores from the two grades that had significant improvements in both groups (2^{nd} and 3^{rd}), there was a statistically significant difference in post-tests cores between the I-THINK and TPS groups, with statistically significantly greater scores in the I-THINK vs. the TPS groups (p = .008). Detailed results and implications will be discussed.

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CALCULUS I TEACHING: WHAT CAN WE LEARN FROM SNAPSHOTS OF LESSONS FROM 18 SUCCESSFUL INSTITUTIONS?

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During classroom observations conducted as part of the Characteristics of Successful Programs in College Calculus (CSPCC) project we observed 73 lessons taught by 65 instructors at 18 institutions. In this poster we present an analysis of nearly 500 math problems enacted in those observed lessons, attending to how the tasks were enacted (lecture, group work, etc.), technology used, representations used, and other mathematical features of the problems.

Keywords: Post-Secondary Education; Instructional Activities and Practices; Curriculum Analysis

Relative to the literature in K-12 mathematics education, there is a dearth of information about what transpires in college classrooms. Although there are some exceptions (Laursen, Hassi, Kogan, & Weston, 2014; Mesa, Celis, & Lande, 2014), we know little about teaching or the nature of the tasks that instructors use in classrooms in lower division courses. One of the goals of the CSPCC project(NSF-DRL REESE #0910240, Bressoud, Rasmussen, Mesa, & Pearson) was to document such work for Calculus I. We observed 73 calculus lessons across18 institutions (including associate-, baccalaureate-, master-, and PhD-granting institutions) identified by the CSPCC project as having successful calculus programs. Our observation protocol included a *Problem Log* with which observers recorded the specific mathematical tasks seen in classes as well how the task was enacted (e.g., lecture or group work), what representations were called for (e.g., symbolic or graphical), what technology was used (e.g., graphing calculators or CAS) and other important features (e.g., presenting multiple solutions or doing a proof; see White & Mesa, 2012). By focusing on the content of the tasks and their enactment, we gain a disciplinary-rich perspective on the nature of Calculus I instruction at these institutions. In this poster we describe the pedagogical and mathematical features of the almost 500 observed tasks. We also report on similarities and differences within and across institutions and across institution types using various analytical techniques (e.g., ANOVAs, odd-ratio estimations, clustering), to account for the nature of the data.

In our analysis thus far we have found no significant difference in the method of task delivery across institutions types—that is, there is a uniform predominance of lecture across all institutions types. When we focus on the mathematical features of the tasks, however, PhD-granting institutions use fewer problems emphasizing symbolic representations than the other institutions and baccalaureate-granting institutions use more problems emphasizing skills and techniques than the other institutions. This analysis will allow us to better understand what teaching in successful institutions looks like and how that varies by institution type.

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SUPPORTING GRADUATE STUDENT INSTRUCTORS IN CALCULUS

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Keywords: Post-Secondary Education; Teacher Beliefs; Metacognition; Design Experiments

Graduate student instructors (GSIs) teaching introductory calculus participated in a semester-long working group to learn to use Peer-Assisted Reflection (PAR) in their recitation sections (Reinholz, in press), an activity which was developed through a three-phase design experiment (Cobb, Confrey, Disessa, Lehrer, & Schauble, 2003). PAR involves students analyzing one another's work and exchanging feedback before revising their solutions, which helps students improve their explanations, collaboration, and persistence (Reinholz, in press). The GSIs met six times during the semester, for one hour each time. This poster focuses on the learning of Beth, a second-year graduate student with two semesters of prior teaching experience.

All six of Beth's lessons were coded for teaching practices. Two lessons were double-coded, with 90% agreement. The two disagreements were on teacher probing for explanations. Teacher position was coded holistically, while all other practices were coded for frequency (see Table 1).

Table 1. Beth's Observed Classroom Practices

Week	Teacher Position	Teacher Probes for Explanations	Teacher Links Student Responses	Students Respond to Each Other	Students Present at the Board
2	Front of room	2	0	1	0
4	Front of room	2	0	0	0
6	Circulating	6	0	0	0
8	Circulating	7	4	9	1
10	Circulating	11	2	5	0
13	Circulating	0	0	21	2

Table 1 shows a dramatic shift in Beth's teaching. In a post-semester interview, Beth noted:

> I've definitely become a lot more reflective about my teaching, and I think about it a lot more than in the past. Just, I question, is this working. I feel like honestly when I first starting teaching, I would just do stuff, and think it's good, oh it's great, and I never really asked, is this working, do I think they are learning this way…compared to past semesters, I got students to come up to the board more and got them to explain their words more.

This poster highlights the potential of PAR to support GSIs to grow as teachers.

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INVESTIGATING TEACHERS' BELIEFS AND TECHNOLOGY INTEGRATION

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Keywords: Teacher Beliefs; Technology; Teacher Education-Inservice

Instructional technology (IT) offers a number of potential benefits toward enhancing student learning, yet many teachers do not effectively integrate IT in their instructional practices, a critical issue in mathematics education. By effective, I mean using IT to increase students' conceptual understanding of mathematics through reflection and communication (Hiebert et al., 1997), as well as through using and connecting mathematical representations (NCTM, 2014). Effective IT integration is *essential* to student learning and a necessary component of effective teaching (Ertmer & Ottenbreit-Leftwich, 2010). However, an important factor that influences teachers' use of IT is their beliefs about IT (Zbiek & Hollebrands, 2008).

This poster reports on the findings of a study that sought to investigate secondary mathematics teachers' beliefs and practices regarding instruction and IT integration, as well as how teachers describe effective IT integration. Rather than relying on Likert scale and multiple-choice questions to investigate beliefs, this study employed a new tool for investigating teachers' beliefs using an online survey implementing constructed-response questions (Philipp et al, 2003). The constructed-response questions provided situational context for investigating beliefs and how teachers described effective IT integration (Holstein & Gubrium, 2011).

Findings suggest that the constructed-response survey, designed to address shortcomings of previous tools used to investigate teacher beliefs, provided new insights into teachers' beliefs about "effective" IT integration, and highlighted that targeting beliefs is a necessary component of professional development aimed at increasing the effectiveness of IT integration. Teachers' responses indicate that there may not be a relationship between a teacher's preferred teaching style and how they use IT or what they believe is an effective use of IT. Survey questions, participants' responses and more detailed findings will be shared during the poster session.

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MAINTAINING COGNITIVE DEMANDS OF ALGEBRAIC REASONING TASKS

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Keywords: Instructional Activities and Practices; Algebra and Algebraic Thinking; Classroom Discourse"

Introduction

As recommended by Proulx and Bednarz (2009), the study's goal is to contribute to research on supporting mathematical thinking. It focuses on practices that maintain cognitive demands of rich tasks. This is a significant focus because meaningful experiences with rigorous curriculums depend on cognitive demands during enactment. This report is part of a grounded theory developed from analysis of classroom video data, artifacts, and interviews. Participating students were in grades 3-5 at six schools. Students worked on pattern finding activities.

Results and Conclusion

Launching with minimal instructions. The way the tasks were launched was critical in maintaining the cognitive demands of the tasks. The best launching practice, judged based on the variety and quality of student responses during instruction, involved giving only minimal instructions that were enough for students to understand what they are expected to do without funneling their thinking. Teachers described this approach as relaxed and a flexibly structured teaching style whereby you give students a task with an attitude of "here it is, see what you can do with it. And just ask them few questions and not trying to lead them, just seeing where they go." Teachers' view of the launching styles that maintained open-endedness of tasks of algebraic thinking had an emphasis on exploring mathematical relationships other than a focus on trying to find an existing specific algebraic generalization. As one teacher explained:

> My goal is to make learning discovery based, let the students work through it, not feel like there is an answer that they have to achieve as their goal, but help them to understand that there might be, and sometimes there is more than one answer.

Workspaces. Workspaces are "opportunities for students to use their own critical thinking." One of the critical practices for creating workspaces was moving from traditional worksheets that provide minimal spaces for exploring to 'work spaces' with minimal words and a lot of space. With 'work spaces,' students explored more, and the quality of their visible thinking improved greatly because they communicated the unspoken expectation of engaging with the mathematics more than the tradition worksheets did.

In conclusion, launching with minimal instructions and providing workspaces are critical in maintaining cognitive demands of rich tasks. This poster presentation will show a striking difference in students' visible thinking between classes with and without these practices.

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ASSESSMENT OF EQUITY IN BELIEFS AND PRACTICES OF TEACHING MATHEMATICS TO AFRICAN-AMERICAN STUDENTS

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Keywords: Equity and Diversity; Teacher Beliefs; Teacher Education-Inservice

Researchers who examine the experiences and opportunities of African-American children in learning mathematics identify certain mindsets and dispositions of teachers that can be productive or unproductive in supporting students' development (e.g., Ladson-Billings, 2009; Milner, 2010). Ladson-Billings (2009) describes teachers' *ways of being*, and the relationships teachers seek to form with African-American students, as critical in fostering students' learning.

The purpose of this study is to create and validate a survey to assess teachers' beliefs and practices in teaching mathematics to African-American students. Beyond our specific study and context, the survey can be used more broadly to raise awareness and analyze change over time of productive and unproductive mindsets at the personal, school, and district level.

The current version of the "Assessment of Equity in Beliefs and Practices of Teaching Mathematics to African-American Students" survey contains 80 questions spanning eight categories: Beliefs about Student Achievement and Motivation, Culturally Responsive Teaching, Perceptions of Challenges of African-American Students, Perceptions of Success of African-American Students, Mathematics as a Civil Right, Missionary Beliefs, Beliefs about Critical Thinking, and Student Behavior. The majority of the categories are derived from the components of culturally relevant pedagogy described by Ladson-Billings (2009) and the mindsets identified by Milner (2010), Martin (2007), and Jackson and Wilson's (2012) literature review. In addition to its emphasis on culturally relevant pedagogy, the instrument asks teachers to reflect on their schools' curriculum and course offerings for students of color.

We intend for the results of the survey to provide an indication of: 1) teachers' *mindsets* that may be more or less productive in supporting the mathematical learning of African-American students, 2) teachers' *awareness of and attribution for successes and barriers* in African-American students' learning of mathematics; and 3) teachers' *feelings of self-efficacy and/or support* (i.e., whether teachers feel equipped and supported) in enacting equitable teaching practices in mathematics. We hope to make a contribution to the field by providing a validated survey that could be used to foster critical reflection and awareness on a personal, school, or district level. By triangulating and correlating survey results with other data (e.g., teacher observations and interviews), we may be able to add specificity to Ladson-Billings' notion of teachers' "way of being" that supports mathematical achievement in African-American students.

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THE IMPACT OF TEACHING EXPERIENCE ON MATHEMATICS GRADUATE TEACHING ASSISTANTS' EFFICACY

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The urgency of improving teaching and learning in undergraduate mathematics education is deepening and graduate teaching assistants (GTAs) are becoming increasingly responsible for taking on this task. However, differences in GTAs training, and GTA actual teaching experience, makes it difficult to assess the GTAs' efficacy and ultimately their effectiveness in the undergraduate classroom. This study addressed the research question; do previous teaching experiences of graduate teaching assistant's impact teacher efficacy.

Graduate students often times enter school with little to no teaching experience. In fact, serving as a GTA may be the very first time these students have the opportunity to teach. Research has also shown that GTAs with more experience have reported higher levels of self-efficacy toward teaching (Prieto & Altmaier, 1994) and have been regarded as more effective by students (Ferris, 1991). More importantly, teaching assistants play vital roles in the mathematics education of undergraduates and may become mathematics professors one day.

Denham and Michael's (1981) Teacher Sense of Efficacy Model is used as a theoretical lens through which this study is framed. Denham and Michael "theorized that a teachers' sense of self efficacy is a strong mediating variable in teacher effectiveness and consequent to student achievement" (Prieto & Altmaier, 1994, p. 482). Denham and Michael's model suggests that a heightened sense of efficacy in teachers should affect their perceived and actual ability to teach more effectively" (Prietto and Altmaier, 1994, p. 483). This theoretical model is vital to the proposed study because it allows associations to be made between levels of efficacy among GTAs in the undergraduate mathematics classroom and increased student achievement.

Using the Teacher Sense of Efficacy Survey – long form (Tschannen-Moran& Woolfolk Hoy, 2001), this correlational study collected data regarding the demographics (i.e. previous teacher training/professional development, teaching experience and future career plans) and teaching beliefs of each the 184 voluntary GTA from the 32 participating mathematics departments classified by Carnegie as research extensive universities.

Based on the literature, it was hypothesized that positive relationships would be found between teaching experience and TE. A weak positive relationship was found among teacher efficacy and K-12 teaching experience. Implications for relationships among teaching experience and teacher efficacy call for more structured teaching assistantships (similar to those in the k-12 experience) during the graduate student phase in mathematics departments in order to enhance the effectiveness of GTA future teaching practices. This information moves the teaching experience forward in that it advocates for greater supervision and a larger amount of responsibility during the GTA phase.

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TEACHING MATHEMATICS THROUGH SOCIAL JUSTICE: UNDERSTANDING TEACHERS' FIRST EXPERIENCES

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Keywords: Instructional Activities and Practices

This study was designed to investigate the experiences of teachers implementing social justice themed lessons for the first time. Teaching mathematics through social justice activities has been proposed as a way to support students in building personal connections with and a sense of relevance when learning mathematics (Gutstein, 2003; 2006; Turner & Strawhun, 2007). Recent research has shown that supporting teachers in teaching math through a social justice lens is not a straightforward process (Bartell, 2013). Our central questions were:

- How do teachers view the role of math and context in social justice-themed activities?
- What are the challenges teachers experience as they implement social justice-themed activities?
- What are the supports that they find helpful as they implement these activities?

To address these questions, the authors designed a multiple-case study to examine teachers teaching mathematics through social justice contexts for the first time. Walther co-planned social justice themed lessons or units with the cooperating classroom teachers. All planning sessions were audio-recorded. After planning, the classroom teachers in each case implemented the lessons with students in their classroom. The lessons were video-recorded, with the authors present, taking field-notes throughout. After the lesson or unit concluded, interviews were conducted with teachers and students to gain further insights into their experiences during the events. Data were analyzed using open-coding in order to identify relevant themes and significant themes across cases.

The authors employed a situated perspective that theorizes one's learning as dependent upon social and cultural contexts, not solely on one's individual cognition that is tied to specific contexts (Lave & Wenger, 1991). Initial findings show that teachers experience tensions between math and context in implementing social justice activities, and they navigate these tensions in different ways. In addition, teachers view social justice in different ways that have implications for their implementation of this approach.

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Chapter 11

Technology

Research Reports

Brief Research Reports

Poster Presentations

MATHEMATICS CLASSROOM INTERACTIONS IN THE VIRTUAL SYNCHRONOUS ENVIRONMENT

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We compared classroom interactions during two episodes in a seventh grade virtual mathematics class. Both episodes were drawn from virtual live lessons utilizing different lesson layouts. We investigated how mathematical knowledge was negotiated in both episodes. The results show that the layout of live lessons had an effect on how students shared their thinking.

Keywords: Classroom Discourse; Technology

Technology is changing communication patterns as it introduces new forms of human interactions, asynchronously through emails, texts, blogs and discussion forums, and synchronously between large spatial distances. More remarkably, technology is altering how people experience education. In the U.S., the number of K-12 full-time virtual distant schools is growing exponentially (Barbour & Reeves, 2009). Mathematics too, is being encountered by K-16 students and teachers in virtual environments. Yet, research studies that carefully examine the content of interactions in such educational settings are rare. Such studies are necessary in order to unpack intricacies of mathematics learning and teaching specific to this type of environment. As Morgan and colleagues (2014) posited, there is an emergent need in the field to investigate online and mobile communication that are unique in mathematics contexts. Our goal of the case study we report here was to address this need. Our research aimed to unpack the nature of interactions taking place in one virtual mathematics classroom.

Background Literature

The utility of computer technology as a cultural tool for mediating knowledge construction has long been recognized (Saxe, 1991). The internet as a new form of technology-based learning medium, has also penetrated educational research in recent years. The range of reported research includes exploration of online teacher professional development, and online language programs (Bairral, 2009; Jonassen, 1986). Previously the main audience for online learning consisted of self-motivated adult professionals who clustered around a common interest. Asynchronous settings such as discussion forums and self-study online learning tools have provided the main context for learning among this population. Researchers and educators who explored the use of *hypermedia* or *hypertext* for online instruction have concluded that: (a) content representation in the web is not linear and can be hyper-linked; and that (b) it is beneficial for novices to access web automated knowledge representations that are pre-constructed by experts (Carlson, 1992). However, K-12 online or blended learning has entered the education arena and grown rapidly in the past five years. In comparison to the adult online learning format, synchronous lessons that are analogous to traditional brick-and-mortar classrooms represent a large part of online K-12 school curriculum. How mathematics is taught and learned in this environment has not been a subject of careful analysis. Arguably the most prominent and unique feature of a synchronous environment is how the teacher and students communicate about and with mathematics in the course of their interactions. As a starting point in developing a systematic research agenda surrounding online mathematics teaching and learning, a focus on unpacking synergistic interaction offers a productive scholarly venue.

Carlson (1992) provided a simple, but precise description of communication in a synchronous setting: "As a simple case example, participants are able to converse in dyads or triads with classmates not in close physical proximity; whereas, in the real classroom multiple, simultaneous discussions quickly degenerate into a cacophony of voices" (p.57). A systematic study of language usage and the internet was conducted within the linguistics discipline. For synchronous group chats in particular, linguists have identified the delaying feature (Crystal, 2001). Crystal posited that lags in conversations interrupt the conversational turn in traditional face-to-face interaction, because in the virtual synchronous environment, turn-taking as seen on the screen was dictated by the software rather than the intentions of the end-users. However, Crystal and other linguists' studies of synchronous group chats were drawn from social networks rather than educational settings. Other features absent from synchronous group chat that are critical to the educational environment, such as facial expressions and physical body language, provide challenges for teachers in receiving cues from learners. The constraints of online interactions can certainly impact the quality of discourse taking place. Yet, we know little about specific features of the environment that may motivate classroom interactions and mathematics learning. To address this topic, we planned to study a seventh grade virtual mathematics classroom as the teacher delivered live lessons over the course of one academic semester. In this report, we chose two different live lesson layouts to study mathematics classroom interactions. Two specific questions guided our data analysis: (a) What is the nature of interactions between the teacher and students, and among students in the virtual synchronous mathematics classroom? (b) How do different settings of synchronous virtual mathematics instruction promote productive interactions and discussions about mathematics?

Methodology

Data was collected from a seventh grade mathematics teacher's class at a full-time virtual school. At the time of data collection, two mathematics classes were combined, and the total number of students was around 70. The researchers observed the teacher on a regular basis. A number of live lesson recordings were collected for virtual classroom interaction research.

Since our plan was to examine how the use of various features of the online lessons impacted modes of communication among the students and the teacher, we selected two episodes from two virtual live lessons to complete a comparative analysis. Each live lesson was an hour long. Both lessons shared a similar structure, consisting of four parts: (a) greeting and pre-assessment of previous content (approximately 15 minutes); (b) teacher-led instruction of the main learning objectives for the day (15 minutes); (c) students solving problem(s) and whole-class discussions about the problem(s) (15-20 minutes); and (d) the teacher offering a whole-class summary or students completing exit questions (10 minutes). The selected episodes concerned the third part of the lessons as students worked on assigned problems and discussed their solutions. Episode 1 was 13 minutes and 32 seconds long. Episode 2 was 10 minutes and 7 seconds long. We selected these two episodes to ground our comparison of the two sessions since the teacher's live lessons relied on different layouts, giving rise to contrasting interactions both between the teacher and students, and between students. The teacher's oral communications were transcribed and textual records were downloaded. Both sets of transcripts were then organized in the order in which they appeared in the screen.

Adobe Connect virtual lesson setting

The software used during both virtual live lessons was Adobe Connect. The software connected the teacher and students in real time in various locations. The students were "participants", and chat pods were the only communication tool they were allowed to use. The teacher was the "host" with access to audio, chat pods, drawing tools, and other presentational features. The teacher communicated with the students using primarily audio, whiteboard drawing and the chat pods.

In Episode 1, the teacher intended for the students to write an algebraic expression representing a real-life situation. The layout of this episode is shown in Figure 1. The teacher utilized three functions of Adobe Connect: the poll question to gather student responses at the beginning, a chat pod labeled "EXPLAIN your thinking", and a "chalkboard" demonstration. In Episode 2, the content of the lesson concerned engaging students in solving two-step equations. In addition to the poll question and "chalkboard", two chat pods were utilized in the lesson: one labeled "QUESTIONS Only", and the other labeled "Responses to QUESTIONS" (Figure 2).

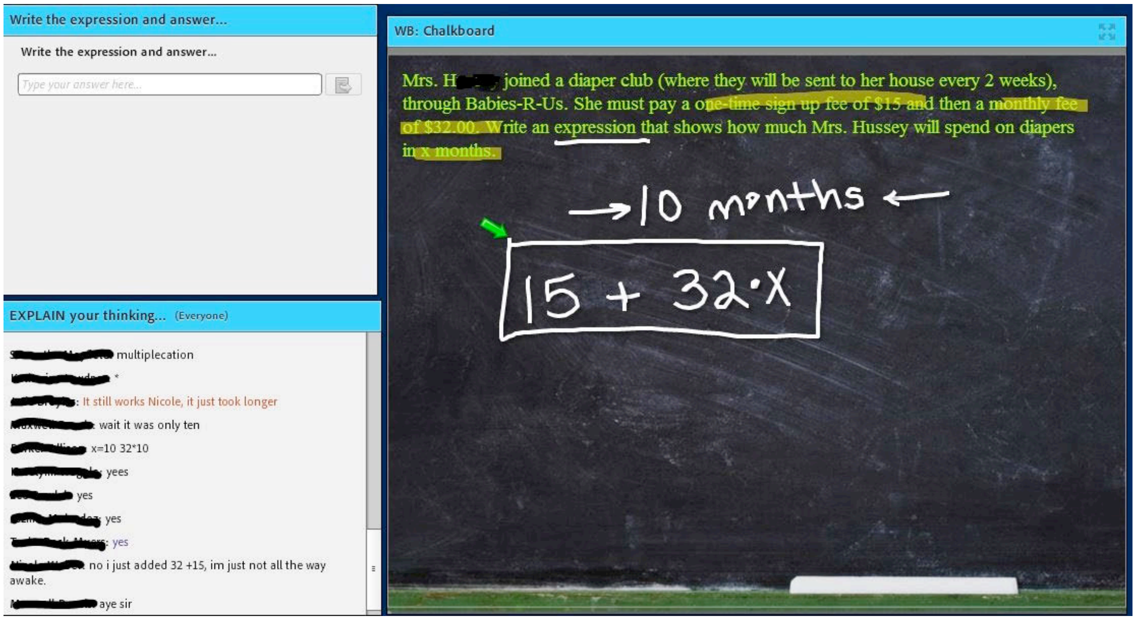

Figure 1:Layout of Episode 1

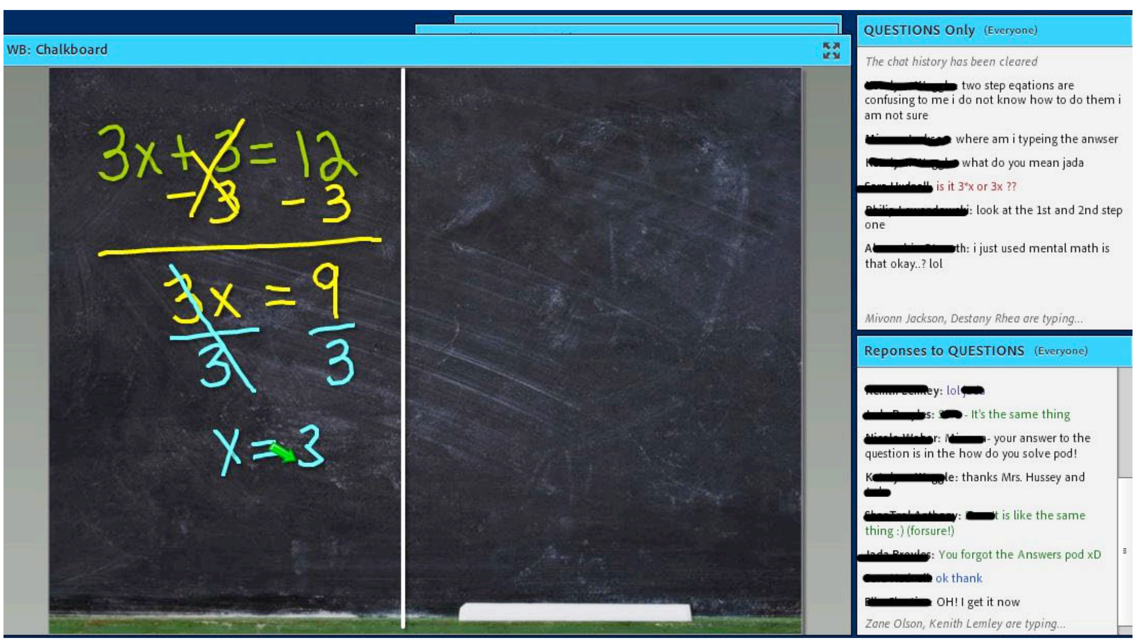

Figure 2:Layout of Episode 2.

Data Coding

The transcript was coded and analyzed in four phases by two researchers. First, we reorganized the individual exchanges by topics of conversation. Second, we segmented both episodes according to the questions that the teacher posed or the type of instructions that the teacher initiated. Then, in each segment, we coded each exchange in three dimensions: (a) whether it was a response or initiation; (b) potential subjects of the exchange (e.g., whether it was addressing the teacher, students or everyone); (c) language functions of each exchange using the Analytical Framework of Peer-Group Interaction (Kumpulainen & Wray, 2002), including experiential (sharing how students experienced the class or the problem), interrogative (asking questions), instructive (providing instructions), affective (sharing feelings and attitudes), judgmental (agreeing or disagreeing), and argumentative (providing arguments). We also added a few language function codes that existed in the data, but not in the Analytical Framework. These codes are observational (providing observations, often about the mathematics), encouraging (students encouraging students), receptive (responding neutrally to a command or idea) and the unclear category. In our last step of data analysis, we identified the mathematical knowledge that was shared and negotiated in both episodes. Relations were mapped out according to the classroom interactions taking place in both cases.

Results

In this section, we will first present findings related to the observed differences between the two episodes according to the number, the sequence, and the nature of exchanges that took place relying on analysis of language functions. We will then discuss how mathematics was shared and negotiated during each discussion session.

Modes of interactions

42 students actively contributed to the chat pods in Episode 1, and 29 in Episode 2. Based on the teacher's instructional initiations, Episode 1 was divided into s segments according to the instructional moves of the teacher. In this episode, only one problem was discussed. Students were given time to solve and comment on the problem in the first two segments. The teacher then raised a few questions for students to consider as they explored the problem. During this time she prepared the virtual whiteboard (segment 3). Once the whiteboard was prepared, the teacher redirected the class to think about the context of the problem, and then led them through solving it (segments 4-5). This part was followed by a Q&A session, which lasted less than one minute (segment 6). Among eleven segments, students contributed 146 exchanges in total, including 20 exchanges that were *initiative* (e.g. "would the first month be 47 because of the 15 dollar fee?"), and 122 exchanges that were *responsive* (e.g. "the one time fee is important"), 112 of which were replies to the teacher's questions. In summary, the mode of interactions in Episode 1was dominated by teacher-student interactions, with a typical pattern of teacher verbal questions, followed by student text responses. The teacher served as the primary initiator of the interactions.

In Episode 2, two group chat pods, the QUESTIONS pod and the EXPLAIN pod, were accessible to students, and two problems were discussed. This episode consisted of five segments. The teacher first reviewed and demonstrated one student's solution (segment 1), followed by a three-minute session (segment 2) devoted to addressing students' questions raised during segment 1. She then posed another similar problem to the class and students were given time to think and respond (segment 3). Then, the teacher again modeled how the problem could be solved on the virtual whiteboard (segment 4) and then devoted over three minutes to respond to students' discussions in the chats (segment 5). In comparison to Episode 1, more evidence of the teacher's facilitation of discussions and student thinking was present in Episode 2; the layout of the "QUESTION" and "Responses to QUESTIONS" pods provided a platform for students to raise questions and invite peer discussions, and for the teacher to facilitate student-student discussions. The teacher spent more than six minutes addressing students' discussions in Episode 2, whereas the teacher spent less than one minute in Q&A during Episode 1. In total, students contributed 65 exchanges with 34 of these exchanges contributed in response to peer comments. This evidenced a higher proportion of student-student interaction compared to the first episode.

Sequence of interaction

In the sessions that we have examined, the sequence of virtual interactions heavily depended on how the conversations were initiated by the teacher (e.g., yes/no questions, comments of the mathematics problem) and the opportunities for responding to the initiation (e.g., the availabilities of chat pods in the layout). When only one chat pod was available in Episode 1, the teacher modeled the solutions of the problem. She was able to maintain a sequential and focused teacher-student interaction by posing closed or dichotomous questions as exemplified in the excerpt below.

123. *T*: If I am paying something everything month, what operation is gonna help achieve that?
124. *Mel*:32*m
125. *Des*: multiplication
126. Mak: multiplication
... ...
140. *Pri*: Multiply
141. *Nic*: I added instead of multiplying OOPS

In the above exchange from lines 124 to 141, students were directly responding to the teacher's question in line 123, though the responses were presented in slightly different ways. Indeed, the interaction was highly sequential in the sense that exchanges could be traced naturally without disruptions. It was also highly spontaneous, since almost twenty exchanges appeared in the chat within a few seconds. Yet, this interaction mode took place only in Episode 1, when dichotomous or closed questions were posed, and responses required short-typed words.

In Episode 2, such a sequential and simultaneous interaction mode was not evident. Instead, interactions between the teacher and students or among students were highly non-sequential and overlapping. Students now had two different chat pods for interaction instead of one, as illustrated in Figures 1 and 2. Because the two chat pods were designated for student questions and student responses to those questions, they promoted student-student interaction in comparison to promoting only teacher-student interaction as in Episode 1. The sequence of interaction became more complex as demonstrated in the excerpt below.

25. (Audio) *T*: Let's do another one I'm gonna throw at you and then we talk about it. Ok?
26. (QUESTION) *Al*: I just used mental math is that okay..?lol
27. (Audio) *T*: How about -2y-4
28. (QUESTION) *Des*: I am confused NO NEVER MIND
29. (QUESTION) *Miv*: what is it called when you put a line in the equal sign
30. (EXPLAIN) *Ken*: xD
31. (Audio) *T*: Try to specific, Des. We will try to help you.
... ...
35. (EXPLAIN) *Sha*: Al-That is fine! As long as you get the right answer. :)

Within these eight exchanges, several conversations were taking place: (a) the teacher was posing a new problem (#25, 27); (b) two students were asking separate questions (#26, 29), one of whom received a student response in #35; and (c) another student was expressing confusion (#28), which caught the teacher's attention in #31. Meanwhile, Ken (#30) may have been responding to either the teacher or the use of the caps lock by Des (#28). We use *linear* to describe the interaction in the Episode 1 excerpt, and *multi-directional*to describe this interaction in the excerpt from Episode 2. Students raised questions in Episode 1, but due to the large number of chat messages in a single chat pod, the questions became invisible; they did not become a source of contemplation or discussion, and never disrupted the lesson flow. In contrast to Episode 1, questions in Episode 2 had a designated place during the teacher's instruction, which made it easier for students and the teacher to notice questions thus providing an available space for group discourse.

Language Functions

Examining the language functions of the texts of student chats, two features distinguished the quality of interactions in the two episodes. First, the proportion of student-self initiated conversations was greater in Episode 2 than in Episode 1. Students openly shared their experiences of the problem (*experiential*), commented on characteristics of the problem (*informative*), and asked questions (*interrogative*). Second, students in Episode 2 engaged in each other's problem solving processes. Some students questioned and answered, commented, and judged each other's mathematics while other students took a lead on sharing information about the problem.(e.g. "*Don't forget, the 4 isn't negative*"). Texts were informative, interrogative, and judgmental inEpisode 2, while texts were mostlyreceptive (e.g., yes, ok), observational, and addressing the teacher in Episode 1.

Taken-and-shared mathematics

In the first half of Episode 1, the teacher asked students to discuss the problems. The teacher's questions provided ample opportunities for diverse student responses, though interactions were one-directional, short, and sequential as the teacher responded to the majority of students' questions or answers. Thus the primary source of knowledge was the teacher who verified answer and confirmed accuracy of responses. Some of the important mathematical information that the individual students provided during the discussion went un-noticed or un-acknowledged. Even more importantly, discussion around these different ideas was not pursued in places where conflicting views on the same question were expressed. Instead, the teacher directed the class, judged the answers, and moved on to a new problem or task. This was partly because student replies in the chat pod were quick, rich and non-uniform, which made it difficult for the teacher to closely follow each contribution. As shown in Figure 3, the content of mathematics discussed became rather linear.

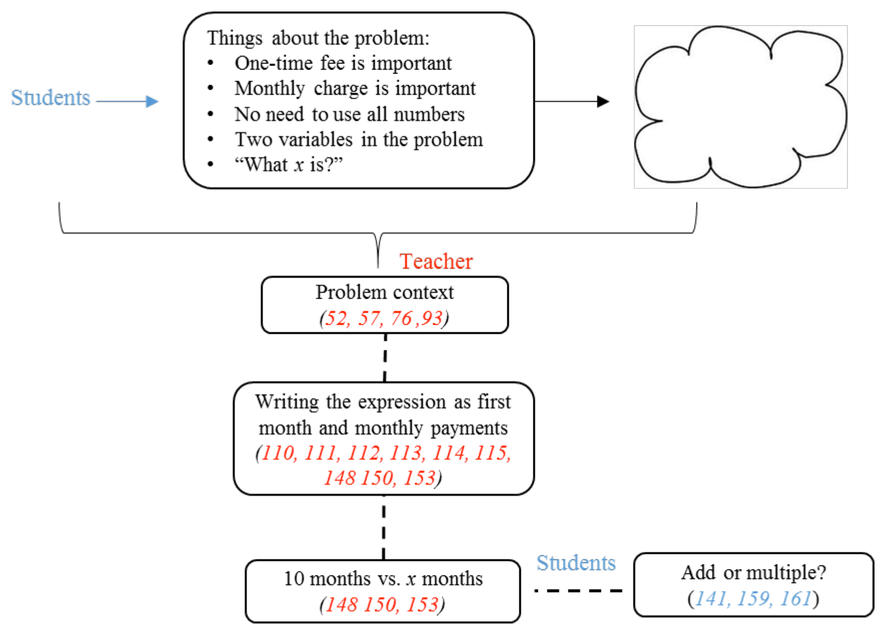

Figure 3: Map of knowledge shared in Episode 1. Blue indicates student contributions. Red indicates teacher contributions.

In Episode 2, mathematical knowledge shared was what we refer to as the "popcorn" style (Figure 4); which characterized features of open and simultaneous discourse. For instance, while the teacher was demonstrating the procedure for solving an equation on the "chalkboard", students'

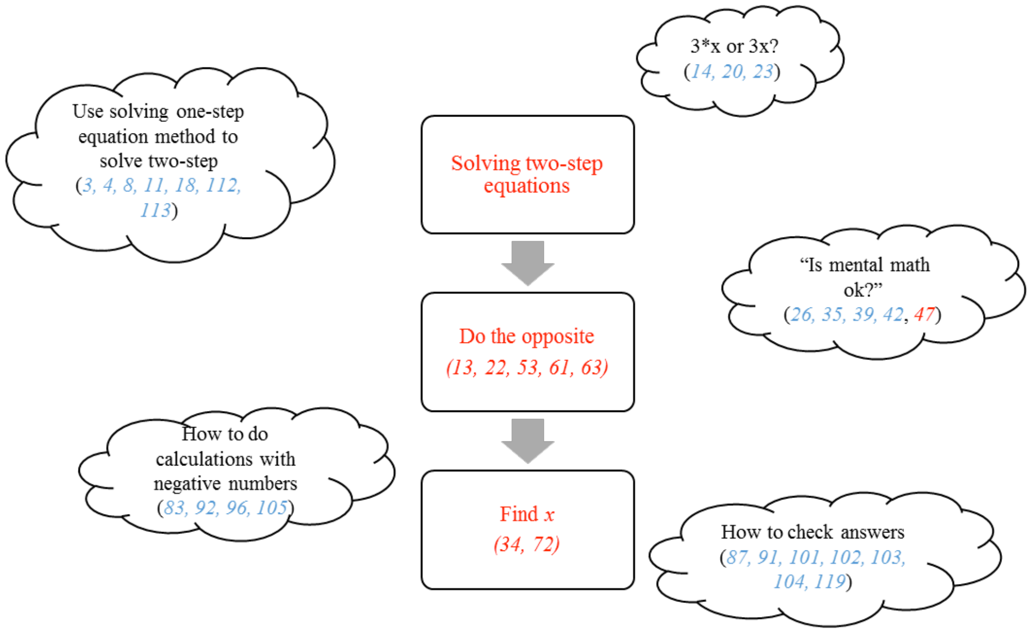

Figure 4: Knowledge shared and negotiated in Episode 2: popcorn style. Blue indicates student contributions. Red indicates teacher contributions.

discussion in the chat pods did not necessarily relate to each step of the teacher's demonstration. The students' comments often corresponded to other issues related to the problem, such as how and when mental arithmetic can be an effective approach to solve two-step equations, negative operations, and the conventions of algebraic expressions, which were separate from the teacher's original agenda of demonstrating the procedure. The content presented by the class was multidirectional. Students who were sitting in front of the computer could follow different "pops" of knowledge shared on the screen and could comment on those that they found interesting or useful. In this episode, students were sharing and negotiating different elements of solving two-step equations and the teacher was no longer the dominating source of knowledge. Instead of acting independently from the students by focusing on her own agenda of the problem's procedure as in Episode 1, the teacher depended on student responses to guide her facilitation of the discussion in Episode 2. The teacher's organization of the chat pods in Episode 2 allowed for greater opportunities for student thinking to be expressed, and for the teacher to have a greater role in facilitating the students' discussion.

Discussion

The purpose of this study was to explore the nature of interactions within online synchronous mathematics lessons resulting from different features of these lessons. By mapping out the transcripts of two episodes, it became evident that classroom interactions in a synchronous environment can be different when the layouts of the classroom were changed. These differences include changes from (1) teacher-students interactions to student-student interactions; (2) a linear conversation sequence to a multi-directional conversation sequence; and (3) students mainly responding to the teacher to interacting with each other. Consequently, these changes effected how students discussed and negotiated mathematics. The literature has documentedthe major feature of virtual interaction as non-sequential and overlapping (Crystal, 2001). Yet, our data showed that a synchronous lesson can be a highly structured lesson guided by the teacher, and communication quite spontaneous and sequenced as it would be in traditional brick-and-mortar classroom interactions. However, as soon as the layout of the lesson changed as in Episode 2, the interactions became non-sequential.

Our results indicate that the teacher's ability to closely monitor the students' interactions and mathematical discussions simultaneously in different chat pods helped effectively initiate and facilitate generative discourse among students. When opportunities were provided for students to negotiate their mathematical knowledge with one another, as seen in Episode 2, the dominating role of the teacher's mathematics was challenged. This issue raises several important questions that merit careful attention from the research community, some of which include: How could mathematics teachers synthesize multiple sources of students' mathematics to advance students' discussions towards building generalizations and identifying important mathematical structures? What type of knowledge might a mathematics teacher need to develop in order to do so? How might teachers manage social interactions using the chat pods or other functions in the environment so as to gear students towards mathematics learning and knowledge construction (Yackel & Cobb, 1996)? With a rapid increase in the desire to converge learning environments towards a virtual paradigm, it seems inevitable that our understanding of how such an educational setting might best be organized and moderated to allow for development of inquiry skills and conceptual understanding of mathematical ideas. The questions we raised here relate explicitly to this agenda.

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A DYNAMIC GEOMETRY-CENTERED TEACHER PROFESSIONAL DEVELOPMENT PROGRAM AND ITS IMPACT

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This study investigated the impact of a dynamic geometry (DG)-centered teacher professional development program on high school geometry teachers' content knowledge and their students' geometry learning. 64 geometry teachers were randomly assigned to an experimental (DG) group and a control group. Both groups received appropriate and relevant professional development. Classroom observation data and the teachers' responses to the implementation questionnaires revealed that most teachers in the DG group were faithful to the DG instructional approach. Teachers in the DG group scored higher on a conjecturing-proving test than did teachers in the control group. The students of teachers in the DG group scored significantly higher than the students of teachers in the control group on a geometry achievement test.

Keywords: Teacher Education-Inservice (Professional Development); *Technology*

Introduction

Dynamic geometry (DG) refers to an active, exploratory study of geometry carried out with the aid of interactive computer software available since the early 1990's that allows for learner knowledge construction and exploration. The most widely used current DG software packages include the Geometers' Sketchpad (Jackiw, 2001), Cabri Geometry (Laborde & Bellmain, 2005) and Geogebra (Hohenwarter, 2001) as well as variations that are applications within handheld graphing calculators or applets on web sites. DG environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena (CCSSI, 2010). With distinguishing features of dragging and measuring, DG software can be used to help students engage in both constructive and deductive geometry (Schoenfeld, 1983) as they build, test and verify conjectures using easily constructible models.

In a funded four-year research project, we conducted repeated randomized control trials to investigate the efficacy of an approach to teaching high school geometry that utilizes DG software as a supplement to regular instructional practices. Our basic hypothesis was that the use of DG software and DG teaching methods that engage students in constructing mathematical ideas through experimentation, observation, data recording, conjecturing, conjecture testing, and proving would result in improved geometry learning experiences for most students. The use of DG software and teaching methods was referred to as the DG approach in the project. The DG software used by the project was the Geometers' Sketchpad (GSP).

In this paper, we report the results from the second year of the project on teacher content knowledge and student achievement. We investigated the impact of the professional development of teachers and their students' geometry achievement in the DG group.The study built upon related research studies on mathematics teachers' professional development (e.g., Carpenter et al., 1989), including those concentrating on technology-centered (and especially DG-centered) professional development (e.g., Meng & Sam, 2011).

Theoretical Framework and Research Questions

An integrative framework (Olive & Makar, 2009) drawing from Constructivism, Instrumentation Theory and Semiotic Mediation was used to guide the study. Within this framework, as teachers and students interact in DG environments, their interactions with the DG technology tool influence the next act by each person, and continue in an interplay between the tool and user. As a user (teacher or student) "drags" an object and observes outcomes from that act, the user adjusts her or his thinking, which in turn influences the next interaction with the tool. Because DG technology allows users to adjust their geometry sketches and the relationships within them, users are transforming the tool, their use of the tool, and their thinking.

This study addresses the following research questions:

- Did teachers in the experimental (DG) group develop stronger conjecturing and proving abilities than did teachers in the control group?
- How well did the teachers implement the DG approach with fidelity in their classrooms?
- Did the students of teachers in the experimental group over a full school year achieve significantly higher scores on a geometry test than did the students of teachers in the control group?

Method

The participants in the study were sampled from the geometry teachers at high schools and some middle schools in Central Texas school districts. The study followed a randomized cluster design, randomly assigning 64 teachers to either an experimental group or a control group receiving relevant professional development, implementing the instructional approaches respectively assigned to them, helping the project staff in administering the pre- and post-tests of the participating students, and participating in other data collection activities of the project.

Professional Development and the DG Treatment

In order to effectively implement the DG approach in their classrooms, teachers must first master the approach. Without professional development, “teachers often fail to implement new approaches faithfully” (Clements et al., 2011, p. 133). So teachers’ professional development (PD) was a critical component of the project. For our PD to be effective, it had to be sustained, rigorous, and relevant to participating teachers, with substantial support from their school districts. Based on these guiding ideas, a weeklong summer institute was offered to the participating teachers in the DG group, followed by 6 half-day Saturday PD sessions during the school year. The PD was planned and implemented collaboratively by project staff that included mathematics and education university faculty members and school-based master teachers selected based on their success as mathematics teachers and their experience with DG software. The project team and master teachers served as partner facilitators for all PD sessions.

The teachers in the experimental (DG) group were actively involved in each PD session and focused on developing their conceptual understanding of mathematics using the DG software as a tool. They worked on challenging problems and developed important geometric concepts, processes, and relationships while building DG skills and teaching methods. They experienced how DG environments encourage mathematical investigations by allowing users to manipulate their geometric constructions to answer "why" and "what if" questions, by allowing them to backtrack easily to try different approaches, and by giving them visual feedback that encourages self-assessment.

Typically, each activity in a PD session consisted of the following instructional events: 1) Presenting a task (exploring concepts/relationships or solving a problem) to the teachers; 2) Requesting teachers to use DG tools to construct the related geometric object or problem situation (with help if necessary) or providing them with a prepared DG environment; 3) Asking teachers what conjecture(s) they can make based on their initial observation; 4) Requesting teachers to use dragging, measuring, and multiple, linked representations to experiment with the constructed or provided DG environment, and observe what characteristics change and what remain the same; 5) Asking teachers to further make and test conjecture(s); 6) Reminding teachers to redo events #4 and #5 in a new aspect or at a higher level, as appropriate; 7) Asking teachers to summarize and reflect on what they have conjectured; and 8) Helping teachers develop explanations to prove or disprove their conjecture(s).

In each PD session, teachers either worked individually at a computer or in small groups. In either case, PD facilitators encouraged teachers to share ideas and help each other. The facilitators circulated, observed (to monitor the progress) asked questions, and provided necessary assistance. They also initiated whole group discussions as needed.

In terms of content, the summer PD sessions concentrated on important and commonly taught topics of high school geometry: triangle congruence and similarity, properties of special quadrilaterals, properties of circles, and geometric transformations. School year follow-up PD aligned with the course scope and sequence determined by the participating school districts.

The PD facilitators modeled what teachers were expected to do with their students in geometry investigations. To help teachers change their instructional practices, their engagement of students, and how they facilitated student learning, mathematical explorations were always followed by discussions on questions such as “How will you teach this content using DG software?” and “How will you lead your students in conjecturing and proving using DG software?” The PD facilitators valued teachers learning from each other and sharing ideas and also sought to provide opportunities to apply new teaching skills. Therefore, teachers were encouraged to present their insights on and experiences with the DG approach and to describe problems they might have experienced or anticipated with other teachers offering suggestions to address the concern. Teachers also prepared lesson plans that they shared with the entire group.

The Control Group

The teachers in the control group taught geometry as they had done before. They also participated in a PD workshop that addressed the same mathematical content as the DG group but without the use of technology. The PD sessions for the control group utilized teaching methods with which teachers were already familiar. The PD facilitators lectured and involved teachers in activity-based instruction. Participants engaged in problem solving without using technology tools. They spent the same amount of time in PD training as the teachers in the DG group. The control group PD was included in the research design to control the variables tied to professional development and ensure both groups experienced sustained, rigorous, and relevant development in high school geometry teaching. Since all teachers participated in PD sessions and all were presented with the same mathematics content, any differences measured between the two groups would be attributed to the presence (or lack of) the interactive DG learning environment (since it was the only instructional difference between the two groups).

Measures and Data Analysis

Measures

A measure of teachers’ conjecturing-proving knowledge. A conjecturing-proving test was developed by the project team to measure teacher knowledge. As a result of a thorough literature review, geometry construct development, item construction, Advisory Board members’ review, and several pilot tests with resulting revisions, a test consisting of 26 multiple-choice items and 2 free-response proofs was produced. The test was administered to the teacher participants as both a pre- and post-test at the PD summer institute.

Teachers' implementation fidelity and classroom observations. The DG approach involves intensive use of dynamic software in classroom teaching to facilitate students' geometric learning. The critical features of the DG approach include using the dynamic visualization to foster students' conjecturing spirit, their habit of focusing on relationships and explaining what is observed, and their logical reasoning desire and abilities. To capture these critical features of the DG approach, two measures of implementation fidelity (the DG Implementation Questionnaire [DGIQ] and the Dynamic Geometry Observation Protocol [DGOP]) were developed. The DGIQ was adapted from a teacher questionnaire developed by the University of Chicago researchers (Dr. Jeanne Century and her colleagues) in an NSF-funded project, based on the critical features of the DG approach. The final version of the DGIQ consisting of six multiple-choice items and ten open-response questions was administered to the teachers in the experimental group six times across the school year. A different version of the questionnaire was administered to the control group teachers (also six times) to examine how they teach geometry without using dynamic technology.

The DGOP was developed to address the critical features of the DG approach. It was adapted from the *Reformed Teaching Observation Protocol* (Sawada et al., 2002). The final version of the DGOP consisted of 25 items with a 4-point Likert response scale from *Never Occurred* to*Very Descriptive* addressing four different aspects: (1) *Description of intended dynamic geometry lesson, (2) Description of implemented dynamic geometry lesson, (3) Assessment of quality of teaching, and (4) Assessment of engagement and discourse*. For the control group, an observation protocol (CGOP) was developed by removing from the DGOP items related to the implementation of DG software functions such as dragging and dynamic measuring. The DGOP or CGOP was administered in 16 Geometry classrooms (8 selected from each group). Each classroom was visited by two observers. Each selected teacher was observed four or five times across the school year.

Student level measures. Two instruments were used for measuring students' geometry knowledge and skills: (for the pre-test) Entering Geometry Test (ENT) used by Usiskin (1981) and his colleagues at University of Chicago; and (for the post-test) Exiting Geometry Test (XGT). The XGT was developed by selecting items from California Standards Tests – Geometry. The final version for XGT had 25 multiple-choice items. (See Jiang et al., 2011 for the details of the two tests.)

All research instruments mentioned above, except the student geometry pre-test, were developed by the project team. For all project-developed measures, the Cronbach's Alpha statistical values were within the acceptable ranges for reliability (e.g., reliability was calculated with Cronbach's alphas of 0.957 and 0.952 for the DGOP and CGOP, respectively.) Item Response Theory (IRT) scoring routines were applied to the DGIQ and students' post-test data providing evidence for the instruments' construct validity.

Data Analysis

Two-level hierarchical linear modeling (HLM), other statistical methods, and the constant comparison method (Glaser & Strauss, 1967) were employed to analyse the quantitative and qualitative data.

Results

Findings about Teacher Content Knowledge from the Conjecturing-Proving Test

The participating teachers completed the conjecturing-proving test at the beginning and end of the summer PD institute. A statistic for teacher content knowledge as measured by the instrument was calculated by adding the number of correct multiple-choice responses with points from free-response items. Average scores were 20.49 on the pre-test and 21.86 on the post-test with an average gain of 1.37. A paired-sample t-test showed that this gain was statistically significant ($p = .003$). These results show that the PD had a positive effect on teachers' conjecturing and proving capabilities. The teachers in the experimental group showed a greater average gain (1.56) than the teachers in the control group (1.18); however, this difference was not statistically significant ($p = .670$).

Findings about DG Approach Teaching from the Classroom Observations

Table 1 provides the results of the DGOP administration measuring the levels of fidelity of the dynamic approach implementation in the DG group. If we focus our attention to the mean scores (with a maximum score of 4) for the DG group, we observe that the three aspects with the highest scores were Good Lesson Design, Use of DG Features, and Teachers' Knowledge. The data provides evidence that the teachers in the DG group demonstrated an intention to implement the DG approach and to some extent they demonstrated knowledge about how to integrate the dynamic approach to teaching geometry. Overall, teachers in the DG group were implementing the DG approach at a moderate level (2.28). In part, this moderate level of implementation was explained by the challenges reported during the school year such as the inaccessibility of computer labs in the first several weeks and the pressure to spend time preparing for the state required tests. However, the majority of the classrooms observed can be described as being faithful to the DG teaching approach.

Table 1: Comparison between DG and Control Groups

Aspect	Sub-aspect	Mean DG	Mean Control	*p*-value
Intended Dynamic Lesson	Good lesson design	2.81	1.85	.032*
	Use of dynamic features	2.75	0.70	.000*
Implementation	Actions beyond use of software	2.06	1.33	.095
Quality of Teaching	Cognitive demand	2.30	1.78	.113
	Teachers' knowledge	2.89	2.84	.924
	Conjecture/Proof	1.93	1.40	.206
Engagement and Discourse		2.37	2.29	.735
Overall DGOP		2.28	1.68	.088

Comparing the two groups, Table 1 also shows the mean values of the CGOP and the *p*-values assessing the significance of the treatment effect computed using a mixed effect ANOVA. Results confirm the efficacy of the DG treatment by showing significant differences in the two aspects related to the intention to implement a dynamic lesson. As a whole, lessons in the DG group had a significantly better design aligned with the DG teaching approach, moving students from initial conjecture, to investigation, to more thoughtful conjecture, to verification and ultimately to proof. Further, lessons in the control group did not use dynamic features in teaching geometry. With respect to the other aspects of the DGOP (or CGOP), the two groups did not differ significantly, however all the DG ratings were higher than those of the control group. Note that most of those aspects assessed elements of the lesson that were not related to the use of dynamic features.

Findings about DG Approach Teaching from the Implementation Questionnaire

The purpose of the DGIQ was to assess the DG group teachers' effectiveness and comfort in using GSP in teaching geometry. Also, the questionnaire results provided the frequency of teacher and student use of GSP. Figure 1 shows how the teachers rated themselves on their effectiveness and comfort in using GSP. Out of 31 teachers who completed the questionnaire, 29% felt that they were at the high level of effectiveness, 61% at the middle level, and 10% at the low level. However, the majority of the teachers (97%) felt very comfortable or somewhat comfortable in using GSP in teaching. Overall, the teachers felt more comfortable than effective in using GSP with only one teacher not feeling comfortable in using GSP in teaching of geometry.

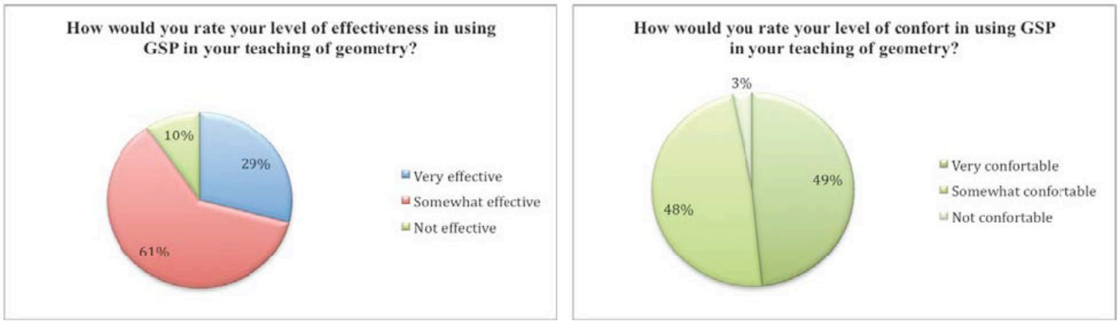

Figure 1: Effectiveness in Using GSP and Level of Comfort in Using GSP

Figure 2 shows average teacher and student use of GSP throughout the school year for those in the DG group. The "average teacher use of GSP" represents the average number of times per week the teacher used the demonstration computer in his/her classroom to do GSP presentations and demonstrations. The "average student use of GSP" represents the average number of times per week students worked in a computer lab doing hands-on explorations with GSP. Out of the 31 teachers who completed the questionnaire, 77% of them used GSP at least one time per week and 38% at least two times per week. However, the student use was lower, with 61% of them using GSP at least one time per week, and only 10% two times per week.

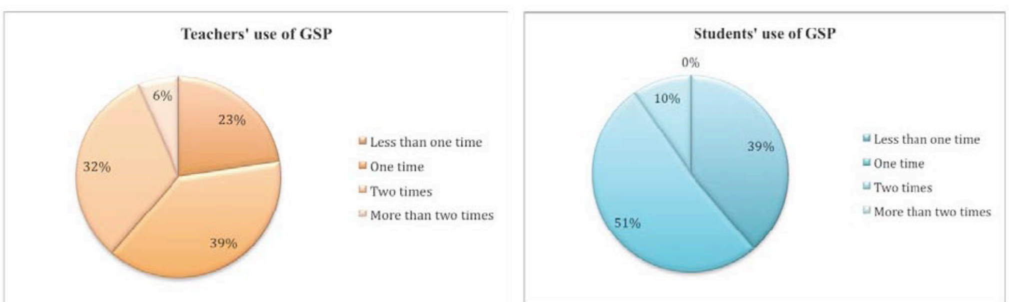

Figure 2: Average Teacher Use of GSP and Average Student Use of GSP per Week

Therefore, in terms of "taking students to the computer lab to do hands-on activities with GSP," the teachers' implementation of the DG approach was at the medium level of intensity. This finding is consistent with data from the classroom observations. However, almost all teachers were positive or enthusiastic in using GSP in geometry teaching. Again, considering the challenges that the teachers experienced during the school year, data supported the conclusion that most of the teachers implemented the DG approach faithfully.

Even though some teachers in the DG group might have not felt as effective in using GSP because of the students' limited use of GSP (one time or less than one time per week), some of them might still be considered very effective if we focus on the ways they used the DG approach. One teacher provided such an example. He felt "somewhat effective" and his students used GSP on average one time per week, but his classroom observations showed very effective use of GSP. During one of the observations, his students were exploring the midsegments of a triangle and their goal was to come up with as many conjectures as possible. Students completed the constructions on their own, made initial conjectures based on their observations, used measurements to confirm their conjectures, and wrote their final conjectures. The teacher circulated among students and provided guiding questions when needed. One student made many measurements but no conjectures. The teacher asked this student, "Do you notice any relationships? What conjectures can you make?" These questions helped the student focus on the objective of the lesson and form conjectures based on the measurements and observations. During the lesson, students also engaged in conversations with one another to discuss their observations and conjectures. Students were actively involved in their learning and the teacher took on the role of a guide by prompting his students through questioning. This lesson not only showed effective use of GSP, but also addressed higher-level thinking.

Findings about Student Achievement from the Geometry Test

Two-level hierarchical linear modeling (HLM) was employed to model the impact of the use of the DG approach on overall student geometry achievement measured by the student post-test (XGT). The model was analysed using student pre-test (ENT) scores as a covariate. The sample of classrooms studied included three different levels of Geometry: Regular, Pre-AP and Middle School (middle school students taking Pre-AP Geometry). Since the classroom expectation and quality of the students in each of these levels were very different, the factor *Class Level* was included in the model. Additionally, the covariate *Years Exp* (number of years of classroom experience) was included in the model. The results of the model indicated that the DG effect was strongly significant ($p = .002$). Comparing the means, the DG group was higher than the control group in each level of Geometry and the effect size (.45) was substantially larger at the Regular Geometry level. (See Jiang et al., 2011 for the details of the HLM analysis results.)

Using the integrative framework (Olive & Makar, 2009) as a lens, further quantitative and qualitative data analysis on the impact of the DG professional development is ongoing.

Discussion

The HLM model taking pretest, class level, and teaching experience into account provided evidence that the students of DG group teachers scored significantly higher than the students of control group teachers on the Exiting Geometry Test. Given that teachers were randomly assigned to the two groups and both groups received comparable sustained, rigorous, and relevant professional development on the same geometry topics, the results of this study provide evidence to support the finding that the DG professional development positively impacted the students' geometry achievement. Both DG and control group teachers demonstrated significant gains on the Conjecturing-Proving Test through the one-week summer PD institute. This result suggests that both the PD sessions designed for the DG group and those designed for the control group had an effect on teachers' conjecturing and proving ability. Although the DG and control teachers did not differ significantly on their mean gain scores, the DG teachers' mean gain score was 32% higher than that of the control teachers. Classroom observation data revealed that lesson plans that the DG group teachers prepared were designed significantly better than the control group teachers' lessons by facilitating students' conjecturing and proving abilities. The teachers' DGOP ratings (overall and in each sub-scale) were consistently higher for the DG group although most of the differences were not statistically significant. In summary, the results of this study suggest that the DG professional development offered to the participating teachers had a significant positive effect on the teachers' mathematics conjecturing-proving content knowledge and their ability to implement a dynamic geometry approach to teaching. The teachers, in turn, helped their students achieve better geometry learning.

Acknowledgments

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DYNAMIC GEOMETRY SOFTWARE AND TRACING TANGENTS IN THE CONTEXT OF THE MEAN VALUE THEOREM

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In this paper we analyze and discuss the postgraduate students' performance related to the tracing of tangent lines to the curve of a quadratic function within Dynamic Geometry Software in the context of Mean Value Theorem. The purpose is to show the possibility of using Dynamic Geometry in promoting learning of such Theorem, based on its geometric interpretation. The theoretical elements adopted in this study are based on the instrumental approach to tool use. The results illustrate the epistemic role of the Dynamic Geometry Technique, as well as the difficulties associated with their paper-and-pencil Techniques.

Keywords: Technology; Advanced Mathematical Thinking; Teacher Education-Inservice

Background

In literature, there is evidence about the influence of using Dynamic Geometry Software (DGS) to encourage students' mathematical thinking (e.g., Guven, 2008; Leung, Chan & Lopez-Real, 2006; Reyes & Santos, 2009). In these studies the possibility of using dynamic geometry is raised to discuss mathematical relationships exploring different cases. In particular, Guven (2008) and Reyes and Santos (2009) show how the dragging and the locus, that emerge in the explorations that let the DGS, promotes the development of conjectures about the mathematical relationships of the objects embedded in the mathematical dynamic model, in the sense of Reyes and Santos. From these studies, and the interest of the community in analyzing the influence of the technological environments in teaching and learning calculus (Ferrara, Pratt & Robutti, 2006), we propose the possibility of using DGS to analyze its potential in promoting the learning of Mean Value Theorem (MVT), based on its geometric interpretation.

In Ferrara, Pratt and Robutti (2006) a study compilation is included about central concepts of calculus such as function, limit, derivative and integral in technological environments; from these backgrounds, we consider that it is important to research the role of technology in the learning of calculus in which the concept of derivate is embedded. Therefore, this paper focuses on studying the use of DGS in learning the MVT through its dynamic modeling.

The understanding of MVT is important because it is the base of contents like the criteria for maximums and minimums. Taking into account the use of DGS allows us to approach the MVT and the mathematical concepts associated, through mathematical dynamic models, and not just in analytic ways, as it is usually presented in textbooks. In this sense, in the dynamic geometry environments, the approach to MVT can be done by tracing tangent lines to the curve of a function in the context of its geometrical interpretation. Thus, the aim of this study is to answer the question: how does the use of dynamic geometry influence the tracing of tangents to the curve of a particular function in the context of the MVT, based on its geometric interpretation?

Theoretical Framework

The theoretical framework adopted in our study is *the instrumental approach to tool use* (Artigue, 2002; Lagrange, 2003, 2005). The use of this approach in dynamic geometry environments is feasible (Leung, Chan & Lopez-Real, 2006). According to Artigue (2002) the instrumental approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). In this sense, there are two possible directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics (Monaghan, 2007). In the former, the focus is the development of mental schemes within the process of instrumental genesis (Drijvers & Trouche, 2008). Within this direction, an essential point is the distinction between artifact and instrument.

In line with Chevallard's theory, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology. According to Chevallard (1999), mathematical objects emerge in a system of practices (*praxeologies*) that are characterized by four components: *task*, in which the object is embedded (and expressed in terms of verbs); *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology.

Artigue (2002) and her colleagues have reduced Chevallard's four components to three: *Task*, *Technique* and *Theory*. The term *Theory* combines Chevallards's technology and theory components. The *Technique* is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values; techniques are most often perceived and evaluated in terms of its pragmatic value, but their epistemic value contribute to the understanding of the objects they involve, that is to say, they are a source of questions about mathematical knowledge (Artigue, 2002, p. 248). According to Lagrange (2003), *Technique* is a way of doing a *Task* and it plays a pragmatic role (in the sense of accomplishing the task) and an epistemic role in that it contributes to an understanding of the mathematical object that it handles during its elaboration; it also promotes conceptual reflection when the technique is compared with other techniques and when discussed with regard to consistency. The consistency and effectiveness of a *Technique*, according to Lagrange (2005) are discussed in a theoretical level; mathematical concepts and properties and a specific language appear.

Our study is in line with the anthropological theory of didactics; thus the focus of this research is the epistemic value of technique. That is, we are interested in studying the students' Techniques they develop within the dynamic geometry environment.

The Study

In the present paper, we discuss and report the results of the designed Activity. Its rationale, the population and the data collection, is detailed bellow.

The Design of the Activity

The design took into consideration the Anthropological line of the instrumental approach. Thus, the three elements *Task, Technique and Theory* were used. The Activity, as Kiernan and Saldanha (2008) note, is a set of questions related to a central Task. In our case, the *Task* is "Drawing a tangent line to the curve of the quadratic function $f(x) = -(x - 3)^2 + 4$ and parallel to a secant line to the curve. The Activity consisted of two phases; the first one involves just working with paper-and-pencil, in order to know the techniques used by the participants in this environment. The second one includes working with the DGS, in order to know how the use of DGS influences and modifies the initial participants' techniques and what other emerges; both phases include technical and theoretical questions. The DGS used was GeoGebra.

The Task consists in given the function $f(x) = -(x - 3)^2 + 4$, participants are asked to plot the curve and draw a secant line to the curve and determine its equation (blue line, Figure 1). Once this part of the Task is completed, participants are asked a Theoretical question related to whether or not a tangent line (red line) to the curve and parallel to the secant line could be traced (i.e., the geometrical interpretation of MVT). If the answer is affirmative, they are asked to trace and determine its equation, first in a paper-and-pencil environment; then, using DGS (with the restriction that differential calculus techniques are not allowed in this environment). The Figure 1 shows a graphic representation of the proposed Task.

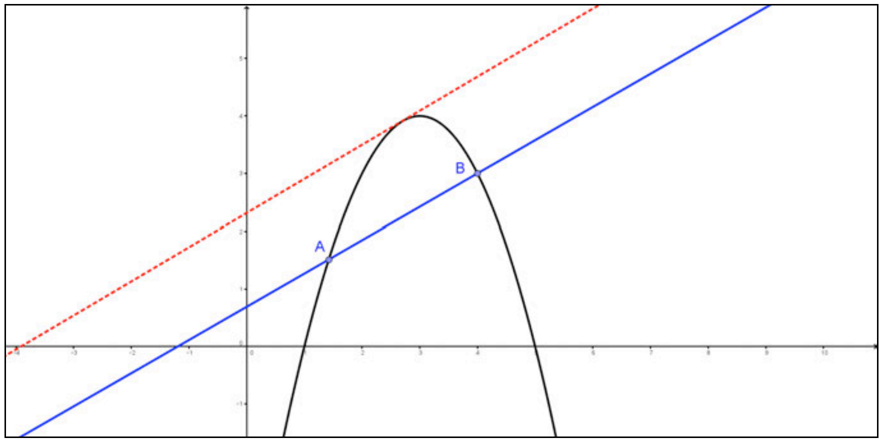


Figure 1: Graphic representation of the Task (graphic interpretation of MVT)

Population

The participants were 16 postgraduate students enrolled in a master program in the teaching of mathematics in Mexico. At the time of collecting data, they were in the 4th semester of the Master's degree. All participants knew GeoGebra and were familiar, at least a year and a half, with this software. All participants, except one, have teaching experience, some of them in university-level, others in senior-secondary-level and just a few in secondary-level. The professional degrees of the participants were among graduates in mathematics, engineers in different areas and one economist.

Implementation of the study

The data collection was carried out in three sessions, during one of the Master's degree courses conducted by one the researchers; each session lasted around 2 hours, which were recorded. The students worked in self-created pairs, in order to promote the dialogue among them and consequently make an audio recording about their own reflections in the use of GeoGebra according to the Task. Each team had a printed Activity, the GeoGebra software installed in their laptops, besides the SCREEN2EXE software which captures the computer screen in order to view the sequence of the students' work with the DGS. In this way, the research data sources include worksheets (printed Activity), the GeoGebra files, SCREEN2EXE files, video recorded files and the researcher's field notes.

Analysis and Discussion of Data

In this report we analyze and discuss the work of four pairs (Teams I, II, III and IV, henceforth), which exemplify the work done by all participants. The analysis conducted is of a qualitative nature inasmuch as we are interested in providing a detailed account of the kinds of Techniques that the participants used to solve the Task in both environments and the Theory they sustain. The analysis makes emphasis in the dynamic geometry techniques which were used by the students; that is to say, we are interested in research the kind of mathematical relations which they identified in the dynamic model of the Task that lead them to solve it.

On the paper-and-pencil work

The paper-and-pencil techniques and the Theory are based on the differential calculus. That is to say, on the usual procedure to determine a tangent line associating the function derivative with the slop of the tangents' family lines to the curve, although the kind of proposed function influenced in their reflections too. About the theory that sustains whether the possibility or not of drawing a tangent line with the required conditions, the participants refer the continuity of the function. Some of them describe it in an explicit way; others in the opposite way. The following Figures show the work of two Teams, that proves what is expressed, once they construct the graphic of the proposed function and pose the secant line equation to the referred function.

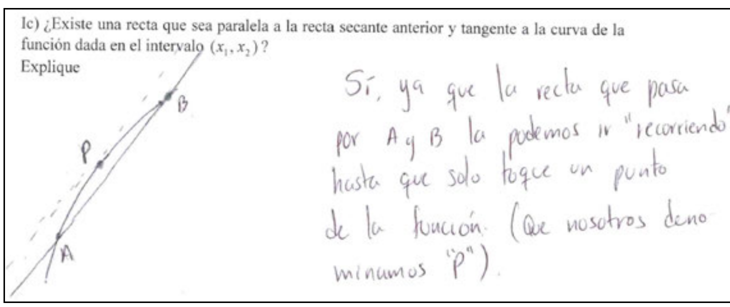
Ic) ¿Existe una recta que sea paralela a la recta secante anterior y tangente a la curva de la función dada en el intervalo (x_1, x_2)?
Explique

1. Si, porque la función es continua en el intervalo.

Ic) Is there a parallel line to the secant and tangent to the curve of the given function in the interval (x_1, x_2)
Explain it
Yes, because the function is a continuous one in the interval.

Figure 2: Theory from Team I

To the question Ic) whether it will be possible to trace a line that it is parallel to the secant and tangent to the curve in a certain interval by the abscissas of the points where the secant line cuts the function of the graphic, the Team I sustains its Technique in the continuity concept of the function (Figure 2). Other Teams do not express their answers in an explicit way, for example, the Team II (see Figure 3). Note that when the students describe that it is possible dragging the secant line, based on a dynamic model of the Task, they demonstrate and idea of continuity of the function.

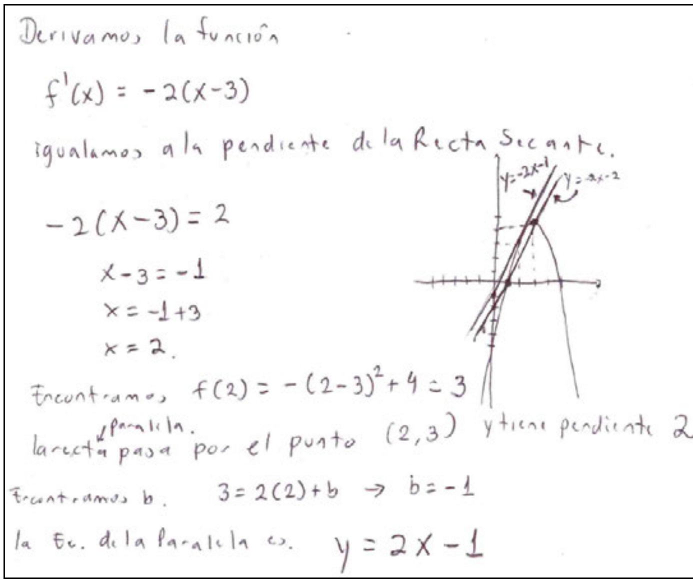


Yes, because we can dragg the line that passes through points A and B until it touches a single point of the function (which we denote by P).

Figure 3: Theory from Team II

Other teams justify their Techniques from their knowledge about the parabola; in particular about one of its specific points, the vertex. The Teams, which worked in this way, propose a parallel secant line to the axis of the abscissas, noticing the vertex of the parabola as the tangent point. However, they also showed ideas about the continuity of the function when they work in the dynamic geometry environment, as later discussed.

Because the participants' previous knowledge (Technique and Theory), it was expected that they would use differential calculus techniques to find and trace the tangent line equation. The analysis of the answers confirms this idea. Once the function is charted and the secant line equation to the curve is determined, those Teams that propose a secant non parallel to the axis of the abscissas use the derivate of the function to find the tangent point from solving the equation $f'(x) = m$, (where m is the slope of the secant line). The Figure 4 shows this Technique (*Derivate Technique*) from Team I.



We derived the [given] function
$f'(x) = -2(x-3)$
We equate to the slope of the secant line
$-2(x-3) = 2$
.
.
.
$x = 2$
We calculate $f(2) = \ldots = 3$

The parallel line passes through the point (2,3) and has slope 2

We find b $3 = 2(2) + b \rightarrow b = -1$

The equation of the parallel line is $y = 2x - 1$

Figure 4: Technique from Team I

In other hand, the Teams that traced the parallel secant line to the axis of the abscissas used an Analytic Geometry Technique taking the maximum of the function like the tangent point. That is to say, they solve the Task using the equation $y = k$, where they identify k like the ordinate to the vertex with coordinates (h,k) of the parabola $y = a(x-h)^2 + k$.

On the Dynamic Geometry work

The work from the Teams in the GeoGebra environment, which was asked not to use differential calculus techniques, shows three characteristics. Some of them use *Algebraic Techniques* provoked by the dynamic model of the Task, and they use the DGS to trace their answers. Other Teams used the geometry dynamic characteristics and specific GeoGebra commands, which we called *Dynamic Geometry Techniques*, to model the Task and look for the mathematical relations that it involves. Others did not consider the use of DGS to explore mathematical relations, because it is taken as obvious. Next, we present examples from each one of these cases.

The Team III was one of those which traced the parallel secant line to the abscissas axis (as it is shown in Figure 5). On the offered explanations by one of the Team members (Student A) it is found that, for them, the answer to the Task is obvious, based on their work that they developed with paper-and-pencil. The following extract illustrates this case.

> Student A: What we did was tracing this [shows a point that was traced on the function of the graphic], and then, traced the parallel [to the secant] […] we moved it, moved it, moved it, move it [they dragged it] until the tangent point was found, which it is easy for us because it is the parabola vertex.

As it is shown in the transcription, the students used parallel and dragging commands as Technique. This let them trace a parallel to the secant line and that passes through a point (traced by them) over the graphic of the function. The mathematical relationship which is shown in their dynamic model is that the parallel line that passes through the parabola vertex fits with the given conditions of the Task. Nevertheless, the tangent point is known in advance, it is not a result from the explorations in the DGS. This is to say, their paper-and-pencil Technique and Theory (their knowledge about the parabola) let them solve the Task in the dynamic environment. The answer that

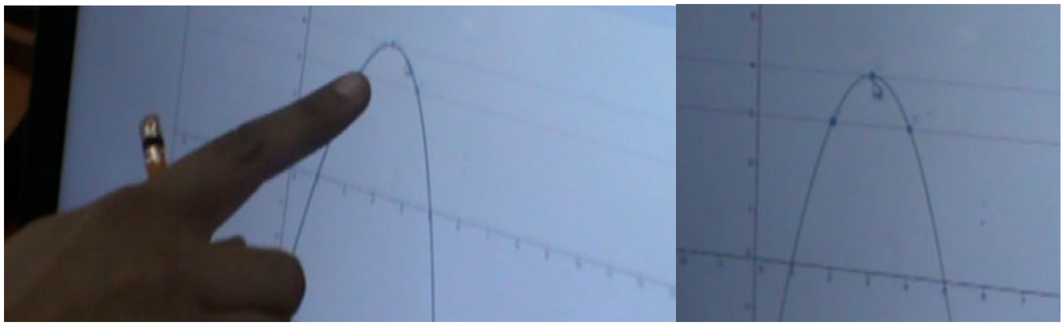


Figure 5: Dragging Technique from Team III

they give is particular, because if the secant line conditions are changed, the line which is proposed by them as the solution will not be the tangent, it just will keep the parallelism condition.

Meanwhile, for the Team I, the dynamic model leads them to try different paper-and-pencil techniques that differ from the calculus. They observed a mathematical relationship in the dynamic model of the Task which led them to the solution; it consisted of a system of equations with the parabola equation *($y = -(x - 3)^2 + 4$)* and the equation of the tangent line *($y = mx + b$)* so that the solution has multiplicity 2 (to fit with the tangent condition), where m is the value of the slope of the secant (in order to fulfill with the parallelism condition). In this way, they calculate the value of the parameter b (of the tangent line). Figure 6 shows this algebraic Technique.

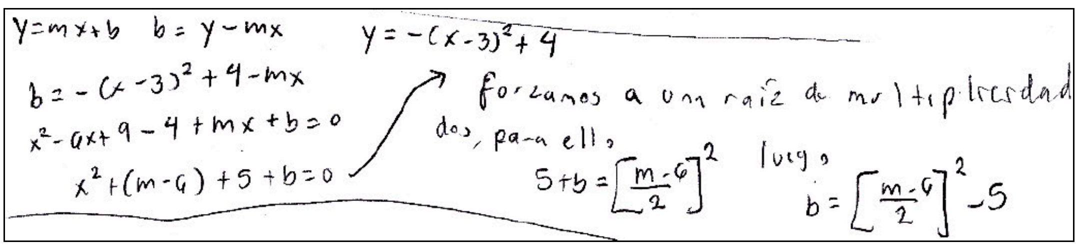
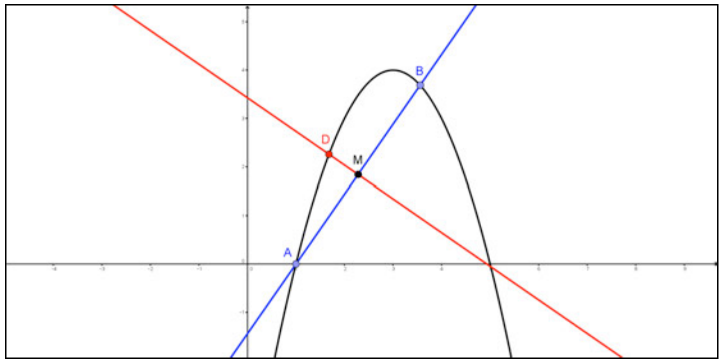


Figure 6: Algebraic Technique from the Team I

The given solution from the Team I is general; however, their answer is not supported by any Dynamic Geometry Technique, but it is an algebraic process. GeoGebra encouraged them to reflect in other alternative paper-with-pencil Technique. Once they found an expression for b in terms of m, they use GeoGebra to graph the equation with these parameters.

The Team II explored in the dynamic model by dragging through a slider as a possible Technique to solve the Task. Nevertheless, this technique lets them an approximation to the solution of the Task. This Team introduced in GeoGebra the requested equation $y = mx + b$ of the line, where m is the same as the value of the secant slope (with this, the condition of parallelism is completed) and associated the parameter b, the ordinate to the origin, with a slider; in order that they manipulate the slider (dragging the line) and propose a solution when, by trial and error, they observe in the graphic representation that the line fulfills the tangent conditions. However, they are aware, using the zoom Technique that their answer is just an approximation.

Finally, the Team IV tried to solve the Task, in the GeoGebra environment, based on the possibilities that the DGS offers. It is interesting how this Team, using the dynamic model, observes mathematical relations in the dynamic construction that they propose in GeoGebra (Figure 7). For this Team, the explorations they did in the DGS let them to conjecture that the middle point M from the segment AB (see Figure 7), the points where the secant (blue line of the Figure) crosses the curve; in particular, the perpendicular bisector (red line) of this segment, lead them to the find the tangent point.



1. A, B en la función 2. Recta que pasa por AB 3. Punto medio entre AB → Perpendicular en este punto. 4. Punto de Intersección entre f(x) y la perpendicular. Sin embargo al comprobar con la ecuación original no es el mismo punto de tangencia calculado analíticamente.	*1. A, B on the function.* *2. Line that passes through AB.* *3. Midpoint point of AB segment, perpendicular [to the AB segment] in this point.* *4. Intersection point between f(x) and the perpendicular.* *However, when comparing with the original equation, we found out that it is not the same tangent point calculated analytically.*

Figure 7: Dynamic Geometry Technique from Team IV

In this case, the students established the tangent point as the intersection of the perpendicular bisector and the function of the curve (Point D in Figure 7). Nevertheless, they found out that this Technique do not lead them to the solution, making comparisons with their initial paper-and-pencil Technique. The most important thing about the work from Team IV is to realize that the dynamic model of the Task let them make conjectures about the midpoint of the segment AB, which, actually, it is related to the solution of the problem.

Conclusions

The offered examples shown in the previous section let us know the influence of the dynamic geometry software, from the participants' Technique and Theory, in the tangent traces. In the paper-and-pencil environment, two types of Techniques are identified, one based on the differential calculus knowledge; the other one, based on the analytic geometry knowledge. Regarding the Theory, it is related to the concept of continuity of the function.

The influence of dynamic geometry is shown in those Teams which use a dynamic model that do not use a parallel secant line to the abscissas axis. Therefore, in one side, the DGS encourages the emergence of paper-and-pencil Techniques based on the explorations in the dynamic model. In other hand, DGS lets them work with their own Techniques and commands from the dynamic geometry and establish relationships between the mathematical objects involved and the emergence of others (for example the perpendicular bisector). It also let them contrast paper-and-pencil Techniques with the software Techniques. According to the instrumental approach, this contrast of Techniques encourages to a reflection in a theoretical level.

In this way, the emergence of new paper-and-pencil Techniques and the reflections about the mathematical relationships provoked by the dynamic models in the use of the DGS show the epistemic role of the dynamic geometry Technique in the trace of tangent lines in the TVM context.

In addition, the results show difficulties associated with their paper-and-pencil Techniques in the sense of holding to this environment, and not exploring the software potentials.

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RELATIONSHIPS BETWEEN PROSPECTIVE MATHEMATICS TEACHERS' BELIEFS AND TPACK

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We examine relationships between prospective teachers' (PTs') beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology, and their knowledge of how to use technology to teach and learn mathematics. We interviewed 4 PTs and used Ernest's (1989) classification of beliefs and Goos, Galbraith, Renshaw, and Geiger's (2003) perspectives of technology to uncover PTs' beliefs. We examined PTs' knowledge of using technology by conducting a task-based interview based on the TPACK framework (Mishra & Koehler, 2006). We found there appeared to be relationships between PTs' beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology and their content knowledge, pedagogical content knowledge, and technological pedagogical content knowledge respectively.

Keywords: Teacher Beliefs; Teacher Education-Preservice; Teacher Knowledge; Technology

One of the guiding principles in the National Council of Teachers of Mathematics' (NCTM, 2014) *Principles to Actions: Ensuring Mathematics Success for All*is the use of tools and technology to explore and make sense of mathematics, reason mathematically, and communicate mathematical thinking. NCTM suggests when tools and technology are used appropriately they support effective teaching and promote meaningful learning. However, some teachers are reluctant to use technology to teach mathematics or do not use it in meaningful ways (Ertmer, 2005). Ertmer (1999) describes two types of barriers to teachers' integration of technology. First-order barriers are obstacles that can be eliminated if money is allocated. These barriers include resources such as access to digital tools, software, Internet, and time to plan and teach technology-based lessons. First-order barriers also include technology training and support, which contribute to teachers' knowledge of technology and how to integrate it into their practice. Second-order barriers are "typically rooted in teachers' underlying beliefs about teaching and learning and may not be immediately apparent to others or even to the teachers themselves" (Ertmer, 1999, p. 51). Second-order barriers are less tangible, more personal, and more deeply ingrained than first-order barriers (Ertmer, 1999). Moreover, research indicates second-order barriers are prevalent among teachers (e.g., Hermans, Tondeur, Valcke, & van Braak, 2008). In order to assist teachers to better overcome both types of barriers, we must understand the relationships that exist among them. In this paper, we share the results of our study in which we examined relationships between prospective teachers' (PTs') knowledge of how to use technology in the teaching and learning of geometry (first-order barriers) and their beliefs about the nature of mathematics, teaching, learning, and the use of technology (second-order barriers).

Related Literature Review and Theoretical Framework

Researchers (e.g., Cooney & Wilson, 1993; Pajares, 1992; Wilkins, 2008) have examined the relationships between beliefs and knowledge and found that one seems to influence the other. Pajares (1992) stated "beliefs influence knowledge acquisition and interpretation, task definition and selection, interpretation of course content, and comprehension monitoring" (p. 328). In other words, teachers' beliefs will influence the ways in which their knowledge is created and their instructional decisions. Conversely, Cooney and Wilson (1993) stated, "beliefs may be dependent on the existence or, perhaps, the absence of knowledge"(p. 150). Therefore, teachers' mathematical knowledge may lead to particular beliefs about the way that mathematical knowledge is best taught (Wilkins, 2008).

Thus, it seems there is a bi-directional relationship between teachers' beliefs and knowledge such that they influence each other. And, both beliefs and knowledge seem to influence teachers' decisions in planning and practice.

Ernest (1989) noted that teachers' approaches to mathematics teaching related profoundly to their system of beliefs. Ernest (1989) provided a model for conceptualizing teachers' beliefs. Ernest's (1989) classifications are organized by the possible ways that teachers may view (1) the nature of mathematics, (2) teaching and (3) learning mathematics. Goos, Galbraith, Renshaw, and Geiger(2003) developed categories, Servant, Master, Partner, and Extension of Self, to describe the different ways that teachers may use technology. We use Goos et al.'s(2003) categories as ways that prospective teachers may view the use of technology for teaching and learning mathematics (see Table 1).

Table 1: Beliefs Classifications

Beliefs about	Classification of beliefs	Description
Nature of Mathematics Ernest (1989)	**Instrumentalist**	Mathematics is a set of facts and rules
	Platonist	Mathematics as a unified body of knowledge that does not change
	Problem Solving	Mathematics as a human creation that is continually changing
Teacher's Role Ernest (1989)	**Instructor**	Goal of instruction is for students to master skills
	Explainer	Goal of instruction is for students to develop conceptual understanding of a unified body of knowledge
	Facilitator	Goal of instruction is for students to become confident problem solvers
Learning Ernest (1989)	**Passive Reception of Knowledge**	Child exhibits compliant behavior and masters skills. Child passively receives knowledge from the teacher
	Active Construction of Knowledge	Child actively constructs understanding. Child autonomously explores self interests
Learning and Teaching Mathematics with Technology Goos, et al. (2003)	**Master**	Dependence on technology, not capable of evaluating the accuracy of the output generated by technology
	Servant	Fast, reliable replacement for mental or pen and paper calculations
	Partner	Cognitive reorganization, use technology to facilitate understanding, to explore different perspectives
	Extension of Self	Incorporate technological expertise as a natural part of mathematical and/or pedagogical repertoire

One framework that has become popular in recent years to describe and examine teachers' knowledge is Mishra and Koehler's (2006) Technology, Pedagogy, and Content Knowledge Framework, known as TPACK. The TPACK framework consists of three main components: Content, Pedagogical, and Technological Knowledge (CK, PK, and TK), and the intersections between and among them, represented as Pedagogical Content Knowledge (PCK), Technological Content Knowledge (TCK), Technological Pedagogical Knowledge (TPK), and Technological, Pedagogical, Content Knowledge (TPCK).

Findings from research on the relationship between teachers' beliefs and their TPACK or integration of technology indicate that there are varying degrees of consistency between the two. Kim, Kim, Lee, Spector, and DeMeester (2013) showed that teachers' beliefs about the nature of content knowledge and learning and about effective ways of teaching influenced their technology integration practices. On the other hand, Chai, Chin, Koh, and Tan (2013) revealed discrepancies between participants' pedagogical beliefs and their TPACK. Chai and colleagues found teachers used a more traditional teaching practice aimed at knowledge acquisition when using technology even though many teachers held constructivist-oriented pedagogical beliefs. The use of a traditional teaching practice is likely due to teachers' lack of knowledge of how to effectively integrate technology into their classroom. Given the mixed results of research studies on the relationships that may exist between teachers' beliefs and their TPACK, additional research is needed to examine whether relationships, in fact, do exist and to describe those relationships. The purpose of this study is to investigate and describe ways in which prospective middle grades mathematics teachers' beliefs relate to their knowledge of technology, pedagogy and content. The research question that guided our study is: What are the relationships among middle grades prospective mathematics teachers' beliefs about the nature of mathematics, teaching, learning, and the use of technology in the mathematics classroom and their level of knowledge of the TPACK components in the context of geometry?

Methods

Our participants were four undergraduate PTs enrolled in a middle grades teacher education program at a university in the southeast United States. We collected data from two semi-structured interviews. In the first interview, we asked participants questions to uncover their beliefs about the nature of mathematics, teaching and learning mathematics, and the use of technology to learn and teach mathematics. In the second interview, we examined participants' TPACK by conducting a task-based interview (Hollebrands & Smith, 2010). The PTs completed four separate tasks in which they analyzed students' work in a Dynamic Geometry Environment (DGE) and created an activity using the DGE to assist students to develop a deeper understanding of the concept or to remedy students' misconceptions (see Figure 1). We analyzed the first interview using Ernest's (1989) classifications of beliefs and Goos et al.'s (2003) perspectives of technology (see Table 1). Independently, each member of the research team read the transcripts and classified each participant's beliefs according to Ernest's (1989) and Goos et al.'s (2003) categories. We shared and discussed our classifications for each belief category and came to an agreement. Next, one member of the research team wrote a brief narrative that described the participant's beliefs and justified our classification of his or her beliefs. The lead researcher performed a member check (Creswell, 2013) by sharing the narrative with each participant. Each participant agreed that we accurately captured his or her beliefs. For the second interview, we used rubrics designed by Hollebrands & Smith (2010) to score the PTs' work on each of the four tasks. The tasks and rubrics were designed to assess the PTs' levels of CK, PCK, TCK, and TPCK (see Table 2). Based on the participants' work on each of the tasks, one of four levels

TASK 1

Suppose students in your middle or high school mathematics class are studying rectangles and squares. They open a dynamic geometry sketch that contains a rectangle and a square, each of which have been constructed. Students are asked to consider properties of rectangles and squares, based on their exploration of the sketch. One pair of students has measured the diagonals and they have noticed they are always congruent. They claim, "quadrilaterals have congruent diagonals."

a. Is this claim always true, sometimes true, or never true? Explain.
b. How would you characterize their current level of geometric understanding?
c. Create a sketch using a dynamic geometry environment that you would like students to use to explore diagonals of quadrilaterals. Be sure to include directions and/or questions you would provide to students as they use this sketch.

Figure 1.Example Task for Examining Participants' TPACK (Hollebrands & Smith, 2010)

Table 2: Rubric Used to Analyze TPACK Interview Task 1

Content Knowledge	Pedagogical Content Knowledge	Technological Content Knowledge	Technological Pedagogical Content Knowledge
A. Responds that the claim is sometimes true. B. Knowledge that there exists at least one quadrilateral for which the diagonals are not always congruent. C. States that for at least the rectangle and square the diagonals are always congruent. D. Provides a correct mathematical justification for why the statement is sometimes true using proofs that involve triangles or other properties.	A. Identifies that the student is able to notice that for a square and a rectangle that the diagonals are always congruent based on their measures. B. Identifies that the student is at level 2 (descriptive) but probably not at level 3. C. Has students consider at least one counterexample of a quadrilateral that has congruent diagonals. D. Asks students to consider at least one example of a quadrilateral that has congruent diagonals.	A. Accurately constructs or draws a quad using a DGE that is a counter-example. B. Uses measures to find the lengths of the diagonals. C. Drags to create multiple examples in a DGE. D. Accurate constructions of the 2 of the following quads: • Square • Rectangle • Parallelogram • Rhombus	A. Uses the DGE technology to focus students on properties of different quadrilaterals and their relationships to the diagonals in the task. B. Creates more than a single example using DGE technology to show the student that they are incorrect in the task. C. Designs an exploration for students by creating accurate constructions and utilizing the measurement and dragging features
Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	Emergent: 0-1 of A – D or no response. Beginner: 2 of A – D Intermediate: 3 of A – D Advanced: All of A – D	Emergent: 0 of A – C or no response. Beginner: 1 of A – C Intermediate: 2 of A – C Advanced: All of A – C

(Beginner, Emergent, Intermediate, Advanced) was assigned for each of the four TPACK categories. We coded each participant's beliefs and TPACK levels individually, then compared codes and reconciled differences.

Findings

Content

We found that participants' level of CK aligned with their views of the nature of mathematics. Ken viewed mathematics as a set of facts, rules, and procedures that are to be utilized to solve problems. He said, "I guess personally I would define it as being able to, again using numbers in a formula to solve problem." Thus, we classified his beliefs about the nature of mathematics as Instrumentalist. In the task interview, he was able to name various shapes but could not consistently identify properties, develop counter-examples, or justify the relationships among properties. We coded Ken's CK as Beginner. In the beliefs interview, Kim, May, and Sue held, in part, a Problem Solving view of mathematics. May said, "I also think math can seem like really rigid and like there is one right answer and this is how you do it. So I would also see the role as a mathematics teacher is kind of dispelling some of those myths about math." Sue initially stated she viewed mathematics as a unified body of knowledge. However, her college courses have influenced her view of mathematics such that she also viewed it as a process of inquiry in which the student creates the mathematics. Thus, Sue seemed to have both Platonist and Problem Solving views. The three PTs viewed mathematics to be, in part, created by the human mind and, thus, is constantly evolving for the learner. The three PTs also displayed high levels of CK. Kim, May, and Sue were able to identify properties of different figures, relationships between figures, and determine whether a figure was rotated or reflected by examining the orientation of the figure. While May and Sue were able to justify why certain conjectures were true, Kim, at times, struggled in this area. Thus, we classified May and Sue having an Advanced level of CK and Kim having an Intermediate level of CK. Thus, the PTs who held a Problem Solving view of mathematics displayed a high level of CK, while the PT who held an Instrumentalist view displayed a low level of CK.

Pedagogy

We found two potential relationships between the PTs' beliefs about teaching and learning and their level of PCK even though their beliefs were not the same. Ken believed students learn mathematics best when the teacher is in control of distributing the content, that is, he believed students learn mathematics through a Passive Reception. During the Task interview, Ken proposed activities that could help students see their mistakes, but would not allow students to determine why they were mistaken. Rather, his activities consisted of telling them what to do. Thus, we coded Ken at a Beginner level of PCK. The three other PTs held, in part, an Active view of learning mathematics. Kim thought students should be given the opportunity to pursue their own solution paths, to solve the puzzle in their own way. May also emphasized students should have ownership of their solutions to problems. She said, "[Students] can figure it out on their own and so I feel like they'd be more likely to understand it because they made that discovery themselves, instead of me just giving it to them." Thus, Kim and May viewed learning as an Active Construction of Knowledge. Sue believed students learn mathematics best through repetition, but she learns best through discovery. Sue seemed to hold both Passive and Active views of learning depending who is doing the learning. On the task interview, Kim displayed an Intermediate level of PCK. She analyzed students' thinking and made reasonable claims about students' understanding and how they came to that understanding. However, she did not consistently create activities that would help students fully understand why a certain conjecture was true or false. Both May and Sue demonstrated advanced levels of PCK. They identified students' levels of understanding and their misconceptions. The examples, counterexamples, and exploration tasks May and Sue developed would help students recognize whether their conjectures were true and allow them to deepen their understanding of the content. Therefore, the PT who held a Passive view about learning mathematics displayed a low level of PCK, whereas the PTs who held more of an Active view of learning displayed high levels of PCK.

There also seems to be a relationship between the PTs' beliefs about teaching and their levels of PCK. Ken believed that good teachers are those who explain mathematical concepts well. He said, "I think in order to be a good math teacher you need to be able to show those students different ways of solving a problem. I think you also need to be able to show them the longer ways of why formulas work." Thus, we coded Ken's view of teaching as an Explainer. Kim and May believed students should be given the opportunity to pursue their own solution paths to become better problem solvers. Thus, they wanted to engage their future students in learning mathematics using exploratory activities. Sue said teachers should facilitate discovery, encourage problem solving, and differentiate learning in the classroom. Therefore, Kim, May, and Sue believed that teachers should be Facilitators. As we mentioned above, Kim's level of PCK was at the Intermediate level, and May's and Sue's were at the Advanced level, all higher than Ken’s. Thus, the PT who viewed the role of a teacher as an Explainer displayed a low level of PCK, whereas the PTs who viewed a teacher as a Facilitator displayed a high level of PCK.

Technology

Comparing the participants’ beliefs about the use of technology in the teaching and learning of mathematics with their levels of TCK and TPCK, two relationships emerge. First, there appears to be a relationship between teachers’ beliefs about the use of technology in the mathematics classroom and their knowledge of how to use it to teach students mathematics (TPCK). Ken, Kim, and Sue’s beliefs about the use of technology in the learning and teaching of mathematics aligned, in part, with the Servant role; technology is to be used to amplify cognitive process, but not change the nature of the activities (Goos et al., 2003). In the task interview, the three PTs struggled to develop technology-based activities to help students overcome a misconception and the activities they created were limited to a certain number of examples or they did not use some of the basic features of the tool. We classified their TPCK at the Beginner level. May viewed technology as a Partner; technology should be used to explore and deepen students understanding of mathematics. In the task interview, May’s activities focused on correcting students' misconceptions and having students understand and consider fundamental properties and relationships that would deepen their understanding of mathematics. We coded May’s level of TPCK as Advanced. Thus, it appears that the PT who viewed technology as a Partner displayed a high level of TPCK while the PTs who viewed technology as a Servant displayed low levels of TPCK.

Second, there does not seem to be a relationship between their beliefs about the use of technology in the teaching and learning of mathematics and their level of TCK. Ken and Kim held very similar views that technology could be used as both a Servant and a Partner. We coded both Ken and Kim at a Beginner level of TCK because they could use multiple tools and features of the DGE (e.g., perform constructions of figures, measure different components of the figures, use the drag feature, and label points), but they did not use them consistently across all four tasks and struggled to accurately construct some figures. May viewed technology as a Partner. During the Task interview, May displayed an Intermediate level of TCK, but she may hold a more advanced level of knowledge because she was unable to use the technology on the final task due to time constraints. Sue’s beliefs about the use of technology aligned with the Servant view. During the task interview, Sue used multiple features of the DGE for the majority of the tasks but struggled performing transformations. Thus, Sue had an intermediate level of TCK. Ken, Kim, and May’s views of technology seemed to correspond with their level of knowledge of how to use technology; the greater the PT’s TCK, the stronger the belief in using it as a way to engage students in learning mathematics. However, Sue’s view of technology and her knowledge of how to use it did not follow this relationship. Thus, we cannot state with any degree of certainty that a relationship exists between teachers’ TCK and their beliefs about technology.

Discussion, Limitations, & Implications

The goal of this study was to investigate the relationships among prospective middle grades mathematics teachers' TPACK in the context of geometry (first-order barriers) and their beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology (second-order barriers). We found that PTs' beliefs about the nature of mathematics relates to their CK, their beliefs about learning and teaching mathematics relates to their PCK, and the use of technology to teach mathematics is related to their TPCK. May was the only PT who displayed high levels in all of the TPACK components. Sue displayed the same levels as May for CK, PCK, and TCK. However, she displayed a low level of TPCK. May and Sue's beliefs were not the same either. May displayed student-centered views of learning and teaching while Sue seemed to have both teacher-centered and student-centered views. In addition, Sue had a servant view of technology. Sue's low level of TPCK is likely related to her beliefs. Sue struggled in developing appropriate technology based activities for students because she had not developed appropriate views of effectively teaching with technology. Kim et al. (2013) found teachers' integration practices differed even though the teachers had access to the same technologies, support, and training. Teachers who held more student-centered views were able to integrate technology more seamlessly into their practices than those with more teacher-centered views. Even though the teachers had a similar knowledge base, the researchers found their integration practices differed and their beliefs seemed to influence the integration. Based on the findings from both our study and Kim et al. (2013), developing a strong knowledge of content, pedagogy, and technology will not ensure that teachers will use technology effectively in the classroom.

Unlike May, Ken's views about the nature of mathematics, learning, and teaching mathematics were teacher-centered and he displayed low levels of TCK and TPCK. This leads us to believe that in order for teachers to use technology effectively, they must develop student-centered views of learning and teaching. However, Kim has student-centered beliefs about teaching and learning, yet she displayed low levels of TPACK. This finding is consistent with Chai et al. (2013) who indicated that while their prospective teachers had developed student-centered perspectives of learning and recognized the advantages of using technology, they struggled designing appropriate technology-based activities. Thus, having student-centered beliefs will not ensure teachers will be able to use technology effectively.

There are two limitations to our study. The first limitation is the number of participants. Perhaps if we had interviewed additional or different participants, we may have found different relationships. The second limitation relates to the underpinnings of the rubric used to determine the level of TPACK knowledge displayed by our participants. Hollebrands & Smith, (2010) developed the rubric based, in part, on what research considers the best practices to teach mathematics with technology, in particular a DGE (e.g. Laborde, Kynigos, Hollebrands, & Staesser, 2006). These practices are based on constructivist principles. Thus, the rubric is likely biased towards participants with student-centered beliefs such that participants who held these beliefs would perform at high level compared to those who held teacher-centered beliefs. In fact, Ken, who held teacher-centered beliefs, did not achieve high levels while May, who holds student-centered beliefs, displayed high levels of TPACK. However, Kim and Sue held, in part, student-centered beliefs and did not display high levels for all TPACK components. Thus, the relationships between the participants' beliefs and TPACK were not predetermined. Even though there are limitations to this study, we believe there are implications for researchers and educators.

As technology becomes more ubiquitous in the classroom, first-order barriers will persist; software developers and curriculum designers are constantly adding new features and creating new tools for the classroom each year. Teachers will need training and support to integrate these updated and new tools into their classroom. However, our findings indicate that just providing knowledge on how to use these tools will not be enough. In order for teachers to use technology effectively, mathematics teacher educators should also focus on developing teachers' beliefs.

In this study, we examined the relationships that may exist between prospective middle grades teachers' beliefs and their TPACK, although we only analyzed their work with a particular tool, a DGE. Future work should examine whether the relationships we found would be the same for other technological tools. In addition, we wondered if the same relationships would appear for prospective secondary mathematics teachers and practicing teachers. We believe further research is needed in this area to provide a clearer picture of the relationships between teachers' beliefs and their TPACK.

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PRESERVICE TEACHERS' LEARNING MATHEMATICS FROM THE INTERNET

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This paper presents the results of a study that examined how preservice teachers used the Internet to find information that would help them better understand one of the Common Core Standards around the Pythagorean Theorem. A brief review of the research around the participants' search strategies, and content understanding is presented first. The connections between the searching strategies used, and the form and quality of the mathematical connections made is the focus of this paper.

Keywords: Technology; Teacher Knowledge; Teacher Education-Preservice

Mathematics teacher educators can not prepare preservice teachers for every piece of mathematics content they will encounter during their career. This means that these future teachers will need to learn mathematics on their own, outside of their teacher preparation programs. Many of these teachers will turn to the Internet to learn about unfamiliar mathematics. This paper presents results from a study that investigated how preservice teachers learned mathematics online, connecting their information seeking strategies and their understanding of the content.

Information Seeking in Digital Environments

Several studies have attempted to describe the behaviors of information seekers as they use digital tools to locate information. These studies have focused on describing two aspects of the searching process; the behavior of the user while seeking resources, and attention to how the user interacted with the resources. (An analogy to a non-digital scenario would be looking at participant behavior while searching for a book in a library, and how the participant interacted with the book once it was located.) Patterns of behavior emerged when looking at how searchers located resources within hyperlinked environments and how searchers located information within particular resources.

Representing and Analyzing Mathematical Connections

Concept maps are visual tools that can be used to illustrate connections between ideas. Consisting of nodes, connecting phrases, and lines, the creator of a concept map can represent their own understanding of a topic, and how they are connecting and relating ideas. Concept maps have been shown to be an effective way to capture student understanding of content in mathematics (Baroody & Bartels, 2000). Concept maps have also been used to track changes in the content understanding of mathematics teachers (Hough, O'Rode, Terman, & Weissglass, 2007).

There have been many attempts in this history of mathematics education to describe ways that learners come to know mathematics. Many frameworks attempt to describe deeper and more richly connected understandings of mathematics and Jon Star (2005) recently contributed a clarification to these approaches, clearly distinguishing between the form of mathematical understanding (i.e. conceptual and procedural) and the quality of that understanding (i.e. rich or superficial). He noted that both forms of mathematical understanding could be understood richly as well as superficially.

Preservice Teachers' Learning of the Pythagorean Theorem from the Internet

This work is part of a project that investigated how seven preservice elementary teachers learned unfamiliar mathematics online. These preservice teachers were enrolled at a large Mid-Western university earning a degree and certification toward teaching at the elementary level. Additionally, these participants were also earning a minor in mathematics and had been taking additional mathematics courses as part of their coursework. In individual sessions, the participants were presented with one of the Common Core State Standards for mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) that they may be responsible for teaching later in their careers (Standard 8.G.B.6 – Explain a proof of the Pythagorean Theorem and its converse). They were asked, via two tasks, to search online for resources that helped them better understand the mathematical content related to this standard. Before, between, and after these tasks, the participants created concept maps representing how they were thinking about content related to the Pythagorean Theorem. The information-seeking behavior of these preservice teachers during their Internet searching was examined as well as the series of concept maps created over the course of these sessions.

Information-Seeking Online

A previous report from Brakoniecki (2014b) looked at the information seeking strategies employed by the seven preservice teachers when searching for information online. All the participants at some point during their searching exhibited a focus on content of the sites they were finding (showing that no person only skimmed the information). Additionally, all participants utilized a timid navigation strategy, sticking close to their search results, and not venturing far from their search results. However, some of the participants focused predominantly on one format of resource, either preferring to spend the majority of their time looking at static resources (e.g. text, pictures, etc.) or dynamic resources (e.g. movies, web applets, etc.). This is in contrast to a different group who split their time locating information with both of these types of resources, focusing both on static and dynamic content. Additionally, there was a distinction in how much time the participants spent attempting to locate information on sites. The three participants who spent the longest time exploring sites were also the participants who exhibited both a focus on static content and a focus on dynamic content, exploring resources with both forms of content. The four participants who spent the least time exploring content in sites, showed a preference for one format of resource over another. This varied by participant where some preferred static resources and others preferred dynamic. Further details on this analysis can be located in Brakoniecki (2014b).

Mathematical Connections

A second paper from this same data set analyzed the concept maps the participants made before, during and after the two Internet searching tasks. These concept maps were analyzed for the structure of the concept maps, the content that was included in the concept maps, and the connections between the content in the maps (Brakoniecki, 2014a). The preservice teachers organized their concept maps in either a hierarchal or radial design in their first map, and almost always chose the same representation for their later iterations where they were asked to redraw their concept maps. When looking at the content the preservice teachers included in their concept maps, overall, the most content appeared in the final concept maps, constructed after participants had performed both searches. A distinction emerged among participants when looking at the connections they included among their content. These connections were analyzed for their form (procedural or conceptual) and quality (rich or superficial). One group of participants included an overwhelming majority of conceptual kinds of connections in their maps, and only toward their final mappings did they include procedural connections. This group also had an absence of richly described links in their concept maps. In contrast to this group, a second group among the participants was found to have more of a balance of between conceptual and procedural links throughout their concept maps. Additionally, every one of their iterations of concept maps contained at least one richly described connection. Further details regarding this study can be located in Brakoniecki (2014a).

Connections Between Information Seeking and Mathematical Connections

The two previous studies found two patterns of behavior when looking at the preservice teachers' information seeking strategies, and two patterns of behavior when looking at the mathematics connections they included in their concept maps. What was surprising was that there was an exact correlation between the groupings of these preservice teachers. The participants who showed a preference for one form of content and spent less time examining content were also the participants who included almost all conceptual links among their content and had a lack of richly described connections. This group of participants seemed to have more of a big idea approach, or a zoomed out orientation, to the content. The participants who preferred a balance of forms of content and spent longer exploring the resource sites were the same preservice teachers who had more of a balance between procedural and conceptual links in their concept maps, as well as always including at least one richly described link in their concept maps. This second group seemed to have more of a detail approach, or a zoomed in orientation, to the content.

Preservice Teacher Understanding of the Standard

Prior to the beginning of these information-seeking tasks, preservice teachers were prompted to imagine that they were teachers in their own classrooms and they were asked what their goals would be for their classes as they worked toward achieving this standard. All of the participants in their responses to this question mentioned goals of having students be able to understand proof or be able to reason through a mathematical argument. Some of the participants also described classroom goals that were unique to the Pythagorean Theorem including details like the Pythagorean Theorem having multiple proofs or multiple representations. Additionally they may have mentioned mathematics that was specific to the Pythagorean Theorem. The division in the preservice teacher behavior to this question mirrored the split in their behavior to the previous tasks. The participants who responded to this preinterview question in the general way were the same participants as the big idea orientation from the prior grouping. The participants who responded to this preinterview question in the specific way were the same participants as the detail orientation in the prior grouping.

It wasn't until the preservice teachers were asked to think about what they wanted to do in their classroom, a pedagogical question that details emerged. This is an interesting point that, looking at content related questions of the preinterview did not show any distinctions among the preservice teachers in this study. Examining the pedagogical content question did show a distinction in the responses. This is important as it helps suggest that the "same" understanding of content does not imply that enacted lessons would also look the same. Teachers' pedagogical goals shape their classrooms just as much as their understanding of the content does.

Future Directions

We do not want preservice teachers to have only a zoomed in orientation or only a detail orientation. Being able to focus on details is an important process for teachers, but so too is the ability to zoom out and look at where the content of a lesson fits with other content. This is sometimes referred to as horizon content knowledge (Ball, Thames, & Phelps, 2008), the knowledge of how the content of a lesson fits in with the content from the day before, or the content of the next unit.

This study raises several questions that point to future directions for research. This study contributes to the conversation that examination of teachers' content knowledge must also be accompanied by an examination of their pedagogical approaches in order to understand their classroom practice. There appears to be a strong connection between the behaviors of participants when searching online for mathematical information, and how they chose to represent their understanding of content, of which classroom goals may influence this behavior (though other influences may yet be discovered). Additional study might also include how preservice teachers might be learning pedagogical content knowledge online. As the internet becomes a more prevalent location for teachers to learn about mathematics and ways to teach mathematics, it's important to better understand the ways in which this resource is being used and how it can be better leveraged for teacher learning.

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MATHEMATICS TEACHING AS LEAN THINKING: A SOFTWARE DEVELOPMENT METAPHOR WHERE TEACHERS LISTEN AND NOTICE

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In this theoretical presentation, we connect research-based mathematics teaching practices with the software development movement known as Lean Thinking. By linking these two worlds, we hope to create educational technology products better aligned with research-based mathematics teaching practices, such as listening to and noticing children's mathematical thinking. The Lean Thinking movement focuses on minimal features, constant listening, and short response cycles as opposed to building out massive features with subsequent rounds of testing. Successful mathematics teachers position themselves the same way: listening to, noticing, and responding to their students' mathematical thinking in small increments with rapid feedback, as opposed to teaching massive units with little opportunity for student voice or formative assessment.

Keywords: Policy Matters; Teaching Beliefs; Teacher Knowledge; Technology

Objective

Current mathematics education research has shown that effective mathematics teaching requires teachers positioning themselves as listeners and noticers of mathematical thinking (Empson & Jacobs, 2008; Jacobs, Lamb, & Philipp, 2010). This focus on listening and noticing appears in multiple forms, from one-on-one clinical interviews (Ginsburg, 1997), orchestrated whole group discussions (Smith & Stein, 2011), or targeted strategy sharing techniques (Kazemi & Hintz, 2014). This shift from positioning teachers as authorities of mathematical knowledge to listeners of children's thinking is a drastic change within modern mathematics pedagogy (National Research Council, 2001). In a similar fashion, the rapidly changing software industry has also undergone massive paradigm shifts. Most notable, the Lean Thinking movement, which positions developers as listeners and responders to end users, has revolutionized not just the software industry, but our increasingly technology-based lives.

In this presentation, we lay a theoretical foundation connecting reform-oriented mathematics education and Lean Thinking. As mathematics educators wary of the over abundance of mathematics education technology, we use language from software development to connect to important ideas within mathematics educators in order to align paradigms in software development with research-based mathematics teaching practices. We hope to help technology developers and mathematics teachers find common ground and language in order to build technology that connects to teachers' practices and attends to teachers' and students' actual needs.

Theoretical Framework

Lean Thinking

The term Lean Thinking comes from Womack and Jones' (1996) profile of Toyota's constantly evolving car production techniques, which created an aura of dependability and quality around the brand. When applied to software development, Lean Thinking focuses on quick response to change, collaboration with end users, and short, frequent timescales for delivery (Highsmith, 2001; Poppendieck & Poppendieck, 2003). Many of us have benefited from Lean Thinking through software products that quickly evolve to meet the needs of its users, such as Facebook and Dropbox (Ries, 2011).

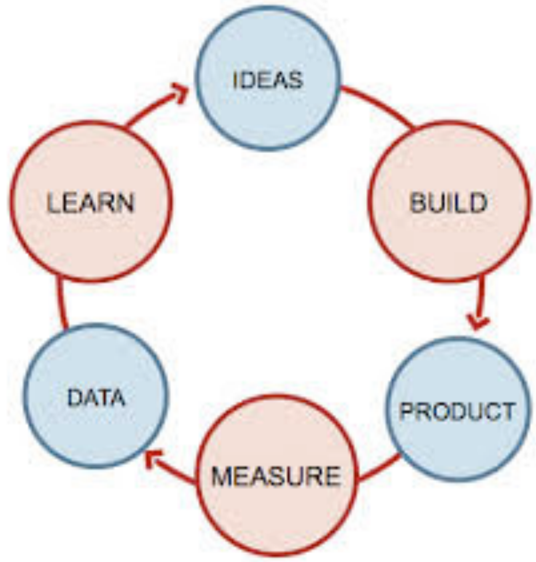

Figure 1. The quick Build-Measure-Learn feedback loop (Ries, 2011).

At the heart of Lean Thinking is a quick Build-Measure-Learn (BML) feedback loop (Figure 1), an ongoing collection of user data. Using the BML loop, developers start with an idea about how a feature might work and then quickly test it with users to see what happens. What sets Lean Thinking apart from traditional software development is the speed of this BML cycle. While a traditional software company rolls out a new version, beta tests, and then collects feedback data, Lean Thinking blasts through dozens of BML loops using a Minimal Viable Product (MVP), a product with only the features developers need to test (Ries, 2011). Often, MVP's are no more than screenshots, with users mimicking the pushing of buttons and telling an interviewer what they would be doing. No time is wasted in developing or building additional features. Through this mindset, developers are able to stay *lean* by focusing their energy on listening and responding to users as opposed to building what they predict a user will want to use.

Mathematics Teaching as Lean Thinking

A close reading of the research in mathematics education shows connections between Lean Thinking and current mathematics teacher education research. First, the research in how children learn mathematics emphasizes a three-step process: 1) Before, in which a teacher connects to a child's prior knowledge; 2) During, in which a student engages in a mathematical tasks; and 3) After, in which a teacher orchestrates a discussion of the strategies that students used to solve a task (Van de Walle, Lovin, Karp, & Bay-Williams, 2014). These three steps align with the Build-Measure-Learn loop. Just as Lean Thinking developers frame themselves as *building services for users* as opposed to *creating finished products*, mathematics education research frames teachers as *building mathematical thinkers* as opposed to *creating standardized memorizers*.

The Before/Build phase acknowledges what a child already knows mathematically in order to build mathematical tasks that connect to this prior knowledge. The During/Measure phase involves a child attempting their own strategies to solve these tasks while the teacher pays careful attention to listen to and notice how a child thinks. Finally, the After/Learn phase involves a teacher taking up children's thinking in order to respond to and orchestrate discussions for students to learn from one another. By being flexible to the many strategies and ideas that students bring up, a teacher can facilitate productive mathematical discussions and position him/herself as learning from the students.

Often, outsiders to education assume mathematics teachers facilitate classes using a dated "I do, we do, you do" approach (Hunter, 1982). But research in mathematics teaching has time and time again shown that an inquiry-based focus on eliciting and unpacking student thinking leads to greater mathematical understanding (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Hiebert et al., 1997; National Research Council, 2001). Like Lean Thinking developers, teachers should not default to a single "right way" to solve a mathematical task, but listen to and extend students' created strategies in order to extend their own thinking. And just as traditional development techniques leads to a lot of wasted effort as developers build features that users will never actually use, research-based teaching practices asks teachers *not* to spend their energy creating lessons that "tell" students how to solve a task (which might be different than how a student conceptualizes the task), but to be flexible of the many ways children solve a task and to respond to this thinking appropriately (Carpenter et al., 1999; Jacobs et al., 2010; Smith & Stein, 2011). Therefore, Lean Thinking is not necessarily a new way of conceptualizing of mathematics teaching practice, but a way to connect research-based mathematics teacher practices with the vernacular of technology and development.

Discussion

We are aware of the challenges of such a marriage of fields, as well as reasons why this merging has been avoided in the past. First, educational technology is often purchased at a district level, a result of the top-down approach used in our nation's school systems. While the primary users of any educational tools are children and teachers, the decisions makers are often administrators. By using language that situates teachers as developers, our hope is for technology developers to see that educational technology must allow for teachers to experiment and learn from their students, something very similar to what they are doing as they build software. Additionally, by positioning teachers as developers, we hope to empower teachers to develop technology solutions for their own practice. We want development to start with the teacher, not the administrator.

Second, forcing children to serve as "beta testers" of an educational intervention traverses ethical boundaries, essentially experimenting unfinished products on real children. However, in our current educational system in which large scale assessments (e.g., PARCC and Smarter Balance) and federal-level standards (e.g., CCSS) are routinely implemented without prior testing, a Lean Thinking approach that focuses on small, quick interventions rather than large scale implementation causes much less harm for children's intellectual development.

In conclusion, we hope that by adapting language from software and technology development into our field, we can (1) help educational technology developers create products that align with research-based mathematics teaching practices and (2) empower mathematics teachers to see themselves as capable of developing and sharing their own technology products. The use of a Lean Thinking paradigm within mathematics education technology is not new, we are just connecting language from two field in the hopes that we can try to make something better together.

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SCIENCE AND MATHEMATICS TEACHERS' PERCEPTIONS OF ONLINE COURSES

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Historically, science and mathematics teacher educators' model best practices in face-to-face settings as a way to teach about learner-centered instruction. This presents a challenge for instructors of online settings. In order to move what is typically considered effective in face-to-face practices to an online model, it is essential for teacher educators to push their thinking about how they can model best practices in new and different ways and to study how teachers make sense of and translate what they learn to the K-12 face to face setting. This study examined science and mathematics teachers' perceptions of their learning in four online courses of an online graduate program that focused on the integration of science and mathematics in the middle grades. Findings validated the teaching practices that were in place or being developed in the teachers' own classrooms as well as the teachers' utilization of various teaching activities.

Keywords: Instructional Activities and Practices; Teacher Beliefs; Teacher Education-Inservice; Technology

Introduction

In the past decade, online education has quickly become a leading force in higher education due to the expanding role of technology and the increased access to the Internet (Bourne & Moore, 2004). With this increased growth, higher education faculty have found that building an effective course in an online environment is much more than just replicating their materials of a face-to-face course to an online setting. Providing effective course content online parallels face-to-face instruction in that considerations must include learner characteristics, course organization, and the preparation of the instructor. Also, moving to an online environment does not exempt the instructor from what is commonly accepted as effective teaching, such as understanding the learners' needs (Dick, Carey, & Carey, 2001).

A strategy for considering how to design effective online instruction is to be informed about learner perceptions in online environments. Existing research about learner's perceptions in online environments is on a myriad of topics that focuses mainly on how the learner understands theory about the online environment and/or how the learner interacts with various aspects of the online environment (Eom, Wen, & Ashill, 2006). What is missing from the literature are learners' perceptions in online environments *when the learner is a teacher*. Teachers as learners in online environments are unique because of the complexity of teaching teachers *about* teaching children, which is typically done in face-to-face environments. Adding to the complexity of teaching teachers in online environments is the specificity when the teachers are teachers of science or mathematics.

Purpose of the Study

The purpose of this study is to gain understanding of science and mathematics teachers' perceptions of online learning experiences over one year in an online graduate program that focused on the integration of science and mathematics in the middle grades. In this particular case, the teachers had taken four different courses over two semesters. Two courses were specific to mathematics education while the other two were specific to science education. Understanding and describing teachers' perceptions of their learning in various coursework is the first step to designing effective courses and creating meaningful learning environments for teachers of science and mathematics. Accordingly, there are two research questions for this study. First, how do science and mathematics teachers perceive online course experiences? Secondly, do science and mathematics teachers perceive their online learning experiences as transferrable to the face-to-face classroom?

Literature Review

There is a body of research (Moore & Kearsley, 2005) that focuses on how the learner interacts with their online environment. However, that research alone is not enough to understand the ultimate success of a learner because it is necessary to consider how the learner perceives their interaction within an online environment. Research studies in online learning have looked at various ways of producing effective and beneficial online class settings through: multiple types of assignments, asynchronous reflection and discussion boards, synchronous conversation, and the use of a variety of media tools (Lebaron & Miller, 2005). Armstrong (2011) & Lin (2011) looked at effective online class settings by studying the role of communication in online learning and found that students perceived their connection in an online class greater than a face-to-face class because they had to take ownership for their learning. Research is now beginning to emerge in the area of online professional development (OPD) programs for teachers (Carey, Kleiman, Russell, Venable, & Louie, 2008; Dash, de Kramer, O'Dwyer, Masters, & Russell, 2012; Dede, Breit, Ketelhut, McCloskey, & Whitehouse, 2006). In two particular studies focusing on mathematics teachers' education, results showed that teacher's perceived their learning as effective as traditional methods of professional development (Groth & Burgess, 2009; Groth, 2007).

Methodology and Context

This study took place within an online Master's of Education program called Integration of Science, Mathematics, and Reflective Teaching (iSMART) which is a two-year program for middle school science and mathematics teachers. [This program is generously funded through the Greater Texas Foundation (http://greatertexasfoundation.org/). The opinions expressed here are those of the author and do not necessarily reflect the views of the Greater Texas Foundation.] The program focuses on the integration of science and mathematics as a means of developing teachers' content knowledge, as well as developing the teachers' technological and leadership skills. The program emphasizes inquiry and reasoning, developing students' skills in becoming effective problem solvers and communicators of mathematical and scientific ideas, and creating learning environments in which the teacher is a facilitator of learning rather than a dispenser of knowledge (Graham & Fennell, 2001). Using reflective collaboration with on-line classmates, the iSMART program seeks to develop in-depth teacher content and pedagogical knowledge through activities of the program including: analyzing theories and models of integration of science and mathematics, analyzing and writing curriculum, studying children's thinking of content, and reflecting on video of own practices. Both asynchronous and synchronous formats are utilized in the program where the teachers meet online for three hours for class one week and then the next week has an asynchronous assignment that builds from the synchronous meeting or sets the stage for the next synchronous meeting.

To obtain the data a researcher-developed survey that combined information from published literature on both online learning and online professional development was given. The survey was constructed of Likert scale items where space for each item was provided for additional comments. The questions were addressing the teachers' perceptions of their online learning in various ways: comparing a methods versus a seminar course, how they viewed their participation within each course, collaboration with their colleagues in each of the four courses, each of the four course structures, the instructors' facilitation and presence, and the perceptions about the ability to transfer the course content. The survey was administered during a week-long face-to-face conference during the summer between the two academic years.

Once the surveys were completed, the surveys were collected and photocopied. The following day the surveys were then re-distributed to participants and they discussed their responses in small groups and were allowed to add to their responses accordingly. In order to best analyze both the Likert and free response data obtained, qualitative data analysis measures were employed. The open-ended responses of each question were coded using a constant comparative method, according to the categories of research significance, as supported by the literature (Merriam, 1998).

Results

A necessary component to understanding the complexity of teaching online graduate courses for middle grades science and mathematics teachers is to analyze the perceptions of the teachers that currently engaging in the coursework. During the coding process the following themes emerged: (a) the instructors' instructional practices, (b) community development within the courses (c) development of content & pedagogical knowledge & (d) ability to transfer knowledge to their own teaching practice.

The teachers emphasized that the format of the class impacted the learning process. Specifically when the structure was more direct instruction (or lecture format), as described by the teachers, less time in conversation occurred, and thus less community was developed which in turn impacted the learning process. In the other courses where the structure was focused on discussion-based, student-led activities, the teachers "enjoyed collaborating because the nature of the courses dealt with how to become stronger teachers/coaches." The teachers' also felt that they could share with colleagues who were in similar teaching situations that they may not otherwise have the opportunity to engage in because the instructors made the online environment a 'safe place' to discuss and aided in the understanding of curriculum they were currently teaching.

Several teachers felt that the coursework was challenging and that the design of the courses offered enough time to process the material in order to understand it as well as to make changes to their teaching practices. The teachers felt that the questioning strategies used by the instructors were highly beneficial to their own pedagogy in their classrooms. Additionally, there were teachers that felt that the way the instruction was structured and modeled assisted them in planning for instruction in their own classrooms.

Implications

Online learning has quickly become a means of delivering graduate education to teachers (Allen & Seaman, 2014). As this study shows, teachers perceive learning in an online setting as effective as a face-to-face class. Teachers are required to continue their professional learning even after they finish their graduate education. Many times this professional learning is place-based (off of the teacher's campus) and does not meet the content criteria they need. However, given the results of this study, that teachers' perceive their learning in an online setting as beneficial, then online professional development (OPD) could be an avenue for future research with teachers' of mathematics and science in the middle grades. Teachers could be in a community of practice while engaging in OPD at their home campus or district location. This type of OPD option for professional development as a venue for research is a new, barely explored, territory, particularly in the areas of science and mathematics education. The overall goal of any educational program, whether it is a graduate program or an OPD program, is to help teachers gain the technology skills and knowledge needed to improve their teaching practice. It is through these future studies that the refinement of teaching teachers about teaching mathematics and science in an online setting can occur.

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MATHEMATICAL MEANING-MAKING THROUGH ROBOT MOTION

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This paper investigates middle school students' interactions while participating in learning activities in which they explored the connections between an arithmetic expression and the motion of a robot along a number line. Results indicate students utilized the technology to form conjectures, justify, and reflect on their experiments as they worked to interpret multiple meanings of the minus sign in the context of the robot's motion.

Keywords: Algebra and Algebraic Thinking; Number Concepts and Operations; Technology

Introduction

Technology-based learning environments can provide novel opportunities for students to build conceptual bridges between real-world phenomena and mathematical formalism. Often, these environments emphasize graphical representations of phenomena. Calculator Based Lab devices, for example, allow students to collect data about phenomena such as breathing patterns and to graph a function that fits that data (Doerr & Zangor, 2000). Motion detectors invite learners to directly experience the translation of physical motion into a graphical representation, and simulation environments can build the same bridge in reverse, allowing students to explore the ways that manipulating the graph of a position-versus-time function affects the motion of an animation (Roschelle, Kaput & Stroup, 2000).

In this paper, we explore a different approach, using the movement of physical robots as a context for students to explore the meaning of arithmetic expressions. We present results from a study in which the research team developed an iPod interface that allowed students to manipulate robot movement along a number line by inputting numbers and operations. Students' interactions with these tools and one another were examined through the framework of cycles of mathematical reasoning (Lehrer & Chazan, 1998) in order to investigate the ways enacting robot motion through mathematical expressions can support student learning.

Why Arithmetic Expressions: Importance and Challenges of Learning Signed Numbers

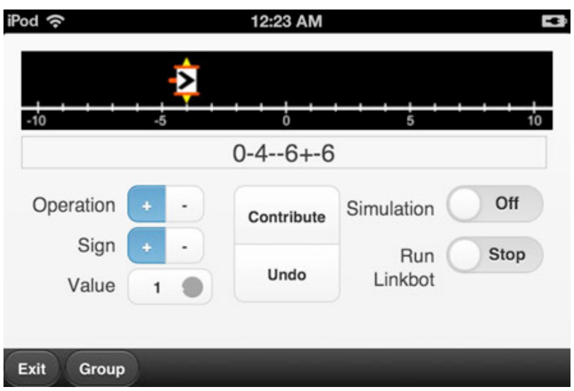
Signed numbers and operations are fundamental algebraic concepts, as the transition from arithmetic to algebra contributes to students' subsequent learning of more advanced mathematics (Vlassis, 2004; Lamb et al., 2012). However, the transition from arithmetic to algebra presents learners with complex cognitive challenges (Herscovics & Linchevski, 1994). Researchers suggest that the challenge of comprehending minus signs may reflect students' attempts to assimilate negative numbers with the characteristics of natural numbers (Vlassis, 2004). The minus sign can represent the *binary* operation of subtraction, as in the expression 5-3 = 2. The same minus sign, however, can also represent the *unary* operation of negation, as in the case of the integer -5 (Lamb et al., 2012). These varied uses of the same symbol may contribute to students' difficulty in grasping its mathematical meaning (Vlassis, 2004). However, understanding the multiple meanings of the minus sign, and developing the ability to flexibly shift among these different interpretations, are critical underpinnings of algebra learning.

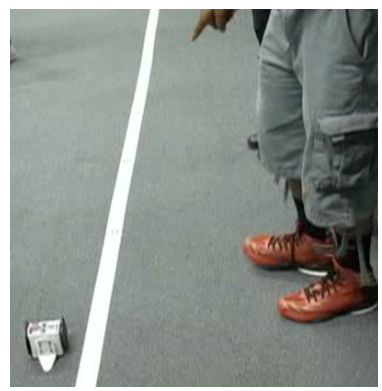
Research Design

We developed a graphical user interface for this study in order to enable students to interact more directly with mathematical representations. The robot moves along a number line in order to enact an arithmetic expression entered by one or more students through handheld devices. The robot behaviors and position are intended to model the mathematical expression in order to provide an accessible medium for students to explore its meaning.

In this activity, students in pairs used handheld devices (iPod touches) to jointly construct integer expressions by entering a series of operations and values into the devices through the graphical user interface (Figure 1). A number line was placed on the floor using velcro tape (Figure 2). After both students constructed and agreed on the inputted expression, the robot moved forward or backward depending upon the sign of the value, and turned around in place when it encountered a negative number. For example, 3+-4-2 indicates that the robot would move from position 3 backward for 4 units, then spin around and move forward towards a negative direction for 2 additional units. The backward motion enacted the *unary* interpretation of the minus sign, and the turning motion enacted the *binary* format of the minus sign. In situations where the previous experiment resulted in the robot facing the negative direction, the software made an adjustment accordingly.

The study took place in two semi-urban K-8 schools in northern California with approximately 130 7th and 8th grade students in six Algebra classes taught by two mathematics teachers. The activity was implemented for one week. Video and audio were recorded with cameras, and additional audio data was recorded from all participating students through the iPods. All student inputs from the iPod devices were captured through the computer server.





Figures 1 and 2: iPod interface and Robot with Number Line

Data Analysis

Analysis of these data was oriented toward three main research questions: 1) How do students understand arithmetic procedures in the context of robot motion? 2) How does using robot enactment of arithmetic expressions support students' understanding of integer and operation though motion metaphor? 3) How does the related robot behavior influence students' understanding of the minus sign's multiple meanings? The research team collected around 50 hours of video and 300 hours of audio of students' interactions with the handheld and robot tools. Next, we present an episode in which one pair of students demonstrated the potential opportunities for reasoning about signed numbers afforded by this learning environment.

Results: Reasoning about Binary and Unary Interpretations of the Negative Sign

In an early cycle of classroom activity with these tools, students were prompted by the teacher to freely explore the robot motions resulting from different expressions. After a few initial attempts to get familiar with the iPod interface and the robots, these two students jointly decided to search for a mathematics expression that would change the direction in which the robot faced by making it spin around 180 degrees. Table 1 provides the log record of the students' inputs in the iPod interface while completing this task.

Table 1. Record of students' inputs "make the robot spin"

Stage	Robot initial position	Student input		Expression
		Operation	Signed Number	
1	1	+	-6	1 + -6
2	-5	+	-7	-5 + -7 + 6
		+	6	
3	-6	-	6	-6 – 6 + 6
		+	6	
	-6	-	8	-6 – 8 + 4
		+	4	
4	-10	+	9	-10 + 9 + -9
		+	-9	
5	-10	-	5	-10 – 5 + 3
		+	3	
	-12	-	3	-12 – 3 + 2
		+	2	
	-13	-	4	-13 – 4 + 2
		+	2	

The episode started with the robot positioned at 1 and facing the positive direction. Student 1 suggested that they "Subtract. Make it a negative 6," then a moment later revised to "subtract 6." However, Student 2 followed the original suggestion, entering plus negative 6. At this stage the students were failing to distinguish between binary and unary meanings of the minus sign. In Stage 2, Student 2 reflected on the previous result and said "just add another, just add a positive, then it might spin." They entered the next expression, -5 + -7 + 6, and voiced optimism that they would "see if this works". The students' long sighs illustrated their disappointment at the unexpected resulting movement. In stage 3, the students then decided to attempt a subtraction with an addition, asserting "maybe we should do a negative and a positive" because "last time we used two positive." The students were now acknowledging that the minus sign has multiple meanings: after inputting -6 + -6, Student 1 asked "Plus a negative sign or is that the opposite? Should be minus a positive?" The two students decided to submit -6 – 6 + 6, and the robot spun. However, Student 1 voiced lingering confusion, stating "But I don't know why it spun." Students then experimented with the expression -6 – 8 + 4 to confirm their prediction. Next, in stage 4, despite Student 1's proposal to "put random stuff in," the fact that the signs and operations of the inputted expression -10 + 9 + -9 were all reversed from the previous attempt suggests this student may in fact have been testing the impact of these variations on the robot's behavior. The robot moved and failed to spin in this experiment. Stage 5 began with the students returning to their previous assumption from stage 3. While seeking ways to trigger the robot behavior of spinning, Student 2 concluded after attempted few more experiments that "so all it is is the negative, the operation" and "so it will have to be the operation instead of the sign".

Discussion

The episode offers evidence of examples where students identified the unique behavior of the robot changing direction as representing the binary meaning of the minus sign, which reveals her discovery that different meanings of the same symbol produced distinctive robot movements. Although this student demonstrated her acknowledgement of the relationship between robot movement and various meanings of the minus sign, the group also drew incorrect inferences. For example, these students had mostly chosen to submit two additional terms, when in fact users only need to input one mathematical term for the robot to spin. We suspect that this may be an artifact of students working in pairs, as each felt the need to each contribute a term.

The environment offered the opportunity to observe the robot motion as a direct enactment of the mathematical expression constructed by students. The activity and technology design provided the appropriate context for students to develop understanding through the cycle of mathematical reasoning (Lehrer & Chazan, 1998), offering learners a tool with which to form conjectures, justify, and reflect on the experiment result, and develop new experimentation to refine their conjectures. Frequent comments such as "Let's try it. Let's see what happens" suggest that students saw their mistakes not as errors, but as necessary elements of their process of exploration. Furthermore, students often justified and reflected on the result of each experiment, as illustrated through student statements such as "What did we do last time?" This experimental cycle repeatedly occurred within a single episode, resulting in students voicing comments such as, "this was actually easy, all it took was experimenting."

Conclusion

The analysis of this episode has enabled the authors to describe students' interactions when offered the opportunity to manipulate robot movement by inputting symbolic expressions. Students used these tools to experiment with signed numbers, using feedback from the technology to confirm or deny their hypothesis, and build up empirical evidence for interpreting the result of their conjecture. Our data demonstrate directly exploring the relationships between arithmetic expressions and robot motion allowed students to treat the environment as an exploratory setting to develop their mathematical reasoning through investigation.

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A COMPARATIVE STUDY OF PROSPECTIVE TEACHERS: VARIOUS WAYS OF RESPONDING TO NEW TECHNOLOGY

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This study examines how prospective teachers in Korea and the U.S. demonstrate Technological Pedagogical Content Knowledge (TPACK) and respond to new technology in mathematics learning. Overall, preliminary findings show that U.S. prospective teachers scored higher in most domains of TPACK than Korean participants. Korean participants demonstrated lower self-efficacy in Pedagogy Knowledge (PK) and Pedagogical Content Knowledge (PCK) when compared to U.S. participants. There were noticeable differences in the ways they evaluated technology and the ways in which they use new technology in instruction.

Keywords: Technology; Teacher Education-Preservice; Teacher Knowledge; Teacher Beliefs

A variety of approaches helpful in guiding teachers in implementing technology in mathematics classrooms have been developed (Kinuthia, Brantley-Dias, & Junor-Clarke, 2010; Polly, 2011). However, the ever-changing nature of technology demands a continued examination of the ways teachers develop successful technology-integrated practices. Researchers have considered a number of areas including teacher knowledge of technology, related pedagogical strategies, and the level of support available for teachers to implement technology in classrooms – we still need more detailed work that describes specific ways that teachers in different cultural and education settings *respond* to new technology and provides insight into effectively guiding teachers to implement appropriate use of technology in classrooms.

Purpose of the Study

The purpose of this study is to examine the ways teachers respond to new technology (e.g. equation solving apps) and how their perceptions of the technology may impact instruction.

While exploring various teacher responses toward new technology, we looked at two groups of prospective teachers in Korea and the U.S. Research questions guiding our study included (1) How do prospective teachers orient their instruction in response to the perceived benefits or hindrances of technology in mathematics instruction? (2) To what degree were prospective teachers in Korea and U.S. similar or dissimilar with regard to ways to respond to new technology?

Perspectives

Research on barriers and challenges affecting technology integration as well as strategies to overcome them have long been studied (Hew & Brush, 2007). Our frameworks include TPACK (Mishra & Koehler, 2006), the role of technology in the learning of mathematics (Goos, Galbraith, Renshaw, & Geiger, 2003), and psychological perspectives associated with teacher knowledge and beliefs, including its impact on teacher change (Ertmer & Ottenbreit-Leftwich, 2010). TPACK and other literature regarding the process of using technology in mathematics classrooms enabled us to conceptualize how teachers perceive the affordances and constraints of technology. Teacher knowledge, as well as attitudes and beliefs (Howard, Chan, & Caputi, 2014) are key attributes that influence teachers' decisions in the classroom. Therefore, understanding these attributes in light of educational technology may provide better insight into how teachers are likely to use technology in their classrooms in the future (Wachira & Keengwe, 2011). With this perspective, we are better able to understand technology integration as participants describe their belief systems about the learning of mathematics and teacher change after experiencing new technology.

Methods

The study setting included two teacher preparation programs in Korea and the U.S. Both programs offer middle grade mathematics education coursework. The program in the U.S. was at a large state university in a southeastern state in the U.S., while the other program was at a mid- sized, private university in a southern city of Korea. The participants included 18 U.S. prospective middle school mathematics teachers (15 female and 3 male) enrolled in a mathematics methods course prior to the semester of full-time student teaching experience in local school districts, and 22 Korean prospective middle school mathematics teachers (15 female and 7 male) enrolled in a mathematics education seminar course prior to student teaching.

The design of the study included three distinct stages: (1) collecting base-line data of belief, skills, and attitudes, (2) providing interventions to experience new technology, and (3) collecting data to document participant perceptions and changes. In the first stage, researchers conducted a survey titled "Survey of Preservice Teachers' Knowledge of Teaching and technology,"(Schmidt, et al. 2009). This survey included 28 Likert items with a scale from 1 (strongly disagree) to 5 (strongly agree), and provided data indicating the level of TPACK demonstrated by the participants. In the second stage, researchers introduced new technology in three separate 30- minute instructional sessions. Each session included a demonstration of PhotoMath, yHomework, and TI-83/84 linear regression demo. PhotoMath is a free app that solves math problems (mostly equations) through smartphone cameras; yHomework is a Math solver app that provides worked-out solutions when equations are entered. The TI-83/84 demo uses the linear regression equation to calculate the slope and the y-intercept of the line containing two points instead of multiple scattered points. Lastly, the third stage included conducting a survey with open response items (see Table 1).

Table 1: Six Open Questions in Our Response Survey

1. Describe perceived benefits of each technology.
2. Describe perceived hindrance of each technology.
3. Describe ways you would use each technology in instruction.
4. Describe situations when students should or shouldn't use each technology.
5. Describe your ideal practice of teaching with technology and how your students engage in the learning of mathematics with technology.
6. Describe ways you want to integrate technology in your instruction (e.g., learning tasks, collaborative projects, homework, assignments) in response to the emerging technology.

The analysis included calculating the mean scores for each student in the TPACK domains. Then we used open-coding on the response survey data to identify patterns related to benefits and hindrance, describing how participants' knowledge, beliefs, and attitudes impact their instruction by comparing the level of TPACK and participants' perceptions of new technology. We first conducted the analysis for the U.S. and Korean participants. Then we looked for any emerging patterns with regard to the participants' beliefs and attitudes, as well as the interaction between TPACK and responses to new technology.

Results

Our preliminary findings include the mean scores for TPACK by group. The response survey also yielded some findings about participants' perceptions of the three technology's affordances and constraints (see Table 2). The table below shows representative comments after the participants experienced each app.

Table 2: Most Common Responses about New Technology

	PhotoMath	yHomework	TI-83/84 "Linear Regression"
Perceived benefits of each technology	• Effective tool for autonomous learning (Korea; 82%) • Easy access to solutions (US; 61%)	• Effective tool for autonomous learning (Korea; 86%) • Display of detailed steps (US; 83%)	• No benefits (Korea; 59%) • Experiencing technology in mathematics (US; 56%)
Perceived hindrance of each technology	• Limitations including errors in producing correct answers (Korea; 64%) • Student cheating (US; 83%)	• Not available for free (Korea; 59%) • Decrease student motivation (US; 61%)	• Prevents opportunity to reason (Korea; 86%) • Too many steps before finding the solution (US; 50%)
How to use each technology in instruction	• Checking answers for practice problems (Korea; 68%) • Not to be used (US; 66%)	• Checking answers for practice problems (Korea; 86%) • Not to be used (US; 61%)	• Illustrating a case of linear regression (Korea; 55%) • Use it for writing linear equations (US; 77%)
Situations when students should or shouldn't use each technology	• Use it for getting help with procedural homework problems (Korea; 45%) • Do not use it at all (US; 72%)	• Use it for identifying mistakes in test corrections (Korea; 45%) • Use it during homework w/ adult supervision (US; 33%)	• Do not use if possible (Korea; 82%) • Use it for prepping standardized tests (US; 72%)

With regard to the degree of which the level of TPACK relates to (1) the ways participants describe an ideal integration of technology in instruction and (2) the ways the new technology influences their instruction, Table 3 shows the themes found among those with high TPACK scores and those with a low TPACK scores.

Table 3: Descriptions of Integration of Technology in Instruction

	Korean Participants	U.S. Participants
Those with higher TPACK score (4 points or above)	• Use technology to enhance mathematical understanding • Use the new technology selectively for those who struggle with procedural problems. • Design assignments which the technology cannot solve	• Use technology to improve problem solving skills • Use the new technology to check answers and find mistakes. • Avoid using the technology in class.
Those with lower TPACK score (2 points or below)	• Use technology to produce accurate solutions • Use the new technology to finish homework assignment	• Use technology to engage students in learning • Not to use the new technology

Discussion

The new technology in the study focused on solving equations and providing worked-out solutions for students. Since the apps solve mostly procedural problems—especially single variable equations—those valuing the teaching of procedures may have felt "threatened" by the emerging technology. It is not surprising that most prospective teachers whose school mathematics experiences were predominantly procedural mathematics would express concerns about students using the new technology mentioned in the study. However, participants expressed conflicting levels of resistance towards the technology depending upon the apps' mathematical capacities. For example, 72% of U.S. participants reported they would not allow students to use PhotoMath, but the same participants said they would encourage students to use yHomework at home if adults are present – we note that yHomework has more capabilities in detailing procedural steps. Also interesting is how most Korean participants considered the apps as similar to calculators and indicated that such apps are limited in their capacity to solve complex mathematics problems.

There was a variety of differences in how the two groups responded to the surveys and evaluated the technologies. First, Korean participants demonstrated lower self-efficacy scores for TPACK than U.S. participants. The cultural differences in expressing self-efficacy may explain this phenomenon (Ertmer & Ottenbreit-Leftwich, 2010). Most Korean participants felt negatively about abusing the linear regression feature of TI calculators; however, U.S. participants were more positive in their view of how such a function of the calculator may improve performance

on standardized tests. Those who had higher TPACK scores also differed between the two countries. Korean participants supported the use of technology to enhance mathematical understanding, while U.S. participants emphasized the potential of technology in serving as a tool for problem solving.

A further analysis of the data including a future study with a large group of participants is in order. We need more detailed work about the persistent patterns underlying prospective teachers' beliefs and skills related to technology integration, and developing ways to increase their abilities in successfully integrating technology in the classroom.

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PROMOTING PRODUCTIVE MATHEMATICAL DISCOURSE: TASKS IN COLLABORATIVE DIGITAL ENVIRONMENTS

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Tasks can be vehicles for productive mathematical discussions. How to support such discourse in collaborative digital environments is the focus of our theorization and empirical examination of task design that emerges from a larger research project. We present our task design principles that developed through an iterative research design for a project that involves secondary teachers in online courses to learn discursively dynamic geometry by collaborating on construction and problem-solving tasks in a cyber learning environment. In this study, we discuss a task and the collaborative work of a team of teachers to illustrate relationships between the task design and productive mathematical discourse. Implications suggest further investigations into interactions between characteristics of task design and learners mathematical activity.

Keywords: Technology; Geometry; Teacher Education-Inservice; Classroom Discourse

Mathematical tasks are central as they convey what mathematics is and what it means to do mathematics. Sierpinska's (2004) considers that "the design, analysis, and empirical testing of mathematical tasks, whether for purposes of research or teaching, is one of the most important responsibilities of mathematics education" (p. 10).Mathematical tasks shape significantly what learners learn and structure their classroom discourse (Hiebert & Wearne, 1993).Such discussions when productive involve essential mathematical actions and ideas such as representations, procedures, relations, patterns, invariants, conjectures, counterexamples, and justifications and proofs about objects and relations among them. Nowadays, these mathematical objects and relations can be conveniently and powerfully represented in digital environments, and many contain functionality for collaboration. However, for such collaborative, digital environments, the design of tasks that promote productive mathematical discussions is an enduring challenge and requires continued theorization and empirical examination (Margolinas, 2013). In this brief research report, guided by the question—What features of tasks support productive mathematical discourse in collaborative, digital environments?—we articulate theoretical and practical principles for designing such tasks for small teams of individuals working in online, collaborative environments.

Our work employs a specific online environment that supports synchronous collaboration and discussions and provides tools for creating graphical and semiotic objects for doing mathematics. The environment, Virtual Math Teams (VMTwG), has a multiuser version of a dynamic geometry environment, GeoGebra. Here, we first indicate our task design to promote potentially productive mathematical discourse among small groups of learners working in VMTwG. Afterward, we present an example of a task along with the mathematical insights a small team of teachers developed discursively as they engaged with it. We conclude with implications and suggestions areas for further research.

Task-Design Principles

Our principles of task design embody particular intentionalities for a virtual synchronous, collaborative environment that has representation infrastructures (GeoGebra) and communication infrastructures (social network and chat features). The intentions are for mathematical tasks to be vehicles "to stimulate creativity, to encourage collaboration and to study learners' untutored, emergent ideas" (Powell et al., 2009, p. 167) and to be sequenced so as to influence the co-emergence of learners instrumentation and building of mathematical ideas. To these ends, sensitive to the infrastructural features of VMTwG, we developed and tested the following seven design principles for digital tasks that are intended to promote productive mathematical discourse by encouraging collaboration in such environments:

- Provide a pre-constructed figure, instructions for constructing a figure, or invitation to construct a figure with particular properties.
- Invite participants to interact with a figure by looking at and dragging objects to notice how the objects behave, relations among objects, and relations among relations.
- Invite participants to reflect on the mathematical meaning of what they notice.
- Invite participants to wonder or raise questions about their noticings and meanings.
- Pose suggestions as hints or new challenges that prompt participants to notice particular objects, attributes, or relationships.
- Provide formal mathematical language that corresponds to awarenesses that they are likely to have explored and discussed or otherwise realized (Hewitt, 1999, 2001).
- Respond with feedback based on participants' work that pose new situations as challenges that extend what participants noticed, wondered, or constructed; invite participants to revisit a task, to generalize noted relationships and to construct justifications and proofs of conjectures; suggest that participants consider the attributes of a situation (theorem, figure, actions such as drag) in order to generate a "what if?" question and explore it.

The hints aim to maintain learners' engagement with a task and the challenges to encourage them to extend what they know. The hints support participants' discourse by eliciting from them statements that reveal what they observe and what mathematical sense or meanings they make of their observations. The challenges provide new, related situations to investigate. Hidden initially, the hints and challenges can be revealed by learners clicking a check box.

These design principals guided how we developed tasks in our research project, a collaboration among investigators at Rutgers University and Drexel University. VMTwG records users' chat postings and GeoGebra actions. The project participants are middle and high school teachers in New Jersey who have little to no experience with dynamic geometry environments and no experience collaborating in a virtual environment to discuss and resolve mathematics problems. The teachers took part in a semester-long professional development course. They met for 28 two-hour synchronous sessions in VMTwG and worked collaboratively on 55 tasks. In the next section, we provide examples of two tasks to illustrate the actualization of our task design.

Task Example and Analysis

Based on our design principles, we developed dynamic-geometry tasks that encourage interlocutors of a team to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, discuss meanings and wonderings associated to their noticings.

During a collaborative session, a team of three teachers worked on identifying the dependencies involved in the constructions of different triangles (see Figure 1). The team dragged the triangles vigorously to explore dependencies among objects in the task. The vertices of first triangle, ABC, were constructed as independent objects, so the team did not belabor discussing it. The second figure is an isosceles triangle DEF. The lengths of DE and DF are equal. Point F is constrained to a hidden circle with radius DE. Points D and E are independent objects. In the following excerpt, concerning the second figure the Team discusses their noticings and the meaning they derive from them:

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

386 ceder: so in the second one, f is dependent on g
387 ceder: I mean d

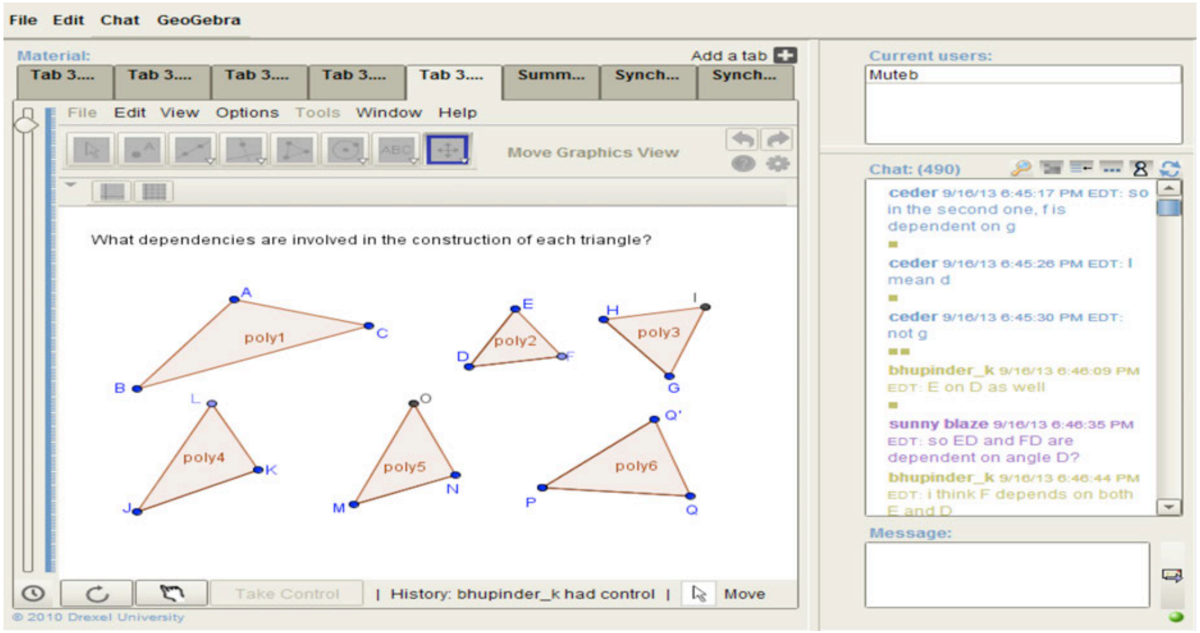

Figure 1: Dynamically examining dependences among different triangles

388 ceder: not g
389 bhupinder_k: E on D as well
390 sunny blaze: so ED and FD are dependent on angle D?
391 bhupinder_k: i think F depends on both E and D
392 ceder: f doesnt look dependent on anything now...am I imissing something?
393 ceder: ok, what am I missing? F can move independently, but when E is moved, F moves, so that makes which one dependent?
394 bhupinder_k: when you move F, ED stays fixed
395 ceder: right, so F is free to move anywhere
396 ceder: but not when E is moved
397 ceder so F is sometimes dependent?

The team discusses dependencies among points, segments and angles. In lines 386 to 388, ceder states that F is dependent on D then dismisses her assertion in line 392. In line 390, sunny blaze asks whether the two line segments are dependent on angle D. Though, week before, the team had already seen and constructed dependent objects, they struggled with this new, more complex situation. The idea of dependency is key and permits interlocutors to identify and build relationships in dynamic constructions. In a latter task, the team uses the concept of dependency to identify relations among objects. The task presents two circles constructed using the same radius, AB, where each endpoint is a center of a circle. Their points of intersections, C and D, were connected to create a perpendicular bisector to radius AB that intersects it at point E. In the session, one teacher states that points C, D, and E are dependent on A and B. Another teacher states that the two circles share the same radius and that dragging the center of one circle affects the size of the other, which makes the circles dependent on the centers. The teachers appropriate the idea of dependency and use it to understand components of the task's constructions.

In these two tasks, the teachers attended to the co-active feedback of the environment to their actions, which enabled them to appropriate the concept of dependency. For the team of teachers, engaging with tasks where dependencies are key relations among geometrical objects was an important step. These tasks triggered a discussion about how to use dependency to create valid geometric constructions.

Discussion and Conclusion

Our focus was to describe how we address task design challenges to promote productive mathematical discourse in an online synchronous environment. In the virtual environment, a teacher or facilitator is present largely as an artifact of the environment's digital tools and most specifically in the structure and content of tasks. An important feature of our task design are principles 2 to 4 since when collaborating interlocutors respond to those prompts they generate propositional statements that can become the focus of their discussions. Their discussions are mathematically productive as their noticings, statements of meaning, and wonderings involve interpretations, procedures, patterns, invariants, conjectures, counterexamples, and justifications about objects, relations among objects, and dynamics linking relations.

Our task-design principles aim to engage interlocutors in productive mathematical activity by inviting them to explore figures, notice properties, reflect on relations and meanings, and wonder about related mathematical ideas. The design provides support through hints and feedback to help learners with certain parts of the tasks. The tasks also include challenges that ask the participants to investigate certain ideas and extend their knowledge. Further investigation is needed to understand how the task-design elements, the affordances of collaborative digital environments, and learners' mathematical discourse interact to shape the development of learners' mathematical activity and understanding.

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EXPLORING THE ENDURING GAP BETWEEN UNDERGRADUATE MATH AND PROFESSIONAL PRACTICE: THE CASE OF COMPUTER PROGRAMMING

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Keywords: Instructional Activities and Practices; Technology; Post-Secondary Education

Several mathematicians and researchers in math education have reported on the disconnection between post-secondary math education and the practices of professional mathematicians. A recent quantitative survey of 302 Canadian mathematicians points to one possible aspect of the gap: while 43% of the participants reported using computer programming in their research, only 18% indicated that they use such technology in their teaching (Buteau et al., 2014). The first statistic highlights the potential that programming may have for doing and learning math. The second inspires the need for further research: why would such a gap exist?

In response to this question, we put forth a qualitative study aimed at exploring the place, role, potential, and constraints of programming in both mathematics research and undergraduate mathematics education. Semi-directed interviews were carried out with 14 mathematicians working within various different mathematical subfields at universities across Canada. The participants were chosen in such a way that they had all used programming in their research or their teaching (or both). They were asked to speak about their own experiences with computer programming, as well as the programming experiences (if any) they provide for their students.

A qualitative analysis of our interviews allowed us to unveil some of the powerful ways in which both applied and pure mathematicians employ programming to accomplish a variety of tasks (e.g., modelling complex phenomena to solve real-world problems, performing calculations and simulations not possible by hand, doing so-called "experimental" mathematics, constructing proofs, developing powerful technological tools). Consequently, it also enabled us to describe the extent of the qualitative gap between how these mathematicians use programming and how this activity is experienced by their students. In the end, the interviewees draw attention to some crucial issues regarding the time required to develop and implement enriching programming activities, the resources available to professors and students in Canadian universities, and the structure and content of undergraduate mathematics programs. Our participants also shared their perceptions of the goals of undergraduate mathematics education, the nature and culture of mathematics departments, and the traditions in the research, teaching, and learning of mathematics. We suggest that some of these visions, traditions, and constraints, which limit the mathematical experience of an undergraduate student, could warrant re-examination.

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POSING PURPOSEFUL QUESTIONS: AN APPROXIMATION OF PRACTICE USING LESSONSKETCH

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An enduring challenge facing teacher educators is creating opportunities to support teacher candidates in enacting mathematics teaching practices, while attempting to manage the complexities of teaching (Darling-Hammond, Hammerness, Grossman, Rust, & Shulman, 2005). One response to this challenge has been to introduce technology as a tool for incorporating representations of classroom instruction. Teacher candidates can then engage with these representations in meaningful ways that serve as *approximations of practice* (Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009).

This research explored 12 teacher candidates' (TCs') responses to a representation of student work in a LessonSketch experience. LessonSketch is an online platform that allows users to explore cartoon teaching scenarios. The goal of this LessonSketch experience was to provide TCs with an opportunity to practice posing purposeful questions that both assess and advance students' mathematical thinking. Using the LessonSketch environment to represent practice allowed the teacher educator to manage some of the complexities of teaching, while still providing the TCs with an opportunity to practice responding in the moment (Herbst, Chazan, Chen, Chieu, & Weiss, 2011).

This poster presents the results from an experience called the Pizza Dough Problem. TCs were first asked to solve the following problem: If pizza dough doubles every five minutes and reaches its maximum volume in thirty minutes, how many minutes does it take for the pizza to reach half its maximum volume? After working on this problem individually and then sharing in small groups, TCs were asked to anticipate potential student solutions and errors. TCs then participated in the LessonSketch experience, in which they were shown three comics depicting both correct and incorrect work from student groups and asked to respond to each group with a question that would further assess students' thinking or develop students' understanding. TCs first wrote questions on their own, and then discussed with a partner and revised their questions.

Preliminary findings show that TCs' questions tended to focus on *assessing* student thinking in response to the first depiction of an incorrect use of proportions and on *advancing* mathematical thinking (e.g., changing parameters) in response to the second depiction where students correctly worked backwards. Further analysis will explore the form, content, and purpose of the questions posed.

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HIGH SCHOOL STUDENTS' USES OF DRAGGING FOR EXAMINING GEOMETRIC REPRESENTATIONS OF FUNCTIONS

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The goal of this study was to investigate seven students' uses of dragging in GSP as they engaged in a sequence of technological activities that utilized a geometric approach to function. Students were paired or grouped so that students with similar backgrounds worked together using paper and pencil and a shared laptop computer. Data collected from the first activity, "Identify Functions," developed by Steketee and Scher (2011) were analyzed. The ways different types of dragging were related to students' thinking about particular aspects of function such as, domain, range, covariation, multiple representations, function notation, invariances, and families of functions, were examined (e.g., Carlson, 1998).

Analysis of students' work revealed two new dragging modalities: dragging a non-draggable point and variable speed dragging. Dragging a non-draggable point is specific to the way in which the activity was created. The instructor made a choice to design the activity with the non-selectable feature to emphasize and connect the action of not being able to select with the idea of independent and dependent variables. It was clear from the work of the students that their use of trying to drag a non-draggable point assisted them in making the association between this dragging type and independent/dependent variables.

Students also used other types of dragging to focus on particular aspects of function. For example, a new type of dragging, variable speed dragging, was used by students to focus on rates of change and co-variation. It was also used, often in conjunction with other representations, when students wanted to provide a more detailed explanation of the relationship between two points. The drag test was also used by some of the students to describe relationship between two points and when students wanted to investigate invariants.

The mathematical backgrounds of the students influenced how students discussed function behavior. In particular a pair of Algebra I students focused on different behaviors, but did not draw on their knowledge of function families or geometric transformations to describe those behaviors. Honors Geometry and Algebra 2 students used the language of geometric transformations and linear functions to describe the behaviors they noticed. Honors Pre-Calculus students drew upon a large repertoire of functions families and function language to notice and describe the behaviors they noticed. These understandings likely influenced the methods of dragging students used and the aspects of function to which they attended. Understanding how students use features of dynamic geometry activities and relationships of these uses to their understandings and aspects of mathematics to which they attend can be useful to designers of instructional materials.

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GETTING FROM HERE TO THERE: EFFECTS OF AN ALGEBRA READINESS INTERVENTION

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Keywords: Algebra and Algebraic Thinking; Technology; Number Concepts and Operations; Middle School Education

Mastering basic algebraic concepts is extremely challenging, and many students never achieve adequate levels of proficiency. Although math instruction often emphasizes memorization of abstract rules, being able to fluently construct, interpret, and manipulate algebraic symbols involves the use of appropriate perceptual processes to *see* expressions and equations as structured objects (Kirshner, 1989; Goldstone, Landy, & Son, 2010). Learning technologies that focus on perceptual-motor training have shown substantial promise (Ottmar, Landy, & Goldstone, 2012), but are underexplored relative to other math interventions.

In this study, we present preliminary findings from a classroom study using *From Here to There (FH2T),* a dynamic computer-based visualization method designed to enhance students' understanding of algebraic notation through building appropriate perceptual strategies. *FH2T* uses discovery-based puzzles to learn mathematical patterns and think more flexibly about numbers and operations. The students' goal is to transform an expression from the starting form (*here*) to the ending form (*there*). Students perform a series of dynamic interactions, including decomposing numbers, combining terms, applying operations to both sides of an equation, and rearranging terms through commutative, associative, and distributive properties.

85 7th grade students participated in a 3-hour study over 6 class periods. This study compares learning gains from 2 dynamic versions of *FH2T* (fluid verses retrieval practice) compared to a control group and explores plausible mechanisms of this learning. Students in the fluid visualizations condition performed 0.20 SD higher than students in both the retrieval practice and control conditions. Next, a significant main effect was found for exposure: for every additional world completed, posttest accuracy scores increased by 0.76 problems (effect size=0.48). Third, a significant interaction between exposure and pre-test scores was found.

Only students who used the more fluid version of *FH2T* demonstrated strong learning gains. These gains seem to be primarily due to practice; however, we cannot tease apart whether this is due to a difference between retrieving explicit rules and perceptual training afforded by the fluid instantiation, or the increased exposure to content that the fluid group received. Future studies should be designed to address these concerns. Overall, these results suggest promise for tablet-based technologies for teaching algebraic content and support that algebra literacy encompasses strong visual-motor routines. Interventions involving the movement of symbolic forms for algebra learning have been receiving widespread attention in recent years; however, this work represents some of the first published outcomes from such perceptual interventions.

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LATTICE LAND: A MATH MICROWORLD FOR CLASSROOM INQUIRY

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Keywords: Advanced Mathematical Thinking; Technology; Geometry; Problem Solving

Based on the constructionist principles of Seymour Papert (1980) and the largely unexplored universe of lattice geometry (Sally & Sally, 2011), *Lattice Land* was designed to be a "restructuration" of the geometry curriculum (Wilensky & Papert, 2010). *Lattice Land* is a colorful, interactive, and mathematically rigorous computer environment for students to discover advanced mathematical thinking through play, conjecture, and experimentation. Unlike most mathematics software, there are no built-in solutions to any set of problems, or any linear pathways a student must take. Together with an inquiry-based curriculum, students at any level of fluency with mathematics will uncover many interesting and mathematically profound results.

This poster introduces some potential explorations in *Lattice Land*: "taxicab" geometry, area optimization, dissection and triangulation, Pick's Theorem, and extensions such as polyomino packing. It is a low threshold, high ceiling exploratory software, a "microworld" (Edwards, 1995) that encourages mathematical discovery and developing *powerful ideas* (Papert, 1980). Even seemingly trivial exercises quickly gain complexity.

For children, *Lattice Land* provides an escape from procedural mathematics. The software backgrounds much of the calculation requirements that often prolong the discovery of more powerful ideas. For more experienced students, the study of lattice geometry shows how familiar mathematic concepts can be constructed and derived in unexpected ways. While users will find that polygon constructions and their properties—distance, perimeter, area—are the same, the derivation of these formulas is tangential to the greater learning goal of being able to reconceptualize mathematics. Additionally, because lattice geometry is unfamiliar to both young mathematicians-to-be and adults trained in traditional school mathematics, *Lattice Land* is an interesting microworld for families to visit together.

Coded in NetLogo (Wilensky, 1999), each model within the *Lattice Land* software suite emphasizes a facet of lattice mathematics. The virtual environment is particularly conducive to exploring abstract mathematics, as it is easy to manipulate and change on the fly. *Lattice Land* is purposely unlike canonical mathematics. As "restructurations" go, *Lattice Land* maintains the cognitive properties of Euclidean geometries, but uses a more approachable platform, backgrounds procedural calculations, and provides a low threshold and playful environment (Wilensky & Papert, 2010). Citizens of *Lattice Land* should also traverse microworlds, and learn how each is deeply connected with the others. I hope this leads the citizens of *Lattice Land* to recognize that mathematics is imbedded in other worlds, as well. Perhaps, even in our own.

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SECONDARY MATHEMATICS TEACHERS' CRITERIA FOR EVALUATING TECHNOLOGICAL TOOLS

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As technology is an essential tool for learning mathematics in the 21st century (NCTM, 2014) and is becoming more ubiquitous in the classroom, mathematics teachers need to be able to select quality technological tools to use with their students. The question that guided our study: What criteria do secondary prospective and in-service teachers use and value most when evaluating technological tools to teach geometry, specifically the triangle inequality theorem?

Our participants were 15 prospective and in-service teachers enrolled in a mathematics education course focused on technology. We placed our participants in 5 groups based on the amount and level of teaching experience and their technology background. During one class meeting, each trio developed a list of criteria, analyzed and evaluated four online applets designed to help students learn the triangle inequality theorem. Then, we conducted a stimulated-recall interview during which each group was asked to rank their list of criteria in order of importance. We video-recorded each group's working during the class activity and interview. After we created transcripts, we coded each group's criteria according to Dick's (2008) descriptions of Pedagogical, Mathematical, and Cognitive fidelities. Some of the groups' criteria did not fit Dick's descriptions and we created two additional codes: Scaffolding and Assistance.

All groups developed criteria coded as Mathematical fidelity, how the technology represents mathematics, and Pedagogical fidelity, how the students interact with the technology. But, none of their criteria was coded as Cognitive fidelity. That is, they do not seem to consider how students will think using the technology or how the technology will reflect how students are thinking about the mathematics. We did notice some difference between the criteria created by teachers with and without teaching experience. The groups of participants with no teaching experience created criteria coded as Assistance, whether the tool provides instructions on how to use it. The groups with teaching experience did not create Assistance criteria. Rather, they focused on whether technology could help students develop an understanding of the content. They created Scaffolding criteria and ranked it as the most important criteria. The groups with no teaching experience ranked criteria coded as Mathematical fidelity very high, whereas the groups with teaching experience ranked other types of criteria higher (i.e. Scaffolding) because they thought that mathematical ideas were innate in the higher ranked criteria.

Even though all groups considered whether mathematics is accurately represented and how students will interact with technology, they did not seem to consider how students think when using technology. By not taking this into account, teachers may not select the best tools for their students. Thus, mathematics teacher educators need to help teachers develop their abilities to examine technologies based on student thinking.

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DIGITAL RESOURCES IN MATHEMATICS: TEACHERS' CONCEPTIONS AND NOTICING

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Mathematics education researchers have explored ways in which teachers construct or supplement their mathematics curriculum (Darling-Hammond, 2006; Remillard, 1999; M. G. Sherin & Drake, 2009). As teachers construct or supplement their mathematics curricula they must find, choose, and decide how to use instructional resources. One available strategy is searching online (Ruthven, 2013) to find digital resources and digital tools. Access to digital tools and resources can be a powerful tool for K-12 teachers. In order to use digital tools and resources effectively, teachers must make decisions in choosing which to use and how to use them as they supplement or construct their curriculum. Decisions depend on the general, pedagogical, and mathematical features that teachers notice. In order to make sense of the features teachers notice and how to support teachers' professional development of their abilities to find, choose, and decide how to use digital tools and resources, this poster presentation presents preliminary frameworks and qualitative analysis results.

Brown (2009) discussed the differences in teachers' skills in "perceiving the affordance of the materials and making decisions about how to use them to craft instructional episodes that achieve her goals" (p. 29). Dietiker, Males, Amador, Earnest, and Stohlmann (2014) defined curricular noticing as "how teachers make sense of the complexity of content and pedagogical opportunities of written curriculum materials" (p. 4), which included attention to analyzing the materials for mathematical content and practices as well as pedagogical practices. To sum up, this type of noticing focuses on what aspects teachers think are important and what aspects teachers perceive in instructional resources.

In this poster presentation, I narrow the focus of these previous ideas on teachers' noticing of features of digital tools and resources designed for use in mathematics teaching and learning. That is, I explore the characteristics of digital tools and resources that teachers decide are important, whether the characteristics are general, pedagogical, or mathematical in nature. I also look to whether the characteristics are superficial or profound in nature.

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INVESTIGATION OF PRE-SERVICE TEACHERS' TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE

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The impact of technology on education has increased; however, technology itself is not guaranteed to support students' mathematical thinking (Heid, 2005). The development of teachers' TPACK (technological pedagogical content knowledge) is critical. In order to help pre-service teachers (PSTs) to gain an understanding and experience about technology integration, case analysis has been adopted in many technology courses, which allows PSTs to enact and examine authentic, content and context specific scenarios.

Dick and Hollebrands (2011) categorized technologies as conveyance or mathematical action tools, based on their purposes within mathematics instruction. Conveyance technologies are used to convey information, whereas mathematical action tools enable students to interact and receive feedback during the performance of mathematical tasks. Many researchers consider mathematical action tools as "cognitive technologies" which Pea (1987) defined as those which help users "transcend the limitations of the mind…in thinking, learning, and problem-solving activities" (p. 91). Pea (1987) articulated cognitive technologies as amplifiers (accelerate processes) and as reorganizers (engage cognitive processes). While amplifiers enhance students' ability to solve problems in an efficient way, but does not change their thinking, reorganizers "modif[y] cognitive processes…which affect our modes of approaching the acquisition of knowledge" (Barrera-Mora & Ryes-Rodriguez, 2013, p. 112).

The purpose of this study is to explore PSTs' TPACK through case analysis completed in an introductory technology integration course in a large Midwest state university. In content specific case analysis, PSTs select technologies, provide descriptions and justifications for their selections and create a lesson plan. The research questions of the study are: What technologies do PSTs select in doing mathematic case analysis with technology integration focus? How do they justify that their technology selections address the case? In what ways do they integrate technology in the content specific case? How do the outcomes of their case analyses provide an understanding about their TPACK?

The data sources of this multi-case study are eight PSTs' case analysis documents. By using qualitative data analysis methods, the researcher identified what kinds of technologies PSTs used and in which purpose these technologies were used. The preliminary findings of the study suggest these PSTs use both kinds of technologies but they choose to integrate technologies as amplifier more than reorganizer. The findings of this study would help future researchers to understand PST's TPACK development at an early stage of their program through case analysis.

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SPECIALIZED CONTENT KNOWLEDGE DEVELOPMENT WITH GSP: A CASE STUDY

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Keywords: Mathematical Knowledge for Teaching; Technology; Geometry

In response to the changes in what and how teachers should be teaching mathematics within K-12 schools, various organizations within the mathematics and mathematics education communities have proposed recommendations and guidelines for teacher education programs and for the mathematical preparation of teachers. The MET II report was the first report which explicitly underlined that prospective middle school teachers need to take mathematics content courses with an emphasis on deep understanding. Considering new insights about the theoretical structure of the mathematical knowledge, these courses were expected to support prospective teachers' subject-matter knowledge development special to their profession, which is called specialized content knowledge by Ball, Thames and Phelps (2008). One possible way to accelerate teachers' development of specialized content knowledge is the use of electronic technologies. In this paper, we present three PSTs' responses to a "square construction" and "inscribed circle of a triangle" tasks, in order to answer the following research question: "How does GSP influence middle grade mathematics PSTs' development of SCK?"

We employed a single case study research methodology with embedded units, as described by Yin (2008). PSTs enrolled in a graduate geometry course at a Southeastern research university in Fall/2013. The geometry course was the case of analysis, while three PSTs served as the embedded units who participated two clinical interviews outside the class. The clinical interviews each involved an open-ended task allowing the participants opportunity to demonstrate their SCK. During the first interview, the three PSTs were asked to complete a square construction task a middle school student started on GSP, and predict what the student was thinking with his/her procedure. During the second interview, two months after the first one, three participants were asked to find the relationship between the area and perimeter of a triangle, and the radius a circle inscribed in the triangle by watching animations built using GSP.

Data from two interviews underlined the necessity of CCK in SCK development. Three focal participants managed to demonstrate SCK as long as they had the required CCK for the event. Simultaneous development of CCK enabled PSTs to demonstrate SCK during interviews. Their ideas on the given tasks with or without the GSP initiated an opportunity for them to develop their SCK. One of the reason for this finding could come from the definition of SCK and its differentiation from CCK. Data also indicated that views about GSP seemed to determine whether GSP would be conducive for a PST's SCK development. If a PST views GSP as a tool for precise measurements and demonstration rather than a learning partner, then the role of GSP in SCK development might be limited. These initial findings provide a starting point for helping teacher educators understand in greater detail how technology should be used for SCK development.

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Chapter 12

Theory and Research Methods

Research Reports

Brief Research Reports

Poster Presentations

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

A CALL TO ACTION: TOWARDS AN ECOLOGICAL-DYNAMICS THEORY OF MATHEMATICS LEARNING, TEACHING, AND DESIGN

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Whereas Natural User Interface technological devices, such as tablets, are bringing physical interaction back into mathematics learning activities, existing educational theory is not geared to inform or interpret such learning. In particular, educational researchers investigating instructional interactions still need intellectual and methodological frameworks for conceptualizing, designing, facilitating, and analyzing how students' immersive hands-on dynamical experiences become formulated within semiotic registers typical of mathematical discourse. We present paradigmatic empirical examples of tutor–student behaviors in an embodied-interaction learning environment, the Mathematical Imagery Trainer for Proportion. Drawing on ecological dynamics—a blend of dynamical-systems theory and ecological psychology—we describe the emergence of mathematical concepts from the guided discovery of sensorimotor schemes.

Keywords: Cognition; Instructional Activities and Practices; Design Experiments; Technology

Introduction: In Search of Action-Oriented Theory of Mathematical Ontogenesis

Whereas commercial production of interactive math apps is booming, extant theory of learning is still a theory-of-learning-with-*paper*(Papert, 2004). In the short term, scarcity of bold research on interactive mathematics learning impedes the formulation of informed policies concerning the integration of technological environments into educational institutions. In the long term, this scarcity is accelerating misalignment between extant theory of learning and emerging practices to which it should apply. As children are learning to move in new ways, so, too, should theory of learning.

A motivation of this paper is that the pedagogical quality and institutional acceptance of action-based learning environments is largely pending on developing informed scholarly and public discourse concerning what it means to learn a mathematical concept and what an instructor's role might be in this process. As such, we are echoing Seymour Papert's consistent call to leverage the technological revolution as an opportunity for deep discussion of the potentially radical changes the educational system should undergo. Similar to Papert, we are optimistic that technological advances in educational media bear the potential of fostering students' deep understanding of mathematical concepts. Complementarily, we submit, these technological advances bear the potential of fostering researchers' deep understanding of learning processes.

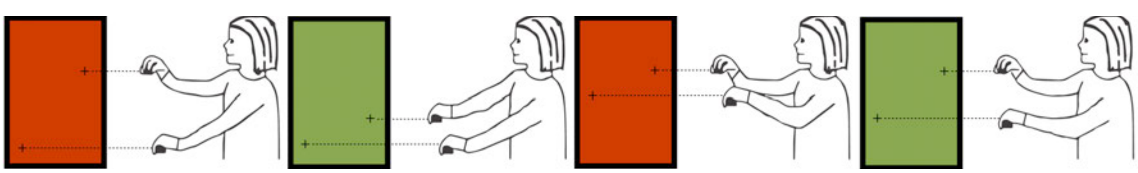
A pedagogical rationale to ground mathematics learning in physical interaction echoes centuries of educational scholarship. We now sketch its recent history. From his cultural–historical psychology perspective, Vygotsky believed that meanings are established through physical interaction. Moreover, he asserted that mature mathematical reasoning tacitly retains and evokes its originary enactive quality (Vygotsky, 1926/1997, pp. 161-163). From a cognitive-developmental psychological perspective, Piaget (1971, p. 6), too, viewed thought as truncated action, emphasizing that "mathematics uses operations and transformations…. which are still actions although they are carried out mentally." Piaget (1968, p. 18)later introduced the notion of *action coordination* as the root of reasoning (see also Nemirovsky et al., 2013).From a philosophy perspective, a resonant view of thought as truncated action has been elaborated by Melser (2004), who puts forth aphylogenetic embodied model of language and reasoning. From an educational-research perspective, Skemp (1976) critiqued math instruction as fostering disjointed "instrumental" knowledge. He promoted an alternative educational program that instead would foster deep "relational" knowledge that resides in non-symbolical dynamical interactions. Similar, Pirie and Kieren (1994) advanced an Enactivist view of knowledge to implicate mathematical reasoning as drawing on dynamical imagery (see Reid, 2014). Decades later, Nathan (2012) denounced mainstream educational practice as still implicitly subscribing to a "formalisms first" epistemology and called to ground mathematical meaning instead in "our direct physical and perceptual experiences" (p. 139). Thompson (2013), too, points to the fundamental problem of mathematics education as the absence of *meaning*, that is, webs of multimodal imagery actions. These inspirational fiats leave us with a set of questions: How do naïve goal-oriented actions give rise to reasoning about immaterial entities? How do students first accept cultural signs? In particular, how might this transpire in discovery-based instruction?

Granted, a number of theoretical frameworks from the learning sciences have been formative in modeling artifact-mediated guided learning of STEM content, such as instrumental genesis (Vérillon & Rabardel, 1995), professional perception (Stevens & Hall, 1998), cultural anthropology (Hutchins, 2014), and semiotic approaches (Radford, 2014). However, these frameworks are not optimally geared to treat the new forms of pedagogical, technological, epistemological, and interactional opportunities created by NUI embodied-interaction learning environments. In particular, extant theoretical frameworks lack analytical specificity for treating sensorimotor schemes—how they emerge, how they are steered, and how they give rise to conceptual knowledge—as the phenomenal core of mathematics learning. And so we present a call to *action* as our *Critical Response to Enduring Challenges in Mathematics Education* (PME-NA 37).

We hasten to note up front that our focus in this paper on fostering motor-action coordinations should not for a moment suggest that we are disregarding or mitigating the formative role of symbols in the development of mathematical knowledge or disavowing the rich theoretical and practical challenges that the symbolic register introduces (Duval, 2006). Rather, we believe that there has not been sufficient focus in the literature on the initial development of action schemes via direct or vicarious interaction with instructional media (but see de Freitas & Sinclair, 2012). And we view NUI technologies as powerful yet under-researched means of fostering those action schemes. Accordingly, this article treats the initial guided construction of mathematically oriented operatory schemes more so than the subsequent signification of these schemes in disciplinary semiotic systems.

Empirical Context: Design-Based Research of the Mathematical Imagery Trainer

The *Kinemathics* project (Reinholz et al., 2011) took on the design problem of students' enduring challenges with proportional relations. We assumed that students have scarce sense of what proportional equivalence is, feels, or looks like. We began by choreographing a bimanual motor-action scheme that enacts proportional equivalence, and then we envisioned, designed, and engineered conditions in which students could learn to move in a new way that emulates this scheme. Our two-step activity plan was for students to: (1) develop a target motor-action scheme as a dynamical solution to a situated problem bearing no mathematical symbolism; and (2) describe these schemes mathematically, using semiotic means we then interpolate into the action problem space.



a.

b. *c.* *d.*

Figure 1. The Mathematical Imagery Trainer for Proportion (MIT-P)

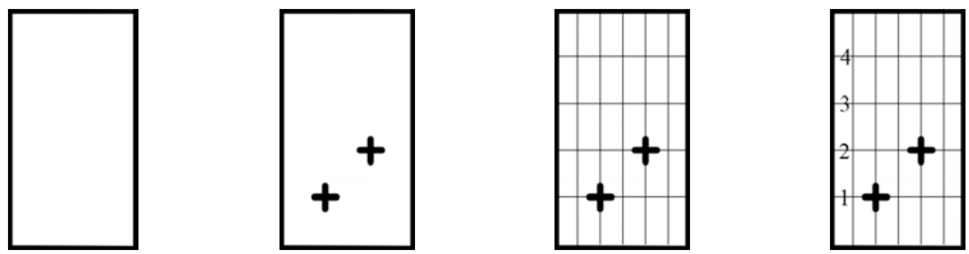
Figure 1 shows the MIT-P set at a 1:2 ratio, so that the favorable sensory feedback (a green background) is activated only when the right hand is twice as high along the monitor as the left hand. This figure sketches out our Grade 4 – 6 study participants' paradigmatic interaction sequence

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

toward discovering an effective operatory scheme: (a) while exploring, the student first positions the hands incorrectly (red feedback); (b) stumbles upon a correct position (green); (c) raises hands maintaining a fixed interval between them (red); and (d) corrects position (green). Compare 1b and 1d to note the different vertical intervals between the virtual objects.

Our design solution was the Mathematical Imagery Trainer for Proportion (MIT-P, Fig. 1). We seat a student at a desk in front of a large, red-colored screen and ask the student to "make the screen green." The screen will be green only if the cursors' heights along the screen relate by the correct ratio (e.g., 1:2). Participants are tasked first to make the screen green and then to maintain a green screen while they move their hands.

The activity advances along a sequence of stages, each launched when the instructor introduces a new display overlay immediately after the student has satisfied a protocol criterion (Fig. 2). The full design includes a ratio table for students to control the cursors indirectly via inserting numbers. (For an iPad version, seewww.tinyurl.com/FreeMITP).



a. b. c. d.

Figure 2. MIT-P display schematics, beginning with (a) a blank screen, and then featuring the virtual objects (symbolic artifacts) that the facilitator incrementally overlays onto the display: (b) cursors; (c) a grid; and (d) numerals along the *y*-axis of the grid. For the purposes of this figure, the schematics are simplified and not drawn to scale.

We implemented the MIT-P design in the form of a tutorial task-based clinical interview with 22 Grade 4 – 6 students, who participated either individually or in pairs, and these sessions were audio–video recorded for subsequent analysis (Reinholz et al., 2011). Our primary methodological approach is for the laboratory's researchers to engage in collaborative ethnographic micro-analysis of selected brief episodes from the entire data corpus (Siegler, 2006), where we focus on the study participants' range of physical actions and multimodal utterance around the available media (Ferrara, 2014). The process is iterative and in dialogue with the learning-sciences literature, leading to the progressive identification, labeling, and refinement of emergent categories (Strauss & Corbin, 1990). New constructs might constitute ontological innovations extending beyond the study context (diSessa & Cobb, 2004). Here we re-analyze our empirical data via a new lens.

Ecological Dynamics

Constructivist pedagogy champions the principle of fostering opportunities for individuals to re-invent cultural–historical knowledge (Kamii & DeClark, 1985). Yet how does this principle play out in learning environments where students are first to re-invent sensorimotor schemes prior to signifying the schemes in a discipline's semiotic register? We sought a theory of learning focused explicitly on the development of physical skill.

Ecological dynamics (Vilar et al., 2012) is a theoretical approach used in sports sciences to study skill acquisition in representative designs of real-game conditions. The framework blends *dynamical systems theory* (Thelen & Smith, 1994)and *ecological psychology* theory of affordances(Gibson, 1977).Applying dynamical systems theory to ecological psychology enables sports scientists to explain the learning of physical skills as the complex and adaptive self-organizing of subject–environment dynamical systems.

Dynamical systems theory is a branch of physics that provides a formal representation of any system evolving over time. The behavior of any living system can be plotted as a trajectory into a state space. In a dynamical systems approach, decision-making and learning processes are modeled not as generating a sequence of disembodied symbolical propositions, such as abstracted inferences and decisions, but as emerging from the agent's goal-oriented, situated, adaptive interactions in the environment (Araújo et al., 2009). The emergent quality of self-organizing complex adaptive systems implies also that learning processes are not linear but stochastic, and the non-linear dynamics of systemic behavior increases with the number of agents, variables, and interactions.

The self-organizing behavior of dynamical systems consisting of human agents (e.g., students) engaged in goal-oriented activity can be affected or "channeled" (Araújo & Davids, 2004, p. 50) by different types of constraints. Newell(1996) identified three sources of constraints affecting the behavior of the system either on a short time scale (i.e., decision making while performing a skill) or a longer time scale (i.e., the process of learning a skill): organism, environment, and task.

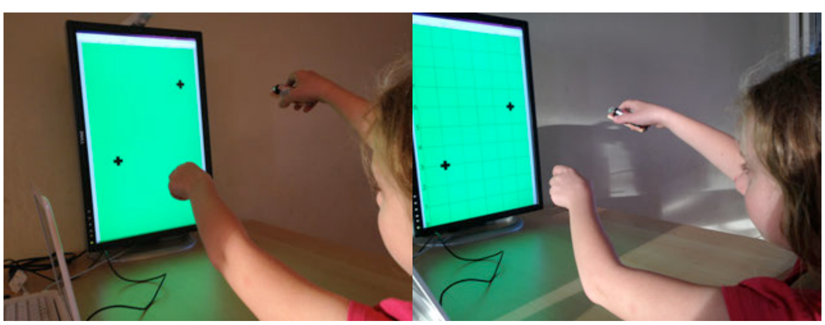
In terms of methodology, ecological dynamics may offer STEM educational research interpretive analytical schemes for modeling the role of instructors' multimodal utterance and actions in shaping students' construction of dynamical enactments. From its systemic view, ecological dynamics regards all forms of intervention, such as physical guidance or metaphoric framing, as productive constraints on the solution of motor-action problems.

The Ecological Dynamics of the Mathematical Imagery Trainer

We now present three ecological-dynamics accounts of children's guided work with an embodied-interaction design for mathematics content, the MIT-P. These concise narratives were selected as appropriate exemplars for showcasing numerous analyses of manipulation, discovery, and coaching in the context of math learning. For continuity, we will treat aspects of student behavior only around the numerical item of a 1:2 ratio.

The Emergence of an Attentional Anchor Mediating System Dynamics

Students typically begin the activity by lifting the controls and, in an attempt to make the screen green, waving them up and down in several different patterns. Eventually, the students discover that their hands "have to be a certain distance" from each other, and yet they attempt to keep this distance fixed. But as they further explore the screen regions, they figure out that "the higher you go, the bigger the distance" (Fig. 3a)Students thus discover, articulate, and empirically validate a systemic interaction principle governing a phenomenon under inquiry: a proposed correlation between two



a. *b.*

Figure 3. (a) A child discovers the vertical interval between the markers as an attentional anchor for making green while moving the hands: the higher it is, the bigger it should be (and vice versa). (b) Once the grid is overlaid, she shifts spontaneously to a new routine.

qualitative properties of *a new object*—the height and size of a linear interval subtended between their hands.

We have been intrigued by students' initial discovery of the interval between their hands as a means of controlling the screen color as well as by the subsequent smooth shift from keeping this interval fixed as they elevate their hands along the screen to varying the interval size in proportion with its elevation. Crafted spontaneously out of thin air, the interval articulates into being, foregrounded from negative space as a new auxiliary stimulus wedged between agent and artifact. The interval coalesces as a ready-to-hand tool for engaging latent correlations in the perceptual field, thus mediating the situated implementation of motor intentionality. It served the students as a spontaneous self-constraint—an order parameter, "steering wheel" (Kelso & Engstrøm, 2006; Newell et al., 2010), or *attentional anchor* facilitating enactment (Hutto & Sánchez-García, 2014).

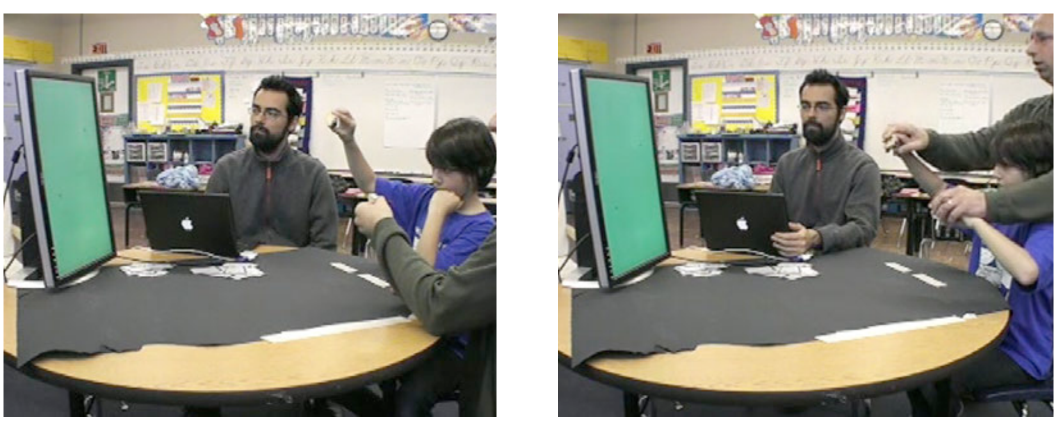
Decomposing and Recomposing an Attentional Anchor in a Reference Field

Once a student discovers the "the higher, the bigger" control strategy oriented on the interval between their hands as a new attentional anchor, the interview protocol proceeds to the next item, the introduction of the grid onto the screen (Fig. 3b). The appearance of the virtual horizontal gridlines materialized the imaginary attentional anchor. The grid's figural qualities immediately relieved students from having to hold the interval between their hands: The attentional anchor was thus electronically reified in the public domain in the form of a perceptually stable, externally present, deictically referable, bounded entity. Yet this frame of reference shifted students abruptly into a new interaction routine: raise the left hand 1 unit, then raise the right hand 2 units, iteratively, to make green. Now the old attentional anchor no longer mediated a goal, so it receded back into negative space.

The theory of ecological dynamics thus offers a view of conceptual development as spontaneous, situated adoption of symbolic artifacts *as action tools*. Symbolic artifacts bear hybrid ontology, in the sense that they are both perceptual and semiotic entities (Uttal, Scudder, & DeLoache, 1997). They are "transitional objects" (Papert, 1980)—both sensory and abstract. We might grab a symbol for its perceptuomotor affordance for action yet only subsequently—as personal and interpersonal situations evolve—leverage its semiotic potential for planning and communicating prospective actions, elaborating reasoning, and supporting argumentation. We kindle then obey new constraints.

Instructor's Multimodal Intervention as Environmental Constraints on Action

Learning is the education of perceptuomotor attention, and teachers can play pivotal roles in this educational process. One expert–novice co-enactment method is to distribute the operation of the



a. *b.*

Figure 4. Hands-on learning may need hands-on teaching:(a) co-manipulating virtual objects as distributed co-enactment; and (b) molding as joint co-enactment.

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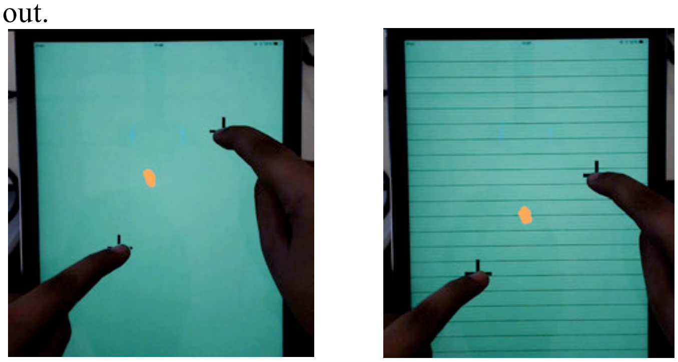
control devices, one person per device. Another method is to co-operate both of the control devices, that is, with both people each handling both devices.

Similar to a pair of athletes in a two-person sport, such as rowing or luge, the tutor and student optimize for effective joint production by continuously and dynamically adjusting each to the other's spatiotemporal actions (Fig. 4). They reach an intimate level of intersubjective sensorimotor coordination by anticipating and closely tracking their mutual actions. Yet as in the martial-arts practice of push-hands, these two participants silently negotiate leadership. The tutor progressively hands over agency to the student, who eventually solo-enacts the new strategy. The tutor-as-dynamical-scaffold fades out.



a. *b.*

Figure 5.Spontaneous evolution of an attentional anchor. The eye gaze (orange spot) hovers over a location on the screen that contains no information in-and-of-itself but only with respect to the dynamical motor-action coordination. A grid then offers a frame of reference for bringing forth the attentional anchor into mathematical consciousness.

Recent results from eye-tracking studies (Fig. 5) confirm our qualitative analyses of the interview data: Just before students articulate a new manipulation strategy, their visual attention tears away from figural constituents on the screen to anchor onto a new location, a higher-order invented "handle" on a structural constellation, that facilitates operable interaction with a dynamical yet invariant conservation (Shayan et al., 2015).

Conclusion

We have introduced an ecological-dynamics view on mathematics teaching and learning. We further presented and interpreted empirical data from implementations of embodied-interaction activities so as to contextualize the ecological-dynamics view and argue for its purchase on enduring research problems germane to the learning sciences in general and to scholarship and application of mathematics education in particular. Based on our findings, we contend that the theory of ecological dynamics offers a useful framework for designing, implementing, and analyzing pedagogical interactions in which students develop fundamental understandings of mathematical notions via solving and reflecting on motor-action inquiry problems. We also explained how an ecological-dynamics view of mathematics learning coheres with, integrates, and extends seminal constructivist and socio-cultural historical perspectives on human learning. From this view, mathematical meanings are cultural constructs that individual agents build by developing and then signifying appropriate motor-action coordinations oriented on discovered attentional anchors. These dynamical coordinations are embodied solutions to physical problems that students encounter when engaging in carefully designed activities.

We wish to position this article as attempting to rekindle essential themes of situated cognition (Greeno, 1998). To our reading, ecological dynamics should offer an effective and comprehensive framework for analyzing socially guided ontogenesis of intelligent participation in cultural practices. In particular, ecological dynamics, with its view of learning as coordinating motor actions in perceptual fields, may replace the cognitive semantics theory of conceptual metaphor (Lakoff & Núñez, 2000) as a more viable means of tracking the subjective and intersubjective emergence of mathematical concepts from situated solving of sensorimotor interaction problems (Gibbs, 2014). Ecological dynamics offers new tools for minding the epistemic gap between action and symbol and thus stands to fill a critical, enduring lacuna in mathematics-education research literature.

Given appropriate cultural mediation, children can learn quite rapidly to move and therefore think in new ways that become signified, elaborated, refined, and reformulated as disciplinary discourse. This thesis suggests *all* children's universal capacity to deeply understand mathematical concepts, regardless of prior academic accomplishment, because it shifts the site of critical mathematical learning away from the symbolic semiotic register toward situated sensorimotor engagement with manipulation problems.

Embodied-interaction activities offer solutions for researchers and teachers alike who wish both to observe mathematical thinking as it is occurring and offer students opportunities to reflect on their actions. Technology-enabled embodied-interaction learning environments transform the practice of mathematical teaching, rendering it similar to coaching in disciplines more readily associated with physical action, such as music, dance, or carpentry. Yet for these instructional devices and methodologies to enter educational institutions, we would all have to rethink multiple aspects of mathematics teachers' professional practice, beginning from epistemology and through to assessment.

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EXPLORING ROLES OF COGNITIVE STUDIES IN EQUITY: A CASE FOR KNOWLEDGE IN PIECES

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Underrepresentation of people of color in mathematics at the postsecondary level warrants more focus on equity issues. The prevalence of cognitive studies at the undergraduate level is met with the call for critical analysis about the kinds of knowledge that get privileged in mathematics education. Connecting to the Funds of Knowledge work, this paper discusses the utility of diSessa's Knowledge in Pieces cognitive framework to uncover productive informal knowledge in learning formal mathematics. Seeing the valorization of knowledge as related to issues of power, a case of a Chicana student's productive sense making about the formal definition of a limit illustrates the way diSessa's framework can help challenge what counts as productive mathematical knowledge and reasoning.

Keywords: Learning Theory; Advanced Mathematical Thinking; Equity and Diversity; Cognition

> Whether parents, teachers, students, or researchers, we all bring valorization of knowledge to our views of what counts as "proper" or "better" approaches to doing mathematics.
>
> –Marta Civil (2014, p. 12)

The President's Council of Advisors on Science and Technology (PCAST) called for 1 million additional college graduates in science, technology, engineering, and mathematics (STEM) fields based on economic forecasts (Executive Office of the President, PCAST, 2012). Within STEM, the number of mathematics degrees being conferred continues to be generally low. People of color, in particular, continue to be severely underrepresented in mathematics. A recent National Science Foundation report shows that historically marginalized population accounts for 20% of the mathematics degrees conferred (NSB, 2014).

Gutiérrez (2002) characterizes equity research as one that explicitly focuses on efforts to understand and mitigate systematic differences in how people experience educational opportunities, particularly differences that privilege one group over another. In the broader area of undergraduate STEM education, some studies have begun to explore issues of marginalization of women and students of color and ways that students manage it in their everyday lives (e.g., McGee & Martin, 2011). However, there has not been a sustained and concerted effort to focus on equity considerations in undergraduate mathematics education research (Adiredja, Alexander & Andrews-Larson, 2015), like there has been in K-12 mathematics education research in recent years (Gutiérrez, 2013).

The need to understand how students make sense of challenging topics in undergraduate mathematics contributed to the prevalence of studies about students' and teachers' individual cognition and practices (Adiredja et al., 2015). Several scholars have argued for critical analysis of the kinds of knowledge and practices that we privilege in teaching and learning of mathematics (Civil, 2014; Gutiérrez, 2013). Considering the socio-political nature of education, Gutiérrez (2013) emphasized the interconnectedness of knowledge and power:

> Knowledge and power are inextricably linked. That is, because the production of knowledge reflects the society in which it is created, it brings with it the power relations that are part of society. What counts as knowledge, how we come to "know" things, and who is privileged in the process are all part and parcel of issues of power.

For example, children of immigrant parents at times would discount their parents' mathematical knowledge as a result of the way things are taught in US schools (Civil and Planas, 2010). This example illustrates one explicit implication of power vis-à-vis the kinds of knowledge students learn to be valued in the classroom. The question then becomes how can we begin to unpack issues of power behind our valorization of mathematical knowledge and approaches, particularly at the undergraduate level?

At the K-12 level, some researchers have responded to this question by trying to leverage and build on cultural aspects of the students' communities in designing curriculum (Moll, Amanti, Neff & Gonzalez, 1992). The goal with the Funds of Knowledge project (Moll et al., 1992) is to help teachers explore and document the often invisible but productive knowledge of non-dominant students. While it is tempting to assert that the nature of mathematics at the undergraduate level is different from that of K-12 mathematics, I posit that the spirit of the Funds of Knowledge work can also be productive at the post-secondary level.

The kind of mathematics that is being learned at the postsecondary level does become increasingly abstract and are often built on prior formal mathematical knowledge. Moreover, the Euro-centricity of mathematics (Joseph, 2010) becomes even more privileged at the post-secondary level. As such one might argue that these realities of mathematics at the post-secondary level limit opportunities for teachers to connect to students' informal knowledge and experiences. For example, in the context of limit in calculus, some researchers have argued that students' intuitive knowledge is an obstacle in learning (Davis & Vinner, 1986), despite the prevalence of such knowledge student thinking about the concept (Monaghan, 1991).

While the Euro-centricity of advance mathematics is a reality, students' intuitive knowledge about mathematics, and more importantly ways that such knowledge might be productive in learning formal mathematics, are largely underexplored in undergraduate mathematics education. Wawro et al. (2012) have shown a case where incorporating students' intuition about using different modes of transportation in traveling is productive to explore concepts in linear algebra. This study suggests that there is room for intuitive knowledge in learning mathematics at the undergraduate level. Recognizing the potential utility of students' informal knowledge, and its potential in disturbing power distribution in the classroom, where can we find such knowledge? What might be able to assist us in uncovering many of the invisible productive knowledge students bring with them to learn formal mathematics?

This paper attempts to offer a response to those questions, particularly in the context of undergraduate mathematics education. This paper argues that the theoretical perspective on epistemology and learning in cognitive studies plays an important role in uncovering knowledge resources students bring into learning formal mathematics. In particular, this paper argues for the utility of the Knowledge in Pieces cognitive framework (diSessa, 1993) to recognize productive knowledge resources students might use in learning formal mathematics, particularly in the face of non-normative language. The analysis shows how this framework has a potential of challenging the existing distribution of power vis-à-vis what counts as productive knowledge.

Theoretical Framework for Cognition

The Knowledge in Pieces (KiP) theoretical framework (diSessa, 1993; Smith, diSessa and Roschelle, 1993) models knowledge as a system of diverse elements and complex connections. The nature of the elements, their diversity, and connections are typical interests for studies using this framework. Characterizing knowledge using generic ideas like "concept" or the commonly used idea of "misconceptions" is viewed as uninformative and unproductive (Smith et al., 1993). Instead, KiP focuses on the context specificity of knowledge to maintain the *productivity* of the particular piece of knowledge. KiP also pays particular attention to the continuity of knowledge, i.e., ways that knowledge gets used or built upon in new contexts. It is common for studies using this framework to uncover productive sense making behind students' use of non-normative language to describe their reasoning (e.g., Campbell, 2011; diSessa, 2014).

One of the main principles of KiP is that knowledge is context specific (Smith et al., 1993). This means that the productivity of a piece of knowledge is highly dependent on the context in which it is used. Context variation can happen as a result of change in the literal problem context, the passage of time, or simply as knowledge is assessed more or less carefully. For example, the knowledge that "multiplication makes a number bigger" is productive in the context of multiplication with numbers larger than one. The knowledge is not productive in the context of multiplication with all real numbers. In contrast to studies that focus on identifying students' misconceptions, KiP focuses on building new knowledge on students' prior knowledge, instead of focusing on efforts to "replace" students' misconceptions (Smith et al., 1993). Adopting this theoretical framework implies that the analysis in this paper will focus on ways that students build on their prior ideas while suspending judgment about their correctness. KiP also posits that students have a lot of intuitive ideas that can be leveraged in instruction. KiP was developed in the context of physics where students have a diversity of intuitive ideas about physics originating from their everyday experience. Some studies have shown that intuitive ideas can also be found in mathematical reasoning as well (e.g., Campbell, 2011).

Mathematical Context and Literature

The formal definition of a limit of a function at a point, also known as the epsilon-delta (ε-δ) definition, is an essential topic in mathematics majors' development that is introduced in calculus. The limit of a function $f(x)$ as x approaches a is L and is written as $\lim_{x\to a} f(x) = L$ if and only if, for every positive number ε, there exists a positive number δ, such that all numbers x that are within δ of a (but not equal to a) yield $f(x)$ values that are within ε of the limit L. This defining property is often written as "for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$". Informally, one might say, "If L is the limit, then for however close one wants $f(x)$ to be to L, one can constrain the x-values so that $f(x)$ would satisfy the given constraint." We return to this intuitive idea shortly.

The formal definition provides the technical tools for demonstrating how a limit works and introduces students to the rigor of calculus. Yet even thoughtful efforts at instruction leave students, including intending and continuing mathematics majors, confused or with at most a procedural understanding about the formal definition (Cottrill, Dubinksy, Nichols, Schwingendorf, and Vidakovic, 1996; Oehrtman, 2008). Many studies assert that students' dynamic conception (the limit is the number that $f(x)$ approaches as x approaches a) is an obstacle in learning the formal definition (Parameswaran, 2006; Williams, 2001). These studies largely focused on the unproductivity of students' prior conception and their sense making.

In the meantime, a small number of studies that focuses on students' sense making of the formal definition (Knapp & Oehrtman, 2005; Roh, 2009; Swinyard, 2011) suggest that students' understanding of a crucial relationship between two quantities featured in the formal definition, epsilon (ε) and delta (δ), warrants further investigation. Davis and Vinner (1986) used the term *temporal order* to describe their relationship. While studies have shown the existence and prevalence of this particular difficulty, its nature is largely underexplored.

The relationship between the quantities δ and ε in the definition can be described using the idea of quality control in manufacturing an item. The conceptual structure at issue can be described as follows: given a permissible error in the measurement of the output (ε), one determines a way to control the input to achieve that result. One does so by determining the permissible error in the measurement of the input (δ) based on the given parameter for the output (ε). In this way, the error bounds follow the following sequential order, error bound for the output, then the error bound for the input. This is because the error bound for the output is given. In some ways, the error bound for the input could be seen as being dependent on the given error bound for the output. Epsilon can be seen as the error bound of the output whereas delta is the error bound for the input. Therefore, δ and

εfollow the order of ε first, and then δ, or δdepends on ε. In this paper, the student discussed this idea of quality control in the context of working at a pancake house that is known to make 5-inch diameter pancakes. Students are given a permissible error for the size of the pancakes, and they were responsible to control the error in the amount of batter.

Data Collection and Analysis Methods

The data presented in this report is a case study from a larger interview study investigating the role of prior (and intuitive) knowledge in student understanding of the temporal order of epsilon and delta within the formal definition (Adiredja, 2014). Participants of the study were calculus students at a large Western public research university. Students were interviewed about their understanding about the temporal order. They were asked a series of questions about the temporal order before and after engaging with the instructional intervention. The instructional intervention, the Pancake Story uses the context of working at a pancake house to leverage the idea of quality control in discussing the formal definition, as explained in the previous section. A video recording of the interview was transcribed following Ochs' (1979) guidelines. Transcripts were organized by turns, marked changes in the speaker. They included non-verbal behaviors, including relevant gazes, laughter and gestures. Turns that discus one mathematical argument make up an episode. The transcript was modified to facilitate reading. Many hedges, and uh-huh's and um-hm's from the interviewer were removed.

Adriana, the focus of the analysis of this paper, was a mathematics and Chicano studies major. She ethnically identified as Chicana. Adriana received an A in her first semester calculus course in high school and in college. She was selected because despite her strong academic background, even after engaging with the Pancake Story, she still initially (and incorrectly) argued that ε depended on δ. Ultimately, Adriana adopted many of the productive resources from the story and used them to reorganize her knowledge and modify her claim. The analysis was interested in understanding how and why she did so. The analysis did not explore ways that Adriana's identity as a Chicana influenced her reasoning about the topic. Her ethnicity and gender were included to better represent her as a student and challenge the common unintended assumption with cognitive studies that the student is a White male student (Nasir, 2013).

The analysis focused on identifying a knowledge entity called *knowledge resources* (Adiredja, 2014), which is defined as a single or a collection of knowledge elements that might be involved in making a single claim from larger ideas that the student used to make her claim. Knowledge resources were assumed to be neutral; they are not correct or incorrect. This theoretical assumption distinguished knowledge resources from larger ideas that were combinations of several knowledge resources. To identify knowledge resources, the analysis exploit any relevant data (e.g., gestures, other parts of transcripts) that might inform the aim to optimally understand the activation of knowledge resources in various contexts. The analysis then generated multiple models (interpretations) of the student's argument in each episode. The analysis then put these models of student thinking in competition with one another. This process of *competitive argumentation* (VanLehn, Brown, & Greeno, 1984) was used to refine interpretations of student thinking. In this paper I only present the final model of each episode that was the result of the process of competitive argumentation.

Results

This paper only presents two of the four episodes of Adriana's sense making: the first and final episode. These episodes illustrate the changes in Adriana's thinking and salient ideas from the Pancake Story. They also show the initial conflict that Adriana faced in aligning the ideas from the story with her prior knowledge. The four episodes occurred on the span of 14 minutes.

The first episode started with the interviewer's asking Adriana about the dependence between$\delta$and ε after they discussed the Pancake Story. Adriana responded with the same [incorrect] claim she made before she engaged with the story. She argued that ε depended on δ because epsilon was with $f(x)$and delta was with x and $f(x)$ depended on x. The bolded texts marked the ideas that from which knowledge resources were identified.

Adriana: [They kinda depend on each other], yeah in a sense because, but more whatever you're getting, like $f(x)$ is always gonna depend on what x you're inputting it. But then, if you want to get something that's within delta [*marks a small interval on the x axis with two fingers*] you need to see if /.../ for example here [*points to the pancake story*] our epsilon here was already set, then that [*points back and forth between* 4.5 *and* 5.5 *in the inequality* $4.5<f(x)-L<5.5$] kind of depended on what we were putting in for x [*points at the same interval around x on the graph*] but..but mostly whatever you're putting in to your x is gonna determine what you get for $f(x)$ [*pause*]. So I'm still saying the same thing like delta depends on epsilon but=

Interviewer: =Delta depends on epsilon? Or epsilon depends..

Adriana: No, yeah, epsilon depends on delta. But, /…/ if epsilon's already set then you'll manipulate your /…/ delta so it's within an error bound and /…/ then continue to manipu- wait [*long pause*] wait, so you're… hm.

Final model of episode 1: Adriana focuses on her prior claim that epsilon depends on delta. She justifies the claim using *functional dependence* and *function slots* resources. She simultaneously brings up many of the productive resources from the story: *domain constraint for a limit, the givenness* of epsilon and *quality control.* However, these resources are in conflict with her prior conception. Adriana's use of the knowledge resource of *functional dependence* can be seen in her statement, "whatever you're putting in to your x is gonna determine what you get for $f(x)$." Analysis of the final episode revealed that in this episode, Adriana thought about epsilon and delta as a range of errors, i.e., x and y values, instead of error bounds for those values. This suggests that along with *functional dependence,* Adriana uses the knowledge resource *function slots*, i.e., the assumption that when two quantities share a functional relationship, one quantity is the x and the other is the $f(x)$ or the y.

Separate from her previous argument, Adriana also mentioned the idea that the epsilon (the error bound for the pancake) was given (*givenness* knowledge resource) in the story, and that she wanted to get "something" that was within delta. That statement reveals Adriana's preference of only considering x values that are close to a in discussing limit problems. She would control that closeness by choosing x values that were within a small delta. This suggests her use of the knowledge resource *domain constraint for a limit*. The last line of the episode suggests that Adriana might have also taken up the idea of *quality control*: for a given specification on the output, one would manipulate the input so the output would be within the specified error bound. The sentence also reveals her use of the *dynamic conception of a limit* when she talked about continuing to manipulate the delta to get x closer and closer to a. She also erroneously talked about wanting delta to be within an error bound, which is consistent with the interpretation that delta in this episode as a range of x values.

In the final episode, Adriana repurposed the *functional dependence* resource to describe the relationship between the errors but not the error bounds. This productive move helped Adriana to align productive resources from the story with Adriana's prior knowledge. She then prioritized the idea of *givenness* of epsilon and *quality control*, which she already knew since episode 1,to conclude that delta depended on epsilon. She also adopted the story's language.

Interviewer: So, do they depend on each other, is it just one way now?

Adriana: Um, see cus I was looking at it like /…/ **the f of x [$f(x)$] depends on the x** and that's how I was like saying that epsilon depends on delta because **epsilon is related to the f of x** [*f*(*x*)]/…/. But that's just saying **the error of the *L* and the f of x [*f* (*x*)] depends on the *a* and *x*** but that's not to say that epsilon depends on delta.

Interviewer: Ok, so?

Adriana: So, I think that delta depends on epsilon now [*laughs*]. Just cus if it's given like this [*reference unclear*] and **you're trying to aim at getting /…/ within a certain error bound**, then you're gonna **try to manipulate your entries /…/ to be within a certain error bound** [*gestures a small horizontal interval with her palms*]

Interviewer: Ok. Alright, so and so you changed your mind it seems? Um, so how did that happen? Why did you change your mind?

Adriana: Because I was **given an epsilon** [*points at the inequality* $4.5<f(x)–L<5.5$] and that's kinda like the main goal. **The main goal is to get the pancake, /…/ and they gave me a constraint** /…/ and /…/ they **didn't give me an error bound for the batter** or for like the *a* or *x*, they didn't give me an error bound. But I know I **want to make it small so that it's within the error bound, the epsilon**. So then **I would kinda base my delta on what was epsilon**.

Final model of episode 4: Adriana uses *functional dependence* to describe the relationship between the errors ("But that's just saying the error of the *L* and the $f(x)$ depends on the *a* and *x,* but that's not to say that ε depends on δ."). To determine the dependence relationship between ε and δ, she prioritizes the resource *givenness* of epsilon and *quality control*, as seen in her statement, "you're trying to aim at getting within a certain error bound, then you're gonna try to manipulate your entries to be within a certain error bound." Adriana also made the productive observation that delta was not given, showing her use of the *givenness* resource with delta as well. More importantly, not only did Adriana treat delta as an error bound, she also stated that she wanted delta to be *small*. By making delta small, Adriana was no longer using the idea of smaller and smaller or "continuing to manipulate" the input errors that suggests the use of *dynamic conception of a limit* in the first episode.

Discussion

In summary, the analysis revealed that Adriana took up many of the productive resources from the story despite its initially looking as if her understanding seemed unchanged. It took effort for Adriana to align the new productive resources from the story with her prior knowledge. After she repurposed the *functional dependence* resource to describe a relationship between the errors, she prioritized the productive resources from the story. Adriana also used the story to make a novel observation about the temporal order (delta was not given). That move and the adoption of the language of the Pancake Story suggest that the story was a rich learning context for Adriana. The story was able to leverage Adriana's prior knowledge about *functional dependence* and *quality control* while reasoning about the temporal order.

The Knowledge in Pieces framework guided the analysis in revealing knowledge resources from the story that were salient to Adriana, as well as those that existed as part of her prior knowledge. Adriana's language in describing her conception in the first episode was not clear. However, analysis of the structure of her knowledge revealed these productive resources. The analysis was also able to recognize the productivity of Adriana's moves because KiP takes seriously the process of reorganization of knowledge. The theoretical assumptions about epistemology and learning make KiP particularly sensitivity to subtle changes in sense making and potential productive roles that students' prior knowledge can play in learning. It challenges the deficit perspective of student thinking and challenges what counts as productive mathematical knowledge and reasoning. More broadly, researchers studying student thinking wield a great deal of power in deciding what kind of knowledge is valuable, and particularly in suggesting implications to practice from the findings of the analysis. For example, cognitive studies that focus on pathologizing students' thinking might have simply characterized Adriana's return to her prior argument about the temporal order as a result of a persistent misconception. Moreover, a lot of the subtle changes and her adoption of many of the productive resources might have been easily overlooked. Thus, not only would it position her and her thinking in a deficit way, it would also fail to recognize her contribution.

The findings also show the utility of intuitive knowledge in building a conceptual understanding of formal mathematics. The spirit of the Funds of Knowledge work can be seen in the way that the Pancake Story leveraged the intuitive notion of quality control to learn about the temporal order within formal definition of a limit. At the same time, the story was designed with the KiP framework's assumptions about the potential productivity of prior knowledge and ways that knowledge is reorganized. In addition to Wawro and colleagues' (2012) work, we see another case where intuitive knowledge can be productive in learning formal mathematics.

In sum, I argue that cognitive studies can contribute to equity issues more directly by addressing issues of power vis-à-vis valorization of knowledge. In this paper, I made a case for the KiP framework, and recognize that there might be other frameworks that can help uncover non-normative but potentially productive ways of thinking with formal mathematics. This type of work would benefit all students, but would particularly benefit non-dominant students whose knowledge are often devalued or unrecognized at the post-secondary level. In the face of underrepresentation and marginalization of non-dominant students more broadly, cognitive research can play an important role in challenging issues of power in mathematics education.

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PROBLEM POSING: A REVIEW OF SORTS

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In this paper we offer a review of sorts of the studies conducted around issues of problem posing in mathematics education research. We first ground the work on problem posing in the seminal work of Polya and of Brown and Walter, which influenced most studies on this subject. We then propose two perspectives taken on problem posing: the implicit and the explicit. These illustrate the varying emphasis concerning the conception of what is meant by problem posing, one being about actual requests for creating a problem and the other about defining the nature of problem solving processes. We conclude by discussing the significance of this categorization for making theoretical advances in problem posing research.

Keywords: Cognition; Problem Solving

Context

Issues about problem posing have been around for a number of years in mathematics education research. This being so, recently there has been a resurgence of studies on the topic, illustrated through Working Groups (e.g. PME 2009, 2011), Special Issues (e.g. ESM, *83*(1)), and books (e.g. Singer, Ellerton, & Cai, 2015). Through this spread of studies on problem posing, however, numerous orientations have been developed, and often one is at a loss in making differences or even finding similarities between the perspectives taken. Far from being a negative aspect of the field, as it shows its richness and enlargement, there is however a need to distinguish and categorize the kind of work being conducted in order to develop clearer views on what problem posing means and how to study it. Other researchers have also attempted classifications in the past (e.g. Voica et al., 2013; Christou et al., 2005). We re-use and deepen these classifications, combining them with the work of Polya (e.g. 1957) and of Brown and Walter (e.g. 2005), who are seen as pioneers on the theme. In addition, we outline another line of studies to which little attention has been paid to, that is, studies focusing on the activity of problem solving defined as an activity of problem posing. Thus, in this paper we extend the current categorization of studies on problem posing, leading us to varied views of what is meant by problem posing in the community of mathematics education researchers.

To do this, we first situate the field on problem posing through an overview of the work of Polya and of Brown and Walter. We then offer a first category of studies, termed the *explicit perspective*, which focuses on explicit requests to students to participate in an activity of composing problems. We then offer a second category of studies called the *implicit perspective*, which focuses on studies that define the activity of problem solving as one of problem posing.

This being said, as expected, we do not claim to offer an exhaustive list of all work ever conducted on problem posing. In this sense, we do not offer a review, but mainly a review *of sorts*. The intention with this review of sorts is to offer fruitful distinctions related to the underpinnings of what is considered an activity of problem posing. Through these distinctions, we aim to take a step forward in the direction of Silver's (2013) suggestion for more developed theoretical frameworks to support studies in problem posing.

Pioneers of Problem Posing: The Work of Polya and of Brown and Walter

Numerous researchers have mentioned being influenced, directly or indirectly, by the work of Polya or Brown and Walter, making them important sources in the problem posing literature. We thus refer to their work as a way of grounding and contextualizing this review of sorts.

George Polya's Problem Posing

Polya's (e.g. 1957) work on problem solving focuses on helping and pushing students to analyse the problems they solve and to think of other interesting problems in relation to them. In so doing, for Polya, teachers help students to consolidate their knowledge, develop their ability to solve problems and improve their solution or their understanding of it. Polya did not used the expression *problem posing* in his work, referring mostly to what he called the *Looking back* technique, which enables students to generate new ideas and investigate possible connections between mathematical problems. Having solved a problem, students are asked to look at what they have done and then to formulate new problems out of it. Polya argues that through this activity, students gain a better understanding of their solutions and increase their solving abilities. To formulate new problems, Polya suggests various heuristics of *Looking back*, four of which are discussed here. For example, consider this problem for students to solve (Figure 1):

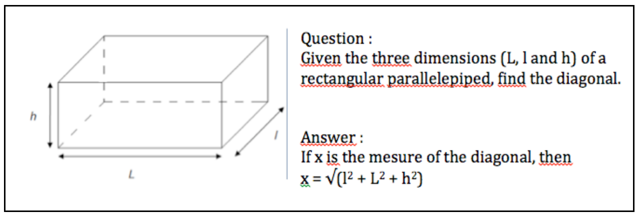

Figure 1: Polya's parallepiped problem (Polya, 1957, p. 7)

A first heuristic consists, once one knows the solution to this problem, of *generating analogous problems*, i.e. similar problems to this one. Polya gives examples of possible formulations: "a) Given the three dimensions of a parallelepiped, find the radius of the circumscribed sphere; b) The base of a pyramid is a rectangle of which the center is the foot of altitude and the sides of its base, find the lateral edges; c) Given the rectangular coordinates (x_1, y_2, z_3), (x_1, y_2, z_3) of two points in space, find the distance of these points." (1957, p. 66) These problems allow students to go back to the initial solution, but for other contexts, which requires them to rethink the solution and not only apply the formula. A second heuristic consists of applying the formula found by *modifying* the problem and its data. For example, the initial problem requires looking for the diagonal of the parallelepiped in relation to its width, length, and height. Another problem can be formulated by asking to find the height of the parallelepiped depending on the diagonal, the length, and the width. This heuristic requires interchanging the role of the various givens of the problem. Polya's third heuristic is *generalizing/specifying*. Generalizing consists in solving the same problem, but for an entire category of numbers or givens. For the above problem, one possible generalization could be: "Find the diagonal of a parallelepiped, being given the three edges issued from an end-point of the diagonal, and the three angles between these three edges." (1957, p. 67), which requires e.g. aiming for algebraic letters to represent the needed values of the problem. Also, a way of specializing the problem would be to look for specific cases, like finding the diagonal of a cube knowing one of its edges. A fourth heuristic is *studying variations*, that is, studying the effect of varying some of the data in the problem. For example, in the analogous problem of the circumscribed sphere, one can vary the radius of the sphere and study its effect on the problem and solution, leading to three possible cases: the sphere is entirely contained in the cube, the sphere is circumscribed in the cube; and finally the sphere encompasses the cube. Polya's heuristics are illustrations of his *Looking back* approach. For him, binding problem posing to problem solving allows students to see the possible mathematical connexions between various problems. By looking back at their solution, by reconsidering and examining the solution and the path they have followed, he argues that students consolidate their knowledge and develop their problem solving skills.

Brown and Walter's Problem Posing

The main goal of Brown and Walter's (e.g. 2005) problem posing is to study mathematics by working on students' questions and reflections on a given topic. For Brown and Walter, questions that arise in the classroom must not only be instrumental (i.e. posed to ensure understanding and performance of what the teacher asks students to do), but rather should help student to develop their mathematical skills, understanding and autonomy. In *The Art of Problem Posing*, Brown and Walter present two perspectives of problem posing: *Accepting* and *What-if-not?* (WIN). These perspectives are to help teachers to develop strategies for using problem posing in class with their students. The *Accepting* perspective refers to students accepting a concept suggested by the teacher (e.g. the concept of prime numbers defined as natural numbers that have exactly two natural divisors), and then finding interesting problems/questions about it. In the case of prime numbers, it could be questions like: *How many prime numbers are there? How to find the next prime number?* They argue that this leads students to explore and work mathematically on a concept, in order to develop an understanding of it.

The WIN perspective consists, on the other hand, in seeing what happens when instead of "accepting" the concept, one contests its characteristics. Brown and Walter suggest various levels of WIN, which they illustrate with the example of the Pythagorean theorem. A first level is for students to list the attributes of the Pythagorean theorem. For example, it may be noted that all number are squared or that the variables are connected by an equal sign. A second level consists of asking a WIN question for each of the attributes listed. For example, *What-if* the variables were *not* connected by an equal sign, but by an inequality? This question opens and becomes a new route to be explored for both students and teacher. A third level, called the *What-if-Not-ing* level, requires combining the negation of two of the attributes listed. In this case, it could be by looking at what happens when the variables are not linked by an equal symbol *and* all numbers are not squared. This also opens a new route to explore. For Brown and Walter, these mathematical explorations allow students to understand the Pythagorean theorem through the importance of its mathematical attributes, as well as developing their ability to formulate questions, explore mathematics, and solve problems. In this sense, the authors argue that after solving a problem, a person does not fully understand the meaning of what he/she has done unless new interconnected problems are formulated and analyzed, which affords a better grasp of the concept worked on. Thus Brown and Walter's problem posing is related to an inquiry process that leads to the exploration of concepts for understanding them better, arguing also for openness toward mathematical questions and thoughts that occur in classrooms.

Conceptualizing Problem Posing: Explicit and Implicit Perspectives

Grounded or not in Polya or in Brown and Walter's ideas, various meanings about problem posing are found in the literature. We distinguish these meanings by suggesting two perspectives. The explicit perspective refers to an explicit request for students to compose problems, whereas the implicit perspective refers to something that occurs implicitly in the activity of problem solving, i.e. every act of problem solving is seen as an activity of problem posing in itself.

The Explicit Perspective: A Pragmatic View of Problem Posing

In the explicit perspective, we distinguish three categories of studies being conducted, highlighting their diverse but complementary nature. We discuss these and give examples for each of them. In our description, we use the word *learners* to refer whether to students or (prospective) teachers who are doing the various kinds of problem posing.

Category 1: To compose a problem without any context or constraint. This first category refers to asking learners to compose a problem without imposing any context or constraints. In short, they need to compose from scratch. This category of problem posing can be linked to what Stoyanova and Ellerton (1996) call a free problem posing situation where students have to formulate new problems in an open situation. For their part, Christou et al. (2005) refer to this kind of problem posing as tasks that require students to pose a problem in general, in free situations. In this category, we find, for example, the work of Ellerton (1986) and of Crespo (2003). In her work, Ellerton asks students to compose a problem that would be difficult, a challenge, for a friend to solve. The students then have a blank card to compose mathematical problems of any kind related to the concepts that they wish. In Crespo's study, elementary students are paired with prospective teachers and correspond one-on-one by sending each other letters. Through this, Crespo aims at placing prospective teachers in an authentic experience of generating problems by asking them to write mathematical problems in their letters for their elementary student. The prospective teachers have no constraints on the type of problem to compose or the mathematical concepts to use.

Category 2: To generate problems from specific constraints. Another category refers to asking learners to generate problems on the basis of specific constraints. Many, if not most, studies conducted on problem posing can be placed in this category. In fact, this category can even be subdivided into three subcategories covering the constraints given to learners for generating problems: (a) generate from a general context; (b) generate from specific constraints; (c) generate from a previously solved problem.

The *generate from a general context* subcategory contains studies that ask learners to generate a problem in a general context. Brown and Walter's (e.g. 2005) *Accepting* perspective of problem posing is an example of this. In the example given above about *Accepting*, the context is the prime numbers, which are introduced to students who then have to generate problems about this mathematical concept without other indications. Work conducted by English (1998) is also of this type, where students have to compose problems in an informal mathematical context free of symbolic representation. For example, she asks students to make up a story problem about what they see in a large photograph of children playing with a set of brightly coloured items. A general mathematical context is then given to learners who have to generate problems from it. The subcategory *generate from specific constraints* refers to studies that ask learners to generate problems within or in relation to specific constraints. Silver (1994) refers to this subcategory as *problem generation*, where the goal is the creation of new problems from a situation prior to any problem solving. This subcategory can also be linked to what Christou and *al.* (2005) call a task that requires students to pose a problem with a given answer, a problem that contains certain information, a question for a problem situation, or a problem that fits a given calculation. Brown and Walter's (e.g. 2005) WIN perspective, as discussed above, takes place in this subcategory as it asks students to generate problems based on a initial mathematical situation using the WIN technique. The WIN technique is seen here as a constraint because it gives insight into the kind of problem students have to generate. Lavy's works (Lavy and Bershadsky, 2003; Lavy and Shriki, 2007), using the WIN technique in class with prospective teachers in a geometry context, is another example of this subcategory. We can also refer to studies of Silver and Cai (1996) and Silver, Mamona-Downs, Leung and Kenney (1996), in which before solving a mathematical task, students are asked to compose three problems that can be solved from the information/data given in an initial problem. When they have composed a new problem based on this one, students are asked to solve eight related problems. The researchers then studied the nature of the composed problems and the relationship between their ability to compose and to solve problems. This kind of problem posing contrasts with the other subcategories because the specific constraints (the technique or the problem) guide the kind of problem that learners would be more likely to compose. The last subcategory, *generate from a previously solved problem*, contains studies asking learners, *after* having solved a specific problem, to create other problems based on this solved problem. The problems then created are modifications of the goals or the conditions of the previously solved problem. Silver et al. (1996) above mentioned study is also in this subcategory, where in another part of their study students have to generate a problem from previously solved ones. Polya's (e.g. 1957) heuristics of the *Looking back* technique (*analogies*, *modifying*, *generalizing-specifying* and *studying variations*) are also examples of this.

Category 3: To transform an initial problem. The third category of problem posing is intricately linked to problem solving strategies, as it contains studies that ask learners to transform an initial problem in order to solve it. This kind of problem posing occurs during the problem solving process, when students are invited, as an efficient solving strategy, to reformulate for themselves the given problem. For example, strategies given to students are to decompose the problem into sub-problems, to simplify or modify the original problem or to solve a related problem. Students use these strategies to achieve one goal: to be able to solve the original problem. The problem posing is then seen as a *means* of solving the given problem. Silver (1994) refers to this category of problem posing as occurring during the process of solving when students must ask themselves "How can I formulate this problem so that it can be solved?" (p. 20). Kilpatrick (1987) mentions that problem posing consists of reformulating an existing problem in order to make it one's own; seeing problem posing (what he calls problem formulating) as an important companion to problem solving. Mason, Burton and Stacey's (1982) *Thinking Mathematically* book explains this approach in detail to help learners solve a problem. In a similar vein, Polya's (1966) video on *Guessing* amply illustrates this category of problem posing. In the video, Polya tells his students that if a problem is too difficult to solve, they should pose easier sub-problems, which could prepare them to solve the bigger problem; examples explored are to consider the problem in 2D instead of 3D, reducing the number of constraints/givens in the problem, and so forth. The aim is that, as they solve these sub-problems, learners gain a better sense of the original problem and prepare themselves for solving it.

The Implicit Perspective: An Epistemological View of Problem Posing

Studies under what we term the *implicit perspective* are less frequently, if ever, accounted for in reviews on problem posing. In this perspective, we integrate studies that conceptualize the problem solving process as events of problem posing. Thus whereas in the first perspective the notion of problem posing was related to *explicit* requests for creating problems through varied contexts, in this second perspective the notion of problem posing happens *implicitly*, without any request, as it defines the activity of problem solving itself. In various ways, work conducted under this lens makes the argument that when students solve problems, they are in fact posing their own problems, as we show below.

Category 1: Problem posing that influences the problem solving path. This category comprises studies that focus on the link between problem posing and problem solving, emphasizing the influence of the posed problems on the solving process. The work of Sevim and Cifarelli (2013) illustrates this. They argue that when solving a problem, a solver creates his/her own goals and purposes. These change as the solver progresses in the solution and also indicate the path of solving that the solver chooses. Armstrong's (2013) work is another example of this, as she records and studies students' questions that arise while they are solving a problem, and which influences the course of the solving. In her work, she looks at the questions that a group of students working together ask themselves when solving a task. For example, students would ask questions like "What is meant by an interval?" "Is it a square root?" or "What if there are x people?" (p. 67), in order to think about the task and arrive at a solution. Each group of students asks different or similar questions, and some groups also ask questions more than once during the same problem solving process. Armstrong made a "tapestry" schema out of this that shows the problem posing path followed by each group when solving their problem. She illustrates that the process used by learners directly influences their problem solving process. Stoyanova and Ellerton (1996) definition of problem posing, "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518), can be linked to this implicit problem posing. In order to solve a problem, learners formulate for themselves meaningful problems on the basis of their mathematical experience; this is not done as a request or as a strategy for solving, but mainly reflects what they do, their solving processes.

Category 2: Gaps between teachers' and students' task. Researchers like Perrin-Glorian, Robert and Rogalski (see e.g. Perrin-Glorian & Robert, 2005; Robert & Rogalski, 2002; Rogalski, 2003) have also worked along those lines to develop meaning about students' problem solving processes. Like Polya, they have not used the expression *problem posing*, but have focused on students' interpretations of the problems given to them, where they formulate for themselves, they pose, what the problem to be solved is. Rogalski (2003) identified various natures that tasks can assume when presented by teachers in the classroom, all of this happening implicitly during the activity of problem solving. First, the teacher prescribes a task to students, which consists essentially in the formulation of the problem. This *prescribed task* may be directly observed, as it consists of the instructions presented by the teacher to the students. The teacher has expectations about the task that the students have to work on: this is the *expected task*. On the other hand, students do not necessarily work on the teachers' prescribed task, but on one that has been redefined from that prescribed task. The *redefined task*, therefore, represents the student's personal representation of the task; somehow his/her *implicit* posing of the task in his/her own terms. Finally, the *effective task* is the actual one to which the student responds, which is not necessarily identical to the one he/she thinks he/she is responding to; this leads to a dynamical interrelation between the *redefined* and *effective* task. These redefined and effective tasks illustrate the problem that the student actually asks/poses him/herself and intends to solve (again, all of this happening implicitly in the solving process). The studies conducted within this framework focus on the gaps between teachers' expectations and students' mathematical activity in solving problems, their problem posing process.

Category 3: Problem posing versus problem solving. Work in this third category is related to Varela's (1996) epistemological definition of problem posing, which he contrasts with problem solving. For Varela, problem solving implies that problems are already in the world, independent of us, waiting to be solved. Varela explains, on the contrary, that we specify the problems that we encounter through the meanings we make of the world in which we live, leading us to recognize things in specific ways. We do not choose problems that are out there in the world independent of our actions. Rather, we bring problems forth, we pose them: "The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but enacted, we bring them forth against a background." (p. 91). The problems that we encounter, the questions that we ask, are thus as much a part of us as they are a part of our environment: they emerge from our interaction with/in it. The problems that we solve are relevant for us as we allow them to be problems. Working in this perspective, René de Cotret (1999) notes that one cannot assert that instructional properties are present in the tasks presented and that these *causally determine* solvers' reactions. As Simmt (2000) explains, it is not tasks that are given to students, but mainly prompts that are taken up by students who themselves create tasks with. Prompts become tasks when students engage with them, when, as Varela would say, they pose them as problems. This posing, as we show in Proulx (2013) about mental mathematics contexts, determines the task solved, hence the strategy developed for it. Students *make* the "wording" or the "prompt" a multiplication task, a ratio task, a function task, an algebra task, and so forth, and solve accordingly, which leads to varied strategies and answers because they often start from "different" posed problems.

In this implicit perspective, students play an important role in what the problem to solve is: not because they have created it, but because a student is always solving his/her own problem, from a given prompt. In this sense it is implicit, that is, it is not an explicitly requested task by someone external to the student (as in the first explicit perspective), but something implicit in the solving process when students engage with the(ir) problem to be solved. In sum, they *implicitly* create a problem in the action of solving the problem; whereas in the first explicit perspective they were explicitly asked to create a problem from various context and data given. Both perspectives and their categories are summarized in Table 1.

Table 1. Summary of Implicit and Explicit Perspectives

Explicit perspective on problem posing (as an explicit request to learners)	**Implicit perspective on problem posing (as defining the problem solving process)**
To compose a problem without any context or constraint	Problem posing that influences the problem solving path
To generate problems from specific constraints	Gaps between teachers' and students' tasks
To transform an initial problem in order to solve it	Problem posing versus problem solving

Final Remarks on these Categorizations of Problem Posing

What do we learn from this? This categorization of work conducted on problem posing is more than a review: it is an extension of the field. By integrating the work under the implicit perspective, which we have seldom encountered in reviews and activities about problem posing (Working Groups, books, Special Issues), we extend what are normally considered as studies in problem posing by opening the way to epistemological considerations about students' mathematical activities. Whereas problem posing as a field is widely known in terms of an activity to plunge students into, as a teaching device or as a strategy for solving problems (see e.g. Voica et al, 2013; Christou et al., 2005), what we have grouped under the implicit perspective is much less known and tackles epistemological issues related to students mathematical activity in itself. Epistemological questions/issues are not new in problem posing work, as some of them can be seen and felt through the work of Stoyanova and Ellerton (1996) and of Kilpatrick (1987), however they oscillate between a view of problem posing as a request on students and as representing students' mathematical activity. The distinction offered here between explicit requests for problem posing and the implicit problem posing activity happening in students' mathematical activity appears fruitful for developing a sharper understanding of what is meant by problem posing and clarifying where one's focus is. As mentioned, this can be felt as a step forward in the direction of Silver's (2013) suggestion for developing finer theoretical frameworks to support studies in problem posing.

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REFRAMING URBAN MATHEMATICS EDUCATION: THE SPATIAL POLITICS OF OPPORTUNITY

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Although equity-oriented discourse is working to move the mathematics education community from achievement-gap rhetoric toward a focus on opportunity gaps, it does not currently recognize the role of space and the politics of space in creating and maintaining opportunity gaps as it relates to mathematics education in urban settings. The purpose of this paper is to engage the task of re-conceptualizing urban mathematics education by proposing a framework for scholarship, policy, and practice. The authors engage scholarship in mathematics education, urban education, critical geography, and urban sociology to substantiate a socio-spatial framework for urban mathematics education, which features a visual schematic that locates mathematics teaching and learning—vis-à-vis a mathematics-instructional triad—within a system of socio-spatial considerations relevant to U.S. urban contexts.

Keywords: Equity and Diversity; Research Methods

An enduring challenge within mathematics education (and in education more broadly) has been to recognize the role that mathematics plays in societal stratification and to address systemic inequities that marginalize populations. “Equity” has been positioned as the key principle for responding to this challenge (National Council of Teachers of Mathematics [NCTM], 2000), and over the last few decades, equity-oriented discourse in mathematics education has developed alongside the emergence of urban mathematics education scholarship. The boundaries between the two domains are difficult to distinguish, thus the two are often conflated or interchanged based on common components related to issues of race, class, power, and status. We argue, however, that contemporary equity discourse has not adequately responded to the particular relationship between the “urban” as a socio-spatial construct and mathematics education. Until these discourses are more clearly framed, neither can fulfill its potential to contribute to the enduring challenge above.

Although we argue that equity and urban mathematics education are separable discourses, we acknowledge that the two are related. Particularly, equity discourse is helping the mathematics education community to move from an achievement gap orientation toward an opportunity gap orientation (Flores, 2007). Considering gaps in opportunity makes room for new analyses of mathematics education related to the ways in which opportunity is constructed within education discourses. In the inaugural issue of the *Journal of Urban Mathematics Education* (*JUME*), Tate (2008) issued a challenge related to conceptualizing urban mathematics education in relation to opportunity:

> The challenge is to build theories and models that realistically reflect how geography and opportunity in mathematics education interact. If this challenge is addressed, the field will be one step closer to making scholarship in urban mathematics education visible. (p. 7)

In a later *JUME* commentary, Rousseau Anderson (2014) returned to the need to consider space in urban mathematics education as “‘place matters’ in the study of urban mathematics education (p. 10). Our aim in this paper is to move further toward recognizing the role of space and the politics of space in creating and maintaining opportunity gaps.

Background

The roots of urban mathematics education as a subdomain of mathematics education extend back at least to efforts during the 1980s (see Tate, 1996), concurrent with the development and publication of the NCTM standards for mathematics curriculum and evaluation (1989) and for mathematics teaching (1991). These developments also coincided with commensurable shifts in research as mathematics education scholarship around the world entered its much-discussed social turn (e.g., Meyer & Secada, 1989; also see Lerman, 2000; Martin & Larnell, 2013; Stinson & Bullock, 2012). For researchers, teachers, policymakers, and education-interested foundations in the United States (e.g., Ford Foundation, National Science Foundation), a crucial new question emerged: How would the then-new vision for school mathematics reform extend to and take shape in urban districts and classrooms (Tate, 2008)? This question remains central in the latest shift to the Common Core State Standards for School Mathematics.

Our aim in this paper is to broaden the discourse in urban mathematics education in ways indicated by Tate's (2008) challenge in the inaugural issue of *JUME*. According to Lou Matthews (2008), *JUME* founding editor-in-chief, the journal was founded "to open up a space in mathematics education that would honor and enrich the work in this domain [i.e., urban mathematics education]" (p. 1). The young journal's growing popularity signals that urban mathematics education has advanced to the point at which we may now begin to evaluate the production of knowledge in this subdomain—and, particularly, the building of "theories and models that realistically reflect how geography and opportunity in mathematics education interact" (p. 7). What has the study of urban mathematics education entailed? What can it become? The purpose of this paper is to take "one step closer" toward addressing these questions and toward new directions for urban mathematics education scholarship and practice.

Overview of the Socio-spatial Framework for Urban Mathematics Education

In the spirit of addressing Tate's challenge (also see Rousseau Anderson, 2014), our objective is to posit a new theoretical framing for scholarship in urban mathematics education—the first of its kind (Figure 1). In this section, we detail the theoretical concepts undergirding the framework. We

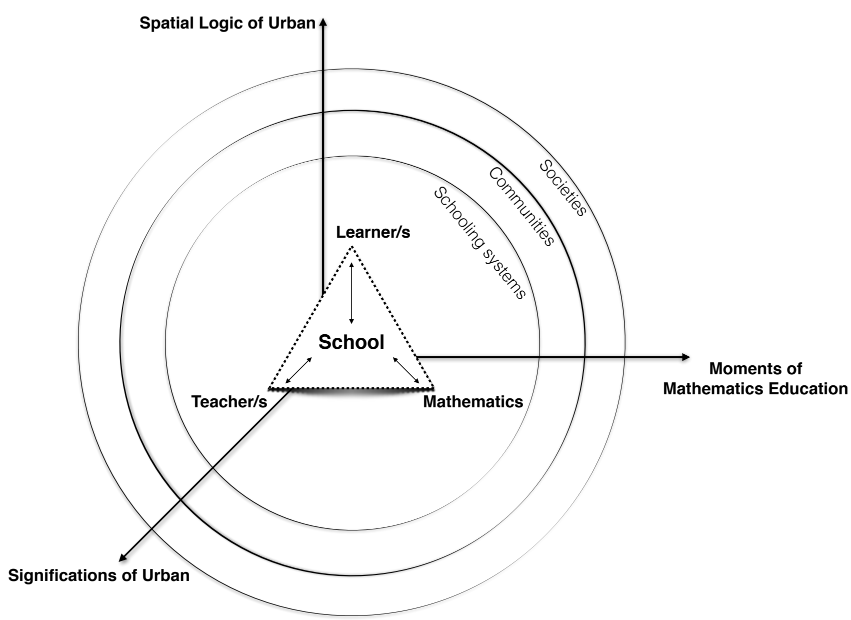

Figure 1. Socio-Spatial Framework for Urban Mathematics Education Scholarship

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

situate this framing squarely (but not entirely) in mathematics education—using as our central unit of analysis the mathematics-instructional triad of teacher(s), learner(s), and mathematics (Cohen, Raudenbush, & Ball, 2003; NCTM, 1991; Stein, Smith, Henningsen, & Silver, 2009). We also incorporate the various theoretical orientations—e.g., cognitivism/behaviorism, constructivism, sociocultural perspectives—that have emerged amid "moments" of mathematics education during the past century (Stinson & Bullock, 2012). We represent these theory-driven moments of mathematics education scholarship as a dimensional axis that intersects with the socio-spatial elements of the framework.

The NCTM Research Committee (Gutstein et al., 2005) argued that, in order for researchers to advance equity in mathematics education, we must "break with tradition, expand boundaries, and cross into fields outside of mathematics education *and* outside education" (p. 96; emphasis original). In this spirit, we extend beyond mathematics education, looking toward the interdisciplinary areas of urban sociology, critical geography, and urban education scholarship to consider the various forces that influence mathematics teaching and learning in urban spaces as well as the social significations that shape interactions in urban settings. We recognize, however, that the task of defining urban has been an overwhelming challenge across disciplines, and our attempt here is to incorporate what is known inasmuch as we can given what is available to us contemporarily (Milner & Lomotey, 2013).

In recent decades, there has been considerable momentum in the humanities and the social sciences to consider space as a social construction that is integral to social analysis (Arias, 2010). This *spatial turn* renders geographic considerations equal to—and mutually constructed with—temporal and social considerations in the analysis of social phenomena (Warf & Arias, 2009). In many ways, this framework represents a spatial turn within mathematics education research in which temporal (i.e., the Moment of Mathematics Education axis), social (i.e., the Significations of Urban axis), and geographic (i.e., the Spatial Logic of Urban axis) elements are taken together as mutually constitutive of urban mathematics education.

To inform the framework with respect to the social meanings that shape urban mathematics education, we draw on Leonardo and Hunter's (2007) typology of significations that circumscribe urban education (also see Martin & Larnell, 2013). We represent that typology as an axis of the framework that intersects with spatial considerations of urban, drawn from scholarship in critical geography (e.g., Soja, 1980; Thrift, 2003). The coordinate representation is intended to signal a socio-spatial dialectic regarding the urban—that is, that social significations and spatial considerations necessarily intersect "to realistically reflect how [spatial] geography and [social] opportunity in mathematics education interact" (Tate, 2008, p. 7). We then add a third axis to situate the socio-spatial elements in relation to the evolution of mathematics education and the theoretical orientations association with these evolutionary "moments" (Stinson & Bullock, 2012).

Mathematics-instructional triad as the central element of the framework

At the center of our framework are interactions among learners, teachers, and mathematics curriculum (see Figure 2). Not only does this center the processes of formal and informal mathematics teaching and learning, but in terms of the diagrammatic representation of the framework, the triad represents a kind of coordinate point with respect to the social, spatial, and mathematics-education "theory-moment" axes. As such, the framing allows for questions that relate mathematics teaching and learning, social contexts, spatial logic, and the evolution of the mathematics education enterprise (also see Weissglass, 2002).

Spatial logic axis of the framework

Most often, discussions of urban space are connected to population density and physical geography (see Milner, 2012). While these elements contribute to our understanding of urban as a means of geographical classification, they are insufficient in that they do not allow for a nuanced

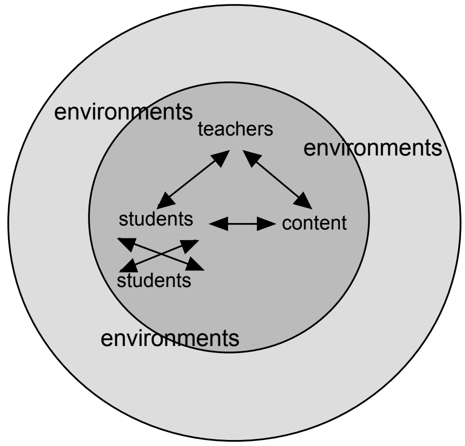

Figure 2. Mathematics-instructional Triad, with Cohen & Ball's (2000) focus on interaction

understanding of space and the non-geographic (i.e., affective) meanings associated therein. To substantiate the spatial aspect of this framing, we draw primarily on human geography and Thrift's (2003) four conceptions of space: (a) empirical-constructing space, or the ways in which space is rendered measurable or objective; (b) interactive-connective space, or the pathways and networks that constitute space; (c) image space, the visual artifacts that we readily associate with space; and (d) place space, or our everyday notion of spaces in which human beings reside (p. 102). These conceptions of space form a spatial logic that is not limited to a geographical sense of urban space and that takes into account meanings associated with space. The strength of articulating four distinctive conceptions of urban allows one to look across their various permutations in ways that provide a nuanced perspective on space.

Social-signification axis of the framework

"Urban" is not simply a geospatial concept; it also carries social and political meanings. Accordingly, urban mathematics education scholarship must engage its social and political dimensions—i.e., relating mathematics teaching and learning to the many ways in which urban can be experienced, influenced, shaped, and contested. Toward offering some conceptual framings, the social-signification axis of our framework includes Leonardo and Hunter's (2007) three significations of urban: urban-as-sophistication (or cosmopolitan space), urban-as-pathological (or urban as "dirty, criminal, and dangerous;" p. 789), and urban-as-authenticity (or the politics of authenticity). This view of urban as more than just physical space also challenges the prevalent use of urban as a proxy descriptor for poor, Black, and Brown populations who inhabit these spaces and disproportionately fall victim to the segregation and concentrated poverty (Darling-Hammond, 2013). Such employment of "urban" ignores the heterogeneity of urban space, its politics, its people, and their experiences (Fischer, 2013).

Theory-Moment Axis of the Framework

Stinson and Bullock (2012) outlined four moments of mathematics education research since its emergence as a research domain. These moments—the process-product moment, interpretivist-constructivist moment, social-turn moment, and socio-political-turn moment—are characterized by particular theoretical orientations—cognitivism, interpretivism/constructivism, sociocultural theories, and theories of power, respectively. These moments are overlapping categorical periods of research, practice, and policy (also see Gutierrez, 2013). These periods have often been indexed by a crisis metaphor (Washington, Torres, Gholson & Martin, 2012).The third axis incorporates these moments and the associated theoretical orientations.

With a third axis in the framing, we attempt to construct what could be called a mathematical-socio-spatial dialectic. That is, we situate the mathematics-instructional triad within the dimensional space of not only the socio-spatial dialectic but also with respect to the ongoing "moments" of mathematics education theory and practice (Stinson & Bullock, 2012; also see Martin & Larnell, 2013). Put differently, the axes represent the intersectionality of geography (or spatiality), social opportunity, and the development of mathematics education, which is what Tate (2008) originally outlined.

Implications: Urban Mathematics Education and Equity

Central to this framework is the understanding that urban mathematics education is a complex domain in its own right. It is *more than just* mathematics education performed with—or on—people who are labeled "urban" based on race and/or class signifiers. Additionally, it is more than just a descriptor for situating traditional or reform-oriented mathematics teaching and learning in certain locales (i.e., the "inner city"). Thus, it is important that we address explicitly the need for a consideration of urban mathematics education that is separate from—yet connected to—prevailing equity discourse in mathematics education. Examining mathematics education in urban spaces through an equity-oriented lens appropriately centers conversations on children of color and their mathematical identities and experiences. However, engagement with the urban in such work is often limited either to contextual descriptors connected to racial demographic and free-and-reduced-lunch data or to situated applications of mathematics curricula or pedagogies in spaces inhabited by people who are largely Black and/or Brown and poor.

As a descriptor in research, "urban" functions as a sort of veil. This veiling allows the researcher to acknowledge race and class in superficial ways that obscure weightier systemic issues related to race and class. This urban-as-veil perspective also frames our collective understanding of urban populations in ways that—perhaps ironically—obscure populations that do not align with notions of urban educational contexts as Black, Brown, and/or poor. The challenge with this veiling is that it allows equity discourse to disengage from the substantive issues in urban education, racism, and classism that inhabit mathematics classrooms and other aspects of the "network of mathematics education practices" (Valero, 2012, p. 374). We propose that this framework for urban mathematics education encourages a more complex understanding of "urban" that attends to the role of place in mathematics education and, additionally, unveils race and class as distinct categories that each warrant significant analysis in their own right.

These examples represent common ways in which equity discourse interacts with the urban in mathematics education. However, these approaches miss possibilities for understanding the implications of place on mathematics teaching-and-learning environments. We propose that engaging the elements of this framework allows equity-oriented mathematics education researchers to remove the urban veil in a way that acknowledges the roles of place, race, and class as distinct and mutually constitutive. Specifically, it aims to position urban mathematics education as a system of connections among mathematics, race, class, power, and the politics of space. This positioning allows mathematics education researchers to explore the interactions between geography and opportunity within a multidimensional framework that acknowledges the political underpinnings of opportunity gaps that equity discourses reveal.

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ANALYZING THE CALCULUS CONCEPT INVENTORY: CONTENT VALIDITY, INTERNAL STRUCTURE VALIDITY, AND RELIABILITY ANALYSIS

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We present findings from an analysis of the Calculus Concept Inventory. Analysis of data from over 1500 students across four institutions indicates that there are deficiencies in the instrument. The analysis showed the data is consistent with a unidimensional model and does not have strong enough reliability for its intended use. This finding emphasizes the need for creating and validating a criterion-referenced concept inventory on differential calculus. We conclude with ideas for such an instrument and its uses.

Keywords: Post-Secondary Education; Assessment and Evaluation; Research Methods; Instructional Activities and Practices

Introduction and Literature Review

As educators and educational researchers, we seek to develop calculus courses effective in building conceptual understanding in addition to procedural fluency, and continually investigate promising new pedagogical strategies. The Mathematical Association of America recommends that all math courses should build conceptual understanding, "mental connections among mathematical facts, procedures, and ideas" (Hiebert & Grouws, 2007, p. 380) by helping "all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind" (Barker et al., 2004, p. 13).

Concept inventories have emerged over the past two decades as one way to measure conceptual understanding in STEM education. These inventories intend to assess student understanding of concepts before entering a course addressing those concepts. Thus, students are required to use common sense and prior knowledge to respond to assessment items. After completing a course, the concept inventory can measure gains in conceptual understanding. Therefore, items should avoid using terminology taught in the course to which students have no prior exposure.

The first concept inventory to make a significant impact in the undergraduate education community was the Force Concept Inventory (FCI), written by Hestenes, Wells, & Swackhamer (1992). Despite the fact that most physics professors considered the Inventory questions "too trivial to be informative" (Hestenes et al., 1992, p. 2) at first glance, students did poorly on the test, and comparisons of high-school students with university students showed modest gains between the two. Of the 1,500 high-school students and over 500 university students who took the test, high school students were learning 20%-23% of the previously unknown concepts, and college students at most 32% (Hestenes et al., 1992, p. 6). Through a well-documented process of development and refinement, the test has become an accepted and widely used tool in the physics community, and has led to changes in the methods of instruction for introductory physics.

The FCI paved the way for the broad application of analyzing student conceptual understanding of the basic ideas in a STEM subject area (Hake, 1998, 2007; Hestenes et al., 1992). Concept inventories exist in a variety of scientific disciplines; including physics, chemistry, astronomy, biology, and geoscience (Libarkin, 2008).

More recently, Epstein (2007, 2013) developed the Calculus Concept Inventory (CCI) for introductory calculus. However, there is a lack of peer-reviewed literature on its development or psychometric analysis. Additionally, several recent analyses call into question the ability of the CCI to measure conceptual understanding. One study showed that the current CCI measured no difference in conceptual understanding between students in a conceptually focused class with frequent student group work and those in a traditional lecture based class, even though other measures indicated that a difference existed (Bagley, 2014). While this result may reflect shortcomings of the conceptually focused class, it may also suggest the inadequacy of the CCI.

The concerns with the CCI motivated us to take a deeper look at how it performs for its original purpose. Specifically, we wanted to determine if the CCI measured gains in conceptual knowledge and to investigate its reliability. In this study, we analyze the results on the CCI from over 1500 students at four institutions to determine whether there is evidence that the CCI, in its current form, exhibits the psychometric properties originally suggested by the author, and to suggest appropriate potential modifications or revisions.

Calculus Concept Inventory

Content Validity

The Calculus Concept Inventory (CCI) was developed by a group of seven individuals to measure topics from differential calculus that they believed were basic constructs (Epstein, 2013). The main purpose of the instrument is to measure classroom normalized gains (change in the class average divided by the possible change in the class average) for the purpose of evaluating the impact of teaching techniques on conceptual learning. The developers of the instrument intended the instrument to measure above random chance at the pre-test setting and to avoid “confusing wording” (Epstein, 2013, p. 7). However, a released CCI item uses terminology, including “derivative” and “f’(x)” (Epstein, n.d.), which is not part of the vocabulary of a first-time calculus student. Such items would be confusing to the student and generate responses around random chance for those items. We seek to determine the extent of the use of such terminology to verify that vocabulary issues do not confound results from the CCI.

Internal Structure Validity

The dimensionality of the CCI is unknown. Epstein (2013) states that the instrument has two primary components, related to functions and derivatives, with a third dimension related to limits, ratios, and the continuum. However, the use of a total percent correct to determine normalized gains implies that the instrument measures a single construct evenly distributed over the 22 items. These two proposed structures of the instrument are contradictory and no details regarding the analysis conducted to support the three-component structure exist. A comprehensive analysis of the internal structure of the instrument is thus necessary to determine whether a unidimensional model is appropriate.

Reliability

Epstein (2013) reports that the CCI has an internal consistency reliability 0.7 for Cronbach’s alpha. This level of internal consistency is at the low end of an acceptable range for an instrument designed to measure differences in means between groups of at least 25-50 individuals. However, there is no such standard for internal consistency necessary for comparing the normalized gains of two different groups. In fact, the use of the normalized gain as a measurement parameter is questionable (Wallace & Bailey, 2010). Instead, the similar types of gains can be measured using ability estimates obtained through item response theory models. Therefore, there is a need to use such models to determine the internal consistency reliability of the CCI.

Methods

Content Validity

Since the CCI was designed to measure normalized gains in conceptual knowledge of calculus, it is given as both a pre-test and a post-test. As such, at both of the sittings, the test should measure conceptual understanding, and not include items requiring vocabulary and notation specific to calculus. Otherwise, students who are repeating calculus would likely have higher pretest scores, regardless of their conceptual understanding of calculus, and would thus likely have lower normalized gains, as seen in previous studies (Epstein, 2013). Therefore, we conducted an analysis of the items to determine which items may contain vocabulary and/or notation not included in any of the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010) that have become accepted as preparation for calculus throughout most of the United States.

Internal Structure Validity

We collected data from approximately 2000 students at four universities at the beginning and the end of a first semester calculus class. We cleaned the data by eliminating subjects with missing data and randomly selecting either a pre-test or post-test for all remaining subjects to avoid dependent samples. This left a sample size of 1792 students with an even distribution of pre-tests and post-tests.

We then used the eigenvalues of the inter-item correlation matrix to determine the expected number of factors related to the instrument, followed by a confirmatory factor analysis based on the predicted number of factors, with a bent toward a unidimensional model. In the eigenvalue analysis, we compared the results from the actual data to results from randomly generated data with the same sample size and with a 20% probability of correct answers, as nearly all of the items on the CCI had five choices.

Reliability

Using the results of the factor analysis, we used an appropriate unidimensional or multidimensional item response theory model to analyze the internal reliability of the instrument and to measure the test information and standard errors for the instrument.

Results

Content Validity

Out of the 22 items on the CCI, nine contained language or notation not included in any standards for courses that are considered prerequisite for calculus. These included the words derivative and concavity, and notation such as f'(x), f''(x), and dy/dx. An additional two items contained language closely related to some precalculus topics; for instance, some students may have exposure to the relationship between velocity and acceleration and the concept of linear approximations. However, these topics are not necessarily included in the courses prior to calculus.

Therefore, the CCI does not satisfy the conditions necessary to measure conceptual understanding for students as they enter a calculus course. However, since all of these language and notation conventions are part of the normal language during the first semester of calculus, including such language on a test at the end of a semester of calculus may measure conceptual understanding. This issue needs to justification by anyone using the standard normalized gains when researching or evaluating first semester calculus courses.

Internal Structure Validity

From the analysis of the eigenvalues from the factor analysis, the CCI has at most two components. Both the first and the second eigenvalue are above the 95% confidence interval for the randomly generated data. However, since the second eigenvalue (1.24) is extremely close to the 95%

confidence interval of the eigenvalue generated by random data 1.1765 +/- 0.04, this second component may or may not actually be present (since a large first eigenvalue will pull up the second eigenvalue).

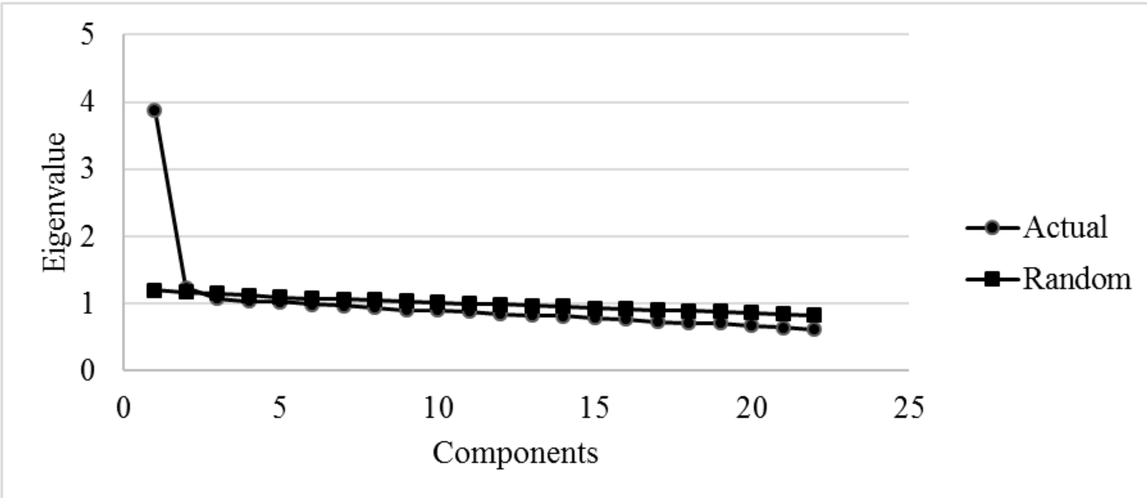

Figure 1: Scree Plot for Calculus Concept Inventory

Table 1: Item CFA Estimates for CCI

	Full CCI		Abbreviated CCI	
Item	Estimate	Standard Error	Estimate	Standard Error
Question 1	1.000			
Question 2	5.776	1.313	1.000	
Question 3	5.537	1.264	0.961	0.065
Question 4	5.649	1.288	0.978	0.065
Question 5	4.574	1.058	0.802	0.062
Question 6	3.243	0.769	0.560	0.053
Question 7	3.497	0.825	0.604	0.055
Question 8	5.055	1.158	0.877	0.062
Question 9	4.792	1.103	0.830	0.062
Question 10	4.693	1.084	0.803	0.062
Question 11	0.816	0.349		

Question 12	2.380	0.610	0.414	0.056
Question 13	4.228	0.985	0.735	0.060
Question 14	3.386	0.803	0.587	0.055
Question 15	3.735	0.880	0.650	0.058
Question 16	2.928	0.704	0.504	0.052
Question 17	5.619	1.282	0.975	0.065
Question 18	1.535	0.412		
Question 19	3.322	0.790	0.570	0.055
Question 20	3.575	0.849	0.617	0.058
Question 21	3.857	0.917	0.661	0.064
Question 22	3.885	0.913	0.678	0.059

Since the scree plot and eigenvalue analysis favors a unidimensional structure, and since the intended use of the instrument is as a one-dimensional inventory, a one-dimensional confirmatory factor analysis model was used to determine model-data fit. The model had 231 degrees of freedom, $p<0.001$, with the item estimates given below. The fit indices were excellent with a Comparative Fit Index (CFI) of 0.936 and a Root Mean Square Error of Approximation (RMSEA) of 0.024 (Hu & Bentler, 1999). Therefore, a unidimensional model is assumed to fit the data well. One notices that three of the items (1, 11, and 18) have significantly lower estimates that the remaining items. If one removes these items, we maintain the unidimensionality of the instrument (CFI: 0.939 and RMSEA: 0.028) and all estimates are approximately equal values. This enables a more appropriate use of number correct to estimate an individual's ability without having to scale the values of certain items.

Reliability

Since the instrument satisfies the unidimensionality assumption, one, two, or three-parameter models can be used to analyze the data. Since the different items are believed to have different discrimination, only the two and three parameter models were used. The three-parameter model did not have good model-data fit on several of the items loading heavily on the construct with the c parameters for the majority of the items significantly below random chance. Therefore, a two-parameter model was determined to be the best fit of the data and the theoretical construct of the inventory. In the analysis of the two-parameter model, three items demonstrated a weak fit, items 1, 11, and 18. These three items also had low loadings in the factor analysis and so were removed from the analysis to determine if the remaining items have an improved fit. The remaining 19 items had a good fit (-2LL of 37258, $p<0.0001$) with the two parameter model. The standard error for the ability estimate of individuals is extremely high with the lowest value of 0.4128 logits and an average error of 0.7307 logits (see Figure 2). For example, if an individual is at the mean in terms of actual conceptual understanding of calculus, as measured by the CCI, the measured score of the person by the inventory has a 68% chance of being within 0.42 logits of the mean. Therefore, the inventory would only be able to differentiate between samples of means if there is a substantial difference between the samples or the sample size approaches 100 students each. Furthermore, in order to use the logit scores one must first transform the percent correct score into logit scores using the results in Table 2.

Figure 2: Test Information Function and Standard Error

Table 2: Transformation of Scores to Logits

Number Correct	Ability Estimate		Number Correct	Ability Estimate		Number Correct	Ability Estimate
0	-3.05		7	-0.13		14	1.47
1	-2.52		8	0.10		15	1.77
2	-1.75		9	0.31		16	2.13
3	-1.28		10	0.53		17	2.60
4	-0.93		11	0.74		18	3.36
5	-0.63		12	0.97		19	4.00
6	-0.37		13	1.21			

Discussion and Conclusion

The purpose of this study was to assess the degree to which the CCI conforms to certain standards for psychometric properties, including content validity, internal structure validity, and reliability. We conclude that the existing CCI does not conform to accepted standards for educational testing (American Educational Research Association, 2014; DeVellis, 2012). We thus argue that there is a need to create and validate a criterion-referenced concept inventory on differential calculus. Such a concept inventory would significantly impact teaching and learning during the first two years of undergraduate STEM students by providing a resource to measure students' conceptual understanding of differential calculus. The work of Carlson, Madison, and West (2010) in developing the Calculus Concept Readiness instrument could serve as a model and foundation for a differential calculus concept inventory. Such an instrument would be useful for instructors for formative and summative assessment during their calculus courses to improve student learning. Researchers and evaluators to measure growth of student conceptual understanding could also use such an instrument during a first semester calculus course to compare gains of students in classrooms implementing differing instructional techniques.

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TOWARD A FRAMEWORK FOR ATTENDING TO REFLEXIVITY IN THE CONTEXT OF CONDUCTING TEACHING EXPERIMENTS

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In this paper, we present an analytical framework for attending to reflexivity in the context of conducting teaching experiments and rationalize its components by appealing to the constructivist foundations on which the methodology is based. To illustrate the importance and utility of this framework, we discuss our analysis of recent mathematics education literature that has employed the teaching experiment methodology. In so doing, we reveal the extent to which researchers' presentations of their models of students' mathematical thinking and learning reflect these researchers' attention to reflexivity. We conclude by reflecting on the implications of researchers attending to reflexivity in the context of conducting teaching experiments.

Keywords: Research Methods; Design Experiments; Learning Theory; Cognition

Steffe and Thompson's (2000) elaboration of the teaching experiment methodology has gained traction in recent years as a scientific methodology for constructing models of students' mathematical thinking and learning. The teaching experiment methodology is grounded in radical constructivist epistemology, and the methods and procedures that comprise it reveal its constructivist roots. However, the constructivist foundations of both the experimental and analytical aspects of the methodology are often overlooked or, at a minimum, are not explicitly addressed in reports of studies that employ teaching experiments. We have especially noticed that while researchers may strictly adhere to the experimental practices of teaching experiments, the analytical practices essential to the methodology are less rigorously observed. We conjecture that researchers' inattention to particular analytical procedures of teaching experiments derive, at least in part, from their lack of attention to, or understanding of, the constructivist foundation on which these analytical procedures are based. A particularly common analytical practice of teaching experiments that researchers often overlook is the necessity of attending to *reflexivity*, a critical discernment and communication of the researcher's role in the constitution of his or her model of students' mathematical thinking and learning. Accordingly, the aim of the present paper is to present a framework for what it means to attend to reflexivity in the context of conducting teaching experiments. We take the position that justifying the need for attending to reflexivity, and conceptualizing what is involved in doing so, results from an understanding of specific constructivist premises that underlie the teaching experiment methodology.

Teaching Experiment Methodology

The principal aim of a teaching experiment is to construct a model of another's mathematical thinking and learning. It is important to note that such a model is not a direct representation of another's mathematics, but is rather a characterization of plausible conceptual operations from which his or her observable actions may have derived. It is thus the goal of a teaching experiment for a researcher to construct a model of another's mathematics that is viable with the researcher's interpretation of his or her observable behaviors. To achieve this goal, Steffe and Thompson (2000) designed the teaching experiment methodology to provide researchers with opportunities to experience and make sense of students' mathematical learning and reasoning, both in real time and retrospectively.

A teaching experiment, Steffe and Thompson explain, is "primarily an exploratory tool, derived from Piaget's clinical interview and aimed at exploring students' mathematics" (2000, p. 273). While the intent of a clinical interview is to understand students' current knowledge, teaching experiments,

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in contrast, are aimed at investigating students' progress over multiple teaching episodes. Teaching experiments, therefore, allow researchers to investigate student learning, which involves the modification of students' current cognitive schemes, as they engage in mathematical activity. In the context of a teaching experiment, the schemes that students construct through spontaneous development are brought forth through exploratory teaching, and the interest of the researcher is to discern how students modify their cognitive schemes as they encounter specific teaching actions.

In a teaching experiment, the researcher generates a major hypothesis at its outset and returns to this major hypothesis retrospectively at the conclusion of the teaching episodes. In addition to testing a main research hypothesis, the researcher continually generates and tests sub-hypotheses within and among teaching episodes. These sub-hypotheses are tentative models of students' mathematical realities that seek to explain the specific actions or utterances the researcher observes. Accordingly, the researcher develops these sub-hypotheses by attending to students' language and actions, and abductively postulating meanings that may lie behind them.

"Teaching" in the context of the teaching experiment methodology is a dynamic interaction informed by an evolving model of students' mathematics. Accordingly, learning how to productively interact with the participant is an instrumental component of conducting a teaching experiment. There are two complimentary types of interaction between the teacher-researcher and the student in a teaching experiment: (1) *responsive and intuitive interaction*, and (2) *analytical interaction*.

In responsive and intuitive interactions, the teacher-researcher is usually not explicitly aware of how or why she acts as she does and the action appears without forethought; the researcher acts without planning the action in advance of the action (Steffe & Thompson, 2000, p. 278).Steffe and Thompson define analytical interaction as "an interaction with students initiated for the purpose of comparing their actions in specific contexts with actions consonant with the hypothesis" (2000, p. 281). Between teaching episodes, the teacher-researcher develops a hypothetical model of student thinking and defines initial goals. The teacher-researcher interacts responsively and intuitively prior to constructing this hypothetical model of student thinking. In the teaching episode that follows the teacher-researcher's development of the hypothetical model of student thinking and the initial goals, he or she interacts in an analytical manner, extending and articulating the initial goals and revising the hypothetical model of student thinking. This process repeats itself until a mature living model of students' mathematics emerges.

Framework for Attending to Reflexivity

Constructing models of students' mathematical thinking and learning in the context of conducting a teaching experiment requires the researcher to construct a model of the mathematical knowledge students bring to the instructional context and to design and/or select mathematical tasks that will allow students to construct the understandings the researcher envisions. Both of these aspects of constructing models of students' mathematical thinking and learning are fashioned by the researcher's theory of learning as well as his or her mathematical knowledge. It is therefore important that the researcher explicate these two aspects of his or her cognition. While conducting a teaching experiment, the researcher engages in responsive and intuitive interaction with students as they engage in the mathematical experiences in order to elicit observable products of their reasoning. Effectively eliciting observable products of students' reasoning requires that the researcher's actions be informed by a model of students' emerging ways of understanding and ways of thinking. The researcher constructs this provisional model through analytical interaction. The researcher's interaction with students plays a significant role in students' behaviors from which the researcher constructs his or her model of the students' mathematical thinking and learning. Explicating the demands of the researcher on constructing models of students' thinking and learning reveals the various ways that attention to reflexivity is warranted in the context of conducting teaching experiments. We illustrate these four areas of attending to reflexivity in Figure 1 and offer concrete recommendations for how a researcher may attend to reflexivity in each of these areas. These recommendations derive principally from the constructivist assertion that researchers do not explain phenomena, rather researchers explain their *interpretation* of phenomena.

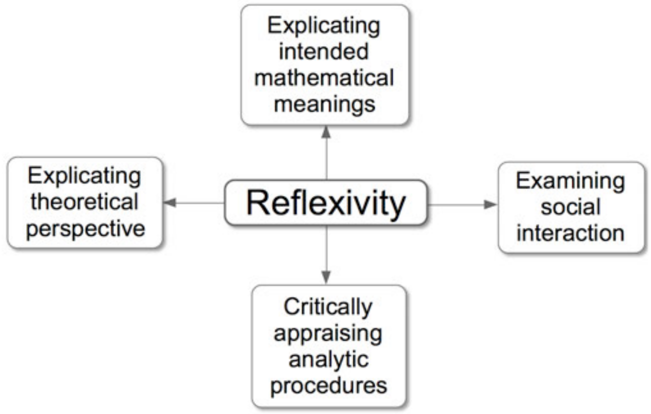

Figure 1. Four Areas of Attending to Reflexivity

Explicating One's Theoretical Perspective

von Glasersfeld (1995) developed the psychological learning theory of radical constructivism as an elaboration of Piaget's genetic epistemology (1971, 1977). The "radical" qualifier emphasizes von Glasersfeld's position that cognitive processing is the foundation of the only reality an organism may come to know. Accordingly, researchers do not have unfettered access to the phenomena they observe and thus must make sense of such phenomena through a variety of interpretative lenses. The purpose of defining and adhering to a theoretical perspective is to attempt to view the world in a systematic and disciplined way that can be communicated. It is important to note that researchers who do not conduct their work by adhering to a particular theoretical orientation have no more direct access to the phenomena they observe than those who do. Not adhering to a particular epistemological stance does not liberate one from perceiving phenomena through a number of subjective interpretative lenses; these lenses are simply not explicit and are thus unavailable to the researcher's conscious awareness. The utility of adhering to an explicit theoretical orientation, then, is that it allows one to become aware of at least some of the interpretative lenses through which he or she views the phenomenon under investigation, thereby giving one agency over these interpretative lenses.

Generally speaking, the role of theory in educational research is to orient and constrain the researcher's attention to those causal variables assumed to be most fundamental to explaining a particular phenomenon, thereby making the complex phenomenon under investigation accessible to empirical study. The theoretical perspective one assumes serves as a lens through which one is able to "control" specific aspects of the phenomenon he or she investigates so as to permit the construction of a viable characterization of a system in a particular state, or of a system undergoing transformation. In other words, researchers adopt theoretical perspectives in an effort to isolate what their theoretical perspective prescribes as the causal variables with the most explanatory power. In this way, our theoretical perspectives impose a set of assumptions and expectations about the phenomena we study that serve "to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (Thompson, 2002, p. 192).

In order for others to ascertain the conceptual origins of a researcher's model of students' mathematical thinking and learning, the researcher must explicate the theoretical suppositions on which her or his work is based. Doing so allows others to discern the interpretive lenses through which the researcher views the phenomenon she studies and to evaluate the researcher's theoretical justification for what she is studying and how she studies it.

Our proposal is for researchers to make explicit their understanding of the background theoretical orientations (e.g., constructivism, sociocultural theory, situated cognition, cognitive information processing theory, embodied cognition) that orients them to framing their work in particular ways, and which conditions how researchers interpret the phenomena they study. It seems to us that, at a minimum, researchers constructing models of students' thinking and learning should address questions like, "What is knowledge? What is learning? What is the process by which one learns? What constitutes evidence of learning? How can one engender learning?" Making explicit one's answers to questions like these allows others to infer aspects of the researcher's role in the constitution of his or her model of students' mathematical thinking and learning.

Explicating Mathematical Meanings

As previously emphasized, a core consideration of conducting a teaching experiment is constructing models of students' initial and emerging mathematical knowledge. In a teaching experiment the researcher interprets what students say and do through the lens of his or her mathematical understandings and creates inferences about students' knowledge through those interpretations. It is in this way that one's own mathematical knowledge very much constitutes an interpretative framework. It is therefore important that a researcher specify what it means to understand the mathematical concept for which he or she is attempting to model students' understandings and to anticipate a multiplicity of ways of understanding this idea. Doing so allows the researcher to expand his or her mathematical interpretative lens so as to accommodate for a variety of students' observable actions in order to construct a viable model of students' mathematical thinking and learning.

Steffe and Thompson (2000) note that the goal of teaching experiments is not that students will come to see an idea as the researcher or teacher does. Instead, it is important that a researcher's model of students' learning represent a reasonable development of a student's thinking given his or her initial mathematical knowledge. At this stage, the researcher might consider a number of questions to explore these issues:

- What is my understanding of the idea that is the focus of the teaching experiment?
- In what way do I intend students understand this idea?
- What understandings do I assume students have at the onset of the teaching experiment? In what ways do I expect these initial understandings to support or inhibit the students' learning the idea that is the focus of the teaching experiment?
- What are the principles on which my design of the activities within the teaching experiment is based?
- How do I anticipate the activities I have developed will support students in constructing the meanings I intend?

Consideration of these questions allows the researcher to expand the interpretative lens through which he or she views the students' mathematics as well as recognize important mathematical understandings that might be surprising or different from his or her own. Addressing these questions also pushes the researcher to articulate the understandings he or she assumes students have at the beginning of the teaching experiment. By documenting these issues during the design of a teaching experiment (often prior to working with students) the researcher creates a record of his or her initial hypotheses. These hypotheses later serve as a means to consider how the researcher's ways of thinking changed in tandem throughout the teaching experiment with the students' mathematical thinking.

Critical Examination of Social Interaction

To suggest that researchers' examination of the social interaction between themselves and the students is an important aspect of attending to reflexivity in the context of conducting a teaching experiment is to suppose that developments in students' knowledge, and thus the models of students' learning that researchers construct to account for such developments, are conditioned by social interaction. We must therefore explain how a researcher's construction of a model of students' mathematical thinking and learning is fashioned by the social interaction that occurs between a researcher and students during the teaching experiment episodes.

From a constructivist perspective, an individual's enacted knowledge is fashioned by his or her understanding of the stimuli inherent to a particular environmental context—stimuli that are often mediated by social interaction. Accordingly, the model of students' mathematical thinking and learning a researcher constructs in the context of a teaching experiment is very much influenced by the social interaction that conditions the evocation of students' mathematical knowledge. Consistent with our recommendation for researchers to explicate their theoretical orientation, we now turn to providing a justification, grounded in constructivist epistemology, for the claim that students' knowledge, and thus the model a researcher constructs to account for its development, is fashioned by the social interaction that occurs in the context of a teaching experiment.

The evocation of specific knowledge is contingent upon an individual interpreting stimuli that activate particular cognitive schemes. In other words, certain knowledge is not made manifest until an individual interprets a certain stimulus in such a way that his or her construction of the stimulus serves as a cue for the enactment of a particular cognitive scheme or a network of related schemes. Therefore, while many believe knowledge is invariantly accessible across time and space, we consider knowledge to be the set of one's cognitive schemes that may become operational in the space of stimuli in which the individual is situated that may enact these schemes. Accordingly, one's ways of perceiving his or her environmental context constitutes a space of stimuli that maintain the potential to make a subset of one's knowledge operational. An individual's knowledge, then, can be thought of as the set of cognitive schemes that *may* become enacted as a consequence of the individual's interpretation of the stimuli inherent to a given environmental context. It is therefore appropriate to say that enacted knowledge is conditioned by the individual's interpretation of his or her environmental context. This is not to suggest that knowledge resides external to the knower since individuals *interpret* and *appraise* their environmental context—interpretations and appraisals that inform the knowledge one employs. To speak of enacted knowledge, then, is to speak of the cognitive schemes that become operational upon one's interpretation of stimuli in a given environmental context that serve to activate such cognitive schemes.

Given our view that knowledge is conditioned by an individual's interpretation of his or her environmental context, we contend that the observable actions students demonstrate in the context of a teaching experiment derive from enacted knowledge that is influenced by their interpretation of their interaction with the researcher. What students make of the researcher's language and actions constitute environmental stimuli that may support or inhibit specific cognitive schemes from being activated. For example, if a student interprets a researcher's questioning as suggesting that the researcher simply wants the student to recite the correct answer, the student may be disinclined to engage in sustained reasoning and sense making. We have ourselves observed students demonstrating very different mathematical knowledge after simply switching the interviewer in a teaching experiment session or task-based clinical interview (cf. Thompson & Thompson 1994). For this reason, it is important for a researcher to discern how students perceive his or her language and actions.

To this end, there are a couple of specific practices to consider. Regularly asking students to verbalize their interpretation of the researcher's questions and statements allows one to obtain artifacts of students' image of their social interaction with the researcher. For instance, asking students questions such as, "What is your understanding of what I just said?" and "Can you tell me in your own words what am I asking you to do?" allows the researcher to gain insight into how students are interpreting their interaction with the researcher. Discerning how a researcher's instructional actions support students in attaining the understandings that the researcher seeks to promote is an essential to constructing viable models of students' mathematical thinking and learning. It is important at this point to reiterate that a researcher's instructional actions do not influence students' thinking. Rather, students' *interpretation* of a researcher's instructional actions influence students' thinking. It is for this reason that, in addition to attempting to elicit observable products of students' reasoning, researchers should consistently attempt to provide occasions for the students to convey their interpretation of the researcher's language and actions.

Critical Examination of Data Analysis

While conducting a teaching experiment, the model a researcher constructs of students' emerging mathematical knowledge is fashioned by his or her ways of perceiving and conceiving students' observable behaviors. Mason's (2002) observation eloquently summarizes this point: "what we learn from an observation is something about the researcher, as well as, perhaps, something about the phenomenon" (p. 181). It is therefore imperative for researchers to document and explicate the decisions and interpretations they make throughout their construction of a model of student's mathematical thinking and learning, and to detail the evolution of this model throughout the research process.

While conducting a teaching experiment, data analysis proceeds in a cyclic fashion whereby the researcher continually generates and refines hypotheses until a stable and viable inductively-derived theory regarding the process by which students construct a desirable understanding of some mathematical idea emerges. Refining provisional hypotheses requires purposeful data collection informed by ongoing analysis. Thus, a hallmark of the teaching experiment methodology is the reciprocal relationship between data collection and analysis; that is, while constructing models of students' mathematical thinking and learning, the data a researcher collects influences the hypotheses he or she constructs, and the hypotheses a researcher constructs informs subsequent data collection. Hence, the boundary between data collection and analysis is necessarily blurred when one conducts a teaching experiment. For this reason it is important for researchers to critically examine how they interpret data during ongoing analysis so that they may ascertain their role on subsequent data collection and, ultimately, on the model of students' mathematical thinking and learning they construct.

Researchers' attention to the role of their interpretation of data on the construction of their model of the process by which students may understand a particular mathematics concept not only clarifies the phenomenon under investigation but also details researchers interaction with the phenomenon. Researchers' documenting their decisions and interpretations during ongoing analysis is important because it produces a form of data about their interaction with students that they can then use in subsequent analyses to bring into conscious awareness the ways in which their interaction with the subject informed their interpretation of students' language and actions.

To discern the role of one's interpretation of data on the model one constructs to account for students' mathematical thinking and learning, a researcher may consider creating artifacts of his or her thinking during ongoing analysis in the form of audio recordings or written memos that focus on the following:

1. Explicating hypothetical conceptual operations that may explain the researcher's interpretation of the students' language and actions throughout testing the viability of an emerging model of students' thinking and learning;

2. Identifying students' specific utterances and actions that contributed to the researcher's construction of these hypothetical conceptual operations;
3. Explaining how the researcher interpreted these utterances and actions so as to make their contribution to his or her model of the students' conceptual operations explicit;
4. Justifying the researcher's instructional actions throughout implementing the instructional sequence and describing his or her interpretation of students' responses to these instructional actions.

It is clear that a researcher's responses to foci (2), (3), and (4) constitute a data set that he or she may retrospectively analyze for purpose of providing insight into the conceptual origins of his or her model of students' mathematical thinking and learning. This data set comprises a record of a researcher's interpretation of the primary data, and the inferences the researcher drew from this data, throughout ongoing and post analysis. One's retrospective analysis of these analytical artifacts is in the service of elaborating a chronology, presented in narrative form, of the development of his or her model of students' thinking and learning by explicating the interactional, institutional, emotional, discursive, theoretical, epistemological, and ontological influences that contributed to its construction (Mauthner & Doucet, 2003). This chronology allows a researcher to present his or her model of student's thinking and learning not just as a product but also as a process, and not by an impersonal machine but by a researcher abounding with subjective interpretative lenses.

Teaching Experiment Literature Analysis

To examine the utility of the analytical framework presented earlier, we examined nineteen of the most highly cited studies in mathematics education that employed a teaching experiment methodology, beginning two years after Steffe & Thompson's (2000) work was published. These articles came from all of the top journals in mathematics education, and covered a wide variety of mathematical topics. We constructed a coding scheme, the components of which corresponded to the four domains of our framework for attending to reflexivity. In our initial coding, we found that less than half (8/19) of these studies met the "explicating mathematical meanings" criterion; just over half (11/19) clearly identified a theoretical perspective; most (18/19) examined social interaction in some way; and roughly half critically examined the data analysis procedures (11/19). While this analysis is in its initial stages, it clearly illustrates important role these components of our framework play in the highest quality studies in our field, yet suggests that there are ways in which the analytical practices of the teaching experiment studies could be expanded. We also recognize that in some cases, the limitations of a journal space and the review process shape the focus of each paper. Yet we think it is important that the field begin to critically appraise what elements must be present in the presentation of results based on a teaching experiment and to include the four components we have presented in this paper.

Conclusion

In this paper, we have proposed four ways in which researchers may attend to reflexivity in the process of conducting teaching experiments in mathematics education. A reviewer of a previous version of this manuscript suggested that each of the four domains of reflexivity we discuss appear as analytical recommendations elsewhere. We respond by noting that such recommendations have not been rationalized with an appeal to the foundational theoretical premises on which specific qualitative methodologies are based. Such rationalizations are essential to supporting researchers in observing the analytical practices that comprise attention to reflexivity in non-superficial ways. It is for this reason that we consider the present paper a contribution, and encourage other researchers to explicate what it means to attend to reflexivity in the context of other qualitative methodologies informed by other theoretical orientations.

Attending to reflexivity is an essential component of the teaching experiment methodology and involves researchers in a systematic and disciplined investigation of how their interpretations and actions influence the models and theories they construct to explain students' mathematical thinking and learning. The recommendations we outlined in this paper assist researchers in becoming consciously aware of at least some of the subjective interpretative lenses through which they perceive the phenomena they study, thereby affording researchers agency over these interpretative lenses. When researchers bring into conscious awareness the lenses through which they make sense of data, they are positioned to communicate the products of their research in way that reveals that their results and conclusions are not about a particular phenomenon, but are instead about their *interpretation* of the phenomenon. Communicating the products of one's research in this way lends a transparency to the research process, thereby inviting others to scrutinize the origins of the models and theories presented in the literature, thereby fulfilling one of the necessary conditions for a scientific enterprise.

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THE NARRATIVE STRUCTURE OF MATHEMATICS LECTURES

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Although lecture is the traditional method of university mathematics instruction, there has been little empirical research that describes the structure of lectures. In this paper, we apply ideas from narrative analysis to an upper-level mathematics lecture. We develop a framework that enables us to conceptualize the lecture as consisting of collections of narratives, to identify connections between the narratives, and to use the narrative structure to identify key features of the lecture. By looking at repetitions of the mathematical concepts across the narratives, graph theory tools provide a means of examining the structure of the lecture. The analysis highlights the demands that students may face to understand the connections between the various mathematical ideas that the instructor introduces.

Keywords: Post-Secondary Education; Research Methods

Lecture is the traditional method of university mathematics instruction. Although there have been a few attempts to describe these lectures (e.g., Dreyfus, 1991), Speer, Smith, and Horvath (2010) noted that, due to the dearth of empirical studies, most of our beliefs about the structure of lectures are based on popular opinion or personal experience.

In general, we view mathematics lectures as consisting of instantiations of mathematical entities (i.e., objects, concepts, and symbols) in service of enacting and modeling broader mathematical processes (such as conjecturing, representing, and justifying) and habits of mind (e.g., looking for patterns). The goal of the current study is to create and use a framework based on narrative analysis to analyze a lecture, identify and describe the various processes and habits of mind that are modeled by the instructor, illustrate the ways the mathematical entities evolve and transform over the course of the lecture, and describe the ways in which the entities and processes are connected to each other.

Theoretical Framework

Narrative analysis, a common analytical technique in literary theory (e.g., Bal, 2009; Holley & Coylar, 2012), considers the sequencing of the elements of the text and how initial elements influence and shape later events (Holley & Coylar, 2012) and focuses on the evolving relationship between characters. Several researchers have described the close relationship between mathematics texts and narratives (e.g., Netz, 2005), and others have directly applied concepts from narrative analysis to analyze aspects of mathematical texts (e.g., de Freitas, 2012; Dietiker, 2012; Andrà, 2013).

Although there are numerous definitions of what a narrative is, they commonly attend to the temporal ordering of events, the way in which these events are connected, and the meaning that is ascribed to the sequence of events by a particular audience(see, e.g., Czarniawska, 2004; Reissman, 2005). We adopt the perspective of Holley and Colyar (2009), who identify narratives as texts in which a reader or observer can identify events, characters, and a plot—and sees connections between the events of the narrative.

Event structure

Bal (2009) describes the interrelated ideas of event, story, and fabula: The narrative text conveys events—"transition[s] from one state to another state" (p. 6); the sequence of events in the text make up the story; and the fabula is the "series of logically and chronologically related events that are caused or experienced by actors" (p. 5). Dietiker (2012) adapted these ideas to apply to mathematical texts:

- A *mathematical event* is "a transition from one mathematical state to another" (p. 15), such as instantiating a mathematical object, creating a representation, or making a conjecture.
- A *mathematical story* is "the [temporal] sequence of events encountered and experienced by a reader throughout a mathematics text" (p. 15).
- The *mathematical fabula* is "a reader's reorganization of the logic around how certain mathematical ideas support or connect the meaning of other mathematical ideas" (p. 16).

Dietiker (2013) thus conceptualizes a mathematical text as a narrative through the lens of comparing the mathematical story and fabula. While the former describes the chronological sequence in which an instructor presents mathematical concepts, the latter describes the logical relationships between the concepts.

Mathematical Characters

In the previous research on mathematical narratives, there has not been a consensus on what constitutes a character (e.g., Dietiker, 2013; Andrà, 2013; de Freitas, 2012). We describe *characters* in a mathematics lecture as including (but not limited to) the mathematical objects and concepts that the instructor instantiates that play a role in the story; these objects can be either general (e.g., the concept of an equivalence class) or specific (e.g., "the equivalence class of 3"). A character is identified by both the underlying concept and its representation (e.g., "the integer a" and "the integer b" are different characters).

Mathematical plot

The plot of a narrative consists of the meaningful connections between the events and informs the construction of a fabula (Polkinghorne, 1988). We define the *plot* of a mathematics narrative to be a description of the way a mathematical process or habit of mind is applied to a collection of mathematical objects or concepts. For example, in the lecture described here, the instructor used integers to illustrate a property of a specific equivalence relation, and then used this example to explain properties of equivalence relations in general. We can describe this process using the habit of mind "articulating a generalization using mathematical language" (Mark, Cuoco, Goldenberg & Sword, 2010).

Identifying Narratives in Mathematics Lectures

Boundaries of narratives. According to Labov (1972), a narrative includes, at a minimum, a sequence of two temporally ordered clauses. The process of identifying the "boundaries" of narratives—where the story begins and ends—can be nontrivial. To address this for the case of mathematics lectures, we identify the initiation and conclusion of a narrative by a shift in the presented content or mode of operation of the class, the use of board space, or natural language speech cues (e.g., the instructor saying "Now…" or "Let's consider…").

Identifying plot and characters. The plot and characters are not found "neatly packaged as such by the narrator" (Emden, 1998, p. 35). Rather, plots and characters are identified through a "tacking procedure" (Polkinghorne, 1988, p. 19) or a process of abduction (Czarniawska, 2004). This begins with the proposal of a potential plot and characters, and then these are compared with the events of the story to see how well they provide a coherent theme.

Framing and embedded narratives. In some cases, narratives are contained within other narratives and function as events within the containing narrative. This can be described by the ideas of *framing narratives* and *embedded narratives*, which describe a hierarchical relationship between two narratives. Following Ryan (1991) and Palmer (2004), we describe the role of a framing narrative as providing a context for its embedded narratives. The plot of the framing narrative provides coherence and focus for the plots of the embedded narratives; thus, the framing narrative influences the way we might read (or observe) an embedded narrative. Conversely, an embedded narrative typically serves as an event in a framing narrative.

Logical connections between events. Although a fabula must be consistent with the plot of a narrative, as Dietiker (2012) notes, "there are many deductive lines of reasoning that can lead to the same conclusion" (p. 16). From a broad perspective, we view two narratives or events as connected when one mathematically builds upon the other. Typical plots of a mathematical narrative include defining and redefining, constructing patterns, and forming and verifying conjectures; such plots may intuitively establish mathematical connections with other narratives by applying these mathematical processes to characters shared between two narratives. Thus, we attempt to capture the idea of mathematical building from one narrative to another by saying that narrative A *connects* to narrative B if a character in narrative B is first formally defined or instantiated in narrative A. Using this definition, the two narratives are connected if the plot of B involves mathematical processes that are applied to the same mathematical objects and concepts that are introduced in the plot of A. These logical connections form the basis for identifying a fabula for a narrative.

Methods

Data Collection

The lectures were taken from a standard junior-level abstract algebra course at a large university in the northeastern United States. The instructor was a tenured professor with a research focus in algebra; he had previously taught the course numerous times. We selected four lectures from the beginning, middle, and near the end of the semester and included instances of presenting definitions, examples, theorems and proofs.

A member of the research team attended each of the lectures and took notes on the instructor's speech and writing. Each lecture was video-recorded; we transcribed the instructor's speech verbatim from the recorded video and used the video to check the accuracy of the researchers' notes. For the analysis presented here, we selected a 45-minute excerpt from one of the lectures; this portion of the lecture included instances of definitions, examples, theorems, and proofs, and the instructor's presentation style was similar to the other lectures. Thus, we view this portion of the lecture as representative of the entire corpus of data.

Identifying Narratives and Plots

To identify characters in the lecture, we began by looking for instances where the instructor instantiated a mathematical object or concept, using the instructor's gestures, speech, and writing to inform this identification.

In order to identify potential framing narratives, we employed a "top-down" approach by identifying a coherent, meaningful plot and its constituent characters. We read the transcript of the lecture holistically and identified broad themes along with the mathematical objects, characters, and representations that were the primary focus of these themes. To identify potential embedded narratives, we employed a "bottom-up" approach by identifying sequential clauses that, when taken together, appeared to describe or apply a mathematical process to a set of characters.

To identify potential plots from these themes and characters, we looked for mathematical processes and habits of mind that the instructor employed, as well as definitions, theorems, and processes that typically play an important role in an abstract algebra class and might serve as a focal point for processes and habits of mind. Then, we worked abductively by identifying the characters and events and then revising the proposed plots so that they provided coherence to the chronologically related elements.

Results and Analysis

Embedded Narratives

The primary focus of our analysis is the collection of framing narratives in the lecture. Consequently, we present here only a summary of the plots of the embedded narratives and other events of the framing narratives, as shown in Table 1.

Table 1: Embedded Narratives

Narrative Number	Description of Plot or Events
Framing Event 0	The instructor identifies the "goals" for the lecture and provides motivation for each goal: examining equivalence relations and introducing the idea of equivalence classes.
1	The instructor recalls and summarizes a narrative from a previous class: "R4" (i.e. for $a,b \in \mathbb{Z}$, $a \sim b$ if $\|a\|=\|b\|$) is an equivalence relation
2	The instructor starts to construct an alternative method for representing an equivalence relation and applies it to the R4 equivalence relation: The set of integers is represented by a large circle; each integer is represented by a labeled dot inside the circle; and a line is drawn between two dots when the corresponding integers are equivalent under ~.
3	The instructor poses the question: Can there be a triangle (i.e. three distinct dots that are all connected to each other) in the dot-and-line diagram?
4	The instructor poses and answers the question: Can there be line segments (i.e. three distinct dots where one dot is connected to the other two) in the dot-and-line diagram?
Framing Event1	The instructor describes R4 as relating pairs of numbers
Framing Event2	The instructor describes motivation for introducing the idea of an equivalence class.
5	The instructor elaborates the dot-and-line diagram by introducing "loops"
6	The instructor poses and answers the question: Does introducing "loops" violate the "no line segments" condition?
Framing Event3	The instructor describes additional motivation for introducing the idea of an equivalence class
7	The instructor generalizes from a "concrete" equivalence relation to an abstract one by translating the dot-and-line diagram for R4 into a dot-and-line diagram for an "abstract" equivalence relation

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Framing Event4	The instructor presents a general, formal definition of equivalence class
8	The instructor generalizes from the previous examples to create a precise mathematical description of equivalence classes for R4
9	The instructor challenges his previous work and works to make more precise his definition of equivalence classes. He poses and answers the question: Are equivalence classes well defined?
10	The instructor poses and answers the question: Are there any equivalence classes under R4 with a single element?
11	The instructor conjectures that [3]=[-3] under R4 (this example is subsequently generalized to all integers). He then tests this conjecture, setting up a proof by contradiction by asking whether [3]≠[-3].
12	The instructor conjectures and proves that [*a*]=[-*a*] for any integer *a* under the equivalence relation R4
13	The instructor conjectures and then sets up a proof that, for integers *a* and *b*, if *a*~*b* then [*a*]=[*b*] under the equivalence relation R4

Framing Narratives

At the beginning of the class, the instructor described the "goals" he had for the day and wrote them on the board: Identifying general properties of equivalence relations, introducing the idea of an equivalence class, and identifying general properties of equivalence classes. Based on this, as well as a holistic reading of the transcript, we identified the three framing narratives as shown in Table 2.

Table 2: Framing Narratives

Framing Narrative	Plot
A	Creating and refining an alternative representation: The instructor develops a diagram for representing a particular equivalence relation
B	The instructor generalizes from a "concrete" example to develop the concept of an equivalence class and construct a precise, abstract definition.
C	The instructor conjectures and then proves that two equivalence classes that share an element are equal

Connections Between Narratives

Figure 1 shows the connections between the various embedded narratives and other events in the lecture; the shape of each vertex indicates the framing narrative of which it was a part, as shown in

Figure 2. In the graph drawing, each vertex corresponds to an embedded narrative or a non-narrative event, labeled with the corresponding number from Table 1. The edges between the vertices

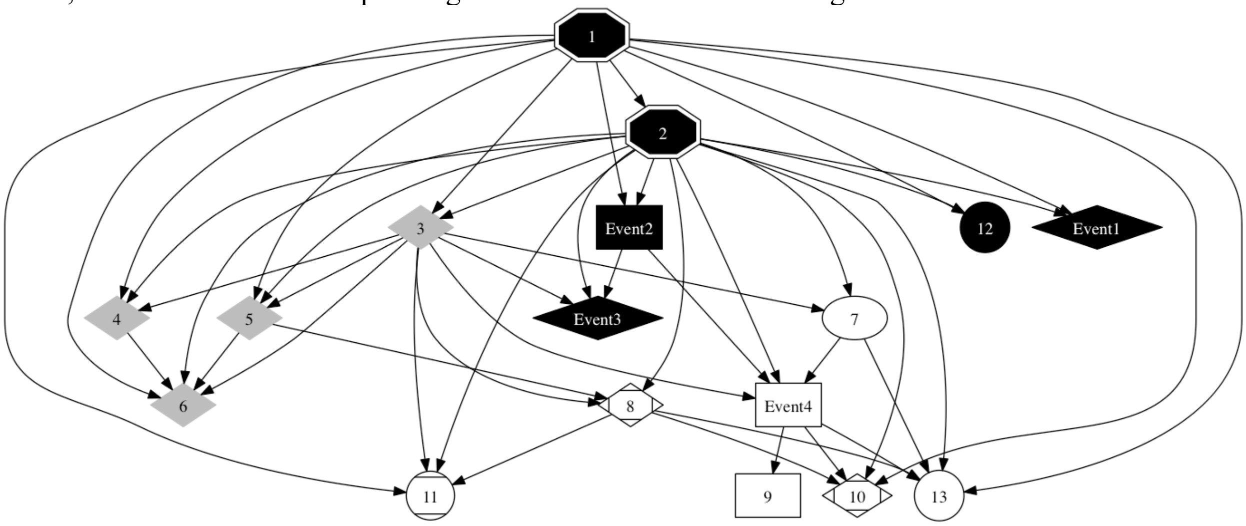

Figure 1: Diagram of Connections Between Embedded Narratives and Events

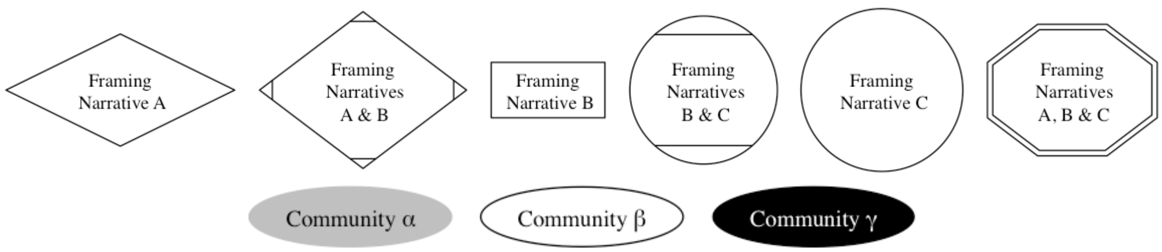

Figure 2: Legend For the Diagram of Connections

represent the connections, and the connection X ➔ Y indicates Y includes characters that were introduced in X.

Structure of the Graph

There are numerous methods for analyzing the graph to identify structural aspects of the lecture. One such method is to identify the *community structure* of the graph; this partitions the graph so that the collection of narratives in each subgraph is relatively densely connected but the connections between the subgraphs are relatively sparse. Using a modularity maximization algorithm, the graph in Figure 1 can be split into the three communities, which are indicated by the shading of the vertices as indicated in Figure 2. There is a close alignment between the communities and framing narratives, which suggests that each framing narrative has relatively dense internal connections and, consequently, relatively strong internal coherence

The graph also displays how "closely related" various pairs of embedded narratives are—i.e., how directly the characters introduced in one narrative are included in a subsequent narrative. This can by identified by analyzing the various paths between vertices. For example, fully understanding the plot of EN10 indirectly required an understanding the characters in embedded narratives 8, 5, 4, and 3 and the relationship between these characters. This "chaining" is reflected in the length of the directed path between embedded narratives 3 and 10.

Another aspect of the lecture highlighted by the graph-theoretic structure is the "centrality" of various embedded narratives, which reveals their narrative significance. There are numerous types of graph-theoretic centrality measures that can be used. For example, the *pagerank centrality* of a vertex measures the number of connections from other vertices and gives a higher weight to connections from other highly connected vertices (Brin & Page, 1998). A narrative with high pagerank centrality has numerous direct and indirect connections to previous narratives; it integrates characters from numerous other narratives and is likely to be culmination of many paths through the graph of embedded narratives. Embedded narratives 6 and 13 had the largest page rank centrality. Both of these narratives integrated the characters from numerous other narratives: EN6 combined R4, the dot-and-line diagram, and the idea of follow-on lines to refine the diagram; EN13 drew upon most of the mathematical objects in the lecture to prove a conjecture about the equality of equivalence classes.

Discussion

In contrast to the common notion of lectures as consisting of formal sequences of definitions, theorems, proofs, and applications (Dreyfus, 1991), the results described here suggest that mathematics lectures may have a significantly more complex structure.

We hypothesize that thinking about parts of a lecture in terms of plots and characters is useful for researchers to assist in identifying the main mathematical ideas of the lecture and the different ways that those ideas may be connected and built upon by the various mathematical objects and concepts. Identifying elements of embedded and framing narratives—including the characters, plots, complications, and resolutions—enables us to describe the temporal development of the important mathematical ideas in the lecture.

In addition to being useful for researchers, identifying the plots and connections between the narratives can help us better understand the challenges that might arise in learning from a lecture. In particular, the complexity of the narrative structure and the numerous habits of mind that are used in the plots suggest reasons why students—who might not possess these habits of mind or be able to quickly identify the structure—might struggle to make sense of a lecture.

Constructing meaning from a text by viewing it as a narrative is a different process than employing a logico-scientific mode of knowing. From a narrative perspective, events are given meaning through abduction; the only way to identify the plot or structural elements of a narrative is by "negotiating and renegotiating meanings by the mediation of narrative interpretation" (Bruner, 1990, p. 67). Consequently apprehending the plot of a framing narrative can involve a complex process in which students must identify the multiple roles that can be played by each embedded narrative, identify potential orderings of these roles, and understand how they fit together.

The analysis of the connections between the embedded narratives revealed several important aspects of the lecture. First, the community structure analysis showed that each framing narrative contained relatively dense connections. If these connections are not made explicit, it may be challenging for students to identify the ways in which the embedded narratives are related to each other. Conversely, these connections may also support sense-making: If students can understand aspects of one narrative, then they can use its connections to develop an understanding of the connected narratives or to identify the plot of a framing narrative. In addition to this structure analysis, the centrality analysis can reveal the ways narratives synthesize ideas, serve as key focal points, and incorporate characters that play significant roles in the narratives.

As long as lecture remains a common form of mathematics instruction, more research is needed to describe lectures themselves as well as the ways students interpret and learn from lectures. The data reported here come from a case study and may not necessarily be representative of all mathematics classes—or even all abstract algebra classes. In order to more completely describe mathematics lectures, it will be essential to gather a larger corpus of data—both more classes from a single instructor, and lectures from other instructors. It will also be essential to collect data on students' interpretations of lectures and to determine what students learn from each lecture and how the narrative structure—and their interpretation of the structure—relates to this learning.

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FLOWCHARTS TO EVALUATE RESPONSES TO VIDEO-BASED PROFESSIONAL NOTICING ASSESSMENTS

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This paper describes a tool developed by the researchers and used to evaluate preservice teachers' professional noticing responses to prompts posed after viewing a video interview of a child solving an arithmetic task. The tool, a series of flowcharts, was designed to increase scoring efficiency and reliability across a team of six raters. Inter-rater reliability increased from 60% to 83% after implementation of the flowchart-scoring tool. This measurement tool and process has the potential to inform assessment strategies in the context of professional noticing frameworks. Streamlining the tool and challenges to generalizability will be addressed.

Keywords: Teacher Education-Preservice; Teacher Knowledge; Assessment and Evaluation

Introduction

Preparing teachers to practice responsive teaching is an enduring challenge of teacher educators. Identifying, measuring, and evaluating responsive teaching practices add to the challenge. Professional noticing has been identified in the literature as a promising practice to provide a foundation for teachers to become responsive practitioners (Goldsmith & Seago, 2011; Santagata, 2011; van Es & Sherin, 2002). This paper describes an innovative tool developed by the researchers and used to evaluate preservice elementary teachers' (PSETs') responses to prompts posed after viewing a video-based measure of professional noticing.

The assessment measured PSETs' responses pre- and post- to a researcher developed instructional module based on professional noticing (Jacobs, Lamb, & Philipp, 2010) within the context of the Stages of Early Arithmetic Learning (SEAL) trajectory (Steffe, Cobb, & von Glasersfeld, 1988). The module decomposes professional noticing into its three interrelated skills, *attending, interpreting, and deciding*, that allows for the skills to be progressively nested and intentionally developed (Boerst, Sleep, Ball, & Bass, 2011). Our results indicate that preservice elementary teachers (PSETs) can grow significantly in the three skills of professional noticing (Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder, 2013).

Theoretical Framework

Teacher Professional Noticing

Following Jacobs, Lamb, and Philipp's 2010 study, and the publication of Sherin, Jacobs, and Philipp's (2011) seminal book, *Mathematics Teacher Noticing*, the construct of teacher professional noticing has garnered the attention of researchers and practitioners alike. In earlier work, Carpenter, Fennema, Levi, and Empson (1999) provided evidence that teachers' *attention* to children's mathematical thinking can affect student learning. Jacobs et al. (2010) built upon the foundation of *attending* to children's thinking by offering a definition that includes three interrelated components, *attending, interpreting,* and *deciding*. Kaiser, Busse, Hoth, König, and Blömeke (2015) report on the challenges of creating valid and reliable instruments for feasibly measuring professional noticing. Determining what teacher competencies to measure and maintaining scoring quality on instruments that do not demand too much time from participants and scorers present the challenge. Kaiser et al. (2015) suggest that beginning with a well-defined theoretical framework is key.

Trajectory-based Mathematics Teaching

Multiple researchers have explored learning trajectory based mathematics teaching (Clements & Sarama, 2009; Sztajn, Confrey, Wilson, & Edgington, 2012) resulting in trajectories of various mathematics concepts. We situated PSET professional noticing development specifically in one such trajectory, SEAL, developed by Steffe and his colleagues (Steffe et al., 1988). SEAL is an early numeracy progression exemplary of "learning trajectories built upon natural developmental progressions identified in empirically based models of children's thinking and learning" (Clements, 2007, p. 45). Video cases illustrating the nuanced behaviors of children's mathematical thinking along the early numeracy progression of SEAL provided a contextualized reflective setting for PSETs to develop the three component skills of professional noticing.

Methodology and Data Sources

To examine PSET growth across the interrelated skills of professional noticing, the authors developed a measure consisting of a brief video clip in which an interviewer poses a comparison, difference unknown task (Carpenter et al., 1999). This clip, while brief, is rich in details of the child's thinking that can be easily attended to, but also includes nuanced details that might be missed by novices thus allowing for a range of scores. The three prompts were drawn from the work of Jacobs et al. (2010) and each focused on one of the three interrelated components, attending, interpreting, and deciding. The prompts were: 1) Please describe in detail what this child did in response to this problem, 2) Please explain what you learned about this child's understanding of mathematics, and 3) Pretend that you are the teacher of this child. What problems or questions might you pose next? Provide a rationale for your answer.

We examined samples of PSET data for each of the professional noticing components for emergent themes (Glaser & Strauss, 1967). The emergent themes were assimilated with researcher-identified key features for each of the components. The themes and key features resulted in benchmarks that defined several ranked response types. The PSETs' professional noticing responses from all implementation and comparison sites were compiled into a spreadsheet and blinded. Scorers did not know if responses were from an implementation or comparison site, nor were they aware if it was pre- or post-assessment data. All data were randomly assigned to deter any biases that may possibly occur when scoring the response.

Initially, three teams of two scorers each rated data sets using the benchmarks determined through the emergent themes. Average inter-rater reliability using this process was 60%. This process was cumbersome, as it required a third scorer when the original scorers' ratings did not agree. Ultimately, more efficient and reliable processes were needed to evaluate the complex professional noticing responses. We drew upon the literature of flow processes (AMSE, 1947) to develop a series of flowcharts to guide the scoring process of each interrelated skill.

Results and Conclusion

The benchmarks developed from the emergent themes of PSETs' responses provided the foundation for the yes/no questions of the flowchart used to guide the raters' scorings. Figure 1 illustrates the flowchart for the interpreting component. The resulting inter-rater reliability averaged 83% for all components across six scorers, a marked improvement from the inter-rater reliability of 60% reached prior to the use of the flowchart scoring tools.

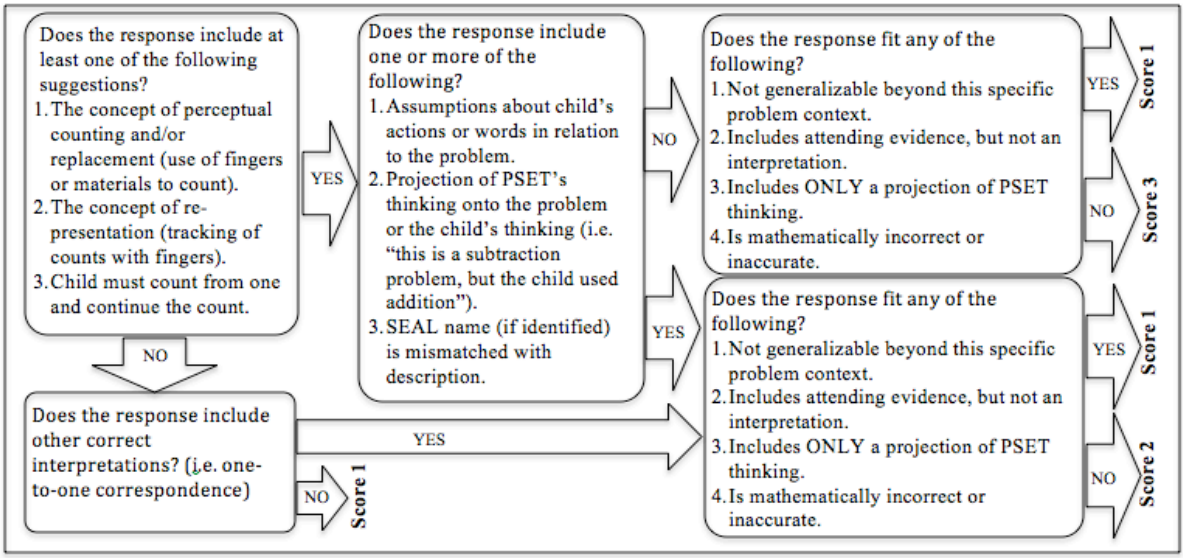

Figure 1: Interpreting Scoring Flowchart

To illustrate the use of the scoring flowchart, one PSET's response to the interpretation prompt is examined.

> "I learned that this student is still using manipulatives or his fingers to count items, but he can also use a counting on strategy to solve problems. I feel that if I was to place this student in one of the SEAL stages, I would have to say he is somewhere between stage 2 and 3. He is still using his senses such as touching (perceptual counting) to count the bears. However, he also counted up from seven to get to eleven."

The criteria for "yes" in response to the first rater question of the scoring flowchart in Figure 1, is satisfied because at least one of the three benchmarks is met. The PSET referenced the child's use of fingers. Following the "yes" route of the flowchart, the PSET response does not include any of the three limitations in the second question, resulting in a "no" response by the rater. The third question, "Does the response fit any of the following [four criteria listed]?" also rates a "no," resulting in a score of 3 for this response.

The scoring flowcharts proved valuable to the scoring process not only for ensuring inter-rater reliability, but also raters could track their responses through the chart with codes such as, "y1nn score 3" for the response above. If questions or discrepancies arose, the subsequent discussion could be focused quickly on the aspect of the response in dissension.

Qualitative data can provide insights on participants' understanding. However, qualitative data is often rich in text, which can pose a challenge to researchers to make meaning from the data (Creswell, 2013). While the open-ended professional noticing responses were concise, our sample included 224 PSETs, which meant 448 responses (due to the pre-post research design) required scoring. It was critical for our research team to establish a systematic process of scoring to make the data analysis process manageable and comprehensible across six scorers. The flowcharts allowed for such systematic processes and increased our inter-rater reliability to 83%. Kaplan and Maxwell (2005) argue that flowcharts can serve multiple purposes including: making data visible, reducing data, and presenting analysis in a holistic form.

The measurement processes developed in this research have the potential to refine the strategies for measurement of responsive teaching practices with respect to noticing frameworks and beyond. Thus, researchers and teacher educators may design or adopt tools to better ascertain individuals' development of key skills. There are limitations with this example, most notably the case specific nature of this assessment. Situational assessments such as this, however, do address the validity question in that what competencies were being measured were definable. Creating an assessment protocol beyond the case level is a challenge, but we argue that this flowchart process can inform similar processes to interpret and analyze future studies of this nature. Used in balance with other measures, this process might further contribute to understanding the complexities of teacher competencies that result in effective responsive teaching practices.

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OBSERVING MATHEMATICS IN COLLECTIVE LEARNING SYSTEMS

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In the past decade there is growing interest in mathematics education research that has emerged from complexity sensibilities that focuses on the collective (group) as a learning system. With this growing emphasis there is a need for methodological tools to study those systems. I am concerned with ways in which we can observe for the emergence of mathematics and mathematical thinking in the collective (rather than the individual). The purpose of this paper is to explore means and methods that can be used to anticipate, describe and account for the mathematics that emerges in collective activity in its own right. What are the markers of collective learning and understanding? How can we observe those markers of collective learning and understanding?

Keywords: Cognition; Learning Theory; Research Methods

I have been struggling with a tension that has emerged for me since first thinking about the class as a complex learning system. How do I observe and account for the mathematics of the collective (the group) rather than the individual? Fifteen years ago I conducted a research study to explore the impact of high activity mathematics within a grade 7 mathematics class. In that class a number of norms were established: working cooperatively with peers; sharing and critiquing solutions with the class; an understanding that the knowledge needed for a task or problem is within the group; and the justification for an answer is in the mathematics rather than with the teacher. One day while the students were writing a quiz (which they were expected to complete independently) one boy got up from his desk, paper in hand and approached a very competent peer to ask a question. I immediately interrupted him to ask what he was doing out of his desk and talking during a test. He replied he was asking Todd what a hexagon was (a fact he needed to proceed with the problem posed in the quiz). When I asked why he would do that during a test, he responded very matter-of-factly, "Well *he* knows." Upon reflection it became clear to me that the student was simply acting within the well established class practice of using all the resources within the group to find out what you need to know in order to proceed with a task or work on a problem.

Reflecting on this event and many others as I worked with the class, my colleagues and I began to theorize the class as a collective learning system. This led me to wonder about the enduring challenges we face as mathematics educators in regards to assessment practices. Why do we have students work in groups on mathematics but then assess each individual's understanding with little or no recognition that their understanding is not only a product of highly interactive group processes but is supported and potentially contingent on the group processes and the knowing of others? Why is it that in spite of using the social group to facilitate understanding we continue to insist students display their understanding isolated from the support of others? Why is it that we do not assess the group's mathematics?

In this paper then, I am not focused on how knowledge is developed by the individual in groups or collectively; rather, I am trying to identify different forms of activity within a group that signal collective learning and understanding. If we can identify markers of the class acting as a single unit (collective activity), we can use those markers to observe that unit's (collective) understanding. As a result we will be better positioned to ask not only how might we assess mathematics learning in school settings differently but what is it that we should be assessing in school mathematics. The purpose my work is to explore the means and methods that can be used to anticipate, describe and account for the mathematics that emerges in collective activity in its own right. What are the markers of collective learning and understanding? How can we observe those markers of collective learning and understanding?

Theoretical Perspective

Learning and Cognition

Maturana and Varela (1992) assert that "knowing is effective action, that is, operating effectively in the domain of existence of living beings" (p. 29). Varela et al. (1991) take this one step further when they claim cognition is perceptually guided action that brings forth a world of significance. These two claims are not discussed in this short proposal but they have been explored by various authors (e.g. February 2015 issue of ZDM). With these definitions I seek to illustrate how a class of mathematics learners with their teacher constitute a cognitive system, which I will henceforth label as a learning system. Of particular relevance here is that the learning system brings forth a world of significance. Hence I am not speaking of the transmission of pre-established knowledge from an expert to the group (teacher to learners). Rather the focus is on the *knowing in action* of the learners and teacher *as a learning system* (or smaller groups with the class). For the purpose of this paper, learning is defined as any transformation that enables the system to maintain its coherence moment by moment in the circumstances in which it finds itself.

Group Learning

There has been a great deal of interest on the learning that emerges through the interaction of students in groups (citations go back at least 30 years). Such research focuses on (for example) the learning of individuals as it arises in mathematical discourse (Sfard, 2001), how it is scaffolded (Goos, et al., 2002) and how it demonstrates the development of socio-mathematical norms (Yackel and Cobb, 1996). More recently, that work is beginning to be theorized in terms of collective learning systems (Rasmussen and Stephan, 2008; Towers, et al., 2009; Stahl, 2009; Hershkovicks et al., 2014; Conner et al., 2014). With this work there is a deliberate shift to investigate the emergent understandings, patterns of interaction, and other phenomena that are not (can not be) attributed to the individual or the sum of the individuals.

Markers of Collective Learning

I would like to propose that there are (at least) two kinds of collective activity we might observe: cumulative and transformative. The former is a result of the combined effort of the group all contributing the same thing. Take for example the lifting of something very heavy like a beam. The capacity to lift is with each individual. However, a beam is far to heavy for a single person to lift. The way to lift the beam is to get enough people to participate in its lifting. Each individual's capacity to lift is added to the cumulative strength of the group to lift the beam. Notice that if an individual is strong enough to lift on her own then additional persons are not needed. The second kind of collective activity is different in that it happens only with a group (at least two people). This kind of activity involves a number of people whose contributions are transformed as they interact with the contributions of others. It is an activity that is not additive or cumulative but transformative. A new form or process emerges from the intersection of the contributions. It is this second kind of collective activity that I am interested in because I believe it is this kind of activity that defines a collective learning system. I ask, what in the mathematics class can only happen when more than one person is present? This suggests a different way to think about collective understanding.

A sports team analogy might be useful to point us in the direction of what to observe for since I am not interested in what the individual learns but rather how the group functions when working on mathematics that enables further activity (maintenance of coherence) the group. Take for example a pass in hockey. One individual sends the puck and another receives the puck. This group activity involves the interaction of individual contributions to create something new. It is not more sending or more receiving, the "send" and the "receive" are transformed into a "pass." Note how without the receiver the "send" might be a shot on goal or it might be a clearing the puck from the zone but it is not a pass. The same act "sending the puck" is only understood as a pass with the interaction between the sender and the receiver. This example has led me to ask what are those individual actions that are transformed to create something new.

What are other examples of collective activity that emerge from the transformation of individual contributions? Rasmussen and Stephan (2008) offer the example of a "fun couple" where neither person would be considered fun on their own. Imagine an act like that of vaudeville comedians, Gracie Allen and George Burns. Their humor emerges from the interaction of Grace who delivers silly lines and George who delivers straight lines. The combination of the silly and straight are transformed into humorous jokes. Neither Grace's lines nor George's are funny on their own; only in the interaction of the lines is the joke created.

The pass in hockey and the joke in the comedy act are two exemplars that have led me to look for the interaction between the contributions of individuals from which something new emerges. I have identified a couple of examples of such transformations observed in clinical interviews. To illustrate, I offer an example from a parent-child dyad working together to understand how to predict the number of possibilities in a geometrical patterning activity. The child was arranging the materials, calling out and counting the patterns. The father was keeping records but at the same time watching his daughter making the tilings and checking for himself that all of the patterns were accounted for. As he recorded he would use words to denote each of his pencil strokes that was replicating the pattern (noted to the right of the transcript).

Kerry: Okay, you could do this one. | = | |
Dan: Okay, What is that one? Okay, one—
Kerry: One, going, going.
Dan: One going, going, going, going. Okay. | = =
Okay, you got your going, going. Yeah, that's the same as your other one, just turned around. Okay, so what do you have?
Kerry: One, one, one [gestures with her finger three times top to bottom over the vertical tiles, then gestures left to right over the two horizontal tiles]
Dan: [repeats as he sketches] One, one, one. | | | =
Kerry: Two sideways.
Dan: [As he draws the two horizontal tiles he utters] Blip blip.
Dan: Okay, what do you call the one sideways or something else?
Kerry: Blip blip.
Dan: Okay the blip-blips.
Kerry: [Calling out the next arrangement] Blip, blip, one, one, one.
Dan: Blip-blip, one, one, one. =| | |

In this case, "blip-blip" emerged from the interaction between the father and child. From their joint action of arranging, calling out the pattern, recording the pattern and calling it back again emerged the utterance "blip-blip." This transformation of their actions and utterances created a new object, the pair of horizontal parallel tiles that must be treated as a unit. From this point forward in the session the blip-blip was integrated with their knowing actions. The emergence and use of a new word out of their collective activity was quite different than the use of language for another pair in the room. At one point in the session the daughter in the other dyad overheard Dan and Kerry talk about looking for mirror images. She almost immediately said, "Oh, I know. Look for mirror images." I observe and interpret the first example (blip-blip) as a transformation of the individual contributions as collective knowing (and object). Whereas in the second example, I see the mimicking of “look for mirror images” as an accumulation within the group of more members doing the same thing.

A “pass” between team players or the creation of a new idea between people working on a mathematical task is but one form of a marker of collective learning. Rasmussen and Stephan (2008) have suggested that in collective activity the content of that activity is not as important as the practices that allow the group to function (p. 203). Hence, I speculate that a second form of marker is a practice within the group. In the vignette shared at the beginning of this paper, we might identify the student going to a classmate to seek information as a practice within that class. The pattern of a question posed to another student, the offering of a response and the listening for the response also may be a marker of collective activity.

The goal of my work in this project is to propose other markers. Once a set of such markers have been proposed, I will then ask what forms of data could be collected that will enable researchers to observe differences in collective learning systems. Is videotape adequate? Are transcripts necessary? Can we remove attributions to individuals (Towers, Martin and Heater, 2013)? What might we learn about the group processes by fast forwarding video-tape and not listening to voices? What value might there be in studying the sound intensity over the period of a class? These questions are the beginning of the second part of this work of trying to identify new methods for studying the collective learning systems known as math classes.

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COGNITIVE NORMALIZATION: A FOUCAULDIAN VIEW OF MATH PEDAGOGY

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Keywords: Classroom Discourse; Cognition; Equity and Diversity; Learning Theories

Background & Theoretical Perspective. This poster presents a theoretical literature review and describes an ongoing study being conducted with these theories. I focus on the text of Foucault (1977) and apply his notion of *normalization* as a meta-lens on mathematics pedagogy.

Previous research shows that normalizing occurs in a variety of educational settings (Gore, 1995; Hardy, 2004), and recent studies demonstrate that normalizing—among other techniques of power—can stymie the implementation of reform-oriented regimes in math and science education (e.g., Donnelly, McGarr, & O'Reilly, 2014). These studies highlight that the normalizing exercise of power impacts learning outcomes by mediating students' *opportunities* to engage the content. I conjecture that normalizing also affects the *quality* of this engagement, by implicitly reproducing non-agentive dispositions toward mathematics content and practices. Furthermore, I maintain that, through discourse, normalization is unfortunately liable to be recreated, and yet it also stands to be renegotiated, reconfigured, and thus ameliorated.

I conceptualize mathematics education as indeed part of a broader form of "disciplinary power" (Foucault, 1977). As such, the practice of mathematics pedagogy *cognitively normalizes* students when it: (a) *evaluates* student action in relation to a cognitive/conceptual norm; (b) *differentiates* among students according to their apparent progress toward this norm; (c) *hierarchizes* students in terms of their "level" of understanding/achievement in relation to this norm; (d) *homogenizes* students when it asserts or demands a group norm; and (e) *excludes* students whose behaviors fall below a minimum threshold and are therefore deemed "abnormal."

Drawing on data from a year-long ethnography of a diverse high-school classroom, this poster presents detailed examples of cognitive normalizing at work in a particular math lesson.

Ongoing Research. My methodology involves using the tools of the learning sciences such as microgenetic analysis of video data (Schoenfeld, Smith, & Arcavi, 1993) with an explicit focus on the classroom availability and individual appropriation of semiotic artifacts (Abrahamson, 2009), to examine the effects of normalizing on the quality of students' engagement and conceptual development. The data corpus consists of 60+ hours of classroom footage, fieldnotes, and student interviews gathered during the 2013-2014 school year. I am currently analyzing two aspects of the entire corpus: (a) teacher-facilitated class discussions; and (b) teacher interventions in group tasks. I am identifying and coding categories representing distinct forms of participation and learning surrounding students' encounters with normalizing power across these two settings. Early findings suggest that in this particular classroom community normalization differentially impacts mathematics learning for women, racial minority students, and students who are non-native English speakers. Findings also reveal teaching tactics that potentially minimize students' inter- and intrapersonal mathematical strife.

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USING DIGITAL LEARNING MAPS TO IMPROVE COHERENCE WITH INTERNET MATERIALS

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Keywords: Learning Trajectories; Assessment and Evaluation; Standards

Research suggests that teachers are creating their own curricula using web-based resources in order to address the Common Core State Standards for Mathematics (CCSS-M) (Davis, Choppin, Roth McDuffie, & Drake, 2013.) The challenge is that this blend of resources often lacks curricular coherence, does not meet the needs of a diverse student body and lacks an assessment method that is well aligned to instruction and curriculum content. Moreover, research on teachers' choices of digital materials states that when under time constraints, teachers can be influenced by superficial affordances rather than deep content concerns (Webel, Krupa, & McManus, in press).

Coherence can be improved by providing teachers with a digital learning map, which affords a visualization of the middle grades content, is aligned to the CCSS-M and linked to high quality materials. This is a critical response to the challenge of aligning curriculum and assessment to CCSS-M in a coherent manner. The map comprises of four fields, nine regions (big ideas), 24 related learning clusters (RLCs) and 65 constructs. It introduces the idea of a RLC (Confrey, Maloney, & Corley, 2014), which shows how closely related constructs can be grouped to create a coherent learning experience. For teachers, the map gives a description of the common expected student behaviors at the construct level and is a guide for them to promote active student exchange. Additionally the map links to free web-based resources aligned to the RLCs. The map also aims to promote personalization as described by Confrey (2014) that does not devolve into individualization.

The poster uses data from a recent design study covering middle school statistics. Teaching activities employed were informed by research on the learning trajectory for statistics. Data was collected over a week using lesson observation videos, pre and posttests, student artifacts and interviews. The data collected was used to elicit information about how students interpreted the concepts of natural and measurement variability, how students coped with the idea of uncertainty and their use of data displays to further their understanding of key ideas in statistics. The results describe observable behaviors of students participating in the study and thus confirm the structure at the construct and stack levels of the learning map. The design study has implications for the teaching of statistics at the middle school level.

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Chapter 13

Working Groups

ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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As the title suggests, this Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. Following on the topics discussed at the Working Group in 2009, 2010, 2011, 2012 and 2013, this year we are going to focus on identity in relation to mathematics education, and the ways in which power, agency, and authority play out in classrooms. Session 1 will consist of a panel discussion with scholars around these issues. They will share their perspectives on the state of the field and elaborate next steps in attending to these and other equity issues in the learning of mathematics, particularly for historically marginalized student groups. During sessions 2 and 3 there will be round table discussions to provide attendees with the opportunity to network, plan potential collaborative work, and/or discuss issues raised by the panel.

Keywords: Equity and Diversity; Affect and Beliefs (and Emotion and Attitudes); Teacher Knowledge; Teacher Education-Inservice (Professional Development)

Brief History

This Working Group builds on and extends the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME scholars graduated from one of three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA) that comprised the DiME Center. The Center was dedicated to creating a community of scholars poised to address some critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has already engaged in important scholarly activities, including the publication of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007), a one-day AERA Professional Development session examining equity and diversity issues in mathematics education (2008), a book on research of professional development that attends to both equity and mathematics issues with chapters by many DiME members and other scholars (Foote, 2010), and a book on teaching mathematics for social justice (Wager & Stinson, 2012) that also included contributions from several DiME members. In addition, several DiME members have published manuscripts in a myriad of leading mathematics education journals on equity in mathematics education. This working group provides a space for continued collaboration among DiME members and other colleagues.

We continue DiME's tradition of discussing current work, hearing from leading scholars in the emerging field of equity and diversity in mathematics education, and opening up this space beyond DiME members in this Working Group. Specifically, the Center historically held DiME conferences each summer. These conferences provided a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. Beginning in the summer of 2008, the DiME Conference opened to non-DiME graduate students and new faculty with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as some not affiliated with an NSF CLT. This was initially an attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition, DiME graduates, as they have moved to other universities, have begun to work with scholars and graduate students including those with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education. Funding for the DiME project has ceased and the PMENA Working Group has become a major way in which to keep the conversation going.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil, 2007; Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Eric Gutstein (Gutstein, 2003, 2006; Gutstein & Peterson, 2013), Jacqueline Leonard (Leonard, 2007; Leonard & Martin, 2013), Danny Martin (Martin, 2000, 2009, 2013), Judit Moschkovitch (Moschkovich, 2002), Rochelle Gutierrez (2002, 2003, 2008, 2013a, 2013b) and Na'ilah Nasir (Nasir, 2002, 2011, 2013; Nasir, Hand & Taylor, 2008; Nasir & Shah, 2011). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and Megan Franke (Kazemi& Franke, 2004), as well as many others outside of the field of mathematics education.

We were pleased for the opportunity offered by the first five years of being a Working Group at PMENA 2009 – 2013 to continue working together as well as to expand the group to include other interested scholars with similar research interests. We have learned through experience that collaboration is a critical component to our work. We were encouraged that our efforts were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past five years.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, prospective teacher education (primarily in mathematics methods classes), factors impacting student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics.

There is much existing research that either focuses on professional development in mathematics (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kazemi& Franke, 2004; Koellner, Jacobs & Borko, 2011; Lewis, 2000; Saxe, Gearhart, & Nasir, 2001; Schifter, 1998; Schifter & Fosnot, 1993; Sherin & van Es, 2003), or professional development for equity (e.g., Grant & Sleeter, 2011; Lawrence & Tatum, 1997; Payne & Smith, 2011; Sleeter, 1992, 1997;). Less research exists, however, which examines professional development or mathematics methods courses that integrate both (Aguirre et al, 2013; Battey & Franke, 2013; Turner et al., 2012). The effects of these separate bodies of work, one based on mathematics and one based on equity, limits both the impact that teachers can have in actual classrooms and students' opportunities to learn mathematics. The former can help us uncover the complexities of children's mathematical thinking as well as the ways in which curriculum can support mathematical understanding in a number of domains. The latter has produced a body of literature that has helped to reveal educational inequities as well as demonstrated ways in which inequities in the educational enterprise could be overcome.

To bridge these separate bodies of work, the Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for traditionally marginalized students to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions that the Working Group will continue to consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

Specific to this year's Working Group focus, we will also consider these research questions:

- What is the role of both teachers' and students' academic and mathematics identity in achievement?
- What is the role of teachers' and students' ethnic and cultural identities in mathematics classrooms?
- How do teachers support students' agency in mathematics classrooms?
- How does agency influence students learning of mathematics?
- How do power and status play out in the mathematics classroom?

Plan for Working Group

The overarching goal of the group continues to be to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education. The PMENA Working Group provides an important forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. Some areas we have identified and intend to continue to examine include: incorporating out of school practices, explicitly examining race (including educator identity around race and teaching about race), analyzing broader social structures (teaching math for social justice; the role of privilege), integrating an equity focus into mathematics education reform efforts (equity and the Common Core) and, for this year, understanding identity, power, and agency. Our main goal for this year, then, is to continue a sustained discussion (theoretical and methodological) around these key issues related to research design and analysis in studies attending to issues of equity and diversity in mathematics education. We will do this by bringing together scholars to share their perspectives on the field, and then provide space and time for smaller groups to discuss, reflect on, and amplify ideas from the presentation.

Our plans for PMENA 2015 will proceed as follows.

SESSION 1:

- Review and discussion of goals of Working Group
- Introduction of participants
- Panel discussion on identity, power, and agency

SESSION 2:

- Round table discussions, networking, and collaboration

SESSION 3:

- Continued round table discussions, networking, and collaboration

Previous Work of the Group

The Working Group met for five productive sessions at PMENA 2009, PMENA 2010, PMENA 2011, PMENA 2012 and PMENA 2013. In 2009, we identified areas of interest to the participants within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Over the past five years subgroups met to consider potential collaborative efforts and provide support. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following were topics covered in the subgroups:

- Teacher Education that Frames Mathematics Education as a Social and Political Activity
- Culturally Relevant and Responsive Mathematics Education
- Creating Observation Protocols around Instructional Practices
- Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms
- A Critical Examination of Student Experiences

As part of the work of these subgroups, scholars have been able to develop networks of colleagues with whom they have been able to collaborate on research, manuscripts and conference presentations.

As a result of our growing understanding of the interests of participants (with regard both to the time spent in the working group and to intersections with their research), we began to include focus topics for whole group discussion and consideration and we continued to provide space for people to share their own questions, concerns, and struggles. With respect to the latter, participants have continually expressed their need for a space to talk about these issues with others facing similar dilemmas, often because they do not have colleagues at their institutions doing such work or, worse yet, because they are oppressed or marginalized for the work they are doing. These concerns, in part, informed the focus topics for whole group discussion and consideration. For example, in 2009 research protocols (e.g., protocols for classroom observation, video analysis and interviewing) were shared to foster discussions of possible cross-site collaboration. In 2012, the Working Group explicitly took up marginalization in the field of mathematics education with a discussion about the negotiation of equity language often necessary for getting published; this was done in the context of the 'Where's the mathematics in mathematics education' debate (see Heid, 2010; Martin, Gholson, & Leonard, 2010). Dr. Amy Parks was invited to join Working Group organizers to share reflections on their experiences. Last year we offered our first panel in which scholars (Dr. Beatriz D'Ambrosio, Dr. Corey Drake, Dr. Danny Martin) shared their perspectives on the state of and new directions for mathematics education research with an equity focus. The success of the panel discussion and feedback from attendees led us to plan to continue this structure again this year, using the working group as a sight for discussion and planning of collaborative work as well as reflection on ideas of senior scholars. Given the focus on new directions for mathematics education research, one of the subgroups of the 2013 Working Group focused on equity and the Common Core. A commentary for the *Journal of Research in Mathematics Education* is currently under review (Bartell, et al., 2015).

Anticipated Follow-up Activities

As has happened following previous years of this working group, we anticipate that scholars who make connections at the working group sessions will maintain contact and at least in some cases, this will lead to collaboration on research questions, conference presentations, manuscripts or research projects.

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COLLECTIVE LEARNING: CONCEPTUALIZING THE POSSIBILITIES IN THE MATHEMATICS CLASSROOM

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The purpose of this working group is to open discussion and foster collaboration amongst researchers and educators on the theoretical and methodological concerns as well as practical implications related to collective learning. The two themes for the working group include: In what ways can we conceptualize classrooms as collective learning systems? And, in what ways can we analyze how the collective, as a coherent entity, learns? Through our work we intend to engage in discussion to further the concept of collective learning in relationship to mathematics teaching and learning, and to explore research methodologies that offer potential insight into classrooms as collectives. Shaping the discussion throughout the three sessions of the working group will be leading scholars in this field who share their current thinking as fodder for smaller groups to discuss, debate, and extend.

Keywords: Cognition; Research Methods

History of the Working Group

Over the past two decades, the working group leaders have individually and in subgroups, been theorizing about, as well as collecting, analyzing, and reporting on data relating to collective action in mathematics classrooms (e.g., Davis & Simmt, 2006; Martin, McGarvey & Towers, 2011; Martin & Towers, 2011; McGarvey & Thom, 2010; Proulx, Simmt & Towers, 2009). Recognizing our overlapping interests and similar theoretical framework in complexity science (e.g., Mitchell, 2009; Waldrop, 1992) and enactivism (e.g., Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991), we came together several times in the past five years to establish points of intersection in our individual programs of research. A major impetus for the work is to more fully understand the ways in which mathematics learning occurs in classrooms viewed as collective systems. That is, how mathematics classrooms can be seen as complex systems in which agents spontaneously interact and adapt to each other, organizing and sustaining learning processes in collaborative ways. Through this working group we would like to expand our discussion to interested PME-NA members to open new opportunities for collaboration and avenues of research. In this working group, our general themes are to conceptualize collective learning and address methodological issues when the 'learning body' consists of a group or classroom as a whole.

Focus Issues

Within mathematics education research, there is an extensive body of literature pointing to group dynamics and discourse (see Francisco, 2013). Indeed, the 1990s and the following two decades were ripe with visions of cooperative learning, collaborative inquiry and communities of practice

(Balacheff, 1991; Bauersfeld, 1995; Cobb, Wood, & Yackel, 1993; Lampert & Cobb, 2003; Lampert, Rittenhouse & Crumbaugh, 1998; McCrone, 2005; Staples, 2007; Yackel & Cobb, 1996). This research helped transform mathematics classrooms: rows of desks where students worked in isolated silence were rearranged into groups where students interacted cooperatively together. Further, this research led to an emergence of efforts to understand, describe and define collective learning systems within the field of education generally, and was taken up specifically in mathematics education (e.g., Bowers & Nickerson, 2001; Cobb, 1999; Cobb, Boufi, McClain & Whitenack, 1997; Crawford, 1999; Goos, 2004; Lave, 1997; Rogoff, 1995; Roth, 2001; Roth & Lee, 2002; Saxe, 2002; Sfard & Kieran, 2001; Stahl, 2006). From this work came a growing realization that acts of cognition often arose from the classroom as a whole and could not be traced back to any one individual. While we do not discount the value of research that explores individual understanding, we recognize that teacher actions and decision making within classroom contexts are often not based on the multitude of individual actions, but on the teacher's sense of the class as a whole of which they are a part (Burton, 1999; Towers, Martin & Heater, 2013).

In order to have a shared framework for discussing collective learning systems, we situate this work within the view that classrooms are complex systems. Complex systems are a particular class of phenomena that include, for example, weather systems, world economies, human interaction, nervous systems and many others. Each complex system arises from the inextricable layering and entanglement of biological, social, societal and environmental sub-systems (Davis & Simmt, 2003; Davis & Sumara, 2006). Events within the systems may be unpredictable in foresight, but are potentially understandable in hindsight. Further, complex systems present possibilities that neither arise from nor are representative of individual agents (Davis & Simmt, 2003). These collective possibilities are established through a self-initiating, self-organizing and self-sustaining process. Another key aspect of complex systems is the dialectical entanglement of the system and its environment. Drawing on Maturana and Varela's (1992) view of an entity's cognition, the system both shapes and is shaped by its surroundings. In this sense, *a collective learning system can be defined as a complex system in which agents spontaneously interact and adapt to each other, organizing and sustaining the group's dynamical processes in a collaborative and collective way.* This description is how we view mathematics classrooms as collective learning systems and it provides the theoretical orientation to situate our discussion on conceptualizing collective learning. In this working group we offer some of our work on complex systems as starting points for discussion (described below), but we make space for alternative perspectives as well as extensions of the work. The first theme we address is: *In what ways can we conceptualize classrooms as collective learning systems?*

We recognize that even if we agree on the importance of viewing classrooms as collective learning systems, researchers describing and analyzing group interaction often report findings based on individual achievement results. Even when group work is considered as a whole, a lack of methodological tools leads to analyses that offer connected, but still individual actions, utterances and understandings. Hence, a second theme of our working group is to address the methodological considerations for viewing groups and classrooms as collectives. *In what ways can we analyze how the collective, as a coherent entity, learns?*

Research methodology for collective systems outside of school contexts is a burgeoning scientific field of study. To begin discussions, we offer studies that provide insight into the adaptive learning and self-organization of complex and collective systems such as insect colonies (e.g., Gordon, 2011; Johnson, 2002), underground webs of fungi and roots in forests (Simard, Martin, Vyse & Larson, 2013), small-world communication networks (Watts & Strogatz, 1998), workplace learning (Garavan & McCarthy, 2008), social movements (Kilgore, 1999), and stock market fluctuations (Ralph, n.d.). These studies and many others, make use of computer models of complex phenomena to create visual expressions and explanations of large volumes of data.

Within mathematics education some approaches to analyzing collective learning have been offered:

- preparing transcripts where all individual identifiers have been removed and the transcripts are analyzed as though there is only one voice (Glanfield, Martin, Murphy & Towers, 2009; Martin, Towers & Pirie, 2006);
- tracing the development of a mathematical idea through a group by attending not to individuals, but to collective utterances and discussion, drawings, working papers and whiteboard markings (Davis & Simmt, 2003; Gordon Calvert, 2001; Namukasa & Simmt, 2003; Proulx, 2003; Simmt, 2011; Thom & Roth, 2009);
- modelling classroom interactions using a dynamic visual network by electronically streaming the observations of conversations turns (Bender-deMoll& McFarland, 2006); and
- analyzing video to track the patterns of physical movement within a class and sound by measuring the overall intensity of amplitude over the period of the lesson.

While these techniques show promise, we seek new and different avenues to explore.

Through this working group we hope to extend the initial forays into conceptualizing, collecting, analyzing and interpreting data from collective learning systems. We seek to enlarge researchers' and educators' understandings of what counts as a 'learner' to include a group as a learning system. We also examine the potential for observational and assessment tools where a group stands as the unit of analysis.

This work has the potential to transform research and teaching practices that address the collective, rather than individualistic learning in schools, and challenge institutional norms that privilege a focus on the individual.

Plan for Engagement

The purpose of this working group is to open discussion and foster collaboration amongst researchers and educators on the theoretical and methodological concerns as well as practical implications related to collective learning. The two themes for the working group include:

In what ways can we conceptualize classrooms as collective learning systems?

In what ways can we analyze how the collective, as a coherent entity, learns?

Through our work we intend to engage in discussion to further the concept of collective learning in relationship to mathematics teaching and learning, and to explore research methodologies that offer potential insight into classrooms as collectives. Shaping the discussion throughout the three sessions of the working group will be leading scholars in this field who will share their current thinking as fodder for smaller groups to discuss, debate, and extend.

Session 1. Theoretical considerations for collective learning

Presentation A. Collective behaviours. What does a collective learning system mean? What might collective learning be? One of the ways to address these questions might be through studying/understanding similar collective phenomena exhibited by insect colonies, vertebrate schools, flocks or plants. Despite the differences, research shows that these systems appear to exhibit similar collective behaviours suggesting the possibility of common underlying mechanisms or patterns. The potential of common mechanisms underlying insect, animal, plant and human collective phenomena might help to conceptualize behaviour along with methodological tools for observing collective learning in mathematics classrooms. For example, studies on ants' foraging behaviour show that ants exhibit collective behaviours by obeying very simple rules: The ants pick up grains at a constant rate, approximately two grains per minute; they prefer to drop them near other grains, forming a pillar; and they tend to choose grains previously handled by other ants, probably because of marking by a chemical pheromone. This example is one of many that will be shared to help conceptualize collective learning in a mathematics classroom.

Focusing questions for small group discussion:

- What interactions might we observe in a mathematics classroom that characterize collective learning phenomenon (e.g., decentralized control)?
- In what ways can foraging, flocking or other simple rules be observed in mathematics classrooms?
- What aspects of interaction signify collective behaviour: students' mathematics productions, mathematical ideas, emergent use of words or expressions, and/or other markers?

Presentation B. Ideas as species. It is not uncommon to encounter suggestions that mathematics "is a living, breathing, changing organism …" (Burger & Starbird, 2005, p. xi) or that it "emerges as an autopoietic [i.e., self-creating and self-maintaining] system" (Sfard, 2008, p. 129). More pointedly, Foote (2007) has argued that mathematics is an adaptive, complex system that is approaching the limits of human verifiability.

Picking up on such notions, this presentation develops the suggestion that *ideas* are *species* – and, more fundamentally, that *thinking* and *evolution* are self-similar processes. It begins with the observation that the words ***idea*** (> Greek *idein*, "to see") and ***species*** (> Latin *specere* "to see") are, in a sense, synonymous, and offers a brief historical trace of how the notions were historically connected.

From there the discussion draws on the following literatures:

- memetics (e.g., Dawkins, 1989; Blackmore, 1999)
- complexity science (e.g., Mitchell, 2009)
- embodiment/biological cognition (e.g., Maturana & Varela, 1979; Varela, Thompson & Rosch, 1991)

This literature serves to support the development of the dynamical similarities of cognition on the level of the knower, knowledge production on the level of society, and evolution on the level of a species. These elements are knitted together in the assertion that mathematical concepts are complex coherences that arise in but transcend the co-entangled activities of a collective of mathematical knowers – that is, an account of the emergence and evolution of mathematical ideas/species is offered.

Focusing questions for small group discussion:

- What mathematics ideas/species might we observe in a school mathematics classroom?
- What are the conditions for a collective learning system from which mathematics transcend?
- What implications arise when mathematical ideas are theorized as species?
- What new possibilities, theoretical and pedagogical, are occasioned when mathematical ideas are conceived as species?

Session 2. Classrooms as Collectives

Presentation C. Mathematical meme ecosystem. In this discussion we bring together *Collective Behaviours* from Presentation A and *Ideas as Species* from Presentation B to bridge our understanding of and consideration for the potential utility for the construct of a mathematical meme ecosystem (MME) in the classroom. Memes as introduced by Dawkins (1989) are ideas (and behaviours) that persist in a social group as individual agents mimic the ideas (and behaviours) of others. We acknowledge the individual and social literature on understanding, and we want to go beyond to focus on the memes, the threads and traces of ideas.

The collective behaviours previously discussed, such as the rules for foraging ants and flocking, allow us now to consider the ways in which teachers are implicated in the collective system as they contribute ideas/species and intentional memes. Moving to the level of an ideational system that rides above the agents in the system and in order to develop the idea of a meme ecosystem, we explore how ideas are picked up and have currency in a classroom setting.

Focusing questions for small group discussion:

- What are some of the memes we can identify in a mathematics classroom?
- How do we observe the emergence of the memes in a mathematic classroom?
- What are the dynamics of the movement and persistence of memes?
- What insights into student learning are gained through conceptualizing mathematics classrooms as mathematical meme ecosystems?
- In what practical ways do memes offer a different sensibility for what it means to teach and learn mathematics in the classroom?

Presentation D. Improvisational coaction. In this presentation we begin by offering the beginnings of a theoretical framework focusing on the collective mathematical activity of learners collaborating in small groups (Martin, Towers, & Pirie, 2006; Martin & Towers, 2009). Influenced by the literature that focuses on improvisational action in the fields of jazz and theatre (e.g., Sawyer, 2003), we demonstrated that it was possible to observe, explain, and account for acts of mathematical understanding that could not simply be located in the minds or actions of any one individual, but instead emerged from the interplay of the ideas of individuals, as these became woven together in shared action. This early theorizing prompted us to develop the theoretical construct of "improvisational coaction" in mathematics learning (Martin & Towers, 2009). Improvisational coaction is a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built upon, developed, reworked, and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual. We have applied these ideas to a breadth of educational contexts from elementary school to workplace learning and in small group and whole-class settings (e.g., Martin & Towers, 2011, 2012; Towers & Martin, 2009, 2014) and most recently have turned our attention to developing methodological advances for studying such mathematical collectivity (Towers & Martin, 2015). In this presentation, we position the structures of improvisational coaction as one potential organizing mechanism that might underlie a mathematical meme ecosystem.

Focusing questions for small group discussion:

- Can improvisational coactions provide a conceptual structure for understanding the organizing mechanisms underlying mathematical meme ecosystems?
- What insights into mathematical meme ecosystems might be gained by analyzing students' improvisational coactions?
- What are some of the indicators/markers of coaction?
- What are the dynamics of mathematical meaning making of a group of agents and how can we observe them?

Session 3. Methodological considerations for studying collective learning in mathematics classrooms

Presentation E. Observing processes of collective learning. In the past two decades the use of group work in schools has become pervasive. Justification for this move range from "that's how people learn" to "that is how people get on in the *real* world." Yet, when it comes to assessment (accountability) the emphasis is always placed on the individual. What does the individual know (stripped of any opportunity to know with others in a group)? We are troubled by educational systems that emphasize group work and students' ability to work together on shared projects, but continue to point back to the individual in accountability and assessment. In spite of decades of research done with pairs, small groups and class-sized groups of students/learners, we as educational researchers have made little progress on understanding *learning* as anything beyond what an individual does. Regardless of our beliefs about the purpose of education, we (collectively) live with the unexamined assumption that at the end of the day we must assess what the individual learned from his or her time in school. As educational researchers we are aware that we have not examined/researched observational practices that could focus on the group as a learning system; hence, our (collective) underlying assumptions face few challenges.

Gharajedaghi (2011) writing about systems in a business context states:

> The obstructions that prevent a system from facing its current reality are self-imposed. Hidden and out of reach, they reside at the core of our perceptions and find expression in mental models, assumptions and images. These obstructions essentially set us up, shape our world and chart our future. They are responsible for preserving the system as it is and frustrate its efforts to become what it can be. (p.159)

With the growing understanding of complexity thinking and the awareness that it offers us a way to consider learning systems at many levels of organization, we are at a point where the methodological obstructions we have faced may be excavated and examined. It is important to study the methods by which we can observe the group as a learning system. From there we can propose methods by which teachers can assess the group's, as well as the individual's learning.

In this work we reflect on identifying some of the processes involved in the collective learning and ask how they might be observed. Initial processes are likely related to the presence of feedback loops and binding processes, the emergence of ideas and practices, the collection/selection of new information, as well as the transformation of the contributions of individuals. So, how do we observe these phenomena when we do not and likely cannot know exactly what they are or what they look like?

Focus questions for small group discussion:

- How might thinking about ecosystems and the means by which these are studied enable deeper examination and critique of methods used in mathematics education to observe collective learning?
- In what ways does such examination and critique present the possibility for the generation of radically different methods of observations to study collective mathematics learning in the classroom?
- What are some of the obstructions that have prevented educators from recognizing the paradox of students learning with a group and being assessed individually?
- What methodological tools do we have or might we imagine for assessing the learning and understanding of a group?
- How might we imagine, develop and study alternative assessments, attending to group learning and knowledge creation?

Final Forum. In this final forum of open discussion we return to each of the previous presentations and insights arising from the small/large group discussions to consider moving the theoretical and methodological issues of collective learning forward. We also return to the two themes posed: *In what ways can we conceptualize classrooms as collective learning systems?* And, *In what ways can we analyze how the collective, as a coherent entity, learns?*

We offer a cycle of inquiry to help frame our discussion that allows us to consider existing/new tools for analysis as well as existing/new data collection (see Figure 1).

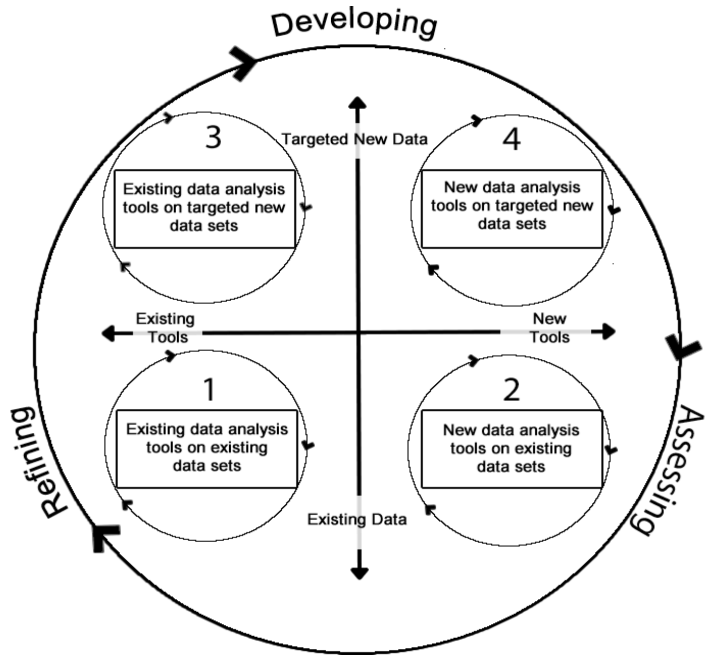

Figure 5: Cycle of inquiry for developing, assessing and refining data collection techniques and data analysis tools.

Anticipated Follow-Up/Future Work

We anticipate that this initial working group discussion at a PME-NA will bring together interested researchers who have wondered about collective learning within a mathematics classroom, what it might look like, and how it might be observed and described. Our hope is to extend this work from the conceptual and methodological to the pedagogical. That is, how might we use insights gained through research to consider how teachers might 'notice' or 'observe' collective learning in their practice.

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CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. The inconsistencies in definitions and use of the terms argumentation, justification, and proof highlight the need for scholarly conversations addressing these (and other related) constructs. Collaboration is needed to move toward, not one-size-fits-all definitions, but rather a framework that highlights connections among them and exploits ways in which they may be used in tandem to address overarching research questions. Working group leaders aim to facilitate discussions and collaborations among researchers and to advance our collective understanding of argumentation, justification and proof, particularly the relationships among these important mathematical constructs. Working group sessions will provide opportunities to engage with a panel of researchers and other participants who approach these aspects of reasoning from different perspectives, as well as to: hear findings from a recent analysis of these constructs in research; reflect on one's own work and position it with respect to the field; and contribute to moving the field forward in this area.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking

Brief History of the Working Group

This is a new working group intended to advance the field's collective understanding of the interrelated objects and processes of argumentation, justification, and proof. We reviewed the prior 20 years of PME-NA proceedings (1995 – 2014) and prior 10 years of PME proceedings (2005 – 2014) to determine whether any previous working groups have focused on these topics. We found one related working group and two related discussion groups; however no group focused on the connections among these three constructs.

A working group on "Learning and Teaching with Proof" was facilitated by Stylianou and Blanton (2004) at PME-NA 26. That working group focused specifically on the development of proof across K-16, whereas the proposed working group focuses on the field's understanding and study of not only proof, but also argumentation and justification, as well as the interrelationships among them. A discussion group on argumentation in mathematics education convened at PME 30 that focused partly on defining and discussing the role of argumentation in mathematics education and research (Schwarz & Boero, 2006). While informative to the current efforts, the focus of that discussion group was distinct from the present working group. Most recently, a discussion group at PME 33 focused on the value of "generic proofs," a particular type of proof presentation (Leron & Zaslavsky, 2009), a topic which is much more specific than this proposed working group. Thus, we classify the present working group as new, while recognizing the contributions of earlier efforts in prior PME-NA and PME meetings. Additionally, none of the leaders of the proposed working group have participated in the aforementioned working groups, providing a further distinction between prior work and the present group. The leaders of this working group are researchers working in different areas of argumentation, justification and proof. Recent collaborations and conversations have led us to consider a need for the field at-large to converse about these interrelated objects and processes, which subsequently led to this newly proposed working group.

Focal Issues

There is a large and growing body of research in mathematics education focused on argumentation, justification, and proof. The research on proof, for example, includes studies on: the role of proof in the discipline; proof in school mathematics and at the undergraduate level; what counts as a proof; proof schemes and categories; teachers' conceptions of proof; students' abilities to write valid proofs; and what teaching proof looks like in classrooms at various levels (e.g., Boero, 2007; Harel & Sowder, 2007; Reid & Knipping, 2010; Stylianou, Blanton, & Knuth, 2009). At the same time, researchers and policy documents have issued calls to engage school children with disciplinary practices such as constructing viable arguments, justifying conclusions, critiquing the reasoning of others, and constructing proof for mathematical assertions (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

Yet, as the field moves forward to strive for maximizing students' learning opportunities for engaging in these disciplinary practices, mathematics educators need to refine their notions of these terms in scholarly activities and in policy documents (Cai & Cirillo, 2014). How, when, and why decisions related to word choices are made (e.g., 'argument' versus 'proof') in curriculum materials, policy documents, and research is an open question. In fact, some researchers have hinted that these choices are not always purposeful. For example, Lynn Steen, a member of the 1989 NCTM Standards Committee, claimed that uncertainty about the role of proof in school mathematics caused NCTM in its first *Standards* (1989) document to resort to, what he called, "euphemisms" such as "'justify,' 'validate,' 'test conjectures,' [and] 'follow logical arguments'" (Steen, 1999, p. 274). Rarely, he stated, did the document use the term 'proof.' Although Steen's comments were published over 15 years ago, we would argue that his proposition, that the role of proof (as well as argumentation and justification) in school mathematics is uncertain, continues to be true today.

Building on NCTM's (2000) document, which recommended that students' experiences with reasoning and proof include making and testing conjectures, judging the validity of arguments, and constructing proofs, the *Common Core State Standards for Mathematics* (*CCSSM*; NGA & CCSSO, 2010) continued this emphasis. Although proof is no longer included as an explicit standard, the authors added attention to argumentation through the third Standard for Mathematical Practice. Proof and justification (or proving, justifying, etc.) are also included, with proof appearing most often in the high school standards. As the field continues to grapple with the meanings and interconnections of argumentation, justification, and proof, the usage of these varying terms could point to potential challenges of implementing and studying this aspect of CCSSM.

One challenge of reading extant research or developing a research agenda related to these disciplinary practices is that the classifications offered differ according to the perspective of the researcher, the focus of the research, and the particular data being analyzed (Reid & Knipping, 2010). Only recently have we begun to see mathematics educators offering explicit definitions of these constructs in their work. This is ironic given the importance of definitions in the field of mathematics itself. A review of the literature reveals much more specific attention to proof than to justification and argumentation in mathematics. Table 1 provides some definitions of the three constructs.

When considering the definitions provided in Table 1, one might notice various things. For example, two of the authors describe proof as an argument. This is interesting given that Cabassut and colleagues (2012) claimed that opposing views exist in the field: On the one hand, it has become customary in mathematics education to use the term 'argumentation' for reasoning which is not yet a

Table 1. Definitions of Proof/Proving, Argumentation, and Justification.

Proof/Proving	Argumentation	Justification/Justify
"the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241)	"mathematical explanation intended to convince oneself or others about the truth of a mathematical idea" (Mueller, Yankelewitz, & Maher, 2012, p. 376)	"to provide sufficient reason for" (National Research Council, 2001, p. 130)
"arguments consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 55)	"discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (Wood, 1999, p. 172)	"an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (Staples, Bartlo, & Thanheiser, 2012, p. 448)
"a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: (1) It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; (2) It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and (3) It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community." (Stylianides, 2007, p. 291)	"the process of making an argument, that is, drawing conclusions based on a chain of reasoning" (Umland & Sriraman, 2014, p. 44)	

proof, on the other hand, other researchers, such as Balacheff (1988) and Duval (2007), believe that argumentation and proof are fundamentally different.

Reid and Knipping argued that while ignoring multiple uses of 'proof' can lead to communicative challenges, the answer is not to insist on one "correct" usage (Reid & Knipping, 2010; Herbst & Balacheff, 2009). In particular, Herbst and Balacheff (2009) stated:

> If the field is in a deadlock as regards to what we mean by "proof," we contend this is so partly because of the insistence on a comprehensive notion of proof that can serve as referent for every use of the word…. We have argued that to make it operational for understanding and appraising the mathematics of classrooms we need at least three meanings of the word. (p. 62)

Awareness of the different uses of proof is an important step in deciphering and making progress in mathematics education research (Reid & Knipping, 2010). In fact, Balacheff (2002) claimed that "research speaks in a very confusing way about the topic" (p. 39).

Conflicting Constructs, Methodologies, and Findings from the Research Literature

In a forthcoming review of the literature, Stylianides, Stylianides, and Weber (forthcoming) asserted that the field has made substantial progress in tackling these issues. Researchers have developed powerful conceptual constructs and methodologies, and our collective understanding of how students perceive proof and how justification, argumentation, and proof are taught in K-12 and university classrooms has very much improved. Nonetheless, there are several places where different researchers have generated conclusions that conflict with one another. Here, we present a few such contradictions.

First, in the debate on how proof should be introduced to students, Marty (1991) proposed teaching students the rules of logic and standard proof techniques while deliberately keeping the content shallow, suggesting that the notion of proof as a justification to prove mathematically meaningful statements should come later. Alibert and Thomas (1991) suggested the opposite; students should initially be engaged in argumentation about the veracity of interesting mathematical statements. In socially negotiating what counts as an acceptable argument, the instructor can lead students to produce the standard norms of proof. Both researchers presented research results that supported their point of view.

Second, as mentioned above, there is a debate as to whether argumentation and proof are activities that are deeply intertwined or fundamentally separate. For instance, numerous authors have proposed that students will be more successful at proof writing if they base their proofs on informal arguments, often involving the inspection of examples and graphical reasoning (e.g., Garuti, Boero, & Lemut, 1998; Raman, 2003; Sandefur, Mason, Stylianides & Watson, 2013; Weber & Alcock, 2004). However, Duval (2007) contended that argumentation and proof occur in different semiotic registers. Due to the different goals and demands of each activity, they must occur separately.

Third, several researchers have presented classroom studies that demonstrate that even young children are capable of writing proofs (e.g., Maher & Martino, 1996). Yet there are numerous studies that suggest that high school geometry students and even advanced mathematics majors struggle to write relatively simple proofs (e.g., Moore, 1994; Senk, 1989). Is it possible that young children have greater competency at proving than most geometry students or mathematics majors? Or are their proofs being evaluated according to different standards?

Fourth, a number of researchers have defined proof to be a convincing justification (cf., Balacheff, 2008). Yet there are other mathematics educators who emphasize that some proofs do not always provide full justification to mathematicians, and mathematicians sometimes are convinced by justifications that they would not call proofs (e.g., de Villiers, 1990; Tall, 1989; Rodd, 2002; Weber, Inglis, & Mejía-Ramos, 2014).

What we contend is that these, and other, inconsistent claims are based on researchers holding different conceptions of argumentation, justification, and proof. What we hope to accomplish in this working group is to develop a better understanding of these differences, an appreciation for what different perspectives of proof can accomplish, and a framework that describes how different perspectives can complement each other, rather than oppose each other and lead to inconsistent findings.

Frameworks for conducting research: The case of proof

In the forthcoming review, Stylianides et al. (forthcoming) described three broad research traditions with respect to proof. One theoretical frame is to view *proving as problem solving*. In these studies, little emphasis is given to the issues of what constitutes a proof, why students engage in this activity, and how they interpret the proofs that they produce. Issues of argumentation and justification are typically ignored. Instead, these studies focus on what competencies are needed to successfully write a proof and design instruction that helps students develop these competencies (e.g., Selden & Selden, 2013; Weber, 2001).

Another school of thought views *proving as convincing* (e.g., Harel & Sowder, 1998). Researchers in this perspective seek to determine what types of justifications students find convincing and attempt to develop instruction that leads students to transform their standards of conviction to those held by mathematicians (e.g., Harel, 2002; Recio & Godino, 2001). Careful attention is paid between arguments that are complete convincing justifications and those that merely increase one's belief in the likelihood of a statement.

A third framework treats *proof as socially embedded activity*. In this perspective, researchers focus on what it is students, from their perspective, are *actually* doing when they engage in proving (e.g., Herbst & Brach, 2006) and how students' and teachers' activities are shaped by social and institutional constraints (e.g., Herbst & Chazan, 2003). Researchers often focus on the cognitive, social, and pedagogical goals that proof can and should play in the classroom beyond being a skill to acquire or a means of conviction (Staples, Bartlo, & Thanheiser, 2012).

In the review, Stylianides and colleagues found that each perspective (a) used different conceptions of argumentation, justification, and/or proof, (b) addressed different questions, (b) developed different theoretical constructs to understand and investigate these questions, and (d) measured the success of instructional interventions in different ways. They also found that considerable progress has been made within each of these perspectives that has enhanced the field's understanding of proof and related constructs. As one can see, many varying conceptions of argumentation, justification, and proof exist in our field. It would be an unrealistic and inappropriate goal of the working group to try to reach a consensus on what argumentation, justification, and proof are. Indeed, in mathematical practice, mathematicians adopt different standards and perspectives on these objects and corresponding processes depending upon their aims and context (Weber, 2014). Rather what we seek to explore is the different ways that researchers conceptualize these constructs (including, but not necessarily limited to, the perspectives above), the consequences of such conceptualizations, and how they might work in tandem to address overarching research questions.

Balacheff (2002) put forth a set of recommendations to help address what he called a research "deadlock" (p. 1). Heeding his recommendations, we will consider the following during the working group: (a) looking for a common lexicon to improve available definitions; (b) engaging with different research programs and their possible contrasts and relationships; (c) considering the theoretical commonalities and divergences, and possibly turning them into research questions; (d) discussing different methodologies, their benefits, and possibly limitations; and (e) acknowledging accepted results or turning objections and differences into research problems.

Plan for the Working Group

The overarching goal of the working group is to facilitate discussion and collaboration among researchers in the field doing work in this area – at various stages of their careers – and to advance our collective understanding of argumentation, justification and proof, and relationships among these important mathematical constructs. Aligned with these goals to advance our collective understanding, we anticipate organizing our time together in the following manner.

Session 1: *Where are we now? Where is the field? Where are you?*

In Session 1, we begin with introductions and discerning interests in the working group, and then provide an overview of the goals for group. We then engage participants in two activities in order to establish some common ground for our collective work and to provide a reflective opportunity for participants to position their own thinking and work with respect to the field. The key activities in Session 1 are as follows:

- We will facilitate Introductions, Define the Problem, and outline Working Group Goals.
- Keith Weber, a co-author of the proof chapter in the forthcoming NCTM handbook, will offer an historical view of the use of these constructs in mathematics and educational research. The talk will highlight convergence in the literature as well as contradictions in definitions and research results. The purpose of the talk will not be to call for convergence, but rather to highlight different traditions and where points of disagreement may lie, and to propose ways in which different traditions may inform each other to advance the field collectively.
- All working group participants produce a diagram or concept map with the terms *argumentation, justification,* and *proof* to elicit personal conceptions and uses of these terms and how they are interrelated. Small groups will then compare and discuss these representations.

Session 2: *How have individual researchers used these constructs to support their work? What choices have they made?*

Session 2 features an interactive panel discussion by three invited math educators who focus on one or more notions of argumentation, justification and proof in their research. The panel of experienced researchers will share their expertise through a facilitated format, beginning with set questions and then evolving into a question-and-answer period and group discussion. Potential guiding questions include the following:

- *What process(es) and object(s) (justifying-justification, argumentation-arguments, and proving-proof) are central to your work? Why did you choose these? How have you conceptualized or defined them for your work?*
- *How do you see these conceptualizations mattering for the work you (we) do with teachers, with students, and/or as researchers as members of the mathematics and/or mathematics education community?*
- *From your perspective, where is the field now with its understanding of these processes and objects, how to foster them in classrooms, and how to support teacher learning of pedagogies to organize student participation in these critical processes?*

Three researchers, whose work is prominent in this area, will serve on the panel:

- Kristen Bieda, Michigan State University, has investigated the interaction between opportunities in curriculum to engage in justification and proof and how teachers enact those opportunities with students in middle school classrooms. Bieda is particularly interested in teachers' goals for justification in their classroom and how they modify and deploy curricular tasks to achieve those goals.
- Anna Conner, University of Georgia, studies the role of the teacher in collective argumentation, specifically how teachers learn to support their students in making mathematical arguments. Within this, Conner also studies teachers' beliefs about proof and how this may influence the argumentation in their classes.
- Pablo Mejía-Ramos, Rutgers University, focuses on the reasoning processes involved in the three main argumentative activities related to the notion of proof in university mathematics: constructing a new proof, reading a given proof, and presenting a known proof. Two of the main goals of Mejía-Ramos's research are (1) to better understand the ways in which different types of students and mathematicians engage in these argumentative activities, and (2) to identify effective strategies for performing such activities.

Samuel Otten, University of Missouri, will moderate the panel discussion.

Session 3: ***Where are you now? What are next steps?***

We anticipate pursuing three general goals in Session 3 and recognize that the particulars of this session will be in response to the first two days. The three goals are:

- Gaining clarity in situating oneself and one's own work within the broader field. Toward this end, we revisit the set of individual diagrams generated in Session 1. Participants will have opportunities to revise their diagrams. We anticipate discussing two or three representations that emerge regarding the relationships among argumentation, justification and proof as "prototypical" views. We will discuss their similarities and differences, consequences for the field, and whether they are complementary or contradictory.
- Generating key questions or suggestions for the field to better leverage the work it is doing. First in small group discussion, and then with the full group, we will develop a handful of key questions that the collective identifies as important to furthering individuals' and the field's work with respect to argumentation, justification and proof. Subgroups may organize around these questions.
- Organizing work on a white paper and identifying next steps. We will share an outline for a white paper, and elicit input for the focus of Year 2 work at the next PME-NA conference. Given the group's interest and the key questions they develop, subgroups can contribute specific sections to the white paper. Subgroups may also organize around a focal process. Finally, a subgroup may plan to take up particular key questions or activities during the year.

Anticipated Follow-up Activities

We anticipate two follow-up activities and products for the first iteration of this working group. First, we have inquired with editors of some of the leading mathematics education journals in the field regarding the potential for publishing a summary of a panel discussion (similar to Sfard, Nesher, Streefland, Cobb, & Mason, 1998), or, alternatively, a research commentary. The commentary might provide recommendations for how mathematics education researchers report what they mean by argumentation, justification and/or proof in their work. Such a commentary may also document key questions for the field as well as share core ideas from discussions and differences in perspectives from organizers, panel members, as well as working group participants.

We also anticipate the development of a white paper from participants of the working group. The paper will have multiple sections documenting ideas from our generative work together, as well as solidifying and extending some of those ideas. A white paper format affords flexibility in how the document is organized and how authorship is assigned (e.g., authorship of sub-sections within the larger white paper). While not suggesting a definitive structure *a priori*, one potential format would include perspectives from developing sub-groups within the working group based on participants' diagrams and particular conceptions of argumentation, justification and proof. This activity will allow for further engagement within and beyond this meeting of the working group. The white paper would be developed over the months following PME-NA 37 with the goal of sharing the document widely (e.g., through our institutions' open-access digital repositories). Thus, such products fulfill our purpose of creating a working group at PME-NA: to engage colleagues in discussion of the different understandings and operations of argumentation, justification and proof. While consensus on these points is not anticipated, and not necessarily a goal, efforts towards consensus regarding how mathematics educators convey their conceptions of these objects and processes is.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC)

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Embodied cognition is growing in theoretical importance and as a driving set of design principles for curriculum activities and technology innovations for mathematics education. The central aim of the EMIC (Embodied Mathematical Imagination and Cognition) Working Group is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education .A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities, practices, and emerging technologies that count as mathematical. EMIC builds upon earlier investigations in professional workplaces, formal K-12 math and science classrooms, mathematics centers, museums, marching bands, bike paths and city streets.

Keywords: Classroom Discourse; Cognition; Informal Education; Learning Theory

Motivations for a New Working Group

Recent empirical, theoretical and methodological developments in embodied cognition and gesture studies provide a solid and generative foundation for the establishment of **an Embodied Mathematical Imagination and Cognition (EMIC) Working Group** for PME-NA 2015 and future meetings. The central aim of EMIC is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education, including, but not limited to studies of mathematical reasoning, instruction, the design and use of technological innovations, and learning in and outside of formal educational settings and across the lifespan.

The interplay of multiple perspectives and intellectual trajectories is vital for the flourishing of embodied mathematical cognition. Partial confluences and differences have to be kept throughout the conversations; this is because instead of being oriented towards a single and unified theory of mathematical cognition, we are aiming towards a philosophical/educational "playground" in which entrenched dualisms, such as mind/body, language/materiality, or signifier/signified are subject to an ongoing and stirring criticism. A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities and emerging technologies that count as mathematical and to envision alternative forms of engagement with mathematical ideas and practices (e.g., De Freitas & Sinclair, 2014). This broadening is particularly important at a time when schools and communities in North America face persistent achievement gaps between groups of students from many ethnic backgrounds, geographic regions, and socioeconomic circumstances (Ladson-Billings, 1995; Moses & Cobb, 2001; Rosebery, Warren, Ballenger & Ogonowski, 2005). There is also need to articulate evidence based findings and principles of embodied cognition to the research and development communities that are starting to generate and disseminate programs for promoting mathematics learning through movement (e.g., Petrick Smith, King, & Hoyte, 2014). Generating, evaluating, and curating empirically validated methods for promoting mathematical development through embodied activities that have engaging social bearing and curricular relevance is an urgent societal objective.

A Brief History

The submitters are proposing a new working group to PME-NA. They draw, however, on earlier interest in this topic from the International PME community (ESM, 2009), and several years of collaboration and intellectual exchange exploring the nature of EMIC from multiple methodological perspectives.

As early as 2003, several of the submitters proposed the idea of a Science of Learning Center to NSF organized around the nascent research on embodied cognition for mathematics education, and worked on a funded NSF "catalyst" grant to begin to curate and publicly share the scholarship distributed across multiple disciplines, such as linguistics, psychology, anthropology, philosophy, and neuroscience. This early work led to, among other things, a 2007 AERA symposium on mathematics learning and embodied cognition entitled, "Mathematics Learning and Embodied Cognition." This early experience led to a 6-year NSF-REESE grant, "Tangibility for the Teaching, Learning, and Communicating of Mathematics," starting in 2008, in which collaboration across three campuses (San Diego State University, University of Wisconsin-Madison, and Vanderbilt) instituted a multidisciplinary approach to the empirical investigation of embodied mathematical thinking, drawing equally on ethnomethodology, phenomenology, and experimental psychology. The three campus-based research teams ran a coordinated series of empirical and design studies that focused on learning the mathematics of space, scale, modality, and motion. The investigations took place in professional workplaces such as medical response teams, formal K-12 math and engineering classrooms, mathematics professors' offices, museums, football fields, bike paths and city streets -- places and contexts where people were learning, communicating, and doing mathematics.

Some of the research findings have been reported in a special issue of *The Journal of the Learning Sciences* (2012), "Modalities of Body Engagement in Mathematical Activity and Learning," and an NCTM 2013 research pre-session keynote panel, "Embodied cognition: What it means to know and do mathematics," as well as a series of academic presentations, book chapters, and journal articles, as well as several masters' theses and doctoral dissertations.

Along with this group, numerous other research efforts formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014; Lakoff & Núñez, 2000; Radford 2009). There is good reason at this point in time, to proclaim that there is a "critical mass" of projects, senior and junior investigators, research findings, and conceptual frameworks to support an on-going community of likeminded scholars within the mathematics education research community.

Focal Issues in the Psychology of Mathematics Education

Emerging, yet still influential, views of thinking and learning as embodied experiences have grown from several major intellectual developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multi-modal interaction, and human thinking as multi-modal simulation of sensory-motor activity (Clark, 2008; Hostetter & Alibali, 2008; Lave, 1988; Nathan, 2014; Varela et al., 1992; Wilson, 2002). These views acknowledge the centrality of both unconscious and conscious motor and perceptual processes for influencing conscious awareness, and of embodied experience as following/producing pathways through social and cultural space. As Stevens (2012, p. 346) argues in his introduction to the *JLS* special issue on embodiment of mathematical reasoning, it will be hard to consign the body to the sidelines of mathematical cognition ever again if our goal is to make sense of how people make sense and take action with mathematical ideas, tools, and forms.

Three major ideas mesh with embodied cognition and serve as useful examples of the ways that embodied cognition perspectives are relevant for the study of mathematical understanding: (1) Grounding of abstraction in perceptuo-motor activity. This conception shifts the locus of "thinking" from a central processor to a distributed web of perceptuo-motor activity situated within a physical and social setting. It suggests that to make meaning people ground seemingly abstract concepts in modality-specific, sensory-motor systems, as an alternative to representing concepts as purely amodal, abstract, arbitrary, and self-referential symbol systems. (2) Cognition is for action. This tenet proposes that things, including mathematical symbols and representations, are understood by the actions and practices we can perform with them, and by mentally simulating the actions and practices that underlie or constitute them. (3) Mathematics learning is always affective: there are no purely procedural or "neutral" forms of reasoning to be learned in a manner detached from the circulation of bodily-based feelings and interpretations surrounding our encounters with them.

Alongside these theoretical developments have been technical advances in multi-modal and spatial analysis, which allow scholars to collect new sources of evidence and subject them to powerful analytic procedures, from which they may propose new theories of embodied mathematical cognition and learning. Just as the "linguistic turn" in the social sciences was largely made possible by the innovation that enabled scholars to collect audio recordings of human speech and conversation *in situ*, growth of interest in multi-modal aspects of communication have been enabled by high quality video recording of human activity (e.g., Alibali et al., 2014; Levine & Scollon, 2004), motion capture technology (Hall, Ma, & Nemirovsky, 2014; Sinclair, 2014), and developments in brain imaging (e.g., Barsalou, in press; Gallese & Lakoff, 2005). New and still largely under-explored advances in location-aware technologies and spatial analysis, such as wearable GPS devices that capture personal space-time paths, expand our opportunities to study human activity and learning in natural settings and at an increasing broader set of spatial and temporal scales (Christensen, 2003; Miller, 2007). A growing suite of *geographic information system* (*GIS*) software tools is now available and can be employed to analyze path structures in relation to other spatially anchored data (e.g., the accessibility of cultural resources that provide children with learning opportunities; Ma, Hall, & Leander, 2010).

Plan for Active Engagement of Participants

The formation of this new scholarly group requires some initial structure and the cultivation of distributed leadership. EMIC scholars will meet for 90 minutes each of three days during the PME-NA 2015 conference. We propose commencing the first gathering on Day 1 of the EMIC working group by engaging participants one or more embodied activities for promoting mathematical reasoning (a sample of the activities we propose is presented in Figure 1). Each of the activities has been used in formal and informal learning settings. The implementation and points of discussion and reflection for each activity will be informed by our prior experiences. One activity (Figure 1a) fosters deeper understanding of the Cartesian coordinate system through physically pushing and pulling handles that control coordinate displays by perceptuomotor integration of the physical and the semiotic. This has been used in interactive museum exhibits. A second activity (Figure 1b) engages participants in a projection activity using Alberti's Window. Participants look through the eyehole to draw projected images on the vertical drawing pane. A third activity uses a walking scale geometry activity (Figure 1c) to facilitate geometric reasoning in new ways by placing participants within the inscriptions (e.g., an isosceles triangle) rather than looking over them on paper from a bird's-eye-view, using everyday objects (string), and participating in large scale imagery that is necessarily collaborative. Finally, an activity engages one's body to experience spatial relations of geometric forms through body mechanics in order to influence their proof practices (e.g., for the Triangle Inequality Theorem).

Day 2 of the working group would invite participants to provide brief (5 to 10 mins) synopses of their research interests and goals for the group. The balance of Day 2 and a portion of Day 3 would be devoted to generating and discussing topics of interest. A partial list of seed topics is presented below. The final time on Day 3 would focus on practicalities, including assigning action items and roles for specific participants.

Sample Seed Topics

- Conceptual blending (Tunner & Fauconnier, 1995) & metaphor (Lakoff & Núñez, 2000)
- Development of spatial reasoning (Uttal et al., 2009)
- Gesture & multimodal instruction (Alibali & Nathan 2012; Alibali et al., 2014Cook et al.; Edwards, 2009)
- Gesture in mathematical imagination (Nemirovsky, Kelton, & Rhodehamel, 2012)
- Perceptuo-motor grounding of abstractions (Barsalou, 2008; Glenberg, 1997)
- Mathematical cognition through action. (Abrahamson, 2014; Nathan et al., 2014)
- Modal engagements (Hall & Nemirovsky, 2012; Nathan et al., 2013)
- Perceptual boundedness (Bieda & Nathan, 2009)
- Perceptuomotor integration (Nemirovsky, Kelton, & Rhodehamel, 2013).
- Progressive formalization (Nathan, 2012;Romberg, 2001) & concreteness fading (Fyfe, McNeil, Son, & Goldstone, 2014)
- Sensuous cognition (Radford, 2009)
- Simulation of sensory-motor activity (Hostetter & Alibali, 2008; Nemirovsky & Ferrara, 2009).
- Use of manipulatives (Martin & Schwartz, 2005)
- Bodily activity among professional mathematicians (Nemirovsky & Smith, 2013)
- Ways spaces are be mathematized by architects and urban designers
- Ways bodies are deployed (or suppressed) in the diversity of conceptual practices we call mathematics

Follow-up Activities

We envision an emergent process for the specific follow-up activities based on participant input. At a minimum, we will develop a list of interested participants and grant them all access to a common, closed discussion forum. We also offer these proposed activities as outcomes for extending the intellectual work of the group beyond the PME-NA meeting. The proposed activities include:

- Develop a multicampus course or MOOC
- Compile/curate literature, activities demonstrating principles of embodied cognition for promoting mathematical thinking, learning, and communicating
- Propose a sharable corpus of video data focused on gesture and action during math learning and instruction (e.g., hosted on Databrary.org)
- Exchange resources/interventions for embodied mathematical cognition for students or teachers, potentially conducting cross-institutional research studies on their use.

- Propose and edit a journal special issue.
- Sponsor the participation of colleagues from outside the community of research in mathematics education (e.g. philosophers, technologists)

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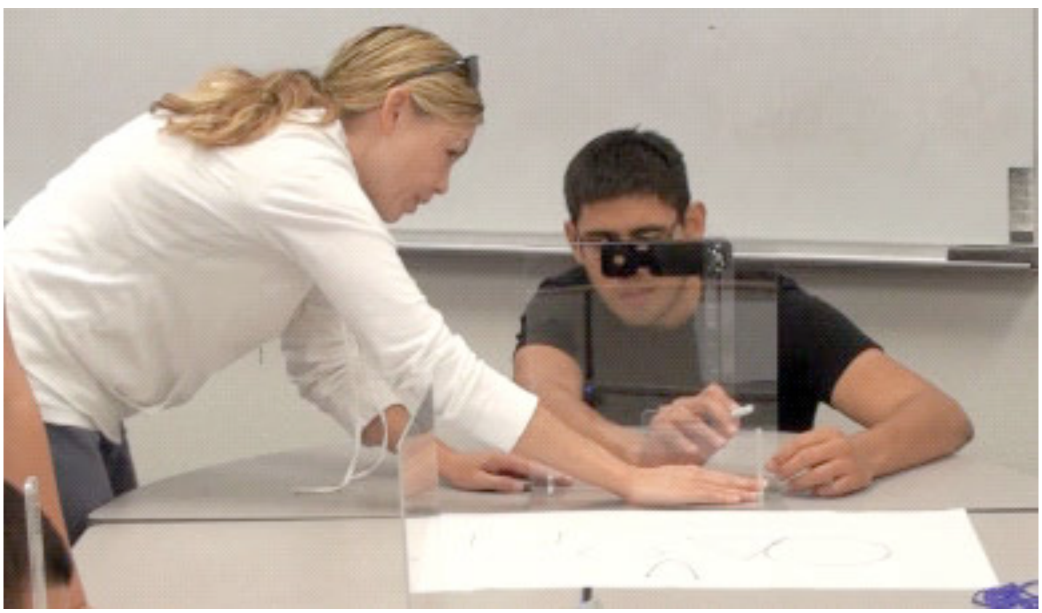

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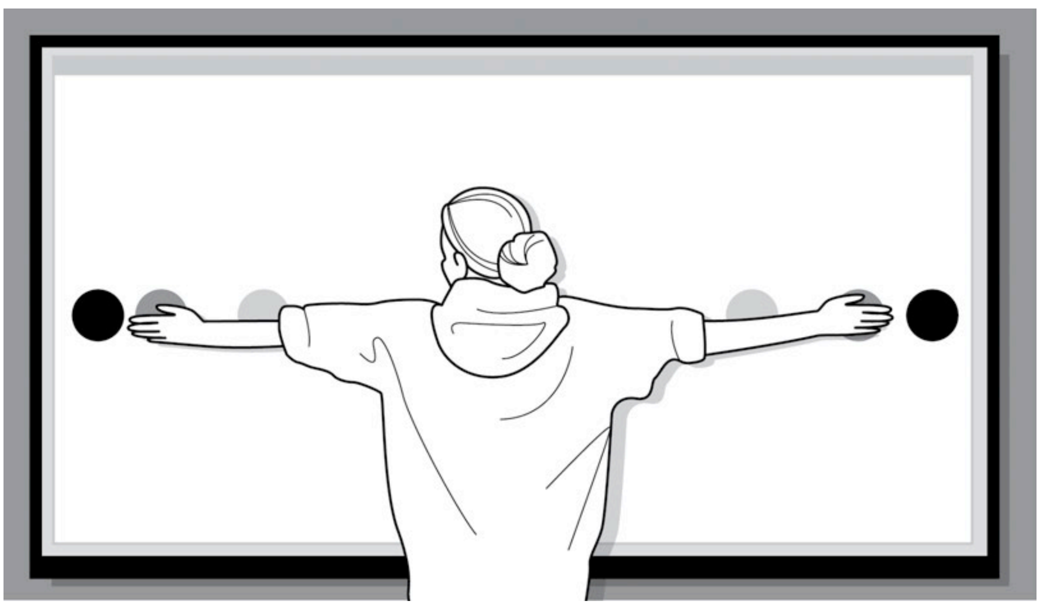

Figure 1. Sample EMIC activities (developed and used by the organizers) to experience embodied mathematics in action. (a) Pushing/pulling handles and mapping images in the Cartesian coordinate system at a museum exhibit. (b) Alberti's Window. (c) Constructing an isosceles triangle at walking scale. (d) Actions to foster geometry proof practices.

EXAMINING SECONDARY MATHEMATICS TEACHERS' MATHEMATICAL MODELING CONTENT KNOWLEDGE

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This is the initial formation of a working group with the purpose of addressing research questions that surfaced during the 38th International Group for the Psychology of Mathematics Education conference in Vancouver, Canada concerning the need to increase the mathematics education community's understanding of secondary teachers' knowledge of mathematical modeling content and pedagogy. In an attempt to develop this knowledge base, this working group will be driven by the following research question: What kinds of knowledge are required for secondary mathematics teachers to facilitate effective and meaningful mathematical modeling instruction? While this question can be dichotomized into subject matter and pedagogical knowledge, the working group will focus explicitly on identifying, distinguishing, and developing secondary teachers' content knowledge. Participants of working group sessions will propose a framework for identifying 1) mathematical modeling knowledge, 2) mathematical content knowledge required/important for modeling 3) modeling knowledge for implementation of mathematical modeling instruction in the secondary mathematics classroom. Efforts will be continued through digital meetings and discussion forums. Findings of the working group and subsequent discussions will be summarized and reported.

Keywords: Modeling; Mathematical Knowledge for Teaching; Teacher Knowledge; Teacher education-Inservice

The Common Core State Standards for Mathematics (CCSSM) regards mathematical modeling as both a content standard and a Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Mathematical modeling describes a problem-solving process associated with ill-posed, less structured tasks often set in real-world contexts (Lesh, Galbraith, Haines, & Hurford, 2013). The modeling process requires the modeler to interpret the task's situation, make assumptions, identify variables, build a mathematical model, manipulate and solve the model, and interpret the solution in the problem's context (Jensen, 2007). Modeling problems have multiple solution paths which are dependent upon the modeler's experiences (Confrey& Maloney, 2007). Teachers' limited experiences with mathematical modeling as learners of mathematics make implementation of this important standard and practice problematic (Goos, 2014).

Mathematics teacher educators are challenged with preparing teachers to understand the intricacies of mathematical modeling and to define effective strategies for classroom implementation. This challenge is greater at the secondary level because past research has focused on implementing mathematical modeling in elementary education (English & Sriraman, 2010) or on undergraduate modeling courses with a pure mathematics lens (Lesh et al., 2013). In addition, literature addressing secondary teachers' modeling cognition or ways to increase their pedagogical knowledge regarding modeling is scarce. The proposed working group will be devoted to describing the mathematical modeling content knowledge secondary mathematics teachers need to facilitate rich opportunities for students.

Formation of the Mathematical Modeling Working Group at PME-NA

We wish to assemble a working group comprised of mathematicians, mathematics educators, mathematics teacher educators, researchers, and psychologists. We encourage participation from scholars and graduate students who are interested in exploring the content knowledge secondary mathematics teachers need to enhance their own mathematical modeling knowledge and how this knowledge is used to develop, implement, and facilitate mathematical modeling as a standard and a practice in the classroom. We propose that members of the working group will divide into smaller groups to provide an intellectual support system, offer critique, and discuss scholarship focused around common research aims. The working group will be committed to the goals of PME-NA with an emphasis on collective reflection, collaborative inquiry, and addressing challenges.

To advance the question driving the working group "*what types of knowledge do secondary teachers need in order to implement mathematical modeling in their secondary classrooms?",* three research areas need to be reviewed: 1) definitions of teachers' knowledge, mathematical modeling, and types of mathematical modeling tasks form the foundation, 2) types of knowledge required for working on mathematical modeling tasks indicate mathematical understanding, and 3) levels of sophistication and mathematical modeling competencies suggest a path for noting progression.

Mathematical knowledge for teaching

Identifying what mathematics teachers need to know to be effective in the classroom has been the source of much debate. Shulman (1986)'s forms of teacher knowledge has provided a comprehensive consideration of the factors that influence teaching performance. Following the work of Shulman (1986), other researchers had examined the nature of teacher knowledge (Chazan& Ball, 1999; Ball, 2000). Ball, Thames, and Phelps (2008) outlined a domain-specific understanding of mathematical knowledge for teaching (MKT) distinguishing teachers' mathematical content knowledge from their pedagogical knowledge. Teachers' *knowledge of content* (CK) has three subcategories of interest: *common content knowledge* (CCK) and *specialized content knowledge (SCK)*, and *horizon content knowledge* (HCK). CCK refers to the mathematic knowledge that educated adults use in their daily lives. SCK refers to the mathematical knowledge teachers employ "in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods for problems" (Hill, Ball, & Schilling, 2008, pp.377-378). HCK refers to teachers' understanding of how mathematics is connected both internally and externally to other subjects, disciplines, and academic grade levels; and it the use of mathematics in everyday life and in the world of work.

What is mathematical modeling?

Mathematical models are mathematical constructs, often equations, intended to explain or predict real-world events or systems going beyond superficial physical attributes to explore structural features, e.g. patterns and relationships (English, 2007;Dossey et al., 2001). Applying mathematics to real world systems is a complex modeling process "of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation" (Confrey& Maloney, 2007, p. 60).

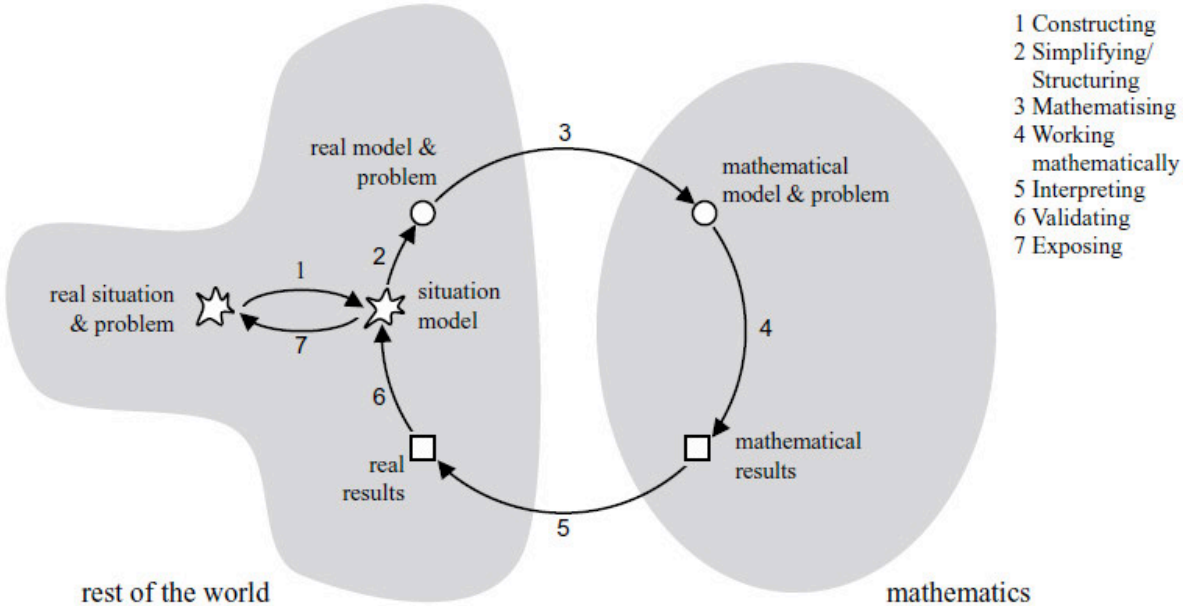

Figure 1: A visual representation of the mathematical modeling cycle (Blum, 2011)

There are many representations of the modeling process leading to the mathematical modeling product needed to solve problems in everyday life. Blum's (2002) description of the modeling process focuses on stages, as shown in Figure 1. In the first stage, the solver interprets the problem's context and *Constructs* a situation model, demonstrating understanding of the problem statement. Variables, assumptions, and relationships are detailed in the second stage, *Simplifying and Structuring,* moving the work from the real world to the mathematical world and leading to the third step, *Mathematization,* where the mathematical model is formed. Mathematizing demands that modelers select relevant information and translate it into an appropriate mathematical statement making it the most difficult step (Voskoglou, 1995). In stage 4, *Working Mathematically,* technical skills are needed to perform calculations and produce mathematical results to be *Interpreted* in the real world context in step 5. *Validating* the model, in stage 6, identifies model limitations, which may suggest the need for revisions and restarts the modeling cycle. When the modeler accepts the solution as reasonable, the last step publishes the results, *Exposing* the findings for others to consider.

Types of mathematical models

Mathematical models, as well as being a powerful tool in the field of applied mathematics, can be categorized by their general characteristics. Some researchers create two general groups: (1) normative models, which use mathematics to establish norms or optimized results such as interest problems, and (2) descriptive models, which use mathematics that describe data about business and physical phenomena (Blum & Niss, 1991). Other classification systems categorize models according to degrees of randomness distinguishing those that ignore random variation from those built upon probabilistic equations and according to levels of theoretical information distinguishing models that are deeply rooted in theoretical principles from those models that are empirically driven (Edwards & Hamson, 2001). The Common Core State Standards for Mathematics (CCSSM, 2010) add probabilistic and statistic models to predict outcomes of events.

Knowledge needed for mathematical modeling

Because mathematical modeling tasks often describe messy real-world situations found in science, engineering, business, art, and literature, modelers need both a broad knowledge of the world and extensive mathematical knowledge to successfully generate a model (Jensen, 2007). During the mathematical modeling process, modelers use organizational and communication skills to collect and make sense of information. Modelers also engage in discussions about the reality of the task situation and the assumptions they make about the situation (Verschaeffel, Greer, & De Corte, 2000; Galbraith & Stillman, 2001).

Understanding the problem context is essential for modelers to link mathematical knowledge to their real world knowledge (Imm & Lorber, 2013). When modelers are unfamiliar with the context of the task, they usually only make general statements about the situation but cannot mathematize the problem because the assumptions made and the variables defined are limited by their lack of knowledge about the problem context (McNair, 2000). Since it is important for the modeler to be familiar with the real world context referred to in the mathematical modeling task, more research is needed to understand how modeling students make connections between reality and mathematics as well as the degree of familiarity needed with the task context in order to mathematize the situation. Mathematical modeling tasks used in secondary mathematics classrooms vary widely in their degree of authenticity as close simulations of real life. More research is needed to determine the relationship between the task's authenticity and modeling performance.

Many types of mathematical knowledge are accessed in solving mathematical problems, and this knowledge can be increased through exposure to different types of modeling activities (Doerr, 2007). Tirosh (1999) detailed a list of mathematical knowledge areas that includes instrumental, formal, relational, conceptual, procedural, algorithmic, visual, intuitive, implicit versus explicit, and elementary versus advanced; suggested considerable overlap between these categories; and recommended that more research is needed. When looking at the mathematical modeling content knowledge needed to successfully generate a modeling solution, research should seek to answer the basic questions of *what* mathematical knowledge is demanded, *why* a particular type of mathematical knowledge is required, and *how* the mathematical knowledge may be applied when working with mathematical modeling situations.

Modeling sophistication and competency

Once a mathematical model is generated, the *level of sophistication* of the modeler can be examined by analyzing the model's degree of complexity, generalizability, effectiveness, and efficiency (Blum, 2002). The level of sophistication depends upon the breadth and depth of mathematics employed by the model as well as the use of constraints which expose deeper understanding of the modeling situation (Zawojewski, 2010). Along the thread of research monitoring modeling progress, mathematical modeling competence is defined as an "insightful readiness to carry through all parts of a mathematical modeling process in a given situation" and should be viewed from a holistic perspective, examining progress in the entire modeling process, as well as a piecemeal perspective, tracking progress through specific individual modeling stages (Blomhoj & Jensen, 2007).

Jensen (2007) defined three dimensions of mathematical modeling competence: the *Degree of Coverage* dimension, which indicates the ability to work independently and monitors the interpretation of the situation; the *Radius of Action* dimension, which tracks the real world and mathematical domains where modelers are able to work; and the *Technical Level* dimension, which monitors the scope and quality of mathematics used in modeling as well as the flexibility in revising models.

Mathematical Modeling Working Group at PME-NA

The framework for the working group is adapted from Jensen's (2007) *expanding box*, which represents growth in competencies while solving mathematical modeling tasks, to a three-dimensional depiction of the three areas of mathematical modeling knowledge for teaching (MMKT), shown in Figure 2. With the three dimensions of MMKT identified, the representation encourages research identifying, tracking, and analyzing mathematical modeling thinking including knowledge, skills, and competencies. It is important to note that difficulties in one dimension can impact performance in other dimensions.

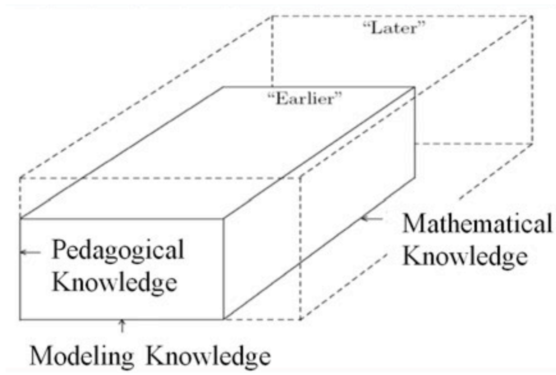

Figure 2: Framework for mathematical modeling knowledge for teaching (Authors, 2015)

Both the *Mathematical Knowledge* and *Modeling Knowledge* dimensions are features of mathematics teachers' content knowledge of mathematical modeling. The first dimension, *Mathematical Knowledge,* tracks growth in modelers' mathematical knowledge and skills as they work with a variety of mathematical modeling tasks. "This dimension represents the size and content of the 'mathematical toolbox.'" (Jensen, 2007) For example, some teachers may exhibit proficiency in algebraic situations but may struggle with probabilistic contexts. Working with real world questions is difficult for less sophisticated modelers who find mathematizing the situation to be challenging, and they often do not incorporate advanced mathematics into their models even though they have completed advanced mathematics coursework (Blomhoj & Jensen, 2007). Jensen's (2007) dimension of *radius of action* maps directly to horizon content knowledge while his technical dimension maps directly to mathematical content knowledge. This dimension provokes many possible discussion questions including the following:

1. What *mathematical content* knowledge is required for secondary mathematics teachers' mathematical modeling instruction?
2. How can progress in teachers' mathematical content knowledge for modeling instruction be monitored?
3. How does general domain-mathematical content knowledge influence mathematical modeling knowledge for teaching?

The second dimension, *Modeling Knowledge*, examines both a teacher's ability to work independently through the modeling process and the teacher's proficiency with different types of modeling problems and contexts. Teachers with a low level of modeling sophistication may struggle stating assumptions, defining variables, and identifying and incorporating key relationships when beginning new modeling tasks; and those with a high level of modeling sophistication may progress more efficiently from translating the real world situation to interpreting the solution. Jensen's (2007) dimension of the degree of coverage maps directly to modeling skills. This dimension provokes many possible discussion questions including the following:

1. What *modeling* knowledge do teachers need to possess to teach mathematical modeling?
   a. What knowledge is needed for working with different types of mathematical modeling tasks?
   b. What knowledge is needed to pose mathematical modeling tasks?
   c. What knowledge is needed to critique a mathematical model?
2. How can progression in modeling knowledge be followed?

The third dimension is *Pedagogical Knowledge*, which monitors teachers' classroom interactions and lesson planning for an increase in teachers' knowledge of students and curriculum with mathematical modeling. Due to the complexity of the main research question, the *Pedagogical Skills* dimension is not a primary focus for the initial working group sessions, but the participants may decide to expand their discussions to include this category.

Plan for Active Engagement of Participants

Members of the working group will actively participate in defining this research arena. The working group will meet three times during the conference and virtually during the course of one year.

Session 1: Foundation

The first session will outline the structure of the working session and lay the foundation for mathematical modeling knowledge for teaching by establishing common definitions and presenting the digital environment as an archive of forum discussions and research ideas that can be shared publicly or privately to promote involvement and collaboration. Sample mathematical modeling tasks will be provided as a vehicle to elicit discussion, establish definitions, and note points of distinction. Members are invited to bring mathematical modeling tasks to share with the group and suggest and generate research questions. The final part of the session will be allocated to introducing the framework that will guide the research focus of the working group and forming research groups based on the interests of the participants.

Session 2: Content Knowledge

The second session will focus on clearly defining mathematical content knowledge needed for mathematical modeling and outlining research goals and objectives to monitor progression in this area. Due to the interdisciplinary nature of mathematical modeling, as well as the various mathematical approaches taken by modelers, the meaning and implications of horizon content knowledge with regards to mathematical modeling will be included in group discussions. Sample tasks and student work samples will be shared to facilitate group discussions regarding mathematical modeling thinking in terms of mathematical knowledge and sophistication in the scope of each groups' research interests and discussion questions.

Session 3: Modeling Knowledge

Using discussion questions, the third session will focus on clearly defining modeling knowledge needed for mathematical modeling and outlining research goals and objectives to monitor progression in this area. Discussions will be focused on how to improve teachers' modeling knowledge holistically and individually.

Post-conference

To sustain the working group's efforts, results of all sessions and meetings will be documented and disseminated to all members. Following the conference, participants will be invited to continue discussing research interests in this area through a web-based discussion forum created, hosted, and monitored by the leaders of the working group and supported by The Ohio State University. The discussion forum will offer an environment for suggesting and reviewing research, posing and critiquing mathematical modeling tasks, posting and discussing teacher and student solutions, and developing an archive of useful and productive mathematical modeling tasks. Working group members will participate in digital meetings throughout the year to review progress on scholarship. Findings of the working group and subsequent discussions will be summarized and reported.

Conclusion

With the lack of literature on developing mathematical modeling content and pedagogy in secondary teachers and the ambiguity of the mathematical modeling strand and mathematical practice as defined by the Common Core State Standards for Mathematics, the mathematics education community is in need of a dedicated group of individuals who will investigate this field. Working group participants will be committed to increasing knowledge on mathematical modeling and sustaining the educational and research goals of PME-NA. A working group at PME-NA devoted to investigating mathematical modeling knowledge for teaching will provoke interesting and productive research questions and studies to expand the field.

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EXPLORING CONNECTIONS BETWEEN ADVANCED AND SECONDARY MATHEMATICS

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The debate in understanding what content knowledge secondary teachers should have in order to provide effective instruction is the main theme of this new Working Group. Our goal is to explore connections between advanced and secondary mathematics as an entry point into this debate. In particular, we will gather interested individuals in an effort to deepen our understanding of existing connections between abstract algebra and secondary mathematics and which of these connections are important for secondary teachers to know and understand. Moreover, we aim to further research in this area by considering possible links between knowledge and understanding of connections and secondary instruction as well as how we might be able to assess teachers' knowledge about connections between advanced and secondary content. We hope to accomplish these goals by discussing important connections between abstract algebra and secondary mathematics, providing opportunities for participants to share their experiences with connections, and engaging in conversations about the impact and assessment of connections in secondary teaching.

Keywords: Advanced Mathematical Thinking; Teacher Education-Preservice; Teacher Education-Inservice; Teacher Knowledge

There has been a longstanding debate in the mathematics and mathematics education communities concerning the knowledge secondary mathematics teachers need to provide effective instruction. Central to this debate is what content knowledge secondary teachers should have in order to communicate mathematics to their students, assess student thinking, and make curricular and instructional decisions. This debate has already led to many fruitful projects (e.g., Connecting Middle School and College Mathematics [$(CM)^2$] (Papick, n.d.); Mathematics Education for Teachers I (2001) and II (2012); Mathematical Understanding for Secondary Teaching: A Framework and Classroom-Based Situations (Heid, Wilson, & Blume, in press)). A common thread in these projects is the belief that mathematics teachers should have a strong mathematical foundation along with the knowledge of how advanced mathematics is connected to secondary mathematics (Papick, 2011). But questions remain about what secondary content stems from advanced connections, which connections are important, and how might knowledge of such connections impact classroom practice. This new working group is being created to further explore these and other related questions.

More mathematics preparation does not necessarily improve instruction (Darling-Hammond, 2000; Monk, 1994). In fact, some research has shown that more mathematics preparation may hinder a person's ability to predict student difficulties with mathematics (Nathan & Petrosino, 2003; Nathan & Koedinger, 2000). Nevertheless, the requirements for traditional certification to teach secondary mathematics across the country continue to include an undergraduate major in the subject. Moreover, questions regarding mathematics teachers' content knowledge and preparation continue to be the topic of concern as evidence from the current conversations around national and international test result of student achievement in mathematics. Therefore, it is important that, as a field, we investigate the nature of the present mathematics content courses offered (and required) of prospective secondary mathematics teachers to gain a better understanding of which concepts and topics positively impact teachers' instructional practice. We hypothesize a valuable starting point would be to develop a better understanding of the nature of the mathematical connections between advanced content taken in undergraduate programs and secondary content taught in schools. Explicit attention to these connections could play a pivotal role in making the mathematics major more meaningful to prospective teachers and could positively impact future teachers' instructional practice.

Research focused on connections between advanced and secondary mathematics has begun to gain traction in the mathematics education community. Such research addresses (a) the connections that exist between particular advanced content courses (e.g., abstract algebra) and secondary mathematics (e.g., Baldinger, 2014; Cofer & Findell, 2007; Usiskin, 2001; Wasserman, 2014) and (b) the impact learning connections between advanced and secondary content has on teachers and their instruction (Baldinger, 2013; Wasserman, 2014).

This new working group will continue and expand on this research in order to explore the following questions:

1. What are the connections between advanced and secondary content?
    a) What connections between advanced and secondary content are important for teachers to know and understand?
2. What is the link to classroom practice?
    a) How does knowledge of connections between advanced and secondary content impact instruction in secondary classrooms?
    b) How can we better support teachers to understand connections between advanced and secondary content and to use pedagogy that employs these connections?
2. How do we assess teachers' knowledge about connections between advanced and secondary content?
    a) How do we determine the depth of teacher knowledge of advanced content and connections to secondary content?
    b) What are indicators that teachers have gained particular understandings?
    c) What do we want teachers to be able to do with this knowledge?

Background

We take the view that connections between advanced mathematics and secondary mathematics encompass both mathematical content and ways of thinking about and engaging with that content. We draw on research around mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), key developmental understandings (Simon, 2006), mathematical practices (e.g., Council of Chief State School Officers [CCSSO], 2010; RAND, 2003) and habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996) to define these connections.

Mathematical knowledge for teaching (MKT) (Ball et al., 2008) describes different domains for the mathematical knowledge teachers draw on in the practice of teaching. Broadly, it encompasses subject-matter knowledge and pedagogical content knowledge. More specifically, subject-matter knowledge entails three domains of mathematical knowledge used in teaching: common content knowledge, specialized content knowledge, and horizon content knowledge. Pedagogical content knowledge includes knowledge of how the content is related to students, to teaching, and to curriculum. These categories were developed based on the mathematical work of teaching at the elementary level, and many scholars have utilized them at the secondary level (e.g., Baumert et al., 2010; Tatto & Senk, 2011), although others have problematized this direct translation to secondary mathematics (e.g., Speer, King, & Howell, 2014). Although these categories may not perfectly capture all that secondary teachers need to learn, they represent a potentially useful framework for thinking about the scope of mathematical knowledge needed for teaching. In the context of connections between advanced mathematics and secondary mathematics, the MKT framework invites the following questions: What knowledge of mathematics does a secondary teacher need that goes beyond the mathematics of the school curriculum? How might advanced mathematics enable a teacher to unpack a secondary mathematics topic? How might an understanding of advanced mathematics influence instructional choices in presenting a secondary mathematics topic?

One useful strategy for exploring teacher learning of connections between advanced mathematics and secondary mathematics is through the construct of “key developmental understandings (KDUs)” (Simon, 2006). A KDU is a "conceptual advance that is important to the development of a concept" (Simon, 2006, p. 365). Silverman and Thompson (2008) use this construct in their framing of teachers’ mathematical knowledge, by proposing that teachers develop MKT of a topic if: (a) they have achieved a KDU that encompasses mathematical understanding of the topic; and (b) they have an understanding of how the topic may evolve instructionally in support of students' reasoning in the K-12 classroom. This highlights questions such as, what are the key developmental understandings around advanced mathematics that have connections to secondary mathematics? How might these particular KDUs support the development of MKT?

It is useful to expand our consideration of what secondary teachers need to know beyond content and concepts to encompass mathematical habits of mind (e.g., Cuoco et al., 1996) and engagement in mathematical practices (e.g., CCSS, 2010). Mathematical habits of mind include looking for patterns, giving precise descriptions, and utilizing visualizations. Mathematical practices include making conjectures, attending to precision, and connecting representations. These are habits and practices that cut across content areas and levels of mathematical study. Engagement in mathematical practices is an explicit feature of school mathematics through their inclusion in the Common Core State Standards. Attending explicitly to mathematical practices in our context invites asking: How is engagement in mathematical practices around advanced mathematics content similar to and different from engaging in those same practices around secondary content?

We draw on these ideas to develop a description of connections between advanced mathematics and secondary mathematics. Connections might be purely mathematical in nature; that is, they might relate directly to subject-matter knowledge. Connections might highlight a relationship between content and students or content and teaching; that is, they might relate to pedagogical content knowledge. Connections might be part of the key developmental understandings that support the development of mathematical knowledge for teaching. Further, connections might go beyond knowledge *of* mathematics and encompass engagement *in* mathematics through the lens of mathematical practices.

History and Goals for Working Group

The purpose of this new working group is to strengthen our collective understanding of connections between advanced and secondary content and their role in secondary mathematics teaching and learning. The members of this working group came into the idea of investigating connections and teacher preparation through various disparate individual and group pursuits. Recently, we began having informal conversations about these ideas based on our experiences. We realized we had common interests and that there were others pursuing similar research. As a result, we saw the opportunity to advance our collective ideas as well as the field’s understanding of connections. For example, one common interest for the working group is the experience of prospective or practicing secondary mathematics teachers as they take advanced mathematics courses, such as abstract algebra. In general, prospective teachers, similar to some math majors, struggle with understanding the material and applying their knowledge (Clark, Hemenway, St. John, Tolias, & Vakilet, 1999; Zazkis & Leikin, 2010). Additionally, many do not understand the purpose of taking advanced mathematics courses (e.g., Cuoco, 2001), especially the relevance to teaching secondary mathematics (e.g., Cuoco & Rotman, 2013a). For example, Broderick (2013) interviewed prospective secondary teachers about the usefulness of their college math courses. He found their comments were consistent with the literature (e.g., G. Hill, 2003), with one caveat. One participant had not passed abstract algebra the first time and went through it again. She found more relevance the second time through and was more satisfied with taking the course. Such findings have led to several efforts to make abstract algebra more accessible and applicable.

Abstract Algebra Textbooks

Based on the need to connect with prospective secondary mathematics teachers, mathematicians have written textbooks in abstract algebra for this specific audience (e.g., Cuoco & Rotman, 2013b; J. Hill, Thron, & Weathers, 2012; Nicodemi, Sutherland, & Towsley, 2007). The preface to Cuoco and Rotman (2013b) states that the textbook is designed for college students who want to teach mathematics in high school. The textbook authors also state that it can serve as a text for standard courses in abstract algebra as well. What distinguishes this text is the authors' assertion that they "have found that the first encounter with groups is not only inadequate for future teachers of high school mathematics, it is also unsatisfying for other mathematics students" (p. xiii). They also specifically "include sections in every chapter, called ***Connections***, in which we explicitly show how the material up to that point can help the reader understand and implement the mathematics that high school teachers use in their professions" (p. xiv, emphasis in original).

Similarly, Nicodemi et al. (2007)'s abstract algebra textbook is written for an audience of both preservice secondary teachers and mathematics majors. They also take the approach to introduce rings and polynomial rings before groups because of the "natural parallel of facts about integers to facts about polynomials" (p. ix), which connects to secondary mathematics. They acknowledge that

> All students appreciate links that connect the abstract mathematics encountered in upper-level courses to familiar, more concrete mathematics of their earlier experiences. Such links are crucial to the future teacher as a bridge between the mathematics they are learning and the mathematics they will teach. (p. ix)

Thus, in most of the sections the authors finish them with a short note to the teacher.

Other textbooks are designed specifically for secondary teachers (e.g., J. Hill et al., 2012). Hill's text focuses on group theory and homomorphisms, but does not consider rings and fields. It also adds references to school mathematics and discusses topics such as cryptography and algebraic coding theory. Thus, while there exists research-based textbook support for courses in connecting abstract algebra to secondary mathematics, J. Hill et al. (2012) show that some textbooks can make choices for which typical abstract algebra topics they cover and do not cover. These choices are also reflected in the research that exists chronicling various implementations of modified abstract algebra courses.

Modified Abstract Algebra Courses

In efforts to make abstract algebra more accessible to secondary teachers, and other majors, certain modifications have been made to the traditional lecture-based format. Courses have been developed with a focus on cooperative learning environments (Barbut, 1987; Cnop & Grandsard, 1998) and including a component of technology (Leron & Dubinsky, 1995; Clark, et al., 1999). For example, Freedman (1983) conducted an abstract algebra course where students were responsible for presentations of the material based on their work with original sources. In another example, Grassl and Mingus (2007) used a new pedagogical model where they conducted an abstract algebra course using team teaching and cooperative learning. They found that this approach provided deeper learning for students as a community of scholars.

Other courses in abstract algebra have been created specifically for prospective and practicing secondary mathematics teachers. These courses focused on connections to high school mathematics (Baldinger, 2013; Cuoco & Rotman, 2013a). The course in Baldinger's study was motivated by studying the concepts necessary to understand a certain proof of the fundamental theorem of algebra. Each concept they studied allowed for opportunities to make connections to secondary mathematics. These topics were studied from an advanced perspective and in non-standard settings. Examples included the natural numbers, polynomials, proofs by induction, and prime factorization. Other connections to the secondary curriculum involved drawing parallels between the integers and polynomial rings.

The tension of content coverage was highlighted in the studies of the implementation of abstract algebra courses as in textbooks. For example, Baldinger (2013) mentioned that the course in her study did not cover groups or rings and gave only a minimal treatment of field extensions. The course in Freedman (1983) took three years to complete. Cnop and Grandsard (1998) also stated they needed to take the class at a slower pace with their modifications. Cuoco and Rotman (2013a) affirmed that their modifications did not "dumb down" the course, which can be an issue when working with prospective teachers. With the potential for issues regarding content coverage, we are brought to the question of what connections can be made to secondary mathematics, while covering the content necessary for teachers and non-teachers alike.

Such research in undergraduate mathematics education has led to the types of questions this working group is interested in exploring, such as what content can best help prospective teachers see relevance to their future teaching and not wonder why they had to take a particular advanced course? If this content can be the focus of an advanced course, could this help people become more effective teachers and foster the development of higher order skills in their secondary students?

Abstract Algebra and Secondary Instruction

Beyond considering abstract algebra courses themselves, it is critical to consider how teachers' knowledge of advanced mathematics influences their teaching practices. G. Hill (2003) describes how a secondary teacher was able to build on the axiomatic approach to abstract algebra in a unit on complex numbers for her secondary students. Wasserman (2014) described some of the ways that prospective and in-service teachers were introduced to ideas of abstract algebra, which drew upon solving simple equations. Wasserman and Stockton (2013) used vignettes to capture some of the potential implications on teaching secondary mathematics in relation to their planned practices.

Wasserman (in press) looked at teachers' practices for teaching some specific elementary, middle, and secondary topics, before and after being learning about introductory concepts in abstract algebra. In identifying the specific topics, he developed a framework that considered the K-12 content areas for which teaching might be influenced by teachers' knowledge of abstract algebra. This provides a different perspective than the traditional listing of connections between abstract algebra and secondary mathematics (e.g., (Z, +) is a group).

Because work with prospective teachers' understanding and practice of the connections between abstract algebra and secondary mathematics is in the emerging stages (e.g., G. Hill, 2003; Wasserman, 2014; Wasserman, in press; Wasserman & Stockton, 2013), additional research is required to evolve and validate this area. A natural next step would also be to assess the prospective teachers' knowledge and practice.

Therefore, this new working group seeks to continue and extend the research described above and to consider the following questions: (a) What are connections between advanced and secondary content? (b) How are these connections linked to instructional practice in secondary classrooms? (c) How do we assess prospective teachers' knowledge about connections between advanced and secondary content?

Rationale and Relevance for PMENA

Drawing meaningful connections between secondary and advanced mathematics has been an enduring challenge for secondary mathematics teachers (e.g., CBMS, 2012). This new working group aims to capitalize on cooperation among mathematicians and mathematics educators to investigate critical issues in the mathematical preparation of prospective secondary mathematics teachers. The working group will promote the exchange of ideas around (a) identifying important connections between abstract algebra and secondary mathematics; (b) exploring the impact of such connections on secondary instruction; and (c) thinking about how teacher educators might be able to assess knowledge and understanding of such connections. For example, in exploring connections between abstract algebra and secondary mathematics, the group will identify key developmental understandings (KDUs) that can be targeted by and connected to specific tasks for prospective teachers.

The exploration of these topics connects directly to the conference theme of *developing critical responses to enduring challenges* by opening a dialogue around the nature of abstract algebra for secondary teachers and how connections to secondary mathematics can be used to enhance the learning of both prospective teachers and other majors. The work will be relevant to the PME-NA audience through the exploration of conceptual ideas such as extending notions of mathematical knowledge for teaching to this advanced level and through practical ideas such as strategies teacher educators might use to assess teacher understanding of connections between advanced mathematics and secondary mathematics.

Organization and Plan for Active Engagement

The overall goal for this new working group is to strengthen our collective understanding of connections between advanced and secondary content and their role in secondary mathematics teaching and learning. We focus on abstract algebra as an entry into this domain, particularly for its rich connection to secondary mathematics topics such as algebra and functions, while also anticipating that the impact of these conversations will prompt research into other content areas (e.g., calculus, linear algebra, and analysis).

Across the three sessions, participants will engage with facilitators about the current state of the research through brief presentations and guided discussions.

Session 1: Exploring connections between abstract algebra and secondary mathematics

In the first session, we will introduce the working group by sharing our definition of connections along with the background and goals of the working group. After this brief presentation, group members will share some of their own work about important connections between abstract algebra and secondary mathematics to allow participants to share and learn about different connections. The discussion will be among the entire group and will focus on the following question: What are the important connections between abstract algebra and secondary mathematics? At the end of this session, the facilitators will create a summary of important connections discussed to help organize and motivate the following sessions.

Session 2: Exploring the impact of connections on secondary instruction

In the second session, we will begin by sharing our summary of the previous session. We will then have a whole group discussion focused on the question: What do we mean by and how do we measure impact on instruction? The second part of this session will use the group's working definition and the important connections from Session 1 to address the following questions:

1. How does knowledge of connections between advanced and secondary content impact instruction in secondary classrooms?
2. How can we better support teachers to understand connections between advanced and secondary content and to use pedagogy that employs these connections?

Participants will be expected to think about important connections they have seen for the teaching of secondary mathematics and how these connections might be applied to secondary mathematics teaching. At the end of the session, the facilitators will create a summary of how important connections might influence teachers' practice to help organize and motivate the final session.

Session 3: Assessing teacher understanding of connections

In the final session, we will begin by sharing our summaries of the previous session. We will engage in a whole group discussion of how group members have assessed or might assess whether teachers have indeed acquired knowledge and understanding of the important connections between abstract algebra and secondary mathematics discussed. Through this conversation, we will collect ideas for assessment and focus on the following questions: (1) How do we determine the depth of teacher knowledge of the connections between abstract algebra and secondary mathematics? (2) What are indicators that teachers have gained particular understandings? (3) What do we want teachers to be able to do with this knowledge? We will conclude the working group by summarizing the ideas shared about important connections, impact on teachers' practice, and ideas for assessment. We will collect contact information from participants to continue the conversation and wrap up by establishing a set of next steps.

Anticipated Follow-up Activities

As a result of this working group, we anticipate three primary follow-up activities. First, we will submit an article to the Notices of the American Mathematical Society. The Notices is the most widely-read journal by professional mathematicians. Our article will provide an opportunity for us to engage mathematics professors in a conversation about the mathematical preparation of secondary teachers. In particular, we will share results from each of the three themes within our working group: (1) specifying mathematical connections between abstract algebra and secondary mathematics, as an opportunity to draw on such examples in abstract algebra courses; (2) recognition that mathematical connections are necessary but insufficient, and sharing ways that, for teachers, abstract algebra may have implications for their *teaching* of secondary mathematics, not just their knowledge of mathematics itself; and (3) discussing ideas about how assessments might inform whether teachers have acquired such desired understandings.

Second, we will prepare an entry for the American Mathematical Society's blog on the Teaching and Learning of Mathematics. We have been invited to contribute such an entry by one of the blog editors who is an author of this working group proposal. This will allow us to reach a wide online audience and provide an overview of connections between abstract algebra and teaching secondary mathematics.

The third anticipated follow-up activity has to do with longer-term work via continued meetings at other conferences and establishing a collective research agenda. The discussion group will provide, based on some of our previous work and interests individually, an opportunity for all of us to get a sense of a bigger picture about abstract algebra for secondary teachers. In the nearer future, this shared perspective will provide opportunities for the group to continue working together, studying and researching more rigorously some of the ideas related to abstract algebra and secondary teaching that arise from the working group. In the farther future, we anticipate branching off into other areas of advanced mathematics (e.g., real analysis). A long-term collective goal is to fundamentally reconsider the teaching of more advanced content courses for teachers in ways that are useful both generally for the teaching of all mathematics majors and also specifically for the unique considerations and professional practices of secondary teachers. With the discussion group as a starting point, building into more rigorous research about connections to secondary mathematics and its teaching, we imagine collectively developing materials for such courses.

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LEVERAGING DIFFERENT PERSPECTIVES TO EXPLORE STUDENT THINKING ABOUT INTEGER ADDITION AND SUBTRACTION

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This is the third meeting of a working group on student thinking about integers. The main goal of this working group includes utilizing different theoretical perspectives and methodologies in small groups to design complementary studies, where student thinking about integer addition and subtraction will be explored. This working group aims to provide a space for participants to capitalize on their differences in theoretical perspectives and methodologies to promote productive scholarly discussion about the same research topic, student thinking about integer addition and subtraction. Participants will actively engage in work that progresses towards these studies, with the intent to develop a monograph that highlights this research.

Keywords: Number Concepts and Operations; Cognition; Research Methods

A Brief History of the Working Group

The first working group on student thinking about integers convened during PME-NA 35 (Lamb et al., 2013). During this working group, facilitators shared perspectives of current research in the field on student thinking about integers. Discussion with these speakers and participants revolved around the presentations and what "Integer Sense" entails. The work initiated at PME-NA 35 continued at the joint PME 38 and PME-NA 36 meetings (Bofferding, Wessman-Enzinger, Gallardo, Salinas, & Peled, 2014). At this meeting, the organizers presented an extensive review of all of the integer articles from the PME and PME-NA proceedings. Further, they shared and discussed perspectives on integer research stemming from seminal work on integers. The group concluded with a discussion on next directions for collaborative research on the teaching and learning of integers. Responding to the need for collaborative research, this working group proposes a collaborative research project that welcomes all perspectives on student thinking about integer addition and subtraction and provides a platform that embraces these collective differences.

Relevance to Psychology of Mathematics Education

Compared to research on whole number addition and subtraction, research on student thinking about integer addition and subtraction is fairly limited. Perhaps for this reason, there is increased interest in student thinking about integer addition and subtraction in our field (e.g., Bishop et al., 2014a, 2014b; Author, 2014; Stephan & Akuyz, 2012). However, research on student thinking about integer addition and subtraction has been conducted, and even represented at PME and PME-NA, for over three decades (e.g., Bell, 1982; Gallardo, 2003; Marthe, 1979; Peled, Mukhodpadhyay, & Resnick, 1989). This points to a need for research on student thinking about integer addition and subtraction to begin building bridges to connect the research. One way to do this will be to discuss similarities and differences when using different theoretical lenses or analyses on similar topics (e.g., Lewis, 2008).

For guidance as a field interested in the growth of research on student thinking about integers, it is helpful to turn to well-established agendas, like the research on student thinking about whole number operations, and reflect on how these agendas have flourished and have became well-connected. Research on student thinking about whole number operations grew and became the most proliferated and well-connected area of research (Mathematics Learning Study Committee, 2001) with researchers taking different perspectives on similar topics. The 1970s involved investigations into how young children counted on or solved different types of word problems (e.g., Jermam, 1970; Steffe & Johnson, 1971). Research on student thinking was often focused on "basic skills," accuracy, and speed (Jermam, 1970; Bright, Harvey, & Wheeler, 1979).

By the 1980s and 1990s, research exploded on student thinking about whole number operations, where most of the scholarly discussion revolved around student-invented strategies and different problem types (e.g., Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Carpenter, & Moser, 1984; Fuson et al., 1997). As this influx in research on student thinking about whole numbers increased, researchers responded to the multiple perspectives on student thinking. For example, Cobb (1985) reacted to three different papers from Baroody (1984), Carpenter & Moser (1984), and Fuson (1984) that appeared in *The Journal for Research in Mathematics Education*. We can now look across these agendas and compare and contrast the findings and perspectives, and even categorize these perspectives by agenda (e.g., Cognitively Guided Instruction (CGI), Conceptually Based Instruction Project (CBI), Problem Centered Mathematics Project (PCMP), Stages of Early Arithmetical Learning (SEAL), Supporting Ten Structures (STST)).

By the 2000s, the similarities and differences of these different theoretical perspectives and research methodologies were embraced and projected the field forward, not only in the area of whole number arithmetic but in many other areas, such as early algebra (e.g., Carpenter, Franke, Levi, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006), understanding of the equal sign (e.g., Jone & Pratt, 2011), rational numbers (e.g., Empson & Levi, 2011; Steffe & Olive, 2010), and even preservice teacher education (e.g., Vacc & Bright, 1999).

If we compare the development of research and research agendas on student thinking about integers to the field of whole numbers, we can learn that embracing multiple perspectives is productive and insightful. Similar to the increased interest in student thinking about whole number arithmetic of the 80s and 90s, we are currently positioned to respond to this increased research interest on student thinking about integers. Drawing upon these productive comparisons and research, this working group aims to establish a space for those interested in researching student thinking about integer addition and subtraction. Based on past participation in the working group, we anticipate that participants will represent a variety of theoretical perspectives, which will fuel a set of complementary studies and continued discussion about similarities and differences in our investigations.

Theoretical Perspectives & Methodological Approaches

Across the PME and PME-NA proceedings and recent journal articles, researchers have presented work around negative integers from a variety of perspectives and using different methodological approaches. Some or all of these may play a role in the working group discussion, studies, and final products. We present a few examples:

Integer sense. Both Gallardo (2002) and Bishop et al. (2014a) point to ways that that we can think about and use negative integers (see Table 1). Gallardo based her framework for interpreting negative integers on historical analyses of the topic and clinical interviews with middle-schoolers. Bishop et al. based their interpretations from the literature and mathematical reflections.

Rather than focusing on the different ways of interpreting integers, Kilhamn (2009) theorized about what number sense is in relation to concepts involving integers. These components include "intuitions about numbers and arithmetic" (p. 331), the "ability to make numerical magnitude comparisons" (p. 332), the "ability to recognize benchmark numbers and number patterns" (p. 333), and "possessing knowledge of the effects of operations on numbers" (p. 334).

Table 1: Comparisons and Interpretations of Gallardo (2002) and Bishop et al. (2014a)

Gallardo (2002) interpretations of negative numbers (p. 179)	Bishop et al. (2014a) interpretations of -5 (p. 20)	Reflections
Subtrahend "where the notion of number is subordinate to the magnitude (for example, in a – b, a is always greater than b where a and b are natural numbers"	"An action of removing 5 from a set"	Removing five from a set matches closely to interpreting a negative number as subtracting a positive number.
Relative or Directed Number "where the idea or opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domains"	"The location on a number line (coordinate plane, etc.) 5 units to the left of, or below, 0" "An action of moving 5 units left or five units down" "A debt of $5 is also a directed number; it is the opposite of a credit of $5."	Placing a negative number on a number line allows one to interpret the negative number as a relative number or a directed number. Debt can be interpreted as direction. Or, -5 can be a relative number that represents a loss of five dollars.
Isolated Number the result "of an operation or as the solution to a problem or equation"	"The integer between -6 and -4"	The negative number may be treated as a symbolic number that has order.
Formal Negative Number "a mathematical notion of negative number, within an enlarged concept of number embracing both positive and negative numbers (today's integers)"	"Describing the equivalence class [(0,5)] in which we define (a, b) to mean a – b, and all other ordered pairs (a,b) such that a + 5 = 0 include (1, 6), (2, 7), (100, 105), and all other ordered pairs (a,b) such at a + 5 = 0 + b for a,b that $\in \mathbb{N}$. [More formally, we can write (0, 5) ~ (a, b).]"	The negative number can be thought of in more formalized ways. For example, -5 is compared to an equivalence class. We can also talk about the additive group of the integers or the ring of the integer and how integers are not a field because the multiplicative inverses are not all integers.

Integer number line and conceptual change. Using an experimental design involving pre- and post-test task-based interviews as well as instruction around different aspects of integer concepts, Bofferding (2014) identified different conceptions children have of integers. Aligned with a conceptual change paradigm (Vosniadou, 1994), the conceptions fall along a continuum where initial ideas about negative numbers arise from students' conceptions of whole numbers and progress to more formal understanding. Although the categories and concepts explored in the research support Kilhamn's (2009) ideas of number sense, Bofferding's interpretation of students' work focuses on how their conceptions arise and become differentiated from their whole number understanding and when planning for instruction focuses on how to effectively bridge learning of whole number and integer concepts. Further exploration of students' developing integer conceptions would benefit from the use of additional research methods, such as a teaching experiment (Steffe & Thompson, 2000) or microgenetic analysis (Siegler, 1996), which could further clarify what portions of the instruction or work with integers influenced students to change their thinking in Bofferding's study.

Conceptual models. Wessman-Enzinger & Mooney (2014) found that when children posed stories for integer addition and subtraction problem types the students' reasoning could be classified into the Conceptual Models for Integer Addition and Subtraction (CMIAS). The five CMIAS described are Bookkeeping, Counterbalance, Translation, Relativity, and Rule. With *Bookkeeping,* the integers are treated as gains and losses, and zero represents neither a gain nor a loss. For example, students posed stories that involved gains and losses of candy bars. And, other students used "needs" and "wants" of various other items, like baseball cards, to represent the negative integers. With *Counterbalance,* the integers are treated as neutralizing each other, and zero represents a status of neutralization. A distinguishing feature of Counterbalance from Bookkeeping is that the quantities always remain present with Counterbalance. For example, consider -2 + 3 = 1, which can be represented by two electrons and three protons, where there is an electrical charge of 1. Although there is an electrical charge of 1, the two electrons and three protons still remain present. With *Translation,* the integers are treated as a vector or with movement. Zero in Translation represents either the position or no movement. With *Relativity* the integers are treated as a comparison to an unknown referent. Zero represents the unknown referent. For example, for -5 + -10 = -15 a student posed the story, "Say you are down five runs in the first inning of a baseball game. And you end up losing by fifteen runs. You would have to have ten runs in the other innings to be down by fifteen runs" (Wessman-Enzinger & Mooney, 2014, p. 203). In this story, the actual score of the game is unknown. The integers are used as relative numbers to the unknown referent, the score of the tied game. Relativity is related to Translation, but both movement and the dual-role of the zero in the model distinguish Translation. Although related to Gallardo's (2002) interpretation of directed number and relative number, the CMIAS distinguishes the use of the integers here. With *Rule* the integers are treated with a procedural rule about signs. Wessman-Enzinger and Mooneys' interpretations of student thinking about integers is that hidden behind the use of contexts are implicit mathematical meanings. And, the utilization of these contexts is isomorphic to mathematical uses of integers. Further investigation into ways that students respond differently to contextualized problems, promoting different CMIAS, could be supported by both task-based interviews (Goldin, 2000) or teaching experiments (Steffe & Thompson, 2000).

Goals of Working Group

During this working group, one of the goals will be to establish small-groups that will work collaboratively together on a research project of their creation. Each of these groups will begin by analyzing similar video data provided by the organizers. This will initiate discussion on ways that using different theoretical perspectives and analysis highlight similarities and differences in student thinking about integer addition and subtraction. However, the main aim of this working group will be to begin formulating research questions and begin designing small studies that are each related to each other. It is the hope of this working group that each of these small studies would be conducted in 2016 and that participants will begin analysis in 2016 as well. Participants could share their initial results at a future meeting so that the group can begin to make comparisons and larger conclusions about student thinking. Additionally, the working group will select a journal to propose a monograph to that illustrates and embraces these different perspectives on making sense of student thinking about integer addition and subtraction.

Extending the Previous Work from Past Working Groups

Both of the previous working groups, PME-NA 35 and PME 38/PME-NA 36, were discussion-oriented. Drawing on colleagues' work and literature reviews, the focal point of each of these discussions was on student thinking about integers and integer addition and subtraction. Each of these working groups concluded with participants eager to work collaboratively on a project. This working group extends the work from these previous groups by presenting a collaborative project that supports connecting our multiple research agendas, inviting all, whether neophytes or experienced researchers, interested in student thinking about integer addition and subtraction to participate.

Plan of Working Group

Plan for Session 1

The first session will begin with a brief overview on the history of the working group for any new members attending. The facilitators will also give a brief update from the joint PME 38 and PME-NA 36 meetings. Then, the session will transition to introductions among participants. The participants will briefly share their interests in integer addition and subtraction research and the general theoretical perspectives and methods they employ. The facilitators will ask participants to break into small groups. The facilitators will show a short video clip of a student solving an integer open number sentence. Discussion will begin with the following question:

1. Using your preferred theoretical perspective on student thinking about integer addition and subtraction (e.g., Metaphorical Reasoning, Mental Models (MM), Ways of Reasoning (WoR), Conceptual Models for Integer Addition and Subtraction (CMIAS)), how would you make sense of or describe the student thinking in this video?

This question will be discussed extensively in small-groups first, and then we will transition to whole-group discussion together. In whole-groups we will discuss the following question:

2. What similarities and differences are present in our discussion about student thinking?

Plan for Session 2

The second session will begin with a brief summary of the previous day's discussion. The facilitators will then describe the proposed collective study where participants will work in small groups to design and implement small studies in the domain of integer addition and subtraction. Together, the group will brainstorm a central, overarching research question to drive the individual studies. The participants will then spend time in their small-groups deciding a topic and research question for their mini-study. At the end of the session the participants will submit their status and progress in a Google survey document so that the facilitators are aware of progress or issues within the group.

Plan for Session 3

The third session will begin with the facilitators engaging the participants in discussion about what was reported in the Google survey document. Discussion will also revolve around expectations for the studies and ways to keep active as a group throughout the year (e.g., scheduling virtual check-ins, meeting at other conferences throughout the year). Then, the remainder of the time will be spent in small groups actively working on the planning and the logistics of the implementation of their small studies.

Anticipated Follow-up Activities

The facilitators will promote active engagement throughout the year in at least two ways. First, the facilitators will provide a shared Google spreadsheet listing our groups' goals to participants. The shared Google spreadsheet will have the small-groups listed as the rows and the collectively-decided group goals (e.g., topic, research questions, theoretical lens, participants, data collection, data analysis) as the columns, which will be modified during the third session. The group will use this shared spreadsheet as a place to report dates when they complete a goal and as a way to see the progress of the other groups in comparison. Second, the small groups will be encouraged to plan at least one Skype session with the facilitators. That way, the facilitators will be aware of each of the group's progress. This will be important for coordinating both the next PME-NA working group session, as well as planning the monograph, where each of these studies become chapters.

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This Working Group will consider multiple aspects of research and practice related to mathematics learning and teaching with English Learners. Our goals for the Working Group include: (1) developing a shared understanding of the research questions, issues, challenges, and contributions that research studies focusing on English learners can make to research in mathematics education and (2) developing a plan for supporting further connections among research projects in the future. We will examine the mathematics education of English learners from a variety of angles, including student, teacher, preservice teacher, school, and community perspectives. Further, we will examine current and future avenues of research around a variety of aspects of mathematics learning and teaching with English learners such as assessment, curriculum, resources, and instructional strategies. By sharing current work and focusing on particular methodologies, we hope to describe the current landscape and build structures to catalyze translational work in this area.

Keywords: Equity and Diversity; Teacher Education-Preservice; Teacher Education-Inservice; Research Methods

Brief History

This group of researchers started working collaboratively as part of the NSF-funded Center for the Mathematics Education of Latinas/os (CEMELA). CEMELA brought together researchers from across the country to collaborate on research focused specifically on critical issues related to Latinos/as in mathematics. Prior to CEMELA, researchers interested in such a focus worked mostly in isolation. In considering issues related to Latinos/as in US schools, the issues of language and English Learners (ELs) are closely related. While not all Latinos/as are English Learners, and not all ELs are Latinos/as, these two groups have significant overlap. For example about 80% of ELs speak Spanish as a first language, and Spanish-speaking ELs appear to struggle on measures of academic achievement (Goldenberg, 2008).

CEMELA expanded the field's knowledge of ELs in mathematics through conducting studies in interdisciplinary teams that helped increase the field's understanding of Latinas/os and mathematics education. CEMELA's research focused on teacher education, research with parents, and research on student learning, resulting in well over 50 publications and presentations. Several of these studies involved the investigation of questions related to the interplay of language, culture, and mathematics education.

Following the conclusion of CEMELA's funding, Zandra de Araujo, Sarah Roberts, Craig Willey, and Bill Zahner continued to meet regularly. These meetings focused on examining intersections among these early career scholars' work related to the mathematics education of ELs. To date these meetings have resulted in a number of national presentations at the annual meetings of the National Council of Teachers of Mathematics, the American Educational Research Association, and PME-NA. Currently, this group is working on several manuscripts and follow-up studies related to the preparation of teachers to work with ELs. This Working Group will provide a space for these scholars to continue their work and further their scholarship. It will also provide a forum to engage other scholars interested in this topic and foster meaningful collaborations among all the participants.

Focal Issues

ELs are the fastest growing group of U.S. students (Verplaeste & Migliacci, 2008). In fact, U.S. schools have seen an increase of 152% in EL students over the past 20 years (National Clearinghouse for English Language Acquisition, 2008). The growing number of ELs across the country provides the basis for all states, cities, schools, and teachers being prepared to attend to the needs of ELs in their mathematics classrooms. No longer is supporting ELs a concern for only those states, like Arizona, Texas and California, with traditionally high numbers of EL students. Instead, with all but ten states across the country seeing increases in their EL populations between 2002-03 and 2011-12 (National Center for Educational Statistics (NCES), 2014), there is increasing pressure for support in addressing the needs of these students.

Despite the rise in ELs, teacher preparation has not kept up with this trend. In 2002, the NCES reported that out of the 41% of teachers who worked with ELs in their classrooms, only 13% received EL-specific professional development. In 2008, Ballantyne, Sanderman, and Levy found that it was "likely that a majority of teachers have at least one English language learner in their classroom," although "only 29.5% of teachers with ELLs in their classes have the training to do so effectively" (p. 9). This misalignment of the realities of today's classrooms and teacher preparation has necessitated examination into effective means of supporting current teachers and preparing prospective teachers to meet the needs of linguistically diverse learners.

The implementation of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO), 2010) will put additional pressure on teachers of students who are still gaining proficiency in English. Teaching aligned with the CCSSM's content standards and the standards of mathematical practice will increase the language demands required to engage in mathematical discourse, which is typically in English. Providing ELs with quality educational experiences is no longer relegated to only language specialists; it is a "mainstream" issue for which all mathematics teachers must be prepared (Bunch, 2013).

Our Working Group aims to provide a response to the critical and enduring challenge of providing quality educational experience for linguistically diverse students in mathematics. This Working Group will bring together researchers from several universities to examine current and past research on ELs in mathematics education while also supporting new collaborations to establish future, imperative research directions. As part of these discussions, we aim to think through logistical and structural arrangements that will facilitate and support these collaborations going forward.

Aims of the Working Group

The Working Group will consider multiple aspects of research and practice related to mathematics learning and teaching with ELs. The aim of this Working Group is to present and discuss several research projects related to mathematics learning and teaching with ELs in order to develop new research questions, to refine data analyses, and to move these research agendas forward.

The larger Working Group will break into several subgroups: a) Student Learning; b) Preservice Teacher Education; c) Inservice Teacher Education; d) Family and Community Resources; e) Curriculum; and f) Language Perspectives. In the following sections, we provide a brief overview of key issues in each of these areas as well as the key research questions that will guide the work of these subgroups.

Student Learning

The subgroup focused on student learning will discuss our past and current research connected to investigating the mathematics learning of ELs. The goal of this discussion is to connect broader research on mathematics learning to our ongoing development of research on ELs, and vice versa. Building upon situated and sociocultural perspectives (Moschkovich, 2002), we start from the premise that ELs, like all students, learn mathematics through a process of appropriating discourse practices, tool use, and perspectives of mathematics. The area of researching student learning in classrooms with a high proportion of ELs is open for further development. In particular, we need better understanding of how research in mathematics education at large is connected with research on the learning of ELs. For example, there is a substantial body of research on student learning of particular mathematical concepts such as rate, ratio, and proportional reasoning (Lobato & Thanheiser, 2002). However, much of the content-focused work in mathematics education is isolated from research on how ELs develop specific mathematical understandings. This has resulted in a gap in the literature where we need to develop more linkages between mathematics education research on student learning of specific topics, and the mathematics education research on student learning of ELs. The following questions will help guide the student learning subgroup's activities during the Working Group's meetings.

- How should learning trajectories in specific content areas be adjusted (if at all) for the learning of ELs?
- What do we know from past and current research about how ELs learn particular key concepts in mathematics?
- How do the needs of ELs in mathematics classrooms vary depending on age, grade, English proficiency, and past educational experiences?
- How do students connect learning to read and use vocabulary to doing mathematics?

Preservice Teacher Education

Preservice teachers typically have few opportunities to think specifically about how they will work with ELs in their mathematics classrooms. Preparation programs often include coursework on teaching ELs that is not specific to the content areas in which preservice teachers will work. Meanwhile, the content courses that preservice teachers do take are often focused solely on content, devoid of exploring how to support ELs in mathematics classrooms. Recently researchers have shared strategies for engaging preservice teachers in working with ELs in mathematics classrooms. For example, Fernandes (2012) suggested a series of content interviews to engage preservice teachers in the process of noticing the linguistic challenges that ELs face and the resources these students bring to their mathematical communication. Additionally, the TEACH MATH (Drake et al., 2010) research team is using tasks in their content courses to support preservice teachers in drawing on students' funds of knowledge (Turner, Drake, Roth McDuffie, Aguirre, Bartell & Foote, 2012). This subgroup will build on this prior work and engage with the following questions during Working Group sessions.

- How are we preparing preservice teachers to work with ELs in their future mathematics classrooms?
- What kinds of clinical experiences best allow preservice teachers to grapple with issues facing ELs and design instructional approaches to support ELs?
- How can we use mathematical discussions with future teachers to facilitate their understanding of ELs?

Inservice Teacher Education

Despite findings that suggest ELs develop academic language in the content areas over a period of about 5-7 years, many ELs are mainstreamed within two years. Thus, it is essential that teachers support ELs in learning content in mainstream mathematics classes while also providing support to develop their academic language. Most inservice teachers have had few, if any, professional learning experiences around working with ELs in mathematics classrooms (Ballantyne et al., 2008). In fact, many mathematics teachers struggle to understand their role in supporting ELs' mathematics language development (Willey, 2013). As the number of ELs grows in classrooms across the country, it is imperative to create and study such learning experiences, while also developing teacher leaders and facilitators to lead these learning experiences.

Much of the prior work on inservice teacher education related to ELs has focused on more general strategies (e.g., sheltered instruction, as in Echevarria & Graves, 1998), such as using visuals, modifying texts or assignments, and using slower speech. We argue there is a need for content specific support for mathematics teachers of ELs. This subgroup will investigate work related to inservice teacher development. The following questions will guide the group's discussions about inservice mathematics teacher education:

- How are mathematics teacher educators supporting inservice teachers to support ELs in mathematics classrooms?
- What resources do teachers bring to professional learning experiences about how to support ELs in their mathematics classrooms?
- How do teachers develop or select mathematics specific supports for ELs?
- How do we develop professional development facilitators to lead professional learning experiences around supporting ELs in mathematics?
- How do facilitators position mathematics teachers to support ELs in their classrooms and instruction?

Family and Community Resources

Families and communities can serve as resources for ELs in their mathematics learning in myriad ways. Families can advocate for their children and provide and support learning experiences both in and out of the classroom. Communities can also provide a wealth of support mechanisms and learning possibilities. For example, teachers can activate student and community resources and can draw on the funds of knowledge of the community by using parents' and students' experiences (Moll, Amanti, Neff, & Gonzalez, 1992). Moll et al. described how students studied candy making and selling within their neighborhood to explore mathematics within this context, such as discussing and analyzing production and consumption. In doing so, the teachers and students acknowledged the value of these community experiences. Additionally, Civil and Bernier (2006) explored the challenges and possibilities of involving parents in facilitating workshops for other parents around key math topics. These studies and others like them illustrate the promising impact of family and community resources in fostering ELs' mathematics learning. The Working Group will explore prior studies on family and community resources as related to ELs in mathematics using the following questions:

- What are the implications of different language policies on parental engagement in mathematics, particularly for ELs?
- How can teachers / teacher education programs build on family and community resources towards the mathematics education of ELs?

Curriculum

Curriculum plays a key role in the teaching and learning of mathematics and teachers play a pivotal role in selecting and enacting curriculum materials for students. The choice of curriculum materials impacts students' opportunities for learning in the mathematics classroom (Kloosterman & Walcott, 2010). Early work on curriculum and English learners focused specifically on the challenges ELs encounter when completing or interpreting word problems (Téllez, Moschkovich, & Civil, 2011). More recent work that centers on both ELs and curriculum has focused on culturally relevant curricula. A rather recent subset of mathematics education, called ethnomathematics, focuses on this area (Barton, 1996; D'Ambrosio, 2006). Other studies have focused on the development of curriculum materials for ELs (e.g., Freeman & Crawford, 2008) or the evaluation of a curriculum's appropriateness for ELs (e.g., Khisty & Radosavljevic, 2010; Lipka et al., 2005). More recently, work related to ELs and curriculum has begun to look at teachers' use of curriculum (e.g., de Araujo, 2012). The Working Group will investigate the intersection of teachers, curriculum, and ELs. The following questions will guide the group's work related to curriculum and ELs.

- How do mathematics curricula and instruction address the needs of English Learners?
- How do ELs interact with curriculum materials? How do English learners learn to read different mathematical texts (textbooks, word problems, etc.)?
- How can teachers' modifications to mathematics curriculum materials support ELs?
- How can teacher educators support teachers' understandings of mathematics curriculum use with ELs?

Language Perspectives

Teachers' and researchers' conceptions of language, second language acquisition, and bilingualism impact teaching and learning mathematics for ELs. The Working Group will consider how perspectives of language, second language acquisition, and bilingualism appear in both theory and practice. We will consider, in particular, how work focused on ELs can draw on current work on language and communication in mathematics classrooms, classroom discourse, and linguistics. Looking for these intersections and connections is crucial; it will ensure that work in mathematics education is both theoretically and empirically grounded in relevant research, and it will prevent researchers from reinventing wheels. The following questions will guide the group's work related to this theme:

- Which constructs do we use to describe and examine language(s) in mathematics classrooms? Although some studies use the term "Academic English" (Scarcella, 2003), it is not clear what this terms means for mathematics, what constitutes "Academic English" in mathematics, or whether other constructs might be more useful for research and practice.
- What constitutes competency in "Academic English" for mathematics in both written and oral modes? If it is the case that "Academic English" is different for different mathematical domains or genres of mathematical texts, then these differences need to be examined.

- How do teachers' views of second language acquisition impact the practice of teaching mathematics for ELs? How can we support teachers in expanding their views of second language acquisition in mathematics classrooms?
- What are students' experiences learning mathematics in their first, second, and both languages? How do different proficiencies in a first language (oral, reading, written, academic English) and previous mathematics instruction in a first language impact students' learning mathematics in English?
- How do students learn to read mathematical texts and use vocabulary to do mathematics? How do ELs learn to read different mathematical texts (textbooks, word problems, etc.)? How can instruction distinguish between children who are competent readers in a first language and those children who are not?

Working Group Plan

The overarching goal of the Working Group is to foster the collaboration of mathematics educators around scholarship related to the mathematics education of English learners. In particular, the following goals will frame the Working Group's activities:

1. Develop a shared understanding of the research questions, issues, challenges, and contributions that research studies focusing on English Learners can make to research in mathematics education
2. Develop a plan for supporting further connections among research projects in the future.

In focusing on current research in this area, with an eye towards further study, the Working Group will address the enduring challenge of providing quality mathematics education for English learners.

During the three sessions, participants will examine and discuss the design of several research studies, analyze sample data collected in at least one of these studies, and discuss future plans for the Working Group. These activities are intended to support participants in a) clarifying research questions, b) refining research tools, methods, and analyses, c) exploring connections among different projects and studies, and d) discussing further collaborations and research on learning and teaching mathematics in classroom with English learners. All work and documents will be shared and distributed via a Google Community. The use of Google Community will allow members to create an institutional memory of activities during the Working Group that members will continue to build upon after the conclusion of the conference.

The planned activities will support these goals in several ways and will be grounded in discussions of sample research designs, data sampling, and sample curricula. The anticipated follow-up activities for this Working Group include planning for a continuation of the Working Group at PME-NA in 2016 and organizing one or more collaborative writing projects on this topic.

Overview of Proposed Working Group Sessions

<table>
<tr><td rowspan="2">SESSION 1</td><td colspan="2">INTRODUCTION, FRAMING THE ISSUES, & CLARIFYING AIMS</td></tr>
<tr><td>ACTIVITIES
1. Introduction and overview of the Working Group including introduction to the Google Community.
2. Brief presentations by panel members from each of the subgroups providing overviews of research projects with specific examples of how researchers have designed the studies. The purpose is to provide an overview of work related to mathematics education of ELs and to introduce the six subgroups in a structured way.
3. Participants will break into subgroups and will analyze and discuss sample data from at least one of the studies presented. This will give participants an opportunity to share their own experiences in designing research studies, collecting data, and analyzing data.
4. Distribution of one or two readings for each subgroup for the next session.</td><td>GUIDING QUESTIONS
1. What research is being done in relation to each of the subgroups?
2. Which aspects of studies focusing on English learners do you find most puzzling? Most useful? Most misunderstood?
3. How have you approached defining the research questions for studies?</td></tr>
</table>

<table>
<tr><td rowspan="2">SESSION 2</td><td colspan="2">PERSPECTIVES, METHODOLOGIES, & INQUIRY</td></tr>
<tr><td>ACTIVITIES
1. Using the questions in the adjacent column, subgroups will discuss the reading(s).
2. Brief whole group discussion that highlights key ideas brought up in the subgroups.
3. Participants work in subgroups to frame new research questions related to their subgroup's focus and develop goals for the final session.
4. Whole group discussion of subgroups' progress.</td><td>GUIDING QUESTIONS
1. What theories and theoretical frameworks have informed the design of your research project(s)?
2. How might your work inform theory in mathematics learning and teaching? How can work on this student population expand our theoretical lenses?
3. What issues and challenges have you faced in designing studies?
4. How have you approached data analysis for studies?</td></tr>
</table>

SESSION 3	PURSUING FURTHER STUDY	
	ACTIVITIES 1. Work time for the subgroups to discuss directions for continued collaboration. Subgroups will also develop next steps as they plan for continued work. 2. Whole group discussion in which subgroups share goals and next steps developed by the subgroups 3. Establish next steps for continued collaboration, including the Google Community	GUIDING QUESTIONS 1. How might other researchers pursue research projects on this topic and what can they learn from the work done so far? 2. What aspects of your research do you expect will be most useful to informing practice (curriculum development, teacher professional development, work with parents, etc.)? 3. How might your work inform not only instructional practices for this population but also instructional practices for other populations?

Follow-up Activities

We anticipate that this Working Group will attract other researchers interested in issues related to the mathematics education of ELs. Therefore, an important component of this initial meeting of the Working Group will be establishing connections with other interested researchers and building opportunities for future collaborations. We will provide space for new researchers to contribute to our work, to suggest new directions, and to add to the growing body of research on mathematics and ELs. At the first meeting of our Working Group, we will establish an online community using Google applications (Plus, Hangout, Groups, Drive, etc.). Google's applications are freely available and allow for a number of collaborative opportunities, including video conferencing, group messaging, collaborative document development, and shared web and social media space. Through this collaborative Google Community, we will organize follow up face-to-face meetings at conferences such as AMTE and the NCTM Research conference. These meetings, both face-to-face and virtual, will allow us to set concrete goals in preparation for the creation of a Working Group proposal for PME-NA 2016 to continue our work.

In addition to these short-term goals, we have several longer-term goals for this Working Group. First, we would like to seek funding for a conference where we can share the results of our work with other researchers and practitioners. A number of the panelists attended the TODOS: Mathematics for All Conference in the summer of 2014, and we expect to have a strong presence at the June 2016 TODOS conference as well. The TODOS conference brings together both practitioners and researchers and provides a venue for the dissemination and discussion of important ideas and issues related to the mathematics education of diverse groups of learners. A related conference specifically focused on ELs would be of great benefit to a multitude of stakeholders as we continue to examine how to best support ELs. In addition to the conference, we will also propose a special edition of a journal focused on issues related to mathematics education and ELs. This would allow for the broader mathematics educational research audience access to current work being done in this area.

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MODELS AND MODELING

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Modeling has recently received much attention as the Common Core State Standards have included it as one of eight core mathematical practices. The Models and Modeling Working Group builds on a long history of collaboration among researchers studying models and modeling. This year's session will focus on building capacity within the broad community of researchers who are interested in theoretical and practical perspectives related to models and modeling. We will begin by engaging with a modeling activity and some closely related materials. This experience will serve to ground our discussions and provide a concrete common reference for participants. We will also draw the group's attention to a new resource: a course-sized repository of such materials, intended to support studies of learning in the areas of data modeling, statistics, and quantification. Next, we will break out into interest groups to discuss research questions and agendas that can make use of available resources and foster cross-institutional collaboration. We will close by sharing these questions and articulating plans for the coming year.

Keywords: Modeling; Problem-Solving; Design Experiments; Research Methods

The *Models and Modeling* Working Group has been a significant presence at the PME-NA Conference since both the Group and the Conference were inaugurated in 1978. Over that time it has supported the pursuit of high-quality research projects and programs, and it has acted as a means of convening a diverse set researchers to work collaboratively on larger projects and research problems. The historical purpose of the Group is to discuss and extend the ways in which models are used both to learn mathematics and applied science and to study those learning processes in action. We propose convening the Group at PME-NA 37 to build capacity within the Models and Modeling Perspective (M&MP) research community and to establish key components of a research agenda for studying modeling at longer timescales. The immediate occasions for this work are the retirement of a founding voice in the community and the publication of a course-sized body of materials to support M&MP research along a variety of dimensions. As a capacity-building effort, the three sessions of the Working Group will both (a) introduce new researchers to the M&MP approach, community, and materials; (b) develop the M&MP research agenda, articulating questions that the community recognizes as important; and (c) identify opportunities for collaborative projects and proposals in the upcoming year.

Significance of the Models and Modeling Perspective

For nearly forty years, M&MP researchers and educators have engaged in design research directed at understanding the development of mathematical ideas. A fundamental principle underlying this work has been that learners' ideas develop in coherent conceptual entities, called *models*. In this context, models are defined as conceptual systems, which are expressed using external representational media or notation systems, and which are used to construct, describe, or explain the behaviors of other systems—often to understand, predict, manipulate, or improve them. Models are thus conceptual and representational tools that have the additional advantage of illuminating how students, teachers, and researchers learn, develop, and apply relevant mathematical concepts(Lesh & Doerr, 2003; Lesh, Doerr, Carmona, & Hjalmarson, 2003).

Under appropriate conditions, these models can be evoked and expressed: in such settings, they can become objects for reflection by learners and collaborative groups, and they can form the basis for rich communication. In particular, when individuals and groups encounter problem situations with specifications that demand a model-rich response, their models are observed to grow through relatively rapid cycles of development toward solutions that satisfy these specifications. Models are thus powerful elements both for creating educational activities and for conducting research into learning.

Originally, the M&MP tradition was focused squarely on investigating the *development of ideas and knowledge* in teachers and students. Thus, the resources and tools produced were first and foremost designed to *study* idea development (as opposed to serving teaching or curricular goals). However, by producing materials that fostered the rapid development of ideas, M&MP designers also laid the foundation for extremely effective instructional sequences addressing big ideas in important mathematical domains. Over time, however, the M&MP community has refined its techniques for creating situations that provoke students to express and improve their models. The results of this work include a body of *Model-Eliciting Activities* (MEAs), which have consistently proven to support students in the modeling cycles described above. In MEAs, students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product—that is, a model, comprising conceptual structures for solving the given problem. These activities give students the opportunity to create, apply and adapt scientific and mathematical models in interpreting, explaining, and predicting the behavior of real-world systems. Extensive research with MEAs has produced accounts of learning in these environments (Lesh, Hoover, Hole, Kelly, & Post 2000; Lesh & Doerr, 2003), design principles to guide MEA development (Hjalmarson & Lesh, 2008; Doerr & English 2006; Lesh, et. al. 2000; Lesh, Hoover, & Kelly 1992) and accounts and reflections on the design process of MEAs (Zawojewski et al., 2008).

An Example MEA: Volleyball

Students are usually introduced to the Volleyball Problem by reading a "math rich newspaper article" that describes a summer sports camp specializing in girls' volleyball. The newspaper article explains how issues arose in the past because it was difficult for the camp councilors to form fair teams that could remain together throughout two weeks of camp. The students' goal in the volleyball problem is to develop a procedure that the camp councilors can use to form teams that are as equivalent as possible—based on information that is gathered during try-out activities that occur during the first day of the camp. After breaking into groups, students are presented with the problem statement:

> **The Volleyball Problem.** Organizers of the volleyball camp need a way to divide the campers into fair teams. They have decided to get information from the girls' coaches – and to use information from try-out activities that will be given on the first day of the camp. The table below shows a sample of the kind of information that will be gathered from the try-out activities. Your task is to write a letter to the organizers where you: (1) describe a procedure for using information like the kind that is given below to divide more that 200 players into teams that will be fair, and (2) show how your procedures works by using it to divide these 18 girls into three fair teams.

Figure 1. The Volleyball Problem

Along with the introduction and statement of the problem, student groups are given tryout data for a sample of 18 players. This data includes some information that is easily represented in tabular form (player's height, measured vertical leap, 40-meter dash time, etc.), as well as some that is not (players' performance in a spiking trial, and brief summative comments about their strengths and weaknesses from the coach of their home team). These data elements are chosen so as to present fundamental challenges to students, involving the nature of data (categorical versus numerical); scaling (e.g., the vertical leap where "large is good" versus the 40-meter dash where "small is good"); units (vertical leap is given in raw inches; height in feet-and-inches); and so on.

Student groups iteratively develop solutions to this problem in the time allotted—usually 60 minutes for this MEA. Afterwards, the class gathers together for a structured "poster session" event. One member in each 3-person group hosts a poster presentation showing the results of their group. The other two students use a Quality Assurance Guide to assess the quality of the results produced by other groups in the class. These instruments are submitted to the teacher and contribute to assessment in various ways, providing evidence for the achievements of both individuals and groups.

Multi-Tiered Design Research

In parallel with student-focused research using MEAs, researchers also have observed that teachers' efforts to understand their students' thinking involve yet another process of modeling: In this case, teachers engage in building models of student understanding. Although these teacher-level models are of a different category from student-level models, students' work while engaged in MEAs does provide a particularly rich context for teachers' modeling processes. Following this line of inquiry, the M&MP community has also produced tools and frameworks that can be useful to teachers in making full use of MEAs in classroom settings, while also providing *researchers* with insights into *teachers'* thinking.

Finally, at a third level of inquiry, researchers' own understandings of the actions and interactions in curricular activity systems (Roschelle, Knudsen, & Hegedus, 2010) involving students, teachers, and other participants in the educational process can also be studied through the lens of model development. Multi-tier design experiments in the M&MP tradition have done precisely this, involving researcher teams in self-reflection and iterative development as well (Lesh, 2002). Therefore, multi-tier design research involves three levels of investigators— students, teachers, and researchers—all of whom are engaged in developing models that can be used to describe, explain, and evaluate their own situations, including real-life contexts, students' modeling activities, and teachers' and students' modeling behaviors, respectively.

Curricular Materials from the Models and Modeling Perspective

Over the past 10 years, M&MP researchers have continued this direction of work in their own teaching and in partnerships with K-12 classroom teachers. Within the domain of statistical thinking in particular, this effort has flourished, producing resources and tools sufficient to support *entire courses* in several versions and including accompanying materials related to learning and assessment aimed at both student development and teacher development. These course materials were initially focused exclusively on data modeling, engaging with most of the content usually covered in an introductory graduate-level course on statistics and probability for education researchers. But as the collection evolved, modules were added dealing with closely related "big ideas" in algebra, calculus, geometry, and complex systems theory. Furthermore, because the courses supported by these materials were designed explicitly to be used as research sites for investigating the interacting development of students' and teachers' ways of thinking, the materials were modularized so that important components could be easily modified or rearranged for a variety of purposes in different implementations. In particular, by selecting from and adapting *the same basic bank of materials*, parallel versions of the course have been developed for: (a) middle- or high-school students, (b) college-level elementary or secondary education students, and (c) workshops for in-service teachers. When these courses have been taught by M&MP researchers familiar with the underlying theory, they have produced astounding "six sigma" gains (Lesh, Carmona, & Moore, 2009).

In the process of developing curricular materials to focus on modeling, research in the M&MP has investigated ways in which MEAs can be integrated within larger instructional sequences or Model Development Sequences or *MDSs* (Lesh et. al., 2003). MDSs offer classroom groups opportunities to unpack, analyze, and extend the models they have produced in MEAs, as well as to connect their ideas with formal constructs and conventional terminology. This unpacking work helps to ensure the lasting retention of concepts at a level of generality required for flexible use and application in novel situations. It also sets the stage for the critical connection between conceptual development (the centerpiece and focus of MEAs), and procedural knowledge that is also required for students to achieve well-rounded competence in any subject area.

Within an MDS, **reflection tools** support students in stepping back from their modeling processes and reflecting on this work as critical observers of both individual and group modeling behavior. We consider these tasks to be core to the learning processes in MEAs: when students interpret situations mathematically, M&MP research expects that they don't simply engage interpretation systems that are purely logical or mathematical in nature. Their interpretations also involve attitudes, values, beliefs, dispositions, and metacognitive processes. Moreover, the M&MP does not treat group roles and functioning as if they were fixed characteristics that determined students' behaviors. Instead, students are expected to develop a suite of problem-solving personae or profiles they themselves can learn to apply as the situation demands. Reflection tool activities encourage groups of learners to turn their attention to describing individual- and group-level processes, functions, roles, conceptions, and beliefs. Tools to support these activities include Ways of Thinking Sheets, various surveys and questionnaires, Concept Maps, Observation Sheets, Self-Reflection Guides, and Quality Assurance Guides for the products created in MEAs.

In **product classification** and **toolkit inventory** activities students continue the work of abstraction, characterizing and classifying the thinking they have done and identifying links among their solutions to different MEAs and between these solutions and the "big ideas" of the course. **Model exploration or Model extension activities** (MXAs) provide model-rich environments for introducing core concepts and skills from the broader curriculum that students need in order to formulate sophisticated models and present them to a mathematical community. In addition, these activities provide students with a vital opportunity to unpack the work they have done in the MEAs. These may use a combination of pointed YouTube videos and interactive simulations in dynamic mathematics software. (Approximately 50 of these YouTube videos have been produced, with

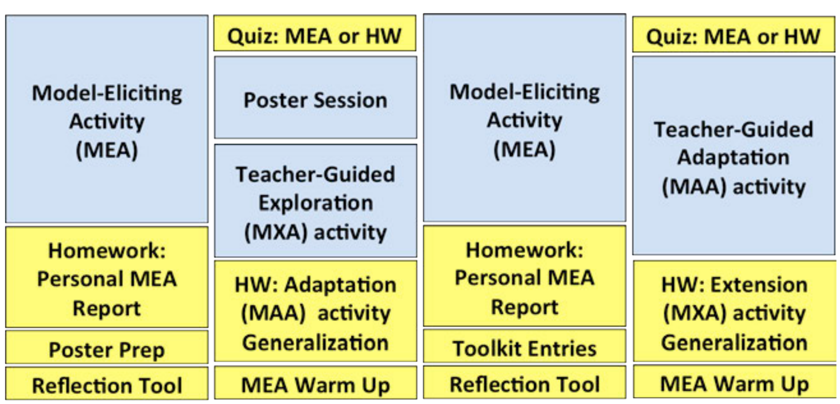

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Figure 2. Example Structure of a Model Development Sequence (MDS). MEAs and other activities represent modular, re-orderable blocks of instructional time.

accompanying simulations. They are currently collected under the ProfRLesh channel.) Finally, **model adaptation activities** (MAAs) allow students to transfer ideas and techniques developed in MEAs to situations calling for similar performances. These MAA activities also provide smaller-timescale modeling scenarios that exercise concepts they have explored in other components of the MDS. They may be pursued individually or in small groups, depending on the nature of the task and the teacher's instructional or assessment goals.

All of these elements of an MDS are designed to be highly modular, to accommodate (as well as to reveal) the needs and intentions of the teacher as they appropriate and adapt the materials for their own use. An example of how this variety of activity types might be laid out in a given unit is shown in Figure 2.

A Course-Sized Resource Repository for Research in Data Modeling and Quantification

Early successes in teaching entire courses with MDS units (e.g., Lesh, Carmona, & Moore, 2009) also demonstrated that these MDSs were highly reconfigurable and re-orderable. The "big ideas" in statistics, data modeling, and quantification could be foregrounded in different orders, leading to a variety of possible longer-term modeling experiences for the students and teacher. In light of these findings, the construction of a course-sized repository became a compelling question for design research. A first-iteration version of this repository will be going live in the course of the Summer of 2015, and several of the activities of the proposed Working Group will focus on making innovative uses of these materials, to gain new illumination on longer-term modeling. At the same time, it is expected that researchers will be involved in the creation of new tools and resources to support research at this longer timescale.

Two important aspects of this effort are (1) a desire to serve the needs of a growing and diversifying M&MP community, and (2) a commitment to flexibility-in-use that we describe as *designing for scale*. In particular, our course materials are presented in modularized and easily modifiable, reconfigurable, and extendable components. Thus, the objective is for teachers and researchers to engage with these materials in a variety of ways, each embodying a unique (teacher- or researcher-level) model of the development of knowledge. The data and evidence produced by this diversity of models will help to support choices among them, to refine emerging assumptions and conjectures, and to iteratively shape the materials of the repository and their presentation. In other words, the development and iterative growth of the research repository should itself be a modeling process. At the same time, we aim for the introduction of this repository of research materials to advance the field by

- facilitating the development, sharing, and testing of new Research Tools for different facets of research into dimensions of learning as they emerge;
- offering a shared setting for the refinement of the *design principles* for research tools to produce evidence about learning at scales higher than the MEA;
- fostering the accumulation of knowledge in teacher and researcher communities, exposing our process and inviting broad participation in constructing the site; and
- encouraging the formation of collaborative communities of teachers and researchers, allowing participants to identify possible collaborators through shared interests in research tools and facets of problems of research or instruction.

Focus Areas of the Models & Modeling Perspective

Here we briefly indicate some of the ways in which M&MP lines of research have contributed in fundamental ways to research on learning, assessment, and teaching. We believe that the publication of a course-sized repository of research materials can facilitate further advances in each of these areas.

Learning

In order to study idea development in the context of real-world problem solving settings, M&MP researchers first used, and then went beyond Piagetian clinical interviews. This spurred an intense period of design research to construct a compelling genre of learning tasks that would (a) stimulate mathematical thinking representative of that which occurs in contexts outside of artificial school settings (Lesh, Caylor, & Gupta, 2007; Lesh & Caylor, 2007); (b) enable the growth of productive solutions through rapid modeling cycles; and (c) leave behind researchable traces of learners' ways of thinking during the process. This line of work produced the notion of Thought-Revealing Artifacts and Model-Eliciting Activities (MEAs) (Kelly & Lesh, 2000; English et. al, 2008; Kelly, Lesh & Baek, 2008). The success of MEAs as a research tool was both enabled by and illustrated by the articulation of a set of six design principles for such activities(Lesh, 2003; Lesh et. al, 2000; Hjalmarson & Lesh, 2008); these principles indicate the key structural and dynamical elements in MEAs as contexts for problem solving. Table below indicates "touchstone" tests for whether each of these six principles has been realized in a given implementation setting.

Table 5. Six MEA Design Principles and Touchstones

Principle	Touchstone Test for its Presence
Reality Principle	Students are able to make sense of the task and perceive it as meaningful, based on their own real-life experiences.
Model Construction Principle	To solve the problem, students must articulate an explicit and definite conceptual system (model).
Self-Evaluation Principle	Students are able to judge the adequacy of their in-process solution on their own, without recourse to the teacher or other "authority figure".
Model Generalizability Principle	Students' solutions are applicable to a whole range of problems, similar to the particular situation faced by the "client" in the MEA.
Model-Documentation Principle	Students generate external representations of their thinking during the problem-solving process.
Simplest Prototype Principle	The problem serves as a memorable representative of a kind of mathematical structure, which can be invoked by groups and by individuals in future problem solving.

Assessment

Although MEAs were developed as contexts for the study of idea development, they also have been shown to be powerful tools to support non-standard approaches to assessment. In part, this potential should be obvious from the model-documentation principle and the emphasis on creating researchable traces of thinking in the form of thought-revealing artifacts. When groups of students produce and negotiate external representations of their emerging thinking on meaningful problems, these artifacts can be expected to tell a rich and compelling story of what these students know and can do (Katims & Lesh, 1994). Furthermore, a common finding of assessment-oriented M&MP research has been that even groups of students whose mathematical abilities are rated low often produce impressive solutions that demonstrate an intuitive grasp of rather sophisticated forms of mathematical reasoning. Finally, when assessment is conceived of as including *formative* assessment, such student performances provide a fertile ground for future instructional work that can build upon intuitive understandings to enable more formalized learning (Borromeo-Ferri & Lesh, 2013; Lesh, Haines, Galbraith, & Hurford, 2010).

Teaching and Professional Development

Because MEAs are designed to engage small groups of learners in autonomous problem-solving work over extended periods of time, they open up the classroom in new ways as a space of reflection for *teachers*. Freed temporarily from the need to facilitate and manage the learning process as it occurs, teachers during MEAs can use the occasion to develop their observational skills and their abilities to detect and interpret their own students' ways of thinking. M&MP researchers have developed a range of Ways of Thinking Sheets and other Reflection Tools to capitalize on these opportunities, giving teachers (and even students) the perspective of analyzing idea development and group functioning.

For teachers, these analyses are compelling because (a) they are authentic and directly relevant to their experience: after all, *their own* students have developed these ideas and constituted these groups; (b) the findings are immediately applicable: analyzing in detail the thinking of one's students in solving a problem instantly suggests instructional strategies that can be enacted as soon as the following class period. These features indicate why viewing MEA implementations as *in-situ* professional development opportunities has been so powerful.

Documentation and Accumulating Shared Knowledge

In addition to the research work described above, the M&MP tradition has also been notable for documenting its progress in design and theory-building, as well as in more conventional, hypothesis-driven research studies. In particular, the community has produced a variety of edited volumes of research over the past four decades, dealing with a range of topics and issues raised in the M&MP work. These include volumes on the design of problem solving tasks, assessment, research design, teacher professional development, the changes in mathematics as a foundation for future work beyond school, and the application of models and modeling perspectives in engineering.

Future Work

Thus, M&MP research has covered an immense amount of ground in the past forty years. Nevertheless, there are also a number of important gaps that need to be addressed in order to maximize the impact that this tradition can have upon mainstream classroom practice. In particular, as described above, in the dimension of student learning much of the M&MP research has framed its interests in terms of *idea development*, where these ideas emerge and interact in the discourse of collaborating small groups and classrooms of learners. This frame is extremely powerful for supporting conceptual analyses of the nature of powerful ideas and the possible relations among them. However, it does not necessarily make claims about student learning or development.

Larger Timescales

When M&MP research engages with questions of the design and implementation of modeling sequences for classroom use that span weeks, months, and even years, it becomes possible (and necessary) to merge the perspective of idea development with that of student learning and cognitive development. Prior work within the M&MP tradition does provide powerful elements of a foundation and framework for this work, yet important theory-building work still remains to be done. Within a 60-minute MEA, idea development in student discourse can exhibit stages of development that Piagetian researchers observe as unfolding in learners' ways of thinking over a span of years. How is this rapid and flexible movement among ideas related to the development of more robust and characteristic ways of thinking?

Work towards creating a research infrastructure to support research at these scales has included an increasing emphasis on a *unit of design* beyond the MEA: at or beyond the MDS level. While any particular classroom realization of an MDS may have the appearance of a fixed curricular progression or teaching trajectory, the design intention is *not* to provide a single, "best" instructional path through the mathematical landscape in question. On the contrary, as with MEAs at the activity level, the goal is to provide a context in which it is possible to study the *range* of possible pathways that can be taken through this terrain, and to understand the fruits that these different choices yield.

Teacher Decision-Making

Though many aspects of design at the MDS level can leverage the history of work at the MEA level, the larger scale introduces new factors as well. Importantly, the teacher's role becomes much more varied and complex at this scale. The observation and analysis of student thinking during MEAs continues to be a critical feature of the teacher's work, but now these observations and findings can be traced as they inform instructional decision making throughout the MDS. In this space, implementations become an extraordinarily rich environment for understanding the values, beliefs, and ways of thinking of *teachers,* on a range of practical, high-consequence matters, such as: (a) what relationships exist between the development of conceptual skills and procedural skills? (b) what is the relative status of student intuitions and official terms, formulations, and algorithms? (c) what documentation or evidence of competence can the activities of an MDS generate at the individual and/or small-group levels, which can factor into classroom assessment practices? (d) how do supports for decision-making about curricular structure and representations of student performance impact teachers' planning behavior? or (e) how can facilitating MEAs, and becoming sensitive and aware of students' ways of thinking, be integrated into teacher preparation programs and communities of teaching practice?

Networks of Connections among Big Ideas

While larger-timescale research requires attention to learners' development and to the curriculum and activity systems represented by students, teachers, classrooms, and schools, it also enables a continuation of the research on "idea development" that characterized MEA-centered work. Indeed, this line of research may only reach satisfying conclusions at this larger timescale. This is because the *meaning* of any particular "big idea" in a subject area derives as much from the relations between that idea and the other ideas of the subject as from the internal structure of that idea in isolation. And, as René Thom once suggested, "the real problem, which confronts mathematical teaching, is not that of rigor, but the problem of the development of 'meaning'…." (Thom, 1973, p. 202). Thus, research at and beyond the level of the MDS, including research at developmental timescales, is alone positioned well to document the way that local modeling efforts in MEA settings can aggregate to support the construction of networks of connections among ideas, yielding structures that afford meaning.

Modeling and Mathematizing: Creating Links to Mainstream Curricular Structures

Finally, carrying the M&MP project forward at larger timescales will produce either connections or collisions with infrastructural elements of curricular structure such as the *Common Core State Standards for Mathematics*, the approaches of leading textbooks, and high-stakes assessments. For teachers it will not be possible to engage in longer-timescale implementations of M&MP materials, without an explicit connection between that work and the dominant structures of the curriculum that constitute the "law" of the instructional land. Fortunately, at least in the *letter* of this "law," modeling occupies a central role. It may be the case that the M&MP's definition of modeling as *mathematizing reality*; that is, "quantifying, dimensionalizing, coordinatizing, or (in general) mathematizing objects, relations, operations, patterns, and regularities which do not occur in pre-mathematized forms" (Lesh, Yoon, & Zawojewski, 2007, p. 346). Nevertheless, if the experience of modeling-as-mathematizing is a powerful one (as we know it is); if logistical and curricular barriers are removed, allowing teachers successfully to adopt materials that encourage this experience (as we believe they can be); and if the resulting curricular sequences are also efficient ways of gaining rich understandings of big ideas (as we believe they are), then it should be possible to make progress in introducing a more authentic version of modeling into the experience of mainstream teachers and students.

Workgroup Outline: Capacity Building

Capacity-building involves establishing both a shared historical sense of interests, theoretical commitments, and methodological approaches on one hand; and a shared research and development agenda for future work on the other. A challenge along the historical dimension of capacity-building in this sense is that the M&MP tradition extends over an extremely long period of time and spans a wide range of subject areas, age-levels, and geographical contexts. To address this challenge, and to lay the foundation for new collaborations, we propose to spend the first day's session engaged in experiencing one MEA and two related MDS activities from the learner's perspective. At prior PME and PME-NA conferences, similar approaches have been successful in orienting participants to the M&MP perspective and grounding discussions in features of student modeling activity.

Agenda-Development: On Day 2 we will break out into facilitated interest groups, based on participants' intended areas of research focus. Importantly, these discussions will actively involve not only the researchers who have played major roles in this development of the M&MP as a field and body of knowledge, but also a range of young researchers new to the field. Pairing these new voices with established researchers will enable an *activation* of the history of the M&MP in forging an agenda for research and development and identifying concrete possibilities for future work and collaborations.

Planning and Identifying Opportunities for Collaboration: On Day 3, interest groups will finalize their discussions on research agendas and report out to the larger group. In this discussion, the group will identify opportunities for organizing collaborative research projects and proposals. "Point people" for each opportunity will be identified, and we will establish the means for sustaining a continued conversation toward acting on these opportunities in the coming year.

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REPRESENTATIONS OF MATHEMATICS TEACHING: STUDYING PRESERVICE TEACHERS' LEARNING FROM WORK WITH REPRESENTATIONS OF PRACTICE

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This working group continues on the theme of prior years, the use of representations of practice in teacher education, and pays particular attention to the design of instruments to observe the learning of preservice teachers from their participation in interventions that use representations of practice.

Keywords: Teacher Education-Preservice; Assessment and Evaluation

Brief History of the Working Group

This is the fifth meeting at PMENA of this Representations of Mathematics Teaching (RMT) working group. The idea of this working group emerged during a series of three-day conferences on representations of mathematics teaching held in Ann Arbor, Michigan every year 2009-2012 (and earlier workshops in 2007 and 2008) organized by ThEMaT (Thought Experiments in Mathematics Teaching), an NSF-funded research and development project directed by Pat Herbst at the University of Michigan and Daniel Chazan at the University of Maryland. ThEMaT originally created animated representations of teaching with cartoon characters to be used for research, specifically to prompt experienced teachers to share the rationality they draw upon while teaching. With the assistance of later funding through a second NSF grant (referred to as ThEMaT II), the Michigan and Maryland groups developed tools and materials to share those animations online and to create further representations of practice, with the purpose of extending the use from its original context of research on the rationality of teaching to the wider context of research and development in teaching and teacher education. These efforts led to the development of the Lesson*Sketch* platform (www.lessonsketch.org), which we see as a prototype of a virtual laboratory for the study of teaching by practitioners and researchers. The Ann Arbor workshops and conferences conducted between 2008 and 2012 were conceived to begin creating a community of researchers and teacher educators who were interested in the use of representations of teaching and the analysis of data collected in response to these representations. These conferences gathered developers and users of all kinds of representations of teaching (including video, written cases, dialogues, photographs, comic strips, and animations) to present their work and discuss issues that might be common to using these representations in teacher education and education research. In 2010 we proposed a working group at PMENA in Columbus, again in Reno in 2011, Kalamazoo in 2012, and Chicago in 2013. These working groups focused on various issues associated with the elaboration and investigation of a pedagogical framework for teacher development that makes use of digital representations of teaching.

In 2013 Chazan and Herbst obtained a new NSF grant (referred as ThEMaT III; Chazan & Herbst, 2013) that enabled the creation of fellowships for mathematics teacher education faculty to engage in a process of creation of practice-based multimedia teacher education materials and their implementation in online or blended teacher education environments. Several earlier attendees of our PME-NA Working Group became fellows, to the point that now there are 12 fellows participating in this project, and they in turn have been recruiting colleagues to form inquiry groups around them. At this point among our fellows and their associates and our projects, we have a community of nearly 40 people working in 23 different states. We are all using the Lesson*Sketch* platform, developed by project ThEMaT II and in operation since 2011 as a prototype of a platform with which to author and disseminate online content for teacher education. The meeting of this year's working group is both an opportunity for fellows and inquiry group members to work face to face and for them to share their work with other individuals who might possibly be interested. The meetings will showcase the work that fellows of the project are doing and engage attendants in discussing how these online experiences with representations of practice can create opportunities for research on preservice teachers learning.

Specifically, this year's work will focus on how these online experiences with digital representations of practice can enable research on preservice teacher's learning of the practices they need to be able to do in order to teach mathematics.

Issues in the Psychology of Mathematics Education that Will Be the Focus of the Work

The work of this group relates to the third of the four enduring challenges in mathematics education identified by conference organizers--the role of assessment in teaching and learning--particularly as it relates to "innovative programs or interventions designed to address an enduring challenge" in the case of preservice teacher preparation. The use of representations of practice in teacher preparation is one of those innovative practices; Herbst, Chazan, Chen, Chieu, and Weiss (2011) provided a conceptualization of these representations of practice and argued for how work with representations of practice can serve to bring a laboratory-like approach to university-based classes for teachers, especially methods classes. This laboratory approach in teacher education, as originally conceived by John Dewey, was meant to engage prospective teachers with problems of practice in a context where they did not yet have to respond to all the demands of actual settings.

Teacher educators have constructed interventions into teacher education that use various representations of practice including students' written work, classroom video records, and storyboards; they have also made claims that such interventions create opportunities for preservice teachers to learn aspects of the work of teaching mathematics--including how to notice students' thinking and how to respond to students. We are interested in ways in which those claims can be verified through assessments of the competencies of preservice teachers who have been involved in such opportunities. The relationship between this working group and the enduring challenge of assessment in teaching and learning concerns therefore the challenge of assessing preservice teachers' learning of the practice of mathematics teaching--specifically, we endeavor to create a space for discussion of what kinds of assessments might well capture the kind of learning that preservice teachers can experience when they engage with representations of practice.

As noted above, the ThEMaT III project (Chazan & Herbst, 2013) has recruited a cadre of mathematics teacher educators who are designing interventions that use digital representations of practice in mathematics teacher education. These activities vary in terms of what they engage preservice teachers in and what kinds of opportunities to learn they might afford. In relation to that activity, we pose the question: How can we assess those opportunities to learn as well as the actual learning that the preservice teachers have done? In the 2010 PMENA discussion paper, Herbst, Bieda, Chazan, and González (2010) briefly reviewed the literature on the use of video records and written cases in teacher education. They noted that classroom scenarios sketched as cartoon animations have begun to be utilized for those purposes and argued that they have affordances that are distinct from those of video and written cases (see also Herbst, Chazan, Chen, Chieu, & Weiss, 2011). They argued that the increased capabilities of information technologies for creating, manipulating, and collaborating over multimedia point to a promising future for teacher development assisted by representations of practice. Yet the features of novel media and their use with digital technologies, for example in online or blended (face-to-face and online) interactions, may require other pedagogical strategies for teacher education that have not been sufficiently identified and explored. This year's gathering of the working group will showcase how four teacher educators affiliated with the ThEMaT III project (known as Lesson*Sketch* fellows) are using the authoring capabilities of the platform to create opportunities to learn for their preservice teachers. The working group will examine how those interventions can provide context for research on teacher learning, particularly as regards to the assessment of teacher learning. The following paragraphs describe the particulars of the instructional modules the four fellows are developing.

Dr. Joel Amidon, from the University of Mississippi, has designed an intervention that gives access to a common space (Azul's classroom) where mathematics problems and problems of practice can be shared, discussed, and analyzed by pre-service teachers (PSTs). The result is a module that can be easily modified to utilize rich mathematical tasks, student work surrounding such tasks, or mathematics teaching scenarios that presented a problem of practice to either the PST or the mathematics teacher educator. Users of the module view tasks, student work, and scenarios through the analysis tools designed by the Teachers Empowered to Advance Change in Mathematics project (see McDuffie, Foote, Drake, Turner, Aguirre, Bartell, & Bolson, 2014). This lens requires users to account for the central mathematics of a task, the design of the task, the learning that can be observed, the actions of the teacher, the funds of knowledge students bring to the task, and the affordances and constraints on power and participation of the students. The template design of this module allows for various questions to be asked such as: (a) What are the features of a teaching scenario that PSTs and MTEs design into a representation? (b) What needs do MTEs seek to meet via this module? How do MTs and PSTs take up the TEACH Math analysis tool beyond the Lesson*Sketch* environment?

Dr. Orly Buchbinder from the University of New Hampshire has designed an intervention that aims at strengthening teachers' subject matter knowledge and understanding of proof as well as pedagogical knowledge for teaching it. The interactive, media-rich module "What can you infer from this example?" aims to support prospective teachers in developing and enhancing these types of knowledge. The module is grounded in research on secondary students' understanding of proof (Buchbinder & Zaslavsky, 2009) and involves multiple components that engage pre-service teachers with proof tasks and representations of classroom interactions around the same task. These representations provide pre-service teachers with opportunities to envision how reasoning and proof activities might unfold in regular classroom environment; anticipate and analyze student thinking; plan and enact possible teacher responses; discuss and evaluate implications of different teacher moves on students' engagement with proving. It is expected that interaction with the module will promote prospective teachers' content and pedagogical knowledge of reasoning and proving. The potential growth of content knowledge may include enhanced proficiency in analyzing the logical structure of mathematical statements and evaluating the validity of different kinds of reasoning in determining whether a certain statement is true or false. The potential growth of pedagogical knowledge will be evident in prospective teachers' proficiency to anticipate students' responses to proof-related tasks, analyze students' reasoning and evaluate whether it is logically sound, and pose follow-up questions that either challenge or support students' reasoning. The evidence will be collected through analysis of prospective teachers' responses as they progress through different module components, analysis of their forum interactions and a questionnaire administered after completion of the module.

Dr. Janet Walkoe from the University of Maryland has designed an intervention to engage pre-service teachers in the practices of questioning and listening to student thinking to interpret their understanding of proportions. There is evidence that a sustained focus on students' thinking has a positive impact on teachers' beliefs and instructional practices (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). The centerpiece of this module is a video of a clinical interview with a middle school student solving a problem related to proportions. Before and after watching the clinical video, preservice teachers examine a representation of a hypothetical classroom dialogue based on the clinical interview. During and following the clinical interview, the pre-service teachers have an opportunity to approximate the practices of listening to students' thinking and interpreting their understanding. After examining the follow up hypothetical classroom dialogue, pre-service teachers approximate the practices of questioning. The goal of the module is to improve pre-service teachers' practices of questioning middle school students about their mathematical thinking.

Dr. Rob Wieman, from Rowan University, is using the notion of psychological schema to describe the changes he seeks to produce in the preservice teachers under his care. Cognitive scientists have developed the idea of schema to help explain how experts make sense of complex situations and decide how to act. Wieman's intervention is designed to assess and change the schemas that teacher candidates use for making sense of and responding to student thinking. He notes that many mathematics teachers and teacher candidates have a relatively simplistic schema for diagnosing student thinking and formulating a response (Carter, 1990). In addition, the schema they do have is informal and idiosyncratic and that they do not describe it using a language that is widely shared in the profession (Carpenter, Fennema, Franke, Levi, & Empson, 2000; Carpenter, Fennema, Peterson, & Carey, 1988). The central activity of the module Wieman has created introduces teachers or teacher candidates to an expanded schema in the context of launching a rich mathematical task. After doing the task themselves, and anticipating student responses, participants are presented with a variety of student reactions. In a series of multiple-choice questions, participants are asked to diagnose student thinking and suggest a response to it. The answer choices represent a set of categories and teacher moves, categories that can act as an expanded schema. Pre- and post-tests measure whether this experience has expanded participants' schema, and whether it has supported groups of participants in developing a common language around student thinking and teacher moves. It is hypothesized that situating the introduction of this specific, expanded schema in the context of a teaching episode, and forcing participants to think about the categories of the expanded schema while making teaching diagnoses and decisions will support participants in expanding their own schema.

These interventions have in common the goal of helping clients, pre-service teachers and in some cases inservice teachers who return for masters degrees, learn to engage in particular practices of teaching. Some of the modules focus on practices that a teacher would typically engage in outside of instruction, such as the analysis of mathematical tasks; while others focus on practices that a teacher would typically engage within instruction, such as listening and responding to students' mathematical thinking. In all cases, the clients have the opportunity to approximate such practices by anticipating what they could do in particular circumstances. In Dr. Amidon's module, pre-service teachers collectively analyze a mathematical task using annotation tools within shared documents. In other instances, the fellows have used the capabilities within the online multimedia environment to capture clients' approximations. For example, in Dr. Walkoe's module, pre-service teachers approximate the practice of listening and responding to students' thinking by answering questions about what they noticed about a student's reasoning in the context of a clinical interview and adding teacher dialogue to a pre-existing classroom depiction. In Dr. Wieman's module, teachers approximate the practices of listening and responding by answering multiple choice questions whose answers represent several categories of student thinking and responding moves respectively. In Dr. Buchbinder's module, teachers approximate the practice of responding to student thinking by elaborating on how a teacher could challenge students' example-based conjectures. While there is a common thread of engaging clients in the approximations of practice, there is much variation across the fellows' project in terms of the goals for learning as well as the types of evidence that could be captured to support claims of client learning. In the cases discussed, fellows look for evidence of learning in the immediate context of PST's response to the activities presented. But evidence of learning could and should be gathered outside of these learning activities as well--in the PST's simulated practices (in front of their peers), as well as in their classroom practice (in their practicum placements). The working group will discuss the ways in which various artifacts collected by the fellows in the context of their clients' navigation of the modules as well as evidence from the work they do outside of the modules could be analyzed to gauge what their clients have learned.

Two sets of resources are planned as potential supports for research on the PST's learning from their participation on the modules. One consists of a set of suggestions for how to code the artifacts that PSTs create when they participate in the online modules created by fellows. Within this set we encounter ways of examining PSTs entries in forums and discussions using elements of systemic functional linguistics to identify instances of reflection and consideration of alternatives, as illustrated by Chieu, Kosko, and Herbst (2015). Within that set we also encounter ways of coding the changes shown by PST's in the representations of practice they create, for example when they use the Depict software to author a classroom scenario; we bring in some lessons learned from a study by Rougee and Herbst (2015) in which storyboard representations of practice were compared with representations realized in the form of text dialogues. These sets of resources for research can support the way the fellows and other users of Lesson*Sketch* are looking at their clients' work on online modules.

A second set of resources takes into consideration that PSTs learning must extend beyond the particular realizations they have in the context of online modules and into their practices with students. How can those practices be inspected for evidence of learning? Obviously one such way is to conduct observations of the instruction that the PSTs lead, but in the working group session we want to give attention to how the authoring and sharing tools of the Lesson*Sketch* platform can provide resources for PSTs to narrate what they did in their lessons. We plan to include two demonstrations in the second meeting of the working group. One of those demonstrations is the use of Depict as a resource for eliciting from PSTs a narration of how their lesson went. Depict is an authoring tool that permits the representation of lessons in the form of storyboards with cartoon characters. We believe that in the event that observations or video recordings of lessons are not available, teacher educators can use Depict in the context of interviews with preservice teachers, representing with it how the lesson developed. But in that context it is important to understand how the activity of jointly depicting a lesson can be described as compatible with an interview--what should the interviewer say and do in order to elicit from the interviewee information about a lesson they taught? A second set of resources comes from another authoring tool, Plan, with which teacher education researchers can create scenario-based questionnaires. Teacher educators can use them to create self reporting instruments that PSTs can use to document what they do. The literature on classroom research has used self report instruments such as surveys (e.g., Blank, 2002), logs (e.g., Rowan, Harrison, & Hayes, 2004), and text based vignettes (e.g., Stecher et al., 2006) to collect information on teachers' practices. For example, in their lesson log, Rowan et al. (2004) asked their respondents to focus on one student and check among options for the question "What was the target student asked to do during the work on number concepts?" and given options such as "Listen to me present the definition for a term or the steps of a procedure" or "explain an answer or a solution method for a particular problem" (p. 123). While some of those statements of practice may be reliably conveyed through text, others might better be conveyed through a combination of text and graphics. Plan is an authoring tool that can be leveraged to author any one of those self report instruments, with the added possibility that the practices that the respondent can be asked to comment on can also be rendered graphically. We envision that the second session will include discussions of self reporting instruments that could be used to get pre-service teachers to document how their classroom teaching performance shows evidence of their learning in the context of the interventions designed by the fellows.

Plan for Engagement

The working group will meet during PMENA in East Lansing to explore how teacher educators can produce evidence of their PSTs learning from their use of representations of practice in practice-based teacher education. This will begin with a presentation by the organizers of the variety of ways that teacher educators have developed materials to carry out practice-based pedagogy within an online multimedia environment. In the first session we will start with a brief and broad description of modules generated by twelve mathematics teacher educators, highlighting the variety of ways the materials provide pre-service teachers opportunities to learn about and engage in instructional practice. Next, the four teacher educators whose work is featured (Amidon, Buchbinder, Walkoe, and Wieman) will each provide a 10-minute presentation to describe the goal and contents of their modules. Each presenter will share materials they have developed to target distinct instructional practices including analyzing mathematical tasks, listening and responding to students' thinking in the context of a launch, listening to and interpreting students' mathematical thinking in the context of a clinical interview, and developing conjectures from examples. They will also provide artifacts of client's approximations of those practices. Following these presentations, during the first meeting of the working group, the group will discuss what are the learning claims that educators would like to make and will engage the attendants in identifying the evidence they would be looking for and in brainstorming the kinds of instruments that should be created to document such learning.

In the second meeting of the working group, members of the Lesson*Sketch* team will illustrate how the tools in the platform can be used in the context of (1) interviewing a PST about a lesson that they taught, (2) creating a lesson log that reports on recurrent practices the PST engages in, and (3) creating a scenario based questionnaire that the PST can use to report on the quality of their practices. They will then engage the audience in trying out the platform in one of those assessment contexts. Attendants will be able to work with a specific module and the fellow who developed the module, and in that context they will be able to try their hand at designing an assessment instrument or activity that could be deployed in the LessonSketch platform.

In the third meeting of the working group, fellows and attendants will share the assessment ideas they developed. There will be an opportunity to discuss features of the platform that enable (or perhaps constrain) the deployment of those assessment instruments.

Anticipated Follow Up Activities

The ThEMaT III project uses inquiry groups centered around the LessonSketch fellows to implement and further develop the modules. Thus, beyond the immediate goal of learning and creating community around common activities, the working group has rather practical goals for follow up activities. We expect that participants will leave the conference poised to use these modules in teacher education and professional development settings or to collect and analyze data regarding how well the materials created windows into teacher learning. We also expect that participants will be interested in reconvening at future PMENA conferences to share information about the effectiveness of interventions that may inform further refinement of these materials for future use. Over time we anticipate this kind of development could converge to have commonly available practice-based curriculum materials that could be used for the purposes of teacher preparation and professional development. We expect that the instruments used to collect this evidence will be made available in repositories such as shared folders and collections in the Lesson*Sketch* platform.

Building on Prior Work

The proposed work builds on prior work of this group in that it continues to address issues on teacher preparation for which representations of practice are usable. It also adds an entirely new direction, possibly useful to other PMENA researchers interested in studying learning, of how to instrument the documentation of learning gains by preservice teachers.

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SPECIAL EDUCATION AND MATHEMATICS

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Approximately 7% of children and adolescents have a mathematical learning disability (MLD) and another 10% show persistent low achievement in mathematics despite average abilities in most other areas. Research on these two groups of students with math difficulties (MD) has traditionally focused on procedural skills. This working group is rooted in a twofold premise: (1) students with MD are capable of and need to develop conceptual understanding and mathematical reasoning skills, and (2) special education instruction and assessment needs to transition toward this focus. Participants will (a) continue to develop and refine the research agenda for the group, (b) brainstorm specific research questions that will address that agenda, (c) explore research methodologies that can answer the potential research questions, (d) discuss collaborations to carry out these studies, and (e) set up a plan for publishing and securing funding.

Keywords: Equity and Diversity; Learning Trajectories; Assessment and Evaluation

Overview of the Working Group

The purpose of our working group is to explore issues of research around the intersection of mathematics education and special education. Substantial work exists that focuses on mathematical cognition, development, and reasoning of students in general education. However, much less is known about the mathematical development of students with disabilities or how to support the learning of these students. The absence of research addressing this subset of students may be due in part to the incompatibility of the theoretical perspectives driving research and practice of mathematics education versus special education. Our working group is designed to create sustainable opportunities for researchers and practitioners interested in learning disability/difficulties and mathematics to move this important dimension of the psychology of mathematics education forward.

Understanding how students with disabilities/difficulties develop mathematics concepts and skills has several implications for both research and practice. First, practitioners in both general and special education can gain essential knowledge of how to approach instruction for diverse learners who may rely upon alternative pathways of understanding mathematics concepts. Second, researchers stand to gain a richer understanding of how cognitive processes involved in learning essential mathematical concepts emerge by studying atypical development. Finally, active study of the development of mathematics concepts and skills for students with disabilities provides both researchers and practitioners with mechanisms for moving toward a methodological focus on pedagogy rooted in assessment of what students with disabilities are capable of learning.

For the purposes of continuing the conversation around mathematics in special education, this group is concerned with students who have significant issues with mathematics, including:

- students with learning disabilities specific to mathematics
- students with cognitive differences in how they understand and process number
- students who are placed in special education and have difficulties with mathematics

We refer to these students as having "math difficulties" (MD) in the remainder of this paper.

History of the Working Group

Our PMENA/PME working group has met three times; each year our group had good participation of both returning and new members. In 2012, 15 researchers (faculty and graduate students) and 2 practitioners met during PME-NA in Kalamazoo, MI. This first meeting was specifically focused on better understanding mathematical learning disabilities (MLD). The working group began with a discussion of the issues around identification and definition of MLD. In particular, the group discussed the unique characteristics of students with MLD (e.g., slow speed of processing despite average reasoning; fundamental issues with number sense; over learning of procedural knowledge at the expense of mathematical reasoning) and implications for instruction and assessment. We took up a theoretical stance that positioned disability as an issue of diversity and considered the origin of the disability as the inaccessibility of instruction rather than a defect within the individual. Members shared videotapes of various students with MLD solving problems in assessment and teaching situations and discussed the need for teachers to target and teach toward the specific mathematical strengths and weaknesses demonstrated by the student. We further discussed at what point(s) the learning paths of students with MLD may differ from what is documented among students in general education, how existing developmental trajectories may or may not fit the population of students with MLD, and the need to expand or further document current trajectories to include students with MLD. Moreover, discussions focused on issues surrounding motivation related to the design and use of instruction, mathematical tools, and mathematical tasks. A rich discussion was held concerning the nature and sequencing of mathematical tasks, the use of concrete and pictorial representations and the extent to which they are and are not supportive of the abstraction of mathematical concepts for this population, and the need for increased research to inform the creation of practitioner tools and resources.

In the first year of our working group our focus was specifically on MLD - those students with a biological and cognitively-based difference in how their brain processes numerical information. Based on our discussions during the first year of our working group we decided to expand from a narrow focus on MLD to a more inclusive focus on students in special education who struggle with mathematics. This not only avoids the definitional issues at the forefront of the field (i.e., the lack of assessments to accurately identify students with MLDs and the resulting conflation of low achievement and MLDs), but also more accurately reflects the diversity of interests of the members of this group.

In the second year of our working group (2013) fourteen participants focused on a collaboration that was a result of the contacts that were made during the first working group. In this collaboration two faculty members worked together on a teaching experiment about fraction knowledge, a compliment to a 2012 funded National Science Foundation (NSF) CAREER project (Hunt, 2012). Their collaboration resulted in each bringing unique expertise; the mathematics education faculty member brought insight into the mathematical thinking of the student, while the special education faculty member brought insight into learning differences. The goal of the teaching experiment was to document how the foundation scheme of unit fractions (1/n) evolves in the mathematical activity of two cases of students with learning disabilities. The students' evolving conceptions were supported by constructivist-oriented pedagogy. Video data segments (i.e., each girl's conceptualization of the multiplicative nature of and inverse relation ($1/m > 1/n$ if $m < n$) among unit fractions; the girls' solutions to novel problems) from this project served as starting points for discussions in the subsequent PME-NA working group meeting. Specifically, working group members used the video segments and descriptions of the collaboration as a springboard for discussing possible research questions and methods of data analysis to employ in future, collaborative work. It is collaborations like these that this working group is designed to foster.

In 2014 we met as a working group at the joint PME and PME-NA conference in Vancouver. There our working group continued to expand to 25 members, now including international members from outside of North America. During the meeting, two working group members (one from math education and one from special education) shared a multiplicative reasoning assessment tool (*Multiplicative Reasoning Assessment Instrument,* Purdue Research Foundation, 2011) resulting from their NSF-funded research project (Xin, Tzur, and Si, 2008). Upon examining this instrument, the group discussed alternative ways for assessing students with MD and implications for intervention development. As a result of our collaboration in Vancouver, we had two main accomplishments. First, as a group we identified three research subgroups: (a) cognitive characteristics of students with MD, (b) interventions for students with MD, and (c) teacher preparation or professional development, that represented the interests of the members. Each research subgroup identified pertinent research questions and an agenda for further collaboration. Second, as a group, we proposed an idea for developing a proposal for a special issue to be published in a special education journal to address the research around the intersection of math and special education. Later in the year, members of the working group have developed a proposal and identified potential contributing works from the working group members. In addition, we have invited a well-known scholar to be part of the guest editorial board for this special issue. Currently, it is in the process of negotiating with a special education journal outlet for potential publication of this special issue

In this coming year we plan to continue and expand collaborations between members of this working group, by focusing discussions around two central themes: (a) math concept development and corresponding methodologies for studying its emergence in students with special needs, and (b) designing research questions and writing a research plan around this topic. We invite interested researchers and educators to participate. We anticipate that there will be several new members to this working group. Our prior working groups' members have continued to grow and generate momentum around research at the intersection of special education and math education, both in North America as well as internationally (PME working group papers 2014, 2015). We hope that the PME-NA working group can provide a synergistic place for an ever-growing number of researchers to come together to begin collaborations around these challenging issues.

Issues Relating to Psychology of Mathematics Education

Historically, special education researchers and teachers focused almost exclusively on students' mastery of procedural skills, such as basic number combinations and ability to execute mathematical algorithms (Jackson & Neel, 2006; Fuchs et al., 2005; Geary, 2010; Swanson, 2007; Kameenui & Carnine, 1998). A recent literature review comparing instructional domains for students with disabilities found that the majority of research conducted in the field of special education addressed basic computation and problem solving, with the primary focus placed on mnemonics, cognitive strategy instruction (e.g., general heuristic four-step strategy: read, plan, solve, and check), or curriculum-based measurement (Van Garderen, Scheuermann, Jackson, & Hampton, 2009). Instructional practices either focused on task analysis (breaking up skills into decontextualized steps that need to be memorized and followed), flash cards, or general heuristics that do not help with domain knowledge learning and concept development (Cole & Washburn-Moses, 2010). In particular, the focus on primarily procedures-driven instruction and rote memorization of skills seems to result in students' incomplete and inaccurate understanding of fundamental mathematics concepts as well as a lack of retention and/or transfer (Baroody, 2011).

Importance of Both Conceptual and Procedural Knowledge

Crucial for rich mathematical understandings that enable retention and transfer of fundamental concepts is the iterative development of conceptual understanding along with procedural proficiency (Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson & Koedinger, 2005). Rittle-Johnson and Alibali (1999) noted that conceptual knowledge supports procedural generalization. In particular, conceptual knowledge could aid children in mindfully avoiding the use of procedures that fail to work in novel situations. Additionally, an ability to understand and manipulate different mathematical representations to conceptually navigate a mathematical context contributes to conceptual understanding and procedural skill (Ball, 1993; Kaput, 1987; Rittle-Johnson et al., 2001). It seems that any investigation into mathematical cognition, whether related to disability or not, must fundamentally engage with issues of conceptual understanding (Hunt & Empson, 2014).

A focus on procedural skills limits students with disabilities' access to the general education curriculum, which is a requirement of the Individuals with Disabilities Educational Improvement Act (Maccini & Gagnon, 2002). In mathematics, access to the general education curriculum means addressing problem-solving, mathematical modeling, higher order thinking and reasoning, and algebra readiness as required by the new Common Core Standards (CCSSI, 2012). To accomplish these Standards, mathematics educators need to actively engage students in making conjectures, justifying and questioning each other's ideas, and operating in ways that lead to deep levels of mathematical understanding (Kazemi & Stipek, 2001; Lampert, 1990; Martino & Maher, 1999; Yackel, 2002).

Conceptual Diagnosis Based Pedagogy

A pedagogical approach to be explored and advanced during this Working Group's meetings is one that focuses on promoting conceptual learning in students with MD. This approach is rooted in a constructivist stance (Piaget, 1985; von Glasersfeld, 1995), particularly the notion of assimilation, which stresses the need to build instruction on what students already know and are able to think/do. That is, teaching needs to be sensitive, relevant, and adaptive to students' available ways of operating mathematically (Steffe, 1990). To this end, teachers must learn how to: (a) diagnose students' available conceptions, and (b) design and use learning situations that both reactivate these conceptions and lead to intended transformations in these conceptions.

Building on Simon (2006)'s core idea of hypothetical learning trajectories, Tzur (2008) has articulated such an adaptive pedagogy, which revolves around the Teaching Triad notion: (a) students' current conceptions, (b) goals for students' learning (intended math), and (c) tasks/activities to promote progression from the former to the latter. Key here is that in designing every lesson one proceeds from conceptual diagnosis of the mathematics students are capable of thinking/doing. That is, assessment methods need to focus on dynamic (formative) inquiry into student understandings, as opposed to on testing correct and incorrect answers per se. This day-to-day diagnosis, obtained via engaging students in solving tasks and probing for their reasoning processes, gives way to selecting goals for students' intended learning. Building on this diagnosis, a mathematics lesson begins with problems that students can successfully solve on their own, which Vygotsky (1978) referred as the Zone of Actual Development (see also Tzur & Lambert, 2011). Recent studies of mathematics teaching in China (Jin, 2012; Jin & Tzur, 2011) revealed a strategic, targeted method, bridging, which is geared specifically toward both: (a) reactivating mathematical conceptions the teacher supposes all students know, and (b) directing their thinking to the new, intended ideas.

Exemplar Research Activities with Students with MD

Since 2008, two members of this working group (one from math education and one from special education) have been working collaboratively on a federal funded grant project (Xin, Tzur, & Si, 2008). This project integrated research-based practices from mathematics education and special education and was aimed to promote multiplicative reasoning and problem solving of elementary students with MD. As an outcome of this collaborative project, the research team has developed an intelligent tutor, PGBM-COMPS. The PGBM-COMPS intelligent tutor draws on three research-based frameworks: a constructivist view of learning from mathematics education (Steffe & D'Ambrosio, 1995), data (or statistical) learning from computer sciences (Sebastiani, 2002), and Conceptual Model-based Problem Solving (COMPS) Xin, 2012) that generalizes word-problem underlying structures from special education.

Rooted in a constructivist perspective on learning (Piaget, 1985; von Glasersfeld, 1995), the PGBM part of the intelligent tutor focused on how a student-adaptive teaching approach (Steffe, 1990), which tailors goals and activities for students' learning to their available conceptions, can foster advances in multiplicative reasoning. This approach eschews a deficit view of students with learning disabilities. Rather, it focuses on and begins from what they do know and uses task-based activities to foster transformation into advanced, more powerful ways of knowing. On the other hand, intelligent computer systems can play an important role in students' learning by effectively modeling their thinking and dynamically recommending tasks tailored to their conceptual profiles. Going hand-in-hand, the COMPS part of the program (Xin, 2012) generalizes students' understanding of multiplicative reasoning to the level of mathematical models. At this stage, students no longer rely on concrete or semi-concrete models for problem solving; rather, the mathematical models directly drive the solution plan.

The collaborative research team has conducted several piloting studies to field test the PGBM-COMPS intelligent tutor with elementary students with MD. The preliminary studies have shown promising results—participating students with MD who interacted with this intelligent tutor not only enhanced their problem solving skills on a researcher-designed criterion test but also a norm-reference standardized test (Xin et al., 2013). In addition, the results of these piloting studies have shown success in promoting students' substantial conceptual advances (e.g., concept of number, multiplicative reasoning).

In another project, another working group member is documenting learning trajectories of elementary school children with MD as they come to understand fractions as quantities (Hunt, 2012). In its first year, the goal of this work is to produce models of children's key developmental understandings, or critical transitions in how children may conceive of a mathematical idea (Simon, 2006), along a carefully sequenced combination of tasks and varying instructional guidance necessary to grow conceptual knowledge not yet well formed (Daro, Mosher, and Corcoran, 2011). The researcher will illustrate varying level of a student's informal notions of fractions, how the mathematical ideas can be elicited, the grappling of ideas a student might experience, and how more solidified notions of mathematics form through a student's activity (i.e., external manipulation or representation; internal mental activity; actions; strategies for problem solving). It is the goal of this research that the mapped trajectories can assist educators looking to individualize instruction for students with MD and improve students' conceptual understanding of fractions.

As part of this grant's Year 1 activities, the research team has documented an initial trajectory from semi-structured interviews with 50 second, third, fourth, and fifth graders with MD. Interviews followed a protocol that established a basis for questioning but also allowed for maximum researcher flexibility to fully examine student thinking. Information pertaining to the children's specific needs was also collected to allow for an examination of any trends occurring across similar cognitive profiles. The constructed trajectory is also being tested with a smaller subset of four students and an expanded group of tasks from which the trajectory is based. Data collected from mini-interventions will undergo analysis (Siegler, 2006) to confirm the robustness of the preliminary trajectory. This grant's Year 2 activities will take up teaching experiment methodologies, much like those used in the collaborative pilot that resulted from this working group, to document how children with MD construct conceptions of fractions.

Working group participants will use artifacts from projects described above as possible starting points to illuminate and further explore possible applications of student-adaptive pedagogy (conceptual diagnosis based) as well as conceptual development trajectories in the design of effective/efficient assessment and intervention programs for students with MD.

We believe such approaches are complimentary and have the potential to become core methodological approaches for teaching and studying the conceptual understandings of students with MD. In a similar way, this working group provides a venue to give and receive feedback on ongoing cutting edge empirical work, which is reshaping how students with MD are researched.

Plan for Working Group

The aim of this working group is to facilitate collaboration amongst researchers and educators concerned with mathematics education for students with MD. The main goal is to promote basic research into how students with special needs think about mathematics and develop mathematical concepts. This working group intends to accomplish the following: (a) continue to develop and refine a research agenda for the group, (b) brainstorm specific research questions that will address that agenda, (c) explore research methodologies that can answer the potential research questions, (d) discuss the logistics of collaborations to carrying out these studies, and (e) embark upon collaborations leading to additional publication and funding opportunities.

These goals are further outlined across sessions as follows:

Session 1: Introductions and Progress-to-date

GOAL: To identify participant's affinity for established sub-groups and identify potential new sub-group possibilities.

- Prior members will briefly introduce the working group's history and describe the collaborations that have emerged in prior years.
- Participants will each introduce themselves and their current research and interest in students with MD.
- As a large group we will discuss whether the previously established sub-groups (i.e., *cognitive characteristics of students with MD, interventions for students with MD, and teacher preparation or professional development*) are representative of the interests of the members. If not, new sub-groups will be created.
- Participants will join the sub-group that is most aligned with their research interests and begin to discuss the overlap in research agenda and current work and potential research agendas the sub-group is interested in exploring.

Session 2: Methodologies / Sharing Data Artifacts

GOAL: Engage in discussions around methodological and analytic approaches to studying MD with examples from working group member's research artifacts.

- Within sub-groups we will:
    - Articulate the overarching research agenda for the sub-group based on the previous day's discussion,
    - Articulate potential research questions that the group would like to address through collaborative work.
    - Explore a variety of methodological and analytic approaches that can be leveraged to address the research questions. To accomplish this, subgroups may:

        - View artifacts (e.g., video, written work, etc.) of work already conducted to highlight possible methodologies for future studies.
        - Discuss which methodologies and analytic approaches best align with the proposed research agenda.
- Within a whole group discussion we will:
    - Share out potential research agendas and methodological discussions from small group discussions

Session 3: Planning and Writing
GOAL: Establish next steps for both the sub-groups and the whole working group.

- Within sub-groups we will:
    - Work on written product of research agenda
    - Develop a project plan for any cross-institutional collaborative work that has developed from the subgroups.
- Within a whole group discussion we will:
    - Share progress and commitments from small group discussion
    - Finalize a plan for individual groups to continue updating progress to the larger group (website).
    - Determine what our next whole group meeting will entail (e.g., PMENA working group for the following year.)

Anticipated Follow-up Activities

Throughout the year, the members of this working group will continue working on research problems of common interest. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations. We will continue our effort in disseminating the collaborative work resulting from this working group to broaden its impact in the field of mathematics education for students with MD.

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TEACHER NOTICING: MEASURING A HIDDEN SKILL OF TEACHING

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The practice of teacher noticing has been the subject of considerable attention within the research community; however, conceptualizations of noticing appear somewhat variable. Similarly, measurements of teachers' noticing capacities tend to vary as well. This working group proposal is organized around continuing the work of past PME teacher noticing working groups (PME-NA, 2013; PME-IG, 2014) and further expanding and focusing effort to address key challenges associated with the measurement of noticing skills.

Keywords: Assessment and Evaluation; Measurement; Teacher Education-Inservice (Professional Development); Teacher Education-Preservice

Introduction

High leverage teaching practices which facilitate the development of students' mathematical development have been the subject of considerable attention (Ball, Sleep, Boerst, & Bass, 2009, Cobb & Jackson, 2011; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). One such practice, professional noticing, involves "instruction that attends closely to children's ideas" (Ball & Forzani, 2011, p.19). Sometimes referred to as teacher noticing, professional noticing [of children's thinking] has been the subject of examination for some time (Erikson, 2011); however, over the past several years, this area of inquiry has experienced renewed interest among scholars (Fisher, Schack, Thomas, Jong, Eisenhardt, Yoder, & Tassell, 2014a; Jacobs, Lamb, & Philipp, 2010; Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder, 2013; Sherin, Jacobs, & Philipp, 2011; Thomas, Fisher, Eisenhardt, Schack, Tassell, & Yoder, 2014/2015).Moreover, professional noticing has been the topic for working group sessions at previous meetings of both the North America and International Groups of the Psychology of Mathematics Education organization (Jacobs, Sherin, & Philipp, 2013; Fisher, Schack, Wilhelm, Thomas, & McNall-Krall, 2014b). The purpose of this proposal is to provide a rationale for maintaining the momentum of past PME-NA working groups focused on this topic while providing the opportunity to extend the scope of activity into the measurement of professional noticing capacities.

History of Professional Noticing Working Group Activities

This working group was initiated by Jacobs, Sherin, and Philipp at the 2013 PME-NA conference in Chicago (2013). At the closing of this event, the working group had identified two areas upon which to focus continued efforts. These areas of focus were: 1) the planning and implementation of a conference on teacher noticing, and 2) the development of a strong internet presence regarding the topic of teacher noticing.

At the 2014 PME-IG conference in Vancouver, the working group continued to make progress on furthering these efforts as well as adding a third goal of the construction of a monograph on teacher noticing (Fisher et al., 2014b). At this conference three teams were formed around the three goals. The need for a fourth team arose during the working session to accommodate those participants who were seeking more information about the construct of teacher noticing.

The teacher noticing conference team decided to apply for funding to support a conference centered around teacher noticing research. This team plans to secure external funding (e.g., AERA Conference Grant), and their goal is to hold the conference as a pre-session to an established STEM education conference. With this manner of organization, attendees could combine travel efforts and attend both conferences in one visit.

The monograph team focused on discussions to extend mathematical teacher noticing to science and include an international context in a monograph. The facilitators of the working session had the good fortune to meet with Jinfa Cai, co-editor, with James Middleton, of the Springer series, *Research in Mathematics Education.* During this meeting we discussed with Dr. Cai the purpose of the monograph as defined by the working group participants. From this we developed a proposed outline of the monograph. The proposed monograph will build upon the work of Sherin, Jacobs, and Philipp's *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes* (2011). At the suggestion of Dr. Cai, we are working to include both seasoned and promising researchers/authors for the chapters of the monograph. Additionally, the chapter authors will include international mathematics and science education researchers. The sections of the book will include a commentary on the chapters within that section. We sent an initial invitation to some of the top researchers in this field, receiving a number of positive responses and chapter abstracts. We disseminated a call for chapter proposals from the participants in the working sessions at both the 2013 and 2014 conferences as well as a broader call for chapters. To date, we have received 31 chapter proposals. The first round of the review process for chapter proposals is underway with each chapter proposal being reviewed by three reviewers. We anticipate submitting a monograph proposal, with chapter recommendations, to the Springer series, *Research in Mathematics Education,* by early summer 2015.

The internet presence team discussed strategies such as a website on teacher noticing, social media groups for discussion, blogs or wikis to post information, and email listservs. At this point, the email listserv has been established that combines the 2013 working group attendees with the 2014 working group attendees and communication has begun with those groups. Additionally, a Facebook page (https://www.facebook.com/groups/182002372007275/) has been created to bolster teacher noticing discussions. Thus far it has been used for the monograph proposal and reviewer calls and will continue to be advertised and used for communications of the monograph and conference teams.

It is our intent that these three teams continue to work synergistically as the chapter authors in the monograph will be invited to speak at the conference, and presentations at the conference may be leveraged to inform monograph materials. All of these efforts will be advertised using social media outlets, email listservs, and websites that will be administered by the technology sub-group.

Measurement of Professional Noticing Capacities

In addition to continuing the processes initiated in previous teacher noticing working groups, this proposed 2015 working group would intentionally address the challenge of noticing measurement. Schack, Fisher, & Thomas (in press) note the varying manner in which noticing has been operationalized in a number of inquiries. For example, some researchers conceive of noticing as the interrelated skills of attending, interpreting, and deciding (Jacobs et al., 2010; Jacobs, Lamb, & Philipp, & Schappelle, 2011) while others focus solely on a single skill such as attending (Star, Lynch, & Perova, 2011).

Moreover, the temporal situation of teacher noticing also changes depending on the inquiry, with noticing positioned as an in-the-moment practice in some studies (Schack et al., 2013; Sherin, Russ, & Colestock, 2011; Stockero, 2014) while other research characterizes the practice of noticing in more retrospective terms (Goldsmith & Seago, 2011). Leatham, Peterson, Stockero, and Van Zoest (2015) characterized instances of student thinking that had significant potential to enhance student learning if teachers acted on them in-the-moment, rather than selecting and sequencing (Smith &

Stein, 2011) the event for later classroom discussion. From these disparities, it follows that measurement of teachers' noticing capacities also appears quite variable within the extant literature. Although the use of video-excerpts seems to be a common element among many measures (Berliner, 1994; Jacobs et al., 2010; Sherin & van Es, 2009; Star & Strickland, 2008), the manner in which such videos are used differs significantly. While some researchers develop inductive scoring rubrics for open-ended teacher responses to video excerpts (Schack et al., 2013), others employ automated scoring mechanisms for similarly open-ended responses (Kersting, Sherin, & Stigler, 2014). The length and context of the videos used for assessment also contributes to the variation in measures. For example, Schack et al. (2013) employed a brief individual interview video excerpt while others have used full class video in terms of both time and context. (Leatham et al., 2015; Star et al., 2011; Star & Strickland, 2008).

Further, some researchers adopt video-based measurement approaches predicated on highly specific progressions of learning (Schack et al., 2013) while others employ more generalized systems of measurement (Goldsmith & Seago, 2011; Santagata, 2011; van Es, 2011). Leatham et al. have focused on mathematically significant events (2015).

Kaiser, Buth, Hoth, König, and Blömeke (2015) have examined the theoretical complexities of video-based measures and identified specific challenges which include the following: 1) Issues related to video perspective (e.g., whether or not the teacher is included in the video or if the video is from the teachers' point of view), 2) Evaluating the "correctness" of responses (e.g., identifying viable theoretical lenses and measurement structures to accommodate unexpected yet "correct" responses), 3) Achieving acceptable psychometric quality (e.g., balancing reliability and validity concerns with reasonable measurement administration protocols and costs). While Kaiser et al., present potential solutions to these challenges, given the variability of scholarship with respect to teacher noticing, additional perspectives on the vexing challenges of measuring teachers' noticing capacities are worthy of exploration.

Plan for Engaging Professional Noticing Assessment Issues

To productively address issues of assessment of professional noticing identified in the previous section, the group will engage in both small and large group discussions and action plans. While an overview of the topics with research and practice-based examples will be presented, we want to primarily take an inductive approach by using the group's collective knowledge about and experience with the assessment issues. Thus, the plans for the sessions are as follows:

Session 1 will include a very brief overview of teacher noticing for participants new to the construct. The primary focus of this session, however, will be on methods and challenges of assessing or evaluating teacher noticing. Several tools, frameworks, rubrics, progressions from various contexts will be shared for discussion with the audience applying one or more to presented scenarios. The facilitators expect discussion around 1) in what contexts has teacher noticing been studied, 2) what tools, frameworks, rubrics, progressions have been used to notice teacher noticing, 3) are there tools that can reliably and validly assess teacher noticing, and 4) what are the potential benefits, challenges, and consequences to assessing teacher noticing? The specific agenda will be narrowed with input from working group participants.

Session 2 will begin with a quick synopsis of session 1, outlining the proposed agenda for studying assessment of teacher noticing. Based on the proposed agenda, participants will work in small groups to continue discussing and reflecting on a specific aspect of assessing professional noticing. This will allow for deeper reflections and connections within the smaller groups. By the end of Session 2, small groups will briefly report on their discussions, progress, and challenges of teacher noticing assessment to the larger group.

Session 3 will focus on developing an action plan for work beyond the conference. Small group goals and activities will be communicated beyond the conference through the already present Teacher Noticing Facebook page. The discussion and goals will inform a future teacher noticing conference for which the 2013/2014 working groups are in the early planning stages. Another possible outcome of the working group might be a compendium of methods used to assess teacher noticing.

Anticipated Follow-up Activities

The primary goal of this working group is to examine and further refine the research on assessment of teacher noticing with the ultimate goal of using such knowledge to support teachers in the development of their professional noticing for the benefit of students' mathematical understanding. The working group will create an environment where participants and small groups can network and create mutually beneficial collaborations that have the potential to further inform future teacher noticing research and practice. It will be encouraged for small groups to think of next steps that are appropriate for their topic, whether they be the development of a measurement tool or multi-site research projects. The group will continue to update information on professional noticing on the Internet, including the Facebook group. Participants will also be kept in communication regarding any updates on the professional noticing monograph, and potentially participating in a focused conference on professional noticing.

Extensions of Previous Working Group Activities

Assessing professional noticing capacities is an increasingly vexing challenge; thus, we think this focus is a timely way to extend previous working group activities. Kaiser, Busse, Hoth, König, & Blömeke (2015) recently highlighted the various theoretical and methodological complexities of video-based assessments of teachers' competence. Even when a theoretical framework guides the development of an assessment, specific content and pedagogical practices need to be selected (Leatham et al., 2015). Kaiser, et al. (2015) point out that there needs to be a balance between reliability and validity by taking feasibility into consideration in terms of the assessment length, grades, and number of classroom situations. They also argue that *noticing* can be linked to other concepts and knowledge frameworks because it has a very action-oriented view of teaching. While they viewed video-based assessments as promising, there were still challenges in developing questions, coding and eliciting responses that reflect practices. To scale up video-based assessments, Kersting, Sherin, & Stigler (2014) used a machine to automatically code short answer responses. The machine scores were highly correlated to human scores and provided insights into task clarity and scoring rubrics.

Teaching mathematics is complex; thus, assessing professional noticing poses many challenges. By engaging in meaningful conversations around assessment, the working group's collective knowledge will be able to get at strategies for addressing and overcoming some of the challenges to measuring noticing capacities.

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Keyword Index

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